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A Local Index Theorem for Families of $\bar{\partial}$-Operators on Punctured Riemann Surfaces and a New Kähler Metric on Their Moduli Spaces

L. A. Takhtajan and P. G. Zograf

Leningrad Branch of the Steklov Mathematical Institute, USSR Academy of Sciences, SU-191011 Leningrad, USSR

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Abstract. We prove a local index theorem for families of $\bar{\partial}$-operators on Riemann surfaces of type (g, n), i.e. of genus g with $n > 0$ punctures. We calculate the first Chern form of the determinant line bundle on the Teichmüller space $T_{g,n}$ endowed with Quillen's metric (where the role of the determinant of the Laplace operators is played by the values of the Selberg zeta function at integer points). The result differs from the case of compact Riemann surfaces by an additional term, which turns out to be the Kähler form of a new Kähler metric on the moduli space of punctured Riemann surfaces. As a corollary of this result we derive, for instance, an analog of Mumford's isomorphism in the case of the universal curve.

Introduction

The Atiyah-Singer index theorem for families of elliptic operators, which plays an important role in modern mathematical physics, is of particular interest for $\bar{\partial}$-operators on complex manifolds. Consider a holomorphic family $p : \mathscr{X} \to B$ of compact complex manifolds over a compact base B, and a holomorphic vector bundle $\mathscr{E} \to \mathscr{X}$. The family $\bar{\partial} = \{\bar{\partial}_b\}_{b \in B}$ of $\bar{\partial}$-operators in the vector bundles $E_b \to X_b$ (restrictions of $\mathscr{E}$ over the fibers $X_b = p^{-1}(b)$, $b \in B$) gives rise (in the sense of K-theory) to the index bundle $\operatorname{ind} \bar{\partial} \in K(B)$ on B with fibers $\ker \bar{\partial}_b - \operatorname{coker} \bar{\partial}_b$ over $b \in B$. The Atiyah-Singer index theorem applied to this special case states that

$$\operatorname{ch}(\operatorname{ind} \bar{\partial}) = p_*(\operatorname{ch} \mathscr{E} \cdot \operatorname{td} T_v \mathscr{X}) . \qquad (1)$$

Here ch denotes the Chern character, $\operatorname{td} T_v \mathscr{X}$ is the Todd class of the vertical tangent (along the fibers of $p : \mathscr{X} \to B$) bundle on $\mathscr{X}$, and $p_* : H^*(\mathscr{X}) \to H^{*-\dim X_b}(B)$ is the operation of "integration along the fibers" (see [1]).

In many applications the bundles $\mathscr{E}$ and $T_v \mathscr{X}$ are Hermitian, so that each of them carries the (canonical) unitary connection compatible with the holomorphic structure. Then by the Chern-Weil formulas $\operatorname{ch}\mathscr{E}$ and $\operatorname{td} T_v\mathscr{X}$ can be realized as closed differential forms on $\mathscr{X}$. When $\operatorname{ind}\bar\partial$ is also a vector bundle, it is quite natural to ask whether there is a connection in $\operatorname{ind}\bar\partial$ such that (1) holds as an equality of corresponding differential forms, i.e. locally on B. More generally, the question is how to express explicitly the Chern character form $\operatorname{ch}(\operatorname{ind}\bar\partial)$ on B in terms of the data $p : \mathscr{X} \to B$ and $\mathscr{E} \to \mathscr{X}$. Such a strong form of (1) is often called a local index theorem, and it makes sense in case of a non-compact base as well.

The situation becomes much simpler when we consider, instead of the index bundle $\operatorname{ind}\bar\partial$, its determinant bundle $\det\operatorname{ind}\bar\partial = \Lambda^{\max}\ker\bar\partial \otimes (\Lambda^{\max}\operatorname{coker}\bar\partial)^{-1}$ on B (where $\Lambda^{\max}$ denotes the maximal exterior power of a vector space). As it was observed by Quillen [2], under rather general assumptions on $p : \mathscr{X} \to B$ and $\mathscr{E} \to \mathscr{X}$ the determinant bundle $\det\operatorname{ind}\bar\partial$ is a holomorphic line bundle on B with a natural metric (Quillen's metric) given by

$$\|\cdot\|_Q = \|\cdot\|(\det\bar\partial^*\bar\partial)^{-1/2}; \tag{2}$$

here $\|\cdot\|$ is the ordinary L^2-norm in $\det\operatorname{ind}\bar\partial$ induced by the metrics in $\mathscr{E}$ and $T_v\mathscr{X}$, $\bar\partial_b^*\bar\partial_b$ is the Laplace operator acting on sections of $E_b \to X_b$, and $\det\bar\partial_b^*\bar\partial_b$ is its zeta function determinant regarded as a function on B. In [2] Quillen studied in detail the family of all Cauchy-Riemann operators (i.e. holomorphic structures in a Hermitian vector bundle $E \to X$ on a compact Riemann surface X). In this case B is an infinite dimensional complex affine space, and the curvature form of the Hermitian line bundle $(\det\operatorname{ind}\bar\partial; \|\cdot\|_Q)$ on B appears to be equal, up to a constant multiple, to the natural Kähler form on B.

A similar result was obtained somewhat later by Belavin and Knizhnik [3] (we follow [4] in exposing of their result). Denote by $B = T_g$ the Teichmüller space of compact Riemann surfaces of genus g, by $\mathscr{X} = \mathscr{T}_g \to T_g$ – the Teichmüller universal curve, by $T_v\mathscr{T}_g \to \mathscr{T}_g$ – the vertical line bundle of the fibration $p : \mathscr{T}_g \to T_g$, and by $\bar\partial_k$ – the family of $\bar\partial$-operators acting on k-differentials on Riemann surfaces (sections of $(T_v^{-k}\mathscr{T}_g)_b \to X_b$). Then for the first Chern form of the determinant line bundle $\det\operatorname{ind}\bar\partial_k$ endowed with Quillen's norm (2) the following formula holds:

$$c_1(\det\operatorname{ind}\bar\partial_k) = \int\limits_{\text{fiber}} (\operatorname{ch}(T_v^{-k}\mathscr{T}_g)\cdot\operatorname{td}(T_v\mathscr{T}_g))_{2,2}\,, \tag{3}$$

where $(\;)_{2,2}$ denotes the $(2,2)$-component of a differential form on $\mathscr{T}_g$, and the integration is taken over the fibers of $\mathscr{T}_g \to T_g$. As one can easily see, formula (3) is a specification of (1) on the level of $(1,1)$-forms. Moreover, if we consider a metric in $T_v\mathscr{T}_g$ which coincides on each fiber of $\mathscr{T}_g \to T_g$ with the Poincaré metric (i.e. Hermitian metric of constant curvature -1), then by a result of Wolpert [5] formula (3) can be rewritten as

$$c_1(\det\operatorname{ind}\bar\partial_k) = \frac{6k^2-6k+1}{12\pi^2}\omega_{\mathrm{WP}}\,, \tag{4}$$

where ω_{WP} is the Weil-Petersson Kähler form on T_g. Such a form of local index theorem is analogous to Quillen's original result. On the other hand, formula (4) can be derived by the methods of Teichmüller theory (see [6]) avoiding a difficult heat kernel technique. The approach of [6] also allows us to prove a local index theorem for families of $\bar\partial$-operators in stable bundles of rank n and degree k on a compact Riemann surface [7] (which provides, for instance, an analytic proof of projectivity of the corresponding moduli spaces for coprime n and k).

Quillen's local index theorem admits various deep generalizations (see [8–10] for details), but in all of them the fibers of $p : \mathscr{X} \to B$ are compact manifolds. Otherwise considerable difficulties occur because of continuous spectrum of the corresponding Laplace operators (along with purely technical difficulties there is a problem how to define properly the determinant of the Laplace operator). However, the methods of [6] work for families of $\bar{\partial}$-operators on non-compact Riemann surfaces as well. The first examples of a local index theorem for families with non-compact fibers were given in our papers [11, 12]. Namely, we have considered the case when $B = T_{g,n}$ is the Teichmüller space of Riemann surfaces of type (g, n), i.e. of genus g with n punctures (cusps), $\mathscr{X} = \mathscr{T}_{g,n}$ is the corresponding universal family (so that the fibers of the fibration $p : \mathscr{T}_{g,n} \to T_{g,n}$ are Riemann surfaces of type (g, n)), and $\mathscr{E} = T_v^{-k}\mathscr{T}_{g,n}$ is the k^{th} power of the vertical line bundle on $\mathscr{T}_{g,n}$ (the case $k = 0, 1$ was treated in [11], and the case $k \geq 2$ in [12]). In this situation the Laplace operator $\Delta_k = \bar{\partial}_k^* \bar{\partial}_k$ associated with the Poincaré metric (i.e. complete Hermitian metric of constant curvature -1) on a Riemann surface X of type (g, n) has n-fold continuous spectrum. To define a regularized determinant of Δ_k consider the Selberg zeta function $Z(s)$ which is given for $\operatorname{Re} s > 1$ by the absolutely convergent product

$$Z(s) = \prod_{\{\ell\}} \prod_{m=0}^{\infty} (1 - e^{-(s+m)|\ell|}), \tag{5}$$

where ℓ runs over the set of all simple closed geodesics on X with respect to the Poincaré metric, and $|\ell|$ is the length of ℓ. The function $Z(s)$ admits a meromorphic continuation to the whole complex s-plane with a simple zero at $s = 1$. For compact Riemann surfaces it was shown in [13] that the determinant of Δ_k defined via its zeta function is equal, up to a constant multiplier depending only on g and k, to $Z'(1)$ for $k = 0, 1$ and $Z(k)$ for $k \geq 2$. Similarly, for Riemann surfaces of an arbitrary type (g, n) we define

$$\det \Delta_k = \begin{cases} Z'(1), & k = 0, 1, \\ Z(k), & k \geq 2. \end{cases} \tag{6}$$

Using this definition, we can calculate the first Chern form of the determinant line bundle $\lambda_k = \det \operatorname{ind} \bar{\partial}_k$ on $T_{g,n}$ endowed with Quillen's metric (2). The result differs from (4) by an additional term in the right-hand side:

$$c_1(\lambda_k) = \frac{6k^2 - 6k + 1}{12\pi^2} \omega_{\mathrm{WP}} - \frac{1}{9} \omega_{\mathrm{cusp}}, \tag{7}$$

where ω_{WP} is the Weil-Petersson Kähler form on $T_{g,n}$ and ω_{cusp} is the symplectic form of a new Kähler metric $\langle , \rangle_{\mathrm{cusp}}$ on $T_{g,n} (n \neq 0)$.

We proceed with exact definitions. Let X be a Riemann surface of type (g, n) equipped with the Poincaré metric ϱ and let Γ be a torsion-free Fuchsian group uniformizing X, i.e. $X \cong \Gamma \backslash H$, where $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is the upper half-plane. Denote by $\Gamma_1, \ldots, \Gamma_n$ the set of non-conjugate parabolic subgroups in Γ, and for every $i = 1, \ldots, n$ fix an element $\sigma_i \in \mathrm{PSL}(2, \mathbb{R})$ such that $\sigma_i^{-1} \Gamma_i \sigma_i = \Gamma_\infty$ where the group Γ_∞ is generated by the parabolic transformation $z \mapsto z + 1$. The

Eisenstein-Maass series $E_i(z,s)$ corresponding to the i^{th} cusp of the group Γ is defined for $\operatorname{Re} s > 1$ by the formula

$$E_i(z,s) = \sum_{\gamma\in\Gamma_i\backslash\Gamma} \operatorname{Im}(\sigma_i^{-1}\gamma z)^s, \qquad i = 1, \dots, n.$$

($E_i(z,s)$ can be meromorphically continued to the whole complex s-plane; for $\operatorname{Re} s = \frac{1}{2}$ the Eisenstein-Maass series $E_i(z,s)$, $i = 1, \dots, n$, form a complete set of eigenfunctions of the continuous spectrum of the Laplace operator Δ_0.) Recall also that the tangent space $T_{[X]}T_{g,n}$ to the Teichmüller space $T_{g,n}$ at the point corresponding to a Riemann surface X can be naturally identified with the space $\Omega^{-1,1}(X)$ of harmonic L^2-tensors on X of type $(-1,1)$ (the Beltrami differentials). The Weil-Petersson metric on $T_{g,n}$ is defined by the formula

$$\langle\mu,\nu\rangle_{\mathrm{WP}} = \int_X \mu\bar{\nu}\varrho,$$

where μ, $\nu \in \Omega^{-1,1}(X)$ are considered as a tangent vectors. Denote by ω_{WP} its Kähler form;

$$\omega_{\mathrm{WP}}(\mu,\bar{\nu}) = \frac{\sqrt{-1}}{2}\langle\mu,\nu\rangle_{\mathrm{WP}}.$$

To define the metric $\langle\,,\rangle_{\mathrm{cusp}}$ set

$$\langle\mu,\nu\rangle_i = \int_X \mu\bar{\nu}E_i(\cdot,2)\varrho, \qquad \mu,\nu \in \Omega^{-1,1}(X), \qquad i = 1, \dots, n. \tag{8}$$

Each scalar product $\langle\,,\rangle_i$ gives rise to a Kähler metric on $T_{g,n}$ (see Sect. 2 below). Their sum

$$\langle\,,\rangle_{\mathrm{cusp}} = \sum_{i=1}^{n}\langle\,,\rangle_i \tag{9}$$

is $\mathrm{Mod}_{g,n}$-invariant Kähler metric on $T_{g,n}$ (where $\mathrm{Mod}_{g,n}$ denotes the Teichmüller modular group) with Kähler form ω_{cusp}.

