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Characterization of States of Infinite Boson Systems

I. On the Construction of States of Boson Systems

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Abstract. In a previous paper [11] it was shown that to each locally normal state of a boson system one can associate a point process that can be interpreted as the position distribution of the state. In the present paper the so-called conditional reduced density matrix of a normal or locally normal state is introduced. The whole state is determined completely by its position distribution and this function. There are given sufficient conditions on a point process Q and a function k ensuring the existence of a state such that Q is its position distribution and k its conditional reduced density matrix. Several examples will show that these conditions represent effective and useful criteria to construct locally normal states of boson systems. Especially, we will sketch an approach to equilibrium states of infinite boson systems. Further, we consider a class of operators on the Fock space representing certain combinations of position measurements and local measurements (observables related to bounded areas). The corresponding expectations can be expressed by the position distribution and the conditional reduced density matrix. This class serves as an important tool for the construction of states of (finite and infinite) boson systems. Especially, operators of second quantization, creation and annihilation operators are of this type. So, independently of the applications in the above context this class of operators may be of some interest.

1. Introduction

For a mathematical explanation of many structural effects, phase transitions, characterizations of equilibrium states, etc. of "large" quantum systems it turned out to be useful to have available mathematical models for infinite quantum systems. There are several approaches to the study of infinite particle systems. In quantum statistical mechanics a common approach is the concept of quasilocal C^*-algebras $\mathscr{A}$. Hereby $\mathscr{A}$ is the (norm completion of the) union of all local algebras representing measurements in bounded areas of the phase space. The local algebras are assumed to be (isomorphic to) the algebras of bounded linear operators on the symmetric Fock space over the bounded region. The state of such a system is given by a positive normalized linear functional on the quasilocal algebra.

Though this approach is fairly general and elegant it is quite difficult to construct in a rigorous way examples of states of infinite systems. Furthermore, even for standard examples it is complicated to calculate explicitly the expectations of numerous physically important measurements. The difficulties are connected for instance with the non-existence of the Lebesgue measure on $\mathbb{R}^\infty$. In classical statistical mechanics one could avoid these troubles by using the well-developed theory of infinite point processes (cf. [5, 6, 18, 27]).

The aim of our investigations