For families of $\bar{\partial}$-operators on punctured Riemann surfaces formula (3) is not valid. We calculate the cuspidal defect

$$\delta_k^{(1)} = c_1(\lambda_k) - \int_{\text{fiber}} (\operatorname{ch}(T_v^{-k}\mathscr{T}_{g,n})\cdot \operatorname{td}(T_v\mathscr{T}_{g,n}))_{2,2}, \tag{10}$$

which appears to be equal to $-\frac{1}{9}\,\omega_{\mathrm{cusp}}$.

Formula (7) has also algebraic geometry consequences. In particular, it gives for the relative dualizing sheaf ω on the universal curve $\mathscr{C}_g = \mathscr{M}_{g,1} = T_{g,1}/\mathrm{Mod}_{g,1}$ the following expression:

$$c_1(\omega) = \frac{4}{3}\,\omega_{\mathrm{cusp}}, \tag{11}$$

and provides an analytic proof of the isomorphism

$$\lambda_k \cong \lambda_1^{6k^2-6k+1} \otimes \omega^{\frac{k(k-1)}{2}}, \tag{12}$$

where λ_k is also considered as a sheaf of $\mathscr{C}_g$. The last formula is an analog of Mumford's isomorphism $\lambda_k \cong \lambda_1^{6k^2-6k+1}$ on the moduli space $\mathscr{M}_g = T_g/\mathrm{Mod}_g$ [14].

The content of this paper is the following. In Sect. 1 we recall necessary facts from the theory of automorphic forms and the spectral theory of the Laplace operator on punctured Riemann surfaces. In Sect. 2 we present basic facts about Teichmüller spaces together with necessary variational formulas, which allow us, in particular, to prove that the metrics $\langle , \rangle_i$ are Kählerian. In Sect. 3 we obtain a formula for the first derivatives of the Selberg zeta function with respect to coordinates on $T_{g,n}$. In Sect. 4 we prove our main result – formula (7). In Sect. 5 we calculate the cuspidal defect (10) in the Atiyah-Singer index theorem and derive from (7) some algebraic geometry consequences.

1. Laplacians on a Punctured Riemann Surface

Let X be a Riemann surface of type (g,n), i.e. $X = \bar{X}\backslash\{x_1, \ldots, x_n\}$, where $\bar{X}$ is a compact Riemann surface of genus g and $x_1, \ldots, x_n$ are pairwise distinct points on $\bar{X}$; we will assume that $2g+n \geq 3$. Then X can be represented as a quotient $\Gamma\backslash H$ of the upper half-plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by the action of a torsion-free finitely generated Fuchsian group Γ. The group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots, A_g, B_g$ and n parabolic transformations $S_1, \ldots, S_n$ satisfying the single relation $A_1B_1A_1^{-1}B_1^{-1}\ldots A_gB_gA_g^{-1}B_g^{-1}S_1\ldots S_n = 1$. The fixed points of the parabolic elements $S_1, \ldots, S_n$ (cusps) will be denoted by $z_1, \ldots, z_n$ respectively. The "images" of the cusps $z_1, \ldots, z_n \in \mathbb{R}\cup\{\infty\}$ under the projection $H \to \Gamma\backslash H \cong X$ are the punctures $x_1, \ldots, x_n \in \bar{X}$. For each $i = 1, \ldots, n$ denote by Γ_i the cyclic subgroup in Γ generated by S_i and choose an element $\sigma_i \in \mathrm{PSL}(2,\mathbb{R})$ such that $\sigma_i\infty = z_i$ and $\sigma_i^{-1}S_i\sigma_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$.

A smooth complex valued function f on H is called an automorphic form of weight $(2\ell, 2m)$ with respect to the group Γ if for any $z \in H$ and $\gamma \in \Gamma$,

$$f(\gamma z)\gamma'(z)^{\ell}\overline{\gamma'(z)}^m = f(z), \qquad (\ell, m \in \mathbb{Z}),$$

(forms of weight $(2\ell, 2m)$ correspond to tensors of type (ℓ, m) on the Riemann surface $X \cong \Gamma\backslash H$). Let $\varrho(z)|dz|^2$ denote the Poincaré metric $y^{-2}(dx^2+dy^2)$ on the upper half-plane H. We denote by $\mathscr{H}^{\ell,m}$ the Hilbert space of automorphic forms of weight $(2\ell, 2m)$ with the natural scalar product

$$\langle f_1, f_2\rangle = \int_X f_1\overline{f_2}\varrho^{-\ell-m+1} = \iint_{\Gamma\backslash H} f_1(z)\overline{f_2(z)}y^{2\ell+2m-2}dx\,dy, \tag{1.1}$$

associated with the Poincaré metric.

For each integer ℓ we consider the Laplace operator $\Delta_\ell = \bar\partial_\ell^*\bar\partial_\ell = -\varrho^{\ell-1}\partial\varrho^{-\ell}\bar\partial$ in the Hilbert space $\mathscr{H}^\ell = \mathscr{H}^{\ell,0}$. Here $\bar\partial_\ell = \bar\partial = \dfrac{\partial}{\partial\bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \sqrt{-1}\dfrac{\partial}{\partial y}\right)$ is considered as an operator from $\mathscr{H}^\ell$ to $\mathscr{H}^{\ell,1}$, and $\bar\partial_\ell^* = -\varrho^{\ell-1}\partial\varrho^{-\ell}$ is the adjoint operator to $\bar\partial_\ell$ in the sense of the scalar product (1.1), acting from $\mathscr{H}^{\ell,1}$ to $\mathscr{H}^\ell$, where $\partial = \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \sqrt{-1}\dfrac{\partial}{\partial y}\right)$. The operator Δ_ℓ is self-adjoint and non-negative in $\mathscr{H}^\ell$. Denote by Ω^ℓ the subspace $\ker\Delta_\ell = \ker\bar\partial_\ell$ in $\mathscr{H}^\ell$, consisting of holomorphic cusp forms of weight 2ℓ. Recall that a holomorphic automorphic form f is called regular if at each cusp z_i is has the following Fourier expansion:

$$f(\sigma_i z)\sigma_i'(z)^\ell = \sum_{k=0}^{\infty} a_k^{(i)} e^{2\pi\sqrt{-1}kz}, \qquad i = 1, \dots, n.$$

If, moreover, $a_0^{(1)} = \dots = a_0^{(n)} = 0$, f is called a cusp form.

Holomorphic cusp forms of weight 2ℓ correspond to meromorphic $(\ell, 0)$-tensors (ℓ-differentials) on a Riemann surface $\bar{X}$ (i.e. meromorphic sections of a line bundle $(T\bar{X})^{-\ell} \to \bar{X}$) which have poles of order not exceeding $\ell - 1$ at the punctures $x_1, \dots, x_n$ and are holomorphic on $X = \bar{X}\backslash\{x_1, \dots, x_n\}$. The Riemann-Roch theorem gives

$$\dim_{\mathbb{C}} \Omega^\ell = \begin{cases} 0, & \ell \le -1, \\ 1, & \ell = 0, \\ g, & \ell = 1, \\ (2\ell - 1)(g - 1) + (\ell - 1)n, & \ell \ge 2. \end{cases}$$

The subspace $\Omega^{\ell,1} = \ker \bar{\partial}_\ell^* = \operatorname{coker} \bar{\partial}_\ell$ in $\mathscr{H}^{\ell,1}$ is the Kodaira-Serre dual of $\Omega^{1-\ell} = \ker \bar{\partial}_{1-\ell}$. We denote by $P_{\ell,1}$ the orthogonal projection of $\mathscr{H}^{\ell,1}$ onto $\Omega^{\ell,1}$. For $\ell < 0$ we have the formula

$$P_{\ell,1} = I - \bar{\partial}_\ell \Delta_\ell^{-1} \bar{\partial}_\ell^* , \tag{1.2}$$

where I is the identity operator in $\mathscr{H}^{\ell,1}$. Moreover, from the equation

$$\Delta_0 \log \varrho = -\frac{1}{2},$$

which means that the Poincaré metric $\varrho|dz|^2$ has a constant negative curvature -1, it follows that

$$\Delta_\ell \bar{\partial}_\ell^* \varrho = \bar{\partial}_\ell^* \varrho \left(\Delta_{\ell-1} + \frac{\ell - 1}{2} \right). \tag{1.3}$$

Now denote by $Q_s^{(\ell)}(z, z')$ the resolvent kernel of the Laplace operator Δ_ℓ on the upper half-plane H; it means that $Q_s^{(\ell)}(z, z')$ is the kernel of the operator $(\Delta_\ell + \frac{1}{4}(s - 2\ell)(s - 1))^{-1}$ (we assume that $\ell \le 0$, $\operatorname{Re} s \ge 1$). The kernel $Q_s^{(\ell)}(z, z')$ is smooth for $z \ne z'$ and is holomorphic in s on the whole complex s-plane. It has an important property that $Q_s^{(\ell)}(\sigma z, \sigma z') = Q_s^{(\ell)}(z, z')$ for any $\sigma \in \mathrm{PSL}(2, \mathbb{R})$ and $z, z' \in H$. For $\ell = 0$ the kernel $Q_s^{(\ell)}$ is given by the explicit formula (cf. [15])

$$Q_s^{(0)}(z, z') = \frac{\Gamma(s)^2}{\pi \Gamma(2s)} \left(1 - \left| \frac{z - z'}{\bar{z} - z'} \right|^2 \right)^s F\left(s, s, 2s; 1 - \left| \frac{z - z'}{\bar{z} - z'} \right|^2 \right), \tag{1.4}$$

where $F(a, b, c; z)$ is the hypergeometric function and $\Gamma(s)$ is the gamma function. At $s = 1$ one has

$$Q_1^{(0)}(z, z') = -\frac{2}{\pi} \log \left| \frac{z - z'}{\bar{z} - z'} \right|.$$

Without writing an explicit expression for $Q_s^{(\ell)}$ with $\ell \le -1$ we give a simple formula

$$y'^{2\ell} \frac{\partial}{\partial z'} Q_1^{(\ell)}(z, z') = \frac{1}{\pi} \cdot \frac{1}{z - z'} \left(\frac{\bar{z} - z}{\bar{z} - z'} \right)^{1-2\ell} \tag{1.5}$$

(see, e.g., [6, Formula (1.6)]).

We denote by $G_s^{(\ell)}(z,z')$, $\ell \leq 0$, $\operatorname{Re} s \geq 1$ the resolvent kernel of the Laplace operator, i.e. the kernel of the operator $(\Delta_\ell + \frac{1}{4}(s-2\ell)(s-1))^{-1}$ on the Riemann surface $X = \Gamma\backslash H$. For $\operatorname{Re} s > 1$ and $z \neq \gamma z'$, $\gamma \in \Gamma$, the kernel $G_s^{(0)}$ is given by the absolutely convergent series

$$G_s^{(0)}(z,z') = \sum_{\gamma\in\Gamma} Q_s^{(0)}(z,\gamma z'), \tag{1.6}$$

which admits term-by-term differentiation with respect to the variables z and z'. The kernel $G_s^{(0)}(z,z')$ with $z \neq \gamma z'$, $\gamma \in \Gamma$, admits a meromorphic continuation in s to the entire complex s-plane and has the following Laurent expansion at $s = 1$:

$$G_s^{(0)}(z,z') = \frac{1}{\pi\left(g+\frac{n}{2}-1\right)} \cdot \frac{1}{s(s-1)} + G_1^{(0)}(z,z') + \underset{s\to 1}{O}(s-1)$$

(see [15, Theorem 2.3]). The kernel $G_1^{(0)}(z,z')$ is called the Green's function of the Laplacian Δ_0 on the Riemann surface $X = \Gamma\backslash H$. One has (see [15, p. 161])

$$-\pi \frac{\partial^2}{\partial z\, \partial z'} G_1^{(0)}(z,z') = -\pi \lim_{s\to 1} \frac{\partial^2}{\partial z\, \partial z'} G_s^{(0)}(z,z') = \Omega(z,z'), \tag{1.7}$$

where $\Omega(z,z')$ is the so-called Schiffer kernel. It is defined as a symmetric bidifferential of the second kind on $\bar{X} \times \bar{X}$ with a double pole of biresidue 1 at the diagonal $z = z'$ and the property

$$\text{v.p.} \iint_{\Gamma\backslash H} \Omega(z,z')\,\overline{\omega(z')}\,dx'\,dy' = 0$$

for every $\omega \in \Omega^1$. The Schiffer kernel does not depend on a marking of X (i.e. on a choice of generators of the Fuchsian group Γ). Moreover, the following formula holds [15, p. 160]:

$$\Omega(z,z') = B(z,z') - \pi \sum_{i,j=1}^{g} (\operatorname{Im}\tau)_{ij}^{-1} \omega_i(z)\omega_j(z'). \tag{1.8}$$

The kernel $B(z,z')$ is the uniquely determined symmetric bidifferential of the second kind on $\bar{X} \times \bar{X}$ with a double pole of biresidue 1 at the diagonal $z = z'$ and zero A-periods with the property that

$$\int_z^{B_i z} B(z,z')\,dz' = 2\pi\sqrt{-1}\,\omega_i(z), \qquad i = 1, \ldots, g;$$

here $\omega_1, \ldots, \omega_g \in \Omega^1$ is the normalized basic of abelian differentials on X, i.e.

$$\int_z^{A_i z} \omega_j(z')\,dz' = \delta_{ij}, \qquad i,j = 1, \ldots, g,$$

where δ_{ij} is the Kronecker symbol. Denote by $\tau = (\tau_{ij})$,

$$\tau_{ij} = \int_{z}^{B_i z} \omega_j(z')\, dz', \qquad i, j = 1, \ldots, g,$$

the period matrix of the marked Riemann surface X. It has a symmetric positive definite imaginary part $\operatorname{Im}\tau = (\operatorname{Im}\tau_{ij})$ (which enters in (1.8)) with the property

$$\operatorname{Im}\tau_{ij} = \int_X \omega_i \bar{\omega}_j, \qquad i, j = 1, \ldots, g.$$

The Green's function of the Laplacian Δ_ℓ with $\ell \leq -1$ is defined simply as $G_1^{(\ell)}(z, z')$; its derivative $\dfrac{\partial}{\partial z'} G_1^{(\ell)}(z, z')$ is given by the absolutely convergent series

$$y'^{2\ell} \frac{\partial}{\partial z'} G_1^{(\ell)}(z, z') = \frac{1}{\pi} \sum_{\gamma \in \Gamma} \frac{1}{z - \gamma z'} \left(\frac{\bar{z} - z}{\bar{z} - \gamma z'} \right)^{1-2\ell} \gamma'(z')^{1-\ell} \tag{1.9}$$

(see, e.g., [6, Formula (1.8)]).

Now for $\ell \leq 0$ set

$$R^{(\ell)}(z, z') = -\frac{\partial}{\partial z} y^{2\ell} \frac{\partial}{\partial z'} (G_1^{(\ell)}(z, z') - Q_1^{(\ell)}(z, z')). \tag{1.10}$$

The following lemma will be used in Sect. 4 in the proof of Theorem 1.

Lemma 1. *The restriction $R^{(\ell)}|_D(z) = R^{(\ell)}(z, z)$ of the kernel $R^{(\ell)}$ on the diagonal $D = \{z = z'\}$ in $H \times H$ is a smooth automorphic form on H of weight 4 with respect to the group Γ whose constant term of the Fourier expansion at each cusp z_i is equal to $\dfrac{\pi}{3}$, i.e.*

$$R^{(\ell)}|_D(\sigma_i z)\sigma_i'(z)^2 = \frac{\pi}{3} + \underset{y \to \infty}{o(1)}.$$

Proof. Due to the Corollary 3.5 in [15]

$$R^{(\ell)}|_D(\sigma_i z)\sigma_i'(z)^2 = -\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\partial}{\partial z} y^{2\ell} \frac{\partial}{\partial z'} Q_1^{(\ell)}(z, z' + k)|_{z'=z} + \underset{y \to \infty}{o(1)}.$$

Using the formula

$$\frac{\partial}{\partial z} y^{2\ell} \frac{\partial}{\partial z'} Q_1^{(\ell)}(z, z') = -\frac{1}{\pi} \cdot \frac{1}{(z - z')^2} \left(\frac{z' - \bar{z}'}{\bar{z} - z'} \right)^{-2\ell},$$

which follows from (1.5), we obtain

$$R^{(\ell)}|_D(\sigma_i z)\sigma_i'(z)^2 = \frac{1}{\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k^2} \left(\frac{2\sqrt{-1}y}{2\sqrt{-1}y + k} \right)^{-2\ell} + o(1) = \frac{\pi}{3} + \underset{y \to \infty}{o(1)}.$$

Recall that the Eisenstein-Maass series $E_i(z, s)$ corresponding to the cusp z_i of the Fuchsian group Γ is defined for $\operatorname{Re} s > 1$ by the absolutely convergent series

$$E_i(z, s) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} \operatorname{Im}(\sigma_i^{-1} \gamma z)^s.$$

It satisfies the differential equation

$$\Delta_0 E_i(z,s) = \frac{s(1-s)}{4}\, E_i(z,s)$$

and has the asymptotic expansion

$$E_i(\sigma_j z, s) = \delta_{ij} y^s + \varphi_{ij}(s) y^{1-s} + \underset{y\to\infty}{O}(e^{-\pi y})$$

near each cusp z_j, $j = 1, \ldots, n$ (see [15, 16]). For any μ, $v \in \Omega^{-1,1}$ set $f_{\mu\bar{v}} = (\Delta_0 + \frac{1}{2})^{-1}(\mu\bar{v}) \in \mathscr{H}^0$. The result below will play an important role in the proof of Theorem 1 of Sect. 4.

Lemma 2. *The function $f_{\mu\bar{v}}$ has the following asymptotics near the cusp z_i of the group Γ:*

$$f_{\mu\bar{v}}(\sigma_i z) = \frac{c_{\mu\bar{v}}^{(i)}}{y} + \textit{exponentially decreasing terms as } y \to \infty,$$

where

$$c_{\mu\bar{v}}^{(i)} = \frac{4}{3}\int_X E_i(\cdot, 2)\mu\bar{v}\varrho, \qquad i = 1, \ldots, n.$$

Proof. Since μ, $v \in \Omega^{-1,1}$, one has $\mu = y^2\bar{\varphi}$, $v = y^2\bar{\psi}$ for some cusp forms φ, $\psi \in \Omega^2$, and hence the function $\mu\bar{v} \in \mathscr{H}^0$ is exponentially decreasing at the cusps $z_1, \ldots, z_n$. Let

$$f_{\mu\bar{v}}(\sigma_i z) = \sum_{k=-\infty}^{\infty} a_k^{(i)}(y)\, e^{2\pi\sqrt{-1}kx}$$

be the Fourier expansion of the function $f_{\mu\bar{v}}$ at the cusp z_i, $i = 1, \ldots, n$. Because $(\Delta_0 + \frac{1}{2}) f_{\mu\bar{v}} = \mu\bar{v}$, each function

$$\frac{d^2 a_k^{(i)}}{dy^2} + \left(4\pi^2 k^2 - \frac{2}{y^2}\right) a_k^{(i)}$$

is exponentially decreasing as $y \to \infty$. The equation

$$\frac{d^2 a}{dy^2} + \left(4\pi^2 k^2 - \frac{2}{y^2}\right) a = 0$$

has a pair of linearly independent solutions $\dfrac{1}{y}$, y^2 when $k = 0$, and

$$\sqrt{y}K_{3/2}(2\pi|k|y) \underset{y\to\infty}{\sim} e^{-2\pi|k|y}, \qquad \sqrt{y}I_{3/2}(2\pi|k|y) \underset{y\to\infty}{\sim} e^{-2\pi|k|y}$$

when $k \neq 0$. Since $f_{\mu\bar{v}} \in \mathscr{H}^0$, increasing solutions cannot occur in the Fourier expansion of $f_{\mu\bar{v}}$ and we immediately obtain for $f_{\mu\bar{v}}$ the above asymptotics.

To evaluate the coefficients $c_{\mu\bar{v}}^{(i)}$ let us use the differential equation $\Delta_0 E_i(z, 2) = -\frac{1}{2} E_i(z, 2)$. Denote by F a canonical fundamental domain of the group Γ in H

with exactly n cusps at the points $z_1, \ldots, z_n$, and set $F^Y = \{z \in F \mid \operatorname{Im}(\sigma_i^{-1} z) \leq Y, i = 1, \ldots, n\}$. We get from Green's formula that

$$\int\limits_X E_i(\cdot, 2)\mu\bar{\nu}\varrho = \iint\limits_{\Gamma\backslash H} (E_i(z, 2)\Delta_0 f_{\mu\bar{\nu}} - \Delta_0 E_i(z, 2) f_{\mu\bar{\nu}}) \frac{dx\,dy}{y^2}$$
$$= \lim_{Y\to\infty} \frac{1}{4} \int\limits_{\partial F^Y} \left\{ E_i(z, 2)\left(\frac{\partial f_{\mu\bar{\nu}}}{\partial y} dx - \frac{\partial f_{\mu\bar{\nu}}}{\partial x} dy\right) \right.$$
$$\left. - f_{\mu\bar{\nu}}\left(\frac{\partial}{\partial y} E_i(z, 2) dx - \frac{\partial}{\partial x} E_i(z, 2) dy\right)\right\}.$$

The last integral can be easily evaluated in terms of Fourier coefficients of the functions $f_{\mu\bar{\nu}}$ and $E_i(\cdot, 2)$:

$$\int\limits_X E_i(\cdot, 2)\mu\bar{\nu}\varrho = -\lim_{Y\to\infty} \frac{1}{4} \sum_{j=1}^n \left(\delta_{ij} Y^2 \cdot \left(-\frac{c_{\mu\bar{\nu}}^{(i)}}{Y^2}\right) - 2\delta_{ij} Y \cdot \frac{c_{\mu\bar{\nu}}^{(i)}}{Y}\right) + \underset{Y\to\infty}{o(1)} = \frac{3}{4} c_{\mu\bar{\nu}}^{(i)},$$

which completes the proof.

2. The Teichmüller Theory and Variational Formulas

Let $T_{g,n}$ be the Teichmüller space of marked Riemann surfaces of genus g with n punctures (we identify it with the Teichmüller space of the marked Fuchsian group Γ uniformizing the Riemann surface X). The Teichmüller space $T_{g,n}$ admits a natural structure of a complex manifold of dimension $3g - 3 + n$. For its description consider in the Hilbert space $\mathscr{H}^{-1,1}(X)$ the subspace $\Omega^{-1,1}(X)$ of harmonic Beltrami differentials; each element $\mu \in \Omega^{-1,1}(X)$ has a form $\mu = y^2\bar{\varphi}$, $\varphi \in \Omega^2(X)$, so $\dim_{\mathbb{C}} \Omega^{-1,1}(X) = 3g - 3 + n$. The space $\Omega^{-1,1}(X)$ is naturally isomorphic to the tangent space $T_{[X]}T_{g,n}$ of the Teichmüller space $T_{g,n}$ at the point $[X]$ representing the (marked) Riemann surface X. In turn, the cotangent space $T_{[X]}^* T_{g,n}$ can be identified with the space $\Omega^2(X)$ of quadratic differentials on X, which is dual to $\Omega^{-1,1}(X)$ with respect to the pairing

$$(\mu, \varphi) = \int\limits_X \mu\varphi, \qquad \mu \in \Omega^{-1,1}(X), \qquad \varphi \in \Omega^2(X).$$

For every $\mu \in \Omega^{-1,1}(X)$ with

$$\|\mu\|_\infty = \sup_{z\in H} |\mu(z)| < 1$$

there exists a unique diffeomorphism $f^\mu : H \to H$ satisfying the Beltrami equation

$$\frac{\partial f^\mu}{\partial \bar{z}} = \mu \frac{\partial f^\mu}{\partial z}$$

and fixing the points $0, 1, \infty$.

Set $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$ and $X^\mu = \Gamma^\mu \backslash H$. Choose a basis $\mu_1, \ldots, \mu_{3g-3+n}$ in the linear space $\Omega^{-1,1}(X)$ and let $\mu = \varepsilon_1\mu_1 + \ldots + \varepsilon_{3g-3+n}\mu_{3g-3+n}$. Then the correspondence $(\varepsilon_1, \ldots, \varepsilon_{3g-3+n}) \mapsto [X^\mu]$ defines complex coordinates in a neighbourhood of the point $[X] \in T_{g,n}$. They are called the Bers coordinates. In the overlapping neighbourhoods of two points $[X]$ and $[X^\mu]$ the Bers coordinates transform complex analytically. The differential of this coordinate change at the point $[X] \in T_{g,n}$ is a linear map $D^\mu : \Omega^{-1,1}(X) \to \Omega^{-1,1}(X^\mu)$,

$$\nu \mapsto D^\mu \nu = P^\mu_{-1,1}\left(\left(\frac{\nu}{1-|\mu|^2}\cdot\frac{\partial f^\mu}{\partial z}\Big/\overline{\left(\frac{\partial f^\mu}{\partial z}\right)}\right)\circ (f^\mu)^{-1}\right),$$

where $P^\mu_{-1,1}$ is the orthogonal projection of $\mathscr{H}^{-1,1}(X^\mu)$ onto $\Omega^{-1,1}(X^\mu)$. With the Bers coordinates $(\varepsilon_1, \ldots, \varepsilon_{3g-3+n})$ in a neighbourhood of the point $[X] \in T_{g,n}$ one can associate $3g-3+n$ vector fields $\dfrac{\partial}{\partial \varepsilon_i}$. At any other point $[X^\mu] \in T_{g,n}$ in this neighbourhood they are represent by the Beltrami differentials $D^\mu \mu_i \in \Omega^{-1,1}(X^\mu)$, $i = 1, \ldots, 3g-3+n$. Further details can be found in [17, 5].

Due to the isomorphism $T_{[X]}T_{g,n} \cong \Omega^{-1,1}(X)$, the scalar product (1.1) defines a Hermitian metric on the Teichmüller space $T_{g,n}$, which is called the Weil-Petersson metric. This metric is Kählerian [17], and its symplectic form will be denoted by ω_{WP};

$$\omega_{\mathrm{WP}}\left(\frac{\partial}{\partial \varepsilon_\mu}, \frac{\partial}{\partial \bar\varepsilon_\nu}\right) = \frac{\sqrt{-1}}{2}\langle \mu, \nu\rangle$$

at the point $[X] \in T_{g,n}$.

In a similar manner the scalar product

$$\langle \mu, \nu\rangle_i = \int_X E_i(\cdot, 2)\mu\bar\nu\varrho, \qquad i = 1, \ldots, n,$$

in $\Omega^{-1,1}(X)$ defines a Hermitian metric on the Teichmüller space $T_{g,n}$, $n > 0$. It turns out that for each $i = 1, \ldots, n$ this metric is also Kählerian (see Lemma 3 below).

Now let us recall the necessary variational formulas. Let $\omega^\varepsilon \in \mathscr{H}^{\ell,m}(X^{\varepsilon\mu})$ be a smooth family of automorphic forms of weight $(2\ell, 2m)$ (i.e. tensors of type (ℓ, m) on the family $X^{\varepsilon\mu} = \Gamma^{\varepsilon\mu}\backslash H$ of Riemann surfaces), where $\mu \in \Omega^{-1,1}(X)$, and $\varepsilon \in \mathbb{C}$ is sufficiently small. We set

$$(f^{\varepsilon\mu})^*(\omega^\varepsilon) = \omega^\varepsilon \circ f^{\varepsilon\mu}\left(\frac{\partial f^{\varepsilon\mu}}{\partial z}\right)^\ell \left(\overline{\frac{\partial f^{\varepsilon\mu}}{\partial z}}\right)^m \in \mathscr{H}^{\ell,m}(X).$$

The Lie derivatives of the family ω^ε in holomorphic and anti-holomorphic tangential directions μ and $\bar\mu$ are defined as follows:

$$L_\mu\omega = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}(f^{\varepsilon\mu})^*(\omega^\varepsilon) \in \mathscr{H}^{\ell,m}(X),$$
$$L_{\bar\mu}\omega = \frac{\partial}{\partial\bar\varepsilon}\bigg|_{\varepsilon=0}(f^{\varepsilon\mu})^*(\omega^\varepsilon) \in \mathscr{H}^{\ell,m}(X).$$

For the density $\varrho(z) = y^{-2}$ of the Poincaré metric, considered as a family of $(1,1)$-tensors, one has

$$L_\mu\varrho = L_{\bar\mu}\varrho = 0 \tag{2.1}$$

for any $\mu \in \Omega^{-1,1}(X)$ (see [17]). For the second variation of ϱ the following formula was obtained in [5]:

$$L_{\mu\bar{\nu}}\varrho = \frac{\partial^2}{\partial\varepsilon_1\,\partial\bar{\varepsilon}_2}\bigg|_{\varepsilon_1=\varepsilon_2=0} (f^{\varepsilon_1\mu+\varepsilon_2\nu})^*(\varrho)$$
$$= \tfrac{1}{2}\,\varrho(\Delta_0+\tfrac{1}{2})^{-1}(\mu\bar{\nu}) = \tfrac{1}{2}\,\varrho\cdot f_{\mu\bar{\nu}}, \qquad \mu,\nu\in\Omega^{-1,1}(X). \tag{2.2}$$

The Lie derivatives of the family $\mu^{\varepsilon\nu} = D^{\varepsilon\nu}\mu \in \Omega^{-1,1}(X^{\varepsilon\nu})$ representing the vector field $\dfrac{\partial}{\partial\varepsilon_\mu}$ on $T_{g,n}$ in a neighbourhood of the point $[X]$, are given by the formulas:

$$\begin{aligned} &L_\nu\mu = 0, \\ &L_{\bar{\nu}}\mu = -\bar{\partial}\varrho^{-1}\bar{\partial}(\Delta_0+\tfrac{1}{2})^{-1}(\mu\bar{\nu}) = -\bar{\partial}\varrho^{-1}\bar{\partial}f_{\mu\bar{\nu}} \end{aligned} \tag{2.3}$$

(see [5, Theorem 2.9]).

The Lie derivatives of the period matrix are given by Rauch's formulas [18]:

$$L_\mu\tau_{ij} = -2\sqrt{-1}\int_X \omega_i\omega_j\mu,$$

$$L_{\bar{\mu}}\tau_{ij} = 0, \qquad \mu\in\Omega^{-1,1}(X), \qquad i,j=1,\ldots,g.$$

This immediately yields

$$L_\mu(\log\det\operatorname{Im}\tau) = \operatorname{tr}((\operatorname{Im}\tau)^{-1}L_\mu(\operatorname{Im}\tau)) = -\int_X\sum_{i,j=1}^{g}(\operatorname{Im}\tau)^{-1}_{\bar{i}\bar{j}}\,\omega_i\omega_j\mu. \tag{2.4}$$

In other words, the $(1,0)$-form

$$\partial\log\det\operatorname{Im}\tau = \sum_{i=1}^{3g-3+n}\frac{\partial\log\det\operatorname{Im}\tau}{\partial\varepsilon_i}\,d\varepsilon_i$$

on $T_{g,n}$ corresponds via the isomorphism $T^*_{[X]}T_{g,n}\cong\Omega^2(X)$ to the family of cusp forms

$$-\sum_{i,j=1}^{g}(\operatorname{Im}\tau)^{-1}_{\bar{i}\bar{j}}\,\omega_i\omega_j.$$

Moreover, for the kernel $B(z,z')$ (see Sect. 1) we have

$$L_{\bar{\mu}}B(z,z') = \frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0} B(f^{\varepsilon\mu}(z), f^{\varepsilon\mu}(z'))\frac{\partial f^{\varepsilon\mu}}{\partial z}(z)\,\frac{\partial f^{\varepsilon\mu}}{\partial z'}(z') = 0$$

for any $\mu\in\Omega^{-1,1}(X)$, since $L_{\bar{\mu}}B$ is a regular bidifferential on $\bar{X}\times\bar{X}$ with zero periods.

Formulas (2.1) and (2.2) play an important role in the calculation of the curvature form Θ of the line bundle $T_v\mathscr{T}_{g,n}\to\mathscr{T}_{g,n}$. Recall that [5] the Teichmüller curve $\mathscr{T}_{g,n}$ is the natural fiber space with the projection $p:\mathscr{T}_{g,n}\to T_{g,n}$; the fibers are Riemann surfaces of type (g,n). Formally, the bundle $T_v\mathscr{T}_{g,n}\to\mathscr{T}_{g,n}$ is defined as $\ker dp\subset T\mathscr{T}_{g,n}$ (the vertical tangent bundle of the fibration $p:\mathscr{T}_{g,n}\to T_{g,n}$). Its restriction to a fiber of the projection p is isomorphic to the tangent bundle of a fiber. Therefore the Poincaré metric on the fibers defines a metric in the line bundle $T_v\mathscr{T}_{g,n}\to\mathscr{T}_{g,n}$. There exists a canonical lifting $TT_{g,n}\to T\mathscr{T}_{g,n}$ [5].

Denoting the corresponding image of $\frac{\partial}{\partial\varepsilon_\mu}$ by τ_μ and the tangent vector field along the fibers by $\frac{\partial}{\partial z}$ we have the following expression for the curvature form Θ (see [5, Formula (5.3)]):

$$\begin{aligned}
&\Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\right) = -\frac{1}{2y^2},\\
&\Theta\left(\tau_\mu, \frac{\partial}{\partial \bar z}\right) = 0,\\
&\Theta\left(\tau_\mu, \bar\tau_\nu\right) = -\tfrac{1}{2}\left(\Delta_0 + \tfrac{1}{2}\right)^{-1}(\mu\bar\nu) = -\tfrac{1}{2} f_{\mu\bar\nu}\,.
\end{aligned} \tag{2.5}$$

Finally, the Lie derivatives for a family $A^\varepsilon : \mathscr{H}^{\ell,m}(X^{\varepsilon\mu}) \to \mathscr{H}^{\ell' m'}(X^{\varepsilon\mu})$ of linear operators are defined by the formulas

$$\begin{aligned}
L_\mu A &= \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} ((f^{\varepsilon\mu})^* A^\varepsilon((f^{\varepsilon\mu})^*)^{-1}),\\
L_{\bar\mu} A &= \frac{\partial}{\partial\bar\varepsilon}\bigg|_{\varepsilon=0} ((f^{\varepsilon\mu})^* A^\varepsilon((f^{\varepsilon\mu})^*)^{-1})
\end{aligned}$$

and are linear operators from $\mathscr{H}^{\ell,m}(X)$ to $\mathscr{H}^{\ell',m'}(X)$. For the families of the operators $\bar\partial_\ell$ and $\bar\partial_\ell^*$ one has

$$\begin{aligned}
L_\mu\bar\partial_\ell &= \mu\bar\partial_{\ell+1}^*\varrho, & L_{\bar\mu}\bar\partial_\ell &= 0,\\
L_\mu\bar\partial_\ell^* &= 0, & L_{\bar\mu}\bar\partial_\ell^* &= \bar\mu\bar\partial_{\ell-1}\varrho^{-1}
\end{aligned} \tag{2.6}$$

(see, e.g., [6], formulas (2.8)). From this it follows that

$$\begin{aligned}
L_\mu\Delta_\ell &= \bar\partial_\ell^*\mu\bar\partial_{\ell+1}^*\varrho,\\
L_{\bar\mu}\Delta_\ell &= \bar\mu\bar\partial_{\ell-1}\varrho^{-1}\bar\partial_\ell\,.
\end{aligned} \tag{2.7}$$

These formulas will be used in the next section. Here we prove the following

Lemma 3. *The metrics $\langle,\rangle_i$, $i = 1, \ldots, n$, on $T_{g,n}$ are Kählerian.*

Proof. We must show that

$$\frac{\partial}{\partial\varepsilon_\mu}\langle\nu,\lambda\rangle_i = \frac{\partial}{\partial\varepsilon_\nu}\langle\mu,\lambda\rangle_i \tag{2.8}$$

for any μ, ν, $\lambda \in \Omega^{-1,1}(X)$ at any point $[X] \in T_{g,n}$. We have

$$\frac{\partial}{\partial\varepsilon_\mu}\langle\nu,\lambda\rangle_i = \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\langle\nu^{\varepsilon\mu},\lambda^{\varepsilon\mu}\rangle_i = \int_X L_\mu E_i(\cdot,2)\nu\bar\lambda\varrho + \int_X E_i(\cdot,2)\nu\overline{(L_{\bar\mu}\lambda)}\varrho\,, \tag{2.9}$$

where we used (2.1) and (2.3). Using the differential equation

$$(\Delta_0 + \tfrac{1}{2})E_i(z,2) = 0$$

and formula (2.7) with $\ell = 0$, we get

$$(\Delta_0 + \tfrac{1}{2})L_\mu E_i(\cdot,2) + \varrho^{-1}\partial(\mu\partial E_i(\cdot,2)) = 0\,,$$

which leads to the formula

$$L_\mu E_i(\cdot,2) = -(\Delta_0 + \tfrac{1}{2})^{-1}(\varrho^{-1}\partial(\mu\partial E_i(\cdot,2)))\,.$$

By means of this formula and formula (2.3) we obtain from (2.9) integrating by parts

$$\begin{aligned}\frac{\partial}{\partial\varepsilon_\mu}\langle\nu,\lambda\rangle_i &= -\int_X(\Delta_0+\tfrac{1}{2})^{-1}(\varrho^{-1}\partial(\mu\partial E_i(\cdot,2)))\cdot\nu\bar{\lambda}\varrho \\ &\quad -\int_X E_i(\cdot,2)\nu\partial(\varrho^{-1}\partial(\Delta_0+\tfrac{1}{2})^{-1}(\mu\bar{\lambda}))\varrho \\ &= -\int_X\partial(\mu\partial E_i(\cdot,2))\cdot f_{\nu\bar{\lambda}}-\int_X\partial(\nu\partial E_i(\cdot,2))\cdot f_{\mu\bar{\lambda}}\,,\end{aligned}$$

since $\partial(\varrho\nu)=0$ for any $\nu\in\Omega^{-1,1}(X)$ and since Beltrami differentials μ and ν are rapidly decreasing at the cusps. The obtained formula is obviously symmetric with respect to μ and ν, which proves (2.8) and the lemma.

Set for $\mu,\nu\in\Omega^{-1,1}(X)$,

$$\langle\mu,\nu\rangle_{\mathrm{cusp}}=\sum_{i=1}^{n}\langle\mu,\nu\rangle_i\,.$$

The metric $\langle,\rangle_{\mathrm{cusp}}$ on $T_{g,n}$ is also Kählerian; we denote by ω_{cusp} its symplectic form, where

$$\omega_{\mathrm{cusp}}\left(\frac{\partial}{\partial\varepsilon_\mu},\frac{\partial}{\partial\bar{\varepsilon}_\nu}\right)=\frac{\sqrt{-1}}{2}\langle\mu,\nu\rangle_{\mathrm{cusp}}\,.$$

The metric $\langle,\rangle_{\mathrm{cusp}}$ is obviously invariant under the action of the Teichmüller modular group $\mathrm{Mod}_{g,n}$.

Finally, let us observe that the scalar products $\langle\mu,\nu\rangle$ and $\langle\mu,\nu\rangle_i$, $i=1,\ldots,n$, for $\mu,\nu\in\Omega^{-1,1}(X)$ can be expressed in terms of the values of the Rankin L-series [19] associated with the cusp forms $\varphi=y^{-2}\bar{\mu}$, $\psi=y^{-2}\bar{\nu}\in\Omega^2(X)$. Indeed, let $\{a_k^{(i)}\}_{k=1}^\infty$, $\{b_k^{(i)}\}_{k=1}^\infty$, $i=1,\ldots,n$, be the Fourier coefficients of the cusp forms φ and ψ of weight 4 at the cusps $z_1,\ldots,z_n$ (see Sect. 1). Then for $\mathrm{Re}\,s>1$ (see [19]),

$$\begin{aligned}\int_X E_i(\cdot,s)\mu\bar{\nu}\varrho &= \iint_{\Gamma\backslash H}\sum_{\gamma\in\Gamma_i\backslash\Gamma}\mathrm{Im}(\sigma_i^{-1}\gamma z)^s\mu(z)\bar{\nu}(z)\frac{dx\,dy}{y^2} \\ &= \iint_{\sigma_i^{-1}\Gamma\sigma_i\backslash H}\sum_{\gamma\in\Gamma_i\backslash\Gamma}\mathrm{Im}(\sigma_i^{-1}\gamma\sigma_i z)^s\mu(\sigma_i z)\overline{\nu(\sigma_i z)}\frac{dx\,dy}{y^2} \\ &= \int_0^\infty\int_0^1 y^{s-2}\mu(\sigma_i z)\overline{\nu(\sigma_i z)}\,dx\,dy \\ &= \int_0^\infty\int_0^1 y^{s+2}\overline{\varphi(\sigma_i z)}\,\psi(\sigma_i z)|\sigma_i'(z)|^4\,dx\,dy \\ &= \frac{\Gamma(s+3)}{(4\pi)^{s+3}}L^{(i)}(\mu,\nu;s+3)\,,\end{aligned}\tag{2.10}$$

where

$$L^{(i)}(\mu, \nu; s) = \sum_{k=1}^{\infty} \frac{\overline{a_k^{(i)}}\, b_k^{(i)}}{k^s}$$

is the Rankin L-series for the cusp forms φ and ψ at the cusp z_i, $i = 1, \ldots, n$. It is absolutely convergent for $\operatorname{Re} s > 4$ and has a simple pole at $s = 4$. Therefore, from (2.10) it follows that

$$\langle \mu, \nu \rangle_i = \frac{4!}{(4\pi)^5} L^{(i)}(\mu, \nu; 5); \qquad i = 1, \ldots, n. \tag{2.11}$$

Moreover, since

$$\operatorname{Res}|_{s=1} E_i(z, s) = \frac{1}{2\pi(2g - 2 + n)},$$

we obtain from (2.10) that

$$\langle \mu, \nu \rangle = \frac{3!}{(4\pi)^3} \left(g - 1 + \frac{n}{2} \right) \operatorname{Res}|_{s=4} L^{(i)}(\mu, \nu; s)$$

for any $i = 1, \ldots, n$ (here Res denotes the residue).

3. First Variation of the Selberg Zeta Function

Recall that the Selberg zeta function $Z(s)$ of a Riemann surface X is defined for $\operatorname{Re} s > 1$ by the absolutely convergent product

$$Z(s) = \prod_{\{\ell\}} \prod_{m=0}^{\infty} (1 - e^{-(s+m)|\ell|}),$$

where ℓ runs over the set of all simple closed geodesics on X (with respect to the Poincaré metric), and $|\ell|$ is the length of ℓ. The function $Z(s)$ has a meromorphic continuation to the whole s-plane with a simple zero at $s = 1$. For the logarithmic derivative of $Z(s)$ one has

$$\frac{1}{2s-1} \frac{d}{ds} \log Z(s) = \frac{1}{4} \iint_{\Gamma \backslash H} \sum_{\substack{\gamma \in \Gamma, \\ \gamma \text{ hyperbolic}}} Q_s^{(0)}(z, \gamma z) \frac{dx\, dy}{y^2}, \tag{3.1}$$

where the sum is taken over all hyperbolic elements of the Fuchsian group Γ uniformizing X; $\operatorname{Re} s > 1$. This formula can be derived from the definition of $Z(s)$ by means of the Selberg transform; see [16] for details.

As it follows from the Teichmüller theory (see Sect. 2), the value of the Selberg zeta function $Z(s)$ at a fixed point s with $\operatorname{Re} s > 1$ is a smooth function on $T_{g,n}$. The next lemma gives an expression for the first derivatives of $Z(s)$ with respect to coordinates on $T_{g,n}$.

Lemma 3. *For any* $\mu \in \Omega^{-1,1}(X)$ *and* $\operatorname{Re} s > 1$ *the following formula holds*

$$\frac{\partial}{\partial \varepsilon_\mu} \log Z(s) = - \int_X \mu \partial \partial' (G_s^{(0)} - Q_s^{(0)})|_D, \tag{3.2}$$

where

$$\partial\partial'(G_s^{(0)} - Q_s^{(0)})|_D(z) = \frac{\partial^2}{\partial z\,\partial z'}\,(G_s^{(0)}(z,z') - Q_s^{(0)}(z,z'))|_{z'=z}$$

(here and in what follows a dash on an operator means that it acts on the variable z').

Proof. First of all, $\partial\partial'(G_s^{(0)} - Q_s^{(0)})|_D$ is a smooth automorphic form of weight 4 for the Fuchsian group Γ, and the integral in the right-hand side of (3.2) is convergent because μ decreases rapidly near the cusps $z_1, \dots, z_n$ of the group Γ. Further differentiating both sides of (3.1) and taking (2.1) into account we obtain that for $\operatorname{Re} s > 1$,

$$\begin{aligned}
&\frac{1}{2s-1}\,L_\mu\left(\frac{d}{ds}\log Z(s)\right)\\
&= \frac{1}{4}\iint\limits_{\Gamma\backslash H} L_\mu\left[\sum_{\substack{\gamma\in\Gamma,\\ \gamma\ \text{hyperbolic}}} Q_s^{(0)}(z,\gamma z)\right]\frac{dx\,dy}{y^2}\\
&= \frac{1}{4}\iint\limits_{\Gamma\backslash H}\left[L_\mu G_s^{(0)}(z,z') - (L_\mu Q_s^{(0)}(z,z') - \sum_{\substack{\gamma\in\Gamma,\\ \gamma\ \text{parabolic}}} L_\mu Q_s^{(0)}(z,\gamma z')\right]\Bigg|_{z'=z}\frac{dx\,dy}{y^2},
\end{aligned} \tag{3.3}$$

where

$$L_\mu Q_s^{(0)}(z,z') = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} Q_s^{(0)}(f^{\varepsilon\mu}(z), f^{\varepsilon\mu}(z'))$$

and

$$L_\mu G_s^{(0)}(z,z') = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} G_s^{\varepsilon\mu}(f^{\varepsilon\mu}(z), f^{\varepsilon\mu}(z'))$$

($G_s^{\varepsilon\mu}$ stands for the resolvent of the Laplace operator Δ_0 on the Riemann surface $X^{\varepsilon\mu} \cong \Gamma^{\varepsilon\mu}\backslash H$). Denote by $G_{s,i}(z,z')$ the resolvent kernel of the Laplace operator on the Riemann surface $\Gamma_i\backslash H$; for $\operatorname{Re} s > 1$,

$$G_{s,i}(z,z') = \sum_{\gamma\in\Gamma_i} Q_s^{(0)}(z,\gamma z'), \qquad i = 1, \dots, n.$$

From the definition of the resolvent and formula (2.7) for $L_\mu\Delta_0$ it follows that

$$\begin{aligned}
L_\mu Q_s^{(0)}(z,z') &= -\,\text{v.p.}\iint\limits_H Q_s^{(0)}(z,z'')\,(L_\mu\Delta_0)''Q_s^{(0)}(z'',z')\,\frac{dx''\,dy''}{y''^2}\\
&= \iint\limits_H \mu(z'')\,\frac{\partial}{\partial z''}\,Q_s^{(0)}(z,z'')\,\frac{\partial}{\partial z''}\,Q_s^{(0)}(z'',z')\,dx''\,dy'', \qquad z \neq z',\\
L_\mu G_{s,i}(z,z') &= -\,\text{v.p.}\iint\limits_{\Gamma_i\backslash H} G_{s,i}(z,z'')\,(L_\mu\Delta_0)''G_{s,i}(z'',z')\,\frac{dx''\,dy''}{y''^2}\\
&= \iint\limits_{\Gamma_i\backslash H} \mu(z'')\,\frac{\partial}{\partial z''}\,G_{s,i}(z,z'')\,\frac{\partial}{\partial z''}\,G_{s,i}(z'',z')\,dx''\,dy'', \qquad z \neq \gamma z',\ \gamma\in\Gamma_i,
\end{aligned}$$

and

$$L_\mu G_s^{(0)}(z,z') = -\,\text{v.p.} \iint\limits_{\Gamma\backslash H} G_s^{(0)}(z,z'')\,(L_\mu \Delta_0)'' G_s^{(0)}(z'',z')\,\frac{dx''\,dy''}{y''^2}$$
$$= \iint\limits_{\Gamma\backslash H} \mu(z'')\,\frac{\partial}{\partial z''}\,G_s^{(0)}(z,z'')\,\frac{\partial}{\partial z''}\,G_s^{(0)}(z'',z')\,dx''\,dy'', \qquad z \neq \gamma z',\ \gamma \in \Gamma,$$

where $(L_\mu \Delta_0)''$ means that the differential operator $L_\mu \Delta_0$ acts in the variable z''. Using now the above expressions for $L_\mu Q_s^{(0)}$, $L_\mu G_{s,i}$, $L_\mu G_s^{(0)}$ and a simple formula

$$\sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ parabolic}}} Q_s^{(0)}(z,\gamma z') = \sum_{i=1}^{n} \sum_{\sigma\in\Gamma_i\backslash\Gamma} (G_{s,i}(\sigma z,\sigma z') - Q_s^{(0)}(z,z')),$$

we derive from (3.3) the following formula:

$$\frac{1}{2s-1} L_\mu \left(\frac{d}{ds}\log Z(s)\right)$$
$$= \frac{1}{4} \iint\limits_{\Gamma\backslash H} \mu(z)\,dx\,dy\,\frac{\partial^2}{\partial z\,\partial z'}\bigg|_{z'=z} \Bigg\{ \iint\limits_{\Gamma\backslash H} G_s^{(0)}(z,z'') G_s^{(0)}(z'',z')\,\frac{dx''\,dy''}{y''^2}$$
$$- \iint\limits_{H} Q_s^{(0)}(z,z'') Q_s^{(0)}(z'',z')\,\frac{dx''\,dy''}{y''^2}$$
$$- \sum_{i=1}^{n} \sum_{\sigma\in\Gamma_i\backslash\Gamma} \left[\iint\limits_{\Gamma_i\backslash H} G_{s,i}(\sigma z,\sigma z'') G_{s,i}(\sigma z'',\sigma z')\,\frac{dx''\,dy''}{y''^2}\right.$$
$$\left.- \iint\limits_{H} Q_s^{(0)}(z,z'') Q_s^{(0)}(z'',z')\,\frac{dx''\,dy''}{y''^2}\right] \Bigg\}$$
$$= \frac{1}{1-2s} \iint\limits_{\Gamma\backslash H} \mu(z)\,dx\,dy\,\frac{\partial^2}{\partial z\,\partial z'}\bigg|_{z'=z}$$
$$\times \frac{d}{ds}\left[G_s^{(0)}(z,z') - Q_s^{(0)}(z,z') - \sum_{i=1}^{n} \sum_{\sigma\in\Gamma_i\backslash\Gamma} (G_{s,i}(\sigma z,\sigma z') - Q_s^{(0)}(z,z'))\right]$$
$$= \frac{1}{1-2s}\,\frac{d}{ds} \iint\limits_{\Gamma\backslash H} \frac{\partial^2}{\partial z\,\partial z'}$$
$$\times \left[G_s^{(0)}(z,z') - Q_s^{(0)}(z,z') - \sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ parabolic}}} Q_s^{(0)}(z,z')\right]\bigg|_{z'=z} \mu(z)\,dx\,dy$$
$$= \frac{1}{1-2s}\,\frac{d}{ds} \iint\limits_{\Gamma\backslash H} \sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ hyperbolic}}} \frac{\partial^2}{\partial z\,\partial z'}\,(Q_s^{(0)}(z,\gamma z'))\bigg|_{z'=z} \mu(z)\,dx\,dy$$

(in this calculation we reversed the order of integration and applied the Hilbert identity to the resolvent kernels). Now let us integrate the last formula over the interval $[s,\infty)$. Since $L_\mu \log Z(s) \underset{s\to\infty}{\longrightarrow} 0$ and

$$\sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ hyperbolic}}} \frac{\partial^2}{\partial z\,\partial z'}(Q_s^{(0)}(z,\gamma z'))\Big|_{z'=z} \underset{s\to\infty}{\longrightarrow} 0$$

uniformly in $z \in H$, we get

$$\frac{\partial}{\partial \varepsilon_\mu}\log Z(s) = -\iint_{\Gamma\backslash H} \sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ hyperbolic}}} \frac{\partial^2}{\partial z\,\partial z'}(Q_s^{(0)}(z,\gamma z'))\Big|_{z'=z} \mu(z)\,dx\,dy\,.$$

Since

$$\frac{\partial^2}{\partial z\,\partial z'} G_s^{(0)}(z,z') = \sum_{\gamma\in\Gamma} \frac{\partial^2}{\partial z\,\partial z'} Q_s^{(0)}(z,\gamma z'),$$

it remains to show that for any $\mu \in \Omega^{-1,1}(X)$

$$\iint_{\Gamma\backslash H} \sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ parabolic}}} \frac{\partial^2}{\partial z\,\partial z'}(Q_s^{(0)}(z,\gamma z'))\Big|_{z'=z} \mu(z)\,dx\,dy = 0,$$

where the sum is taken over all parabolic elements of the group Γ. We have

$$\begin{aligned}
&\sum_{\substack{\gamma\in\Gamma,\\ \gamma \text{ parabolic}}} \frac{\partial^2}{\partial z\,\partial z'}(Q_s^{(0)}(z,\gamma z'))\Big|_{z'=z} \\
&= \sum_{i=1}^{n} \sum_{\sigma\in\Gamma_i\backslash\Gamma} \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{\partial^2}{\partial z\,\partial z'}(Q_s^{(0)}(z,\sigma^{-1}S_i^k\sigma z'))\Big|_{z'=z} \\
&= \sum_{i=1}^{n} \sum_{\sigma\in\Gamma_i\backslash\Gamma} \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{\partial^2}{\partial z\,\partial z'}(Q_s^{(0)}(\sigma_i^{-1}\sigma z,\sigma_i^{-1}\sigma z'+k))\Big|_{z'=z} \\
&= \sum_{i=1}^{n} \sum_{\sigma\in\Gamma_i\backslash\Gamma} \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{\partial^2 Q_s^{(0)}}{\partial z\,\partial z'}(\sigma_i^{-1}\sigma z,\sigma_i^{-1}\sigma z+k)\,(\sigma_i^{-1}\sigma)'(z)^2 \\
&= \sum_{i=1}^{n} \theta_s^{(i)}(z),
\end{aligned}$$

where

$$\theta_s^{(i)}(z) = \sum_{\sigma\in\Gamma_i\backslash\Gamma} \psi_s(\sigma_i^{-1}\sigma z)\,(\sigma_i^{-1}\sigma)'(z)^2,$$

$$\psi_s(z) = \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{\partial^2 Q_s^{(0)}}{\partial z\,\partial z'}(z,z+k)\,.$$

From formula (1.4) for the kernel $Q_s^{(0)}(z,z')$ it is not difficult to deduce that $\psi_s(z)$ is a bounded function on H, depending only on $y = \operatorname{Im} z$. On the other hand, any $\mu \in \Omega^{-1,1}(X)$ is equal to $y^2\bar{\varphi}$, where $\varphi \in \Omega^2(X)$ is a cusp form of weight 4 for the group Γ. Since incomplete theta series (including the automorphic forms $\theta_s^{(i)}$, $i = 1, \ldots, n$) are orthogonal to cusp forms with respect to the scalar product (1.1) (see, e.g., [16]), we have

$$\int_X \theta_s^{(i)} \mu = 0$$

for any $\mu \in \Omega^{-1,1}(X)$ and $i = 1, \ldots, n$, which completes the proof.

Taking into account that

$$\frac{\partial}{\partial \varepsilon_\mu} \log Z'(1) = \lim_{s \to 1} L_\mu \log Z(s),$$

we get from (3.2) at the limit $s \to 1$,

$$\frac{\partial}{\partial \varepsilon_\mu} \log Z'(1) = \int_X R^{(0)}|_D \mu, \tag{3.4}$$

where

$$R^{(0)}|_D(z) = -\frac{\partial^2}{\partial z\, \partial z'} (G_1^{(0)}(z,z') - Q_1^{(0)}(z,z'))|_{z'=z}$$

(see Sect. 1). The first derivatives of $Z(s)$ on $T_{g,n}$ for integer $s = 2, 3, \ldots$ can be expressed in a similar way in terms of the corresponding Green's functions. Namely, we have

Lemma 4. *For any integer $k \geq 1$ and $\mu \in \Omega^{-1,1}(X)$*

$$\frac{\partial}{\partial \varepsilon_\mu} \log Z(k+1) = \int_X R^{(-k)}|_D \mu,$$

where

$$R^{(-k)}|_D = -\partial \varrho^k \partial' (G_1^{(-k)} - Q_1^{(-k)})|_D .$$

Proof. First we will prove that for $\ell = -k \leq -1$, $\operatorname{Re} s \geq 1$,

$$\int_X \mu \partial \varrho^{-\ell} \partial' (G_s^{(\ell)} - Q_s^{(\ell)})|_D = \int_X \mu \partial \varrho^{-\ell-1} \partial' (G_{s+1}^{(\ell+1)} - Q_{s+1}^{(\ell+1)})|_D . \tag{3.5}$$

We observe that $\partial' G_s^{(\ell)}$ (respectively $\partial' Q_s^{(\ell)}$) is the kernel of the operator $(\Delta_\ell + \frac{1}{4}(s-2\ell)(s-1))^{-1} \bar{\partial}_\ell^*$ on the Riemann surface X (respectively on the upper half-plane H). Further, it follows from (1.3) that

$$\left(\Delta_{\ell+1} + \frac{(s-2\ell-1)s}{4}\right)^{-1} \bar{\partial}_{\ell+1}^* \varrho = \bar{\partial}_{\ell+1}^* \varrho \left(\Delta_\ell + \frac{(s-2\ell)(s-1)}{4}\right)^{-1},$$

in other words, it means that

$$\partial' (G_{s+1}^{(\ell+1)} - Q_{s+1}^{(\ell+1)}) = -\varrho^\ell \varrho' \partial \varrho^{-\ell} (G_s^{(\ell)} - Q_s^{(\ell)}) .$$

We have

$$\begin{aligned}
&\int_X \mu(\partial\varrho^{-\ell}\partial'(G_s^{(\ell)} - Q_s^{(\ell)}) - \partial\varrho^{-\ell-1}\partial'(G_{s+1}^{(\ell+1)} - Q_{s+1}^{(\ell+1)}))|_D \\
&= \int_X \mu(\partial\varrho^{-\ell}\partial'(G_s^{(\ell)} - Q_s^{(\ell)}) + \partial\varrho^{-1}\varrho'\partial\varrho^{-\ell}(G_s^{(\ell)} - Q_s^{(\ell)}))|_D \\
&= \int_X \mu((\varrho\partial' + \varrho'\partial)\varrho^{-1}\partial\varrho^{-\ell}(G_s^{(\ell)} - Q_s^{(\ell)}))|_D \\
&= \int_X \mu\varrho\partial(\varrho^{-1}\partial\varrho^{-\ell}(G_s^{(\ell)} - Q_s^{(\ell)})|_D) = 0,
\end{aligned}$$

because for any $\mu \in \Omega^{-1,1}(X)$ one has $\partial(\mu\varrho) = 0$; this proves (3.5). Now starting from Lemma 3 with $s = k+1$ and applying formula (3.5) k times we arrive at the assertion of the lemma.

4. Quillen's Local Index Theorem

Recall that the determinant line bundle of the family of operators $\bar\partial_\ell$ by definition is

$$\lambda_\ell = \det \operatorname{ind} \bar\partial_\ell = \Lambda^{\max} \ker \bar\partial_\ell \otimes (\Lambda^{\max} \operatorname{coker} \bar\partial_\ell)^{-1},$$

where $\ell \in \mathbb{Z}$ and $\Lambda^{\max}$ denotes the maximal exteriour power of a vector space. There is a canonical metric $\|\cdot\|$ (L^2-metric) in the holomorphic line bundle λ_ℓ on $T_{g,n}$ associated with the scalar product (1.1). In the next lemma we calculate the curvature form $\Theta^{(\ell)}$ of the canonical (unitary) connection in the Hermitian holomorphic line bundle $(\lambda_\ell, \|\cdot\|)$. (Because of canonical isomorphism between λ_ℓ and $\lambda_{\ell-1}$ it is sufficient to consider $\ell = k > 0$).

Lemma 5. *For any $\mu, \nu \in \Omega^{-1,1}(X)$ we have*

$$\Theta^{(1)}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar\varepsilon_\nu}\right) = -\frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu}\log\det\operatorname{Im}\tau\,, \tag{4.1}$$

where τ is the period matrix of a marked Riemann surface X. For $k \geq 2$,

$$\begin{aligned}
&\Theta^{(k)}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar\varepsilon_\nu}\right) \\
&\quad = -\operatorname{Tr}\left(((-\mu\bar\nu + (1-k)\,(L_{\mu\bar\nu}\varrho)\varrho^{-1})I + (L_\mu\bar\partial_{1-k})\Delta_{1-k}^{-1}\cdot(L_{\bar\nu}\bar\partial_{1-k}^*))P_{1-k,1}\right),
\end{aligned}$$

where $P_{1-k,1} : \mathscr{H}^{1-k,1}(X) \to \Omega^{1-k,1}(X)$ is the orthogonal projection, I is the identity operator in $\mathscr{H}^{1-k,1}(X)$, and Tr *denotes the trace of an operator.*

Proof. A normalized basis of abelian differentials $\omega_1, \ldots, \omega_g$ determines g global holomorphic sections of the bundle $\ker\bar\partial_1$ over $T_{g,n}$. Therefore, the L^2-norm of the canonical section $\omega_1 \wedge \cdots \wedge \omega_g$ of the line bundle λ_1 is equal to $(\det\operatorname{Im}\tau)^{1/2}$ which leads to formula (4.1). The proof of (4.2) consists of simple linear algebraic calculations and can be found in [6, Lemma 1].

Recall that Quillen's metric $\|\cdot\|_Q$ in the determinant line bundle λ_k is defined as follows

$$\|\cdot\|_Q = \begin{cases} (Z(k))^{-1/2}\|\cdot\|, & k \geq 2, \\ (Z'(1))^{-1/2}\|\cdot\|, & k = 1. \end{cases}$$

In the next theorem we compute the first Chern form $c_1(\lambda_k, \|\cdot\|_Q)$ of the Hermitian holomorphic line bundle $(\lambda_k, \|\cdot\|_Q)$ on $T_{g,n}$.

Theorem 1. *For $k \geq 1$,*

$$c_1(\lambda_k, \|\cdot\|_Q) = \frac{6k^2 - 6k + 1}{12\pi^2}\,\omega_{\mathrm{WP}} - \frac{1}{9}\,\omega_{\mathrm{cusp}}\,, \tag{4.3}$$

where ω_{WP} is the Weil-Petersson Kähler form and ω_{cusp} is the Kähler form of the metric $\langle,\rangle_{\mathrm{cusp}}$.

Proof. Since

$$c_1(\lambda_k, \|\cdot\|_Q) = \begin{cases} \dfrac{\sqrt{-1}}{2\pi}\,(\Theta^{(k)} - \bar\partial\partial \log Z(k)), & k \geq 2, \\ \dfrac{\sqrt{-1}}{2\pi}\,(\Theta^{(1)} - \bar\partial\partial \log Z'(1)), & k = 1, \end{cases}$$

(where ∂ and $\bar\partial$ denote the components of the exterior derivative operator $d = \partial + \bar\partial$ on $T_{g,n}$) it is sufficient to prove that for every integer $m \geq 0$ and $\mu, \nu \in \Omega^{-1,1}(X)$,

$$\begin{aligned} \frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu} \log Z(m+1) = &-\Theta^{(m+1)}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar\varepsilon_\nu}\right) \\ &+ \frac{6m^2 + 6m + 1}{12\pi}\,\langle\mu,\nu\rangle - \frac{\pi}{9}\,\langle\mu,\nu\rangle_{\mathrm{cusp}}\,, \end{aligned} \tag{4.4}$$

where for $m = 0$ $Z(m+1)$ should be replaced by $Z'(1)$. In the main, the proof of this formula follows the proofs of Theorem 2 in [6] and Theorem 2 in [11]. By Lemma 4,

$$\frac{\partial}{\partial\varepsilon_\mu} \log Z(m+1) = \int_X R^{(-m)}|_D\,\mu\,,$$

where

$$R^{(-m)}(z,z') = -\frac{\partial}{\partial z}\,y^{-2m}\,\frac{\partial}{\partial z'}\,(G_1^{(-m)}(z,z') - Q_1^{(-m)}(z,z'))$$

and D denotes the diagonal $z' = z$ in $X \times X$. Therefore

$$\begin{aligned} \frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu} \log Z(m+1) &= \frac{\partial}{\partial\bar\varepsilon}\bigg|_{\varepsilon=0} \int_{X^{\varepsilon\nu}} (R^{(-m)})^{\varepsilon\nu}|_D\,\mu^{\varepsilon\nu} \\ &= \int_X (L_{\bar\nu}(R^{(-m)}|_D)\mu + R^{(-m)}|_D L_{\bar\nu}\mu)\,. \end{aligned} \tag{4.5}$$

Because the kernel $R^{(-m)}$ is regular on the diagonal D in $H \times H$, we have $L_{\bar\nu}(R^{(-m)}|_D) = (L_{\bar\nu}R^{(-m)})|_D$. Let us calculate the contribution to (4.5) of the variations of the kernels $\partial\varrho^m\partial' Q_1^{(-m)}$ and $\partial\varrho^m\partial' G_1^{(-m)}$ separately. From (1.5) we derive that for $z' \neq z$

$$\frac{\partial}{\partial z} y^{-2m} \frac{\partial}{\partial z'} Q_1^{(-m)}(z,z') = -\frac{1}{\pi} \cdot \frac{1}{(z-z')^2} \left(\frac{z'-\bar z'}{\bar z - z'}\right)^{2m}. \tag{4.6}$$

As it is shown in [6, Sect. 4.4],

$$\lim_{z'\to z} L_{\bar\nu}\left(\frac{\partial}{\partial z} y^{-2m} \frac{\partial}{\partial z'} Q_1^{(-m)}(z,z')\right) = \frac{3m+1}{12\pi} y^{-2}\overline{\nu(z)},$$

therefore

$$\int\limits_X \mu L_{\bar\nu}(\partial\varrho^m\partial' Q_1^{(-m)})|_D = \frac{3m+1}{12\pi}\langle\mu,\nu\rangle. \tag{4.7}$$

Because the variation of the kernel $\partial\varrho^m\partial' Q_1^{(-m)}$ contributes a finite amount to (4.5) the same is true for the kernel $\partial\varrho^m\partial' G_1^{(-m)}$, which allows us to vary it outside the diagonal D, and then we can pass to the limit as $z' \to z$. The kernel $\partial\varrho^m\partial' G_1^{(-m)}$ for $m>0$ is the kernel of the operator $-\varrho^m\bar\partial^*_{1-m}\Delta^{-1}_{-m}\bar\partial^*_{-m}$ from $\mathscr{H}^{-m,1}$ to $\mathscr{H}^{1,m}$. As it is shown in [6, Sect. 4.5],

$$\int\limits_X L_{\bar\nu}(\partial\varrho^m\partial' G_1^{(-m)})|_D\mu = -\operatorname{Tr}((-\mu\bar\nu I + (L_\mu\bar\partial_{-m})\Delta^{-1}_{-m}(L_{\bar\nu}\bar\partial^*_{-m}))\cdot P_{-m,1}). \tag{4.8}$$

For $m=0$ we have

$$\partial\partial' G_1^{(0)} = -\frac{1}{\pi}\Omega,$$

where Ω is the Schiffer kernel (see (1.7)). Therefore using (1.8) we obtain

$$\begin{aligned}
&\int\limits_X L_{\bar\nu}(\partial\partial' G_1^{(0)})|_D\mu \\
&= -\int\limits_X \left(L_{\bar\nu}\left(\frac{1}{\pi}B(z,z') - \sum_{i,j=1}^g (\operatorname{Im}\tau)^{-1}_{ij}\omega_i(z)\omega_j(z')\right)\right)\Bigg|_{z'=z}\mu(z)\,dx\,dy \\
&= -\frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu}\log\det\operatorname{Im}\tau - \int\limits_X\sum_{i,j=1}^g(\operatorname{Im}\tau)^{-1}_{ij}\omega_i\omega_j L_{\bar\nu}\mu,
\end{aligned}$$

where we have also used (2.4) and the fact that $L_{\bar\nu}B = 0$ (see Sect. 2). By formula (2.3) the last integral vanishes identically and we obtain that

$$\int\limits_X L_{\bar\nu}(\partial\partial' G_1^{(0)})|_D\mu = -\frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu}\log\det\operatorname{Im}\tau. \tag{4.9}$$

Now we turn to the term $R^{(-m)}|_D L_{\bar\nu}\mu$ in the integrand of (4.5). Let F be a canonical fundamental domain of the group Γ in H such that its cusps are exactly $z_1,\ldots,z_n \in \mathbb{R}\cup\{\infty\}$. We set

$$F^Y = \{z\in F \mid \operatorname{Im}\sigma_i^{-1}z \le Y,\ i=1,\ldots,n\}$$

and

$$C_i^Y = F\cap\{z\in H \mid \operatorname{Im}\sigma_i^{-1}z = Y\},\qquad i=1,\ldots,n.$$

With the help of (2.3) we obtain that

$$\begin{aligned}\int\limits_X R^{(-m)}|_D L_{\bar\nu}\mu &= -\iint\limits_F R^{(-m)}|_D \bar\partial\varrho^{-1}\bar\partial(\Delta_0+\tfrac{1}{2})^{-1}(\mu\bar\nu)\,dx\,dy\\ &= \frac{1}{2\sqrt{-1}}\lim_{Y\to\infty}\iint\limits_{F^Y} R^{(-m)}|_D\frac{\partial}{\partial\bar z}y^2\frac{\partial f_{\mu\bar\nu}}{\partial\bar z}\,dz\wedge d\bar z\\ &= \frac{\sqrt{-1}}{2}\lim_{Y\to\infty}\iint\limits_{\partial F^Y} R^{(-m)}|_D\,y^2\frac{\partial f_{\mu\bar\nu}}{\partial\bar z}\,dz\\ &\quad+\iint\limits_F\frac{\partial}{\partial\bar z}(R^{(-m)}|_D)\,y^2\frac{\partial f_{\mu\bar\nu}}{\partial\bar z}\,dx\,dy\\ &= I_1+I_2,\end{aligned}$$

where $f_{\mu\bar\nu}=(\Delta_0+\frac{1}{2})^{-1}(\mu\bar\nu)$ (see Lemma 1). For the integral I_1 we have, due to Γ-invariance of the integrand, that

$$\begin{aligned}I_1 &= \frac{\sqrt{-1}}{2}\lim_{Y\to\infty}\sum_{i=1}^n\int\limits_{C_i^Y}R^{(-m)}|_D\,y^2\frac{\partial f_{\mu\bar\nu}}{\partial\bar z}\,dz\\ &= -\frac{\sqrt{-1}}{2}\lim_{Y\to\infty}\sum_{i=1}^n\int\limits_0^1(R^{(-m)}|_D\circ\sigma_i)(z)\,(\operatorname{Im}\sigma_i z)^2\frac{\partial f_{\mu\bar\nu}}{\partial\bar z}(\sigma_i z)\sigma_i'(z)\,dx\\ &= -\frac{\sqrt{-1}}{2}\lim_{Y\to\infty}Y^2\sum_{i=1}^n\int\limits_0^1(R^{(-m)}|_D)\,(\sigma_i z)\sigma_i'(z)^2\frac{\partial}{\partial\bar z}(f_{\mu\bar\nu}(\sigma_i z))\,dx,\\ &\quad z=x+\sqrt{-1}Y.\end{aligned}$$

Using Lemma 1 and 2 we obtain that

$$\begin{aligned}I_1 &= -\frac{\sqrt{-1}}{2}\lim_{Y\to\infty}Y^2\left(\sum_{i=1}^n\frac{\pi}{3}\cdot\frac{\sqrt{-1}}{2}\left(-\frac{1}{Y^2}\right)c_{\mu\bar\nu}^{(i)}+o\left(\frac{1}{Y^2}\right)_{Y\to\infty}\right)\\ &= -\frac{\pi}{12}\sum_{i=1}^n c_{\mu\bar\nu}^{(i)} = -\frac{\pi}{9}\langle\mu,\nu\rangle_{\mathrm{cusp}}\,.\end{aligned}\tag{4.10}$$

Let us now proceed with the integral I_2. First of all, for $m=0$ we have $I_2=0$ since $R^{(0)}|_D$ is holomorphic. In this case combining formulas (4.5), (4.7), (4.9), and (4.10), we obtain that

$$\frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu}\log Z'(1)=\frac{\partial^2}{\partial\varepsilon_\mu\,\partial\bar\varepsilon_\nu}\log\det\operatorname{Im}\tau+\frac{1}{12\pi}\langle\mu,\nu\rangle-\frac{\pi}{9}\langle\mu,\nu\rangle_{\mathrm{cusp}}\,,$$

which proves (4.4) for $m=0$. For $m\geq 1$ we observe first that

$$\begin{aligned}\frac{\partial}{\partial\bar z}R^{(-m)}|_D(z) &= -\left(\frac{\partial}{\partial\bar z}+\frac{\partial}{\partial\bar z'}\right)\left(\frac{y}{y'}\right)^{2m}\frac{\partial}{\partial z}y^{-2m}\frac{\partial}{\partial z'}(G_1^{(-m)}-Q_1^{(-m)})|_{z'=z}\\ &= -\frac{m}{2}y^2y'^{-2m}\frac{\partial}{\partial z'}(G_1^{(-m)}-Q_1^{(-m)})|_{z'=z}\end{aligned}$$

(see [6], Sect. 4.6). It follows from this that

$$I_2 = -\frac{m}{2} \iint\limits_F y^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D \frac{\partial f_{\mu\bar{\nu}}}{\partial \bar{z}} \, dx\, dy$$

$$= \frac{m}{2} \cdot \frac{\sqrt{-1}}{2} \lim_{Y\to\infty} \int\limits_{\partial F^Y} y^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D f_{\mu\bar{\nu}} \, dz$$

$$+ \frac{m}{2} \iint\limits_F \frac{\partial}{\partial \bar{z}} \left(y^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D \right) f_{\mu\bar{\nu}} \, dx\, dy$$

$$= I_3 + I_4 .$$

The boundary integral I_3 can be calculated in the same way as I_1. We have

$$I_3 = \frac{m}{2} \frac{\sqrt{-1}}{2} \lim_{Y\to\infty} \sum_{i=1}^{n} \int\limits_{C_i^Y} y^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D f_{\mu\bar{\nu}} \, dz$$

$$= -\frac{m}{2} \frac{\sqrt{-1}}{2} \lim_{Y\to\infty} \sum_{i=1}^{n} \int\limits_0^1 (\operatorname{Im} \sigma_i z)^{-2m} \partial' (G_1^{(-m)} - Q_1^{(-m)})|_D (\sigma_i z) f_{\mu\bar{\nu}}(\sigma_i z) \sigma_i'(z) \, dx ,$$

$$z = x + \sqrt{-1} Y .$$

Note that $y^{-2m} \partial' (G_1^{(-m)} - Q_1^{(-m)})|_D$ is a smooth automorphic form of weight 2 with respect to the group Γ. According to [15, Corollary 3.5], we have

$$(\operatorname{Im} \sigma_i z)^{-2m} \partial' (G_1^{(-m)} - Q_1^{(-m)})|_D (\sigma_i z) \sigma_i'(z)$$

$$= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} y^{-2m} \frac{\partial}{\partial z'} Q_1^{(-m)}(z, z'+k)|_{z'=z} + \underset{y\to\infty}{o(1)}$$

$$= \frac{1}{\pi} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k} \left(\frac{2\sqrt{-1} y}{2\sqrt{-1} y - k} \right)^{2m+1} + \underset{y\to\infty}{o(1)} = \underset{y\to\infty}{O(1)},$$

where we used (1.5), the formula

$$\sum_{k=1}^{\infty} \frac{1}{(z+k)k} = \frac{1}{z} \frac{\Gamma'(z)}{\Gamma(z)} + \frac{C}{z} + \frac{1}{z^2} ,$$

and the Stirling formula. Taking now Lemma 2 into account we conclude that $I_3 = 0$.

In order to calculate the integral I_4 we observe that

$$\frac{\partial}{\partial \bar{z}} \left(y^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D \right) = \left(\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}'} \right) y'^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D$$

$$= \frac{\partial}{\partial \bar{z}} y'^{-2m} \frac{\partial}{\partial z'} (G_1^{(-m)} - Q_1^{(-m)})|_D .$$

The kernel $\frac{\partial^2}{\partial \bar{z}\, \partial z'} G_1^{(-m)}$ is the kernel of the operator $\bar{\partial}_{-m} \Delta_{-m}^{-1} \bar{\partial}_{-m}^* = I - P_{-m,1}$ (see formula (1.2)) in the space $\mathscr{H}^{-m,1}$, where I is the identity operator, and $P_{-m,1}$ is the orthogonal projection of $\mathscr{H}^{-m,1}(X)$ onto $\Omega^{-m,1}(X)$. Moreover, from (1.5) it follows that for $z' \neq z$,

$$y'^{-2m} \frac{\partial^2}{\partial \bar{z}\, \partial z'} Q_1^{(-m)}(z, z') = \frac{2m+1}{\pi} \cdot \frac{1}{(\bar{z}-z')^2} \left(\frac{\bar{z}-z}{\bar{z}-z'} \right)^{2m}.$$

Therefore

$$\begin{aligned} I_2 = I_4 &= \frac{m}{2} \iint_F \left(-y^{-2m} P_{-m,1}(z,z) + \frac{2m+1}{4\pi y^2} \right) (\Delta_0 + \tfrac{1}{2})^{-1} (\mu\bar{\nu})\, dx\, dy \\ &= -\frac{m}{2} \operatorname{Tr}(y^2 f_{\mu\bar{\nu}} P_{-m,1}) + \frac{(2m+1)m}{8\pi} \iint_F (\Delta_0 + \tfrac{1}{2})^{-1}(\mu\bar{\nu}) \frac{dx\, dy}{y^2} \\ &= -m \operatorname{Tr}((L_{\mu\bar{\nu}}\varrho)\varrho^{-1} P_{-m,1}) + \frac{(2m+1)m}{4\pi} \langle \mu, \nu \rangle, \end{aligned} \tag{4.11}$$

where we used formula (2.2) and the formula

$$\iint_F (\Delta_0 + \tfrac{1}{2})^{-1}(\mu\bar{\nu}) \frac{dx\, dy}{y^2} = 2 \iint_F \mu\bar{\nu} \frac{dx\, dy}{y^2} = 2\langle \mu, \nu \rangle,$$

which follows from the equality $\Delta_0(\Delta_0 + \frac{1}{2})^{-1} = I - \frac{1}{2}(\Delta_0 + \frac{1}{2})^{-1}$. Combining the results of computations (formulas (4.7), (4.8), (4.10), (4.11)) we obtain that

$$\begin{aligned} \frac{\partial^2}{\partial \varepsilon_\mu\, \partial \bar{\varepsilon}_\nu} \log Z(m+1) &= \operatorname{Tr}((-\mu\bar{\nu} I + (L_\mu \bar{\partial}_{-m}) \Delta_{-m}^{-1} (L_{\bar{\nu}} \bar{\partial}_{-m}^*) - m\varrho^{-1}(L_{\mu\bar{\nu}}\varrho)) P_{-m,1}) \\ &\quad + \frac{6m^2 + 6m + 1}{12\pi} \langle \mu, \nu \rangle - \frac{\pi}{9} \langle \mu, \nu \rangle_{\text{cusp}}. \end{aligned}$$

Finally setting $m = k - 1$ and taking Lemma 5 into account we arrive at the assertion of the theorem, i.e. formula (4.3).

5. Concluding Remarks

Here we will calculate the cuspidal defect

$$\delta_k^{(1)} = c_1(\lambda_k, \|\cdot\|_Q) - \int_{\text{fiber}} (\operatorname{ch}(T_v^{-k} \mathscr{T}_{g,n}) \cdot \operatorname{td}(T_v \mathscr{T}_{g,n}))_{2,2} \in \Omega^{1,1}(T_{g,n}), \qquad k \geq 1,$$

where integration is taken over the fibers of $\mathscr{T}_{g,n} \to T_{g,n}$.

Theorem 2. *We have*

$$\delta_k^{(1)} = -\tfrac{1}{9}\, \omega_{\text{cusp}}. \tag{5.1}$$

Moreover,

$$\delta_k^{(1)} \left(\frac{\partial}{\partial \varepsilon_\mu}, \frac{\partial}{\partial \bar{\varepsilon}_\nu} \right) = \frac{1}{\sqrt{-1} \cdot 3 \cdot 4^4 \pi^5} \sum_{i=1}^n L^{(i)}(\mu, \nu; 5) \tag{5.2}$$

for any μ, $\nu \in \Omega^{-1,1}(X)$, where $L^{(i)}$ is the Rankin L-series for the cusp z_i (see Sect. 2).

Proof. By definition,

$$\begin{aligned}
&\operatorname{ch}(T_v^{-k}\mathscr{T}_{g,n})\cdot\operatorname{td}(T_v\mathscr{T}_{g,n})\\
&= e^{-\frac{k\sqrt{-1}}{2\pi}\Theta}\cdot\frac{\sqrt{-1}}{2\pi}\Theta\left(1-e^{-\frac{\sqrt{-1}}{2\pi}\Theta}\right)^{-1}\\
&= 1+\left(\tfrac{1}{2}-k\right)\frac{\sqrt{-1}}{2\pi}\Theta+\frac{6k^2-6k+1}{12}\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)^2+\dots,
\end{aligned}$$

where Θ is the curvature form of the Poincaré metric in the line bundle $T_v\mathscr{T}_{g,n}\to\mathscr{T}_{g,n}$ (see Sect. 2). Using (2.5) we have

$$\begin{aligned}
\int_{\text{fiber}}\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)^2\left(\frac{\partial}{\partial\varepsilon_\mu},\frac{\partial}{\partial\bar{\varepsilon}_\nu}\right) &= \frac{\sqrt{-1}}{2\pi^2}\int_X \tfrac{1}{2}(\Delta_0+\tfrac{1}{2})^{-1}(\mu\bar{\nu})\varrho\\
&= \frac{\sqrt{-1}}{2\pi^2}\int_X \mu\bar{\nu}\varrho-\frac{\sqrt{-1}}{2\pi^2}\int_X \Delta_0(f_{\mu\bar{\nu}})\varrho\\
&= \frac{\sqrt{-1}}{2\pi^2}\langle\mu,\nu\rangle,
\end{aligned}\tag{5.3}$$

where we also used Lemma 2 and Green's formula to make sure that the last integral in (5.3) vanishes identically. Rewriting (5.3) in the form

$$\int_{\text{fiber}}\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)^2=\frac{1}{\pi^2}\omega_{\mathrm{WP}},$$

(cf. Corollary 5.11 in [5]) and taking (4.3) into account we obtain (5.1). Formula (5.2) follows now from (5.1) and (2.11).

Note that the cuspidal defect

$$\delta_k^{(0)}=\dim\operatorname{ind}\bar{\partial}_k-\int_{\text{fiber}}(\operatorname{ch}(T_v^{-k}\mathscr{T}_{g,n})\cdot\operatorname{td}(T_v\mathscr{T}_{g,n}))_{1,1}$$

in the Atiyah-Singer index theorem is equal to $-\frac{n}{2}$. Indeed,

$$\begin{aligned}
\delta_k^{(0)} &= (2k-1)(g-1)+(k-1)n-\left(\frac{1}{2}-k\right)\cdot\frac{\sqrt{-1}}{2\pi}\int_X\Theta\\
&= (2k-1)(g-1)+(k-1)n-(2k-1)\left(g+\frac{n}{2}-1\right)=-\frac{n}{2}.
\end{aligned}$$

Finally we present some algebraic geometry consequences of Theorem 1. First of all, because all bundles and metrics on $T_{g,n}$ considered here are invariant under the action of the Teichmüller modular group $\operatorname{Mod}_{g,n}$, formula (4.3) holds also on the moduli space $\mathscr{M}_{g,n}=T_{g,n}/\operatorname{Mod}_{g,n}$ (in the sense of orbifolds). Consider the universal curve $\mathscr{C}_g=\mathscr{M}_{g,1}$ and denote by ω the relative dualizing sheaf on $\mathscr{C}_g$, i.e. the line bundle dual to the vertical tangent bundle (along fibers of projection $p:\mathscr{C}_g\to\mathscr{M}_g$) on $\mathscr{C}_g$. Further, let us denote by $[\omega_{\mathrm{WP}}]$, $[\omega_{\mathrm{cusp}}]\in H^2(\mathscr{C}_g,\mathbb{R})\cong\mathbb{R}^2$ the cohomology classes of the closed $(1,1)$-forms ω_{WP} and ω_{cusp} on $\mathscr{C}_g$. Theorem

1 means that for the first Chern class $c_1(\lambda_k)$ of the line bundle λ_k on $\mathscr{C}_g$ the following formula holds:

$$c_1(\lambda_k) = \frac{6k^2 - 6k + 1}{12\pi^2} [\omega_{\mathrm{WP}}] - \frac{1}{9} [\omega_{\mathrm{cusp}}] . \tag{5.3}$$

From the exact sequence of sheafs

$$0 \to p^*(\Omega^1(\mathscr{M}_g)) \to \Omega^1(\mathscr{C}_g) \to \omega \to 0 ,$$

(where Ω^1 denotes the sheaf of holomorphic 1-forms and $p^*(\Omega^1(\mathscr{M}_g))$ is the inverse image of the sheaf $\Omega^1(\mathscr{M}_g)$) it follows that

$$\lambda_2 = p^*(\tilde{\lambda}_2) \otimes \omega$$

on $\mathscr{C}_g$ (here $\tilde{\lambda}_2$ denotes the determinant line bundle $\det\operatorname{ind}\bar{\partial}_2$ on $\mathscr{M}_g$). This formula together with Mumford's isomorphism $\tilde{\lambda}_2 \cong \tilde{\lambda}_1^{13}$ on $\mathscr{M}_g$ [14] and with the fact that $\lambda_1 = p^*(\tilde{\lambda}_1)$ yields the isomorphism

$$\lambda_2 \cong \lambda_1^{13} \otimes \omega . \tag{5.4}$$

Combining (5.3) for $k = 1, 2$ and (5.4) we get

$$\begin{aligned} \frac{1}{\pi^2} [\omega_{\mathrm{WP}}] &= 12 c_1(\lambda_1) + c_1(\omega) , \\ \frac{4}{3} [\omega_{\mathrm{cusp}}] &= c_1(\omega) , \end{aligned} \tag{5.5}$$

i.e. $\frac{1}{\pi^2} [\omega_{\mathrm{WP}}]$, $\frac{4}{3} [\omega_{\mathrm{cusp}}] \in H^2(\mathscr{C}_g, \mathbb{Z})$ are integral cohomology classes. In particular, it follows from (5.5) that for any compact Riemann surface X of genus $g \geq 2$, imbedded into $\mathscr{C}_g$ as a fiber a projection $p : \mathscr{C}_g \to \mathscr{M}_g$, we have

$$\frac{4}{3} \int_X \omega_{\mathrm{cusp}} = \frac{1}{\pi^2} \int_X \omega_{\mathrm{WP}} = 2g - 2 .$$

Substituting (5.5) into (5.3), we obtain the formula

$$c_1(\lambda_k) = (6k^2 - 6k + 1) c_1(\lambda_1) + \frac{k(k-1)}{2} c_1(\omega) ,$$

which leads to the isomorphism

$$\lambda_k \cong \lambda_1^{6k^2 - 6k + 1} \otimes \omega^{\frac{k(k-1)}{2}} \tag{5.6}$$

on $\mathscr{C}_g$, because according to Harer's result for $g \geq 3$ the Picard group $\operatorname{Pic}(\mathscr{C}_g) = H^2(\mathscr{C}_g, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ [20]. The isomorphism (5.6) is analogous to Mumford's isomorphism $\lambda_k \cong \lambda_1^{6k^2 - 6k + 1}$ on $\mathscr{M}_g$ (see [14]).

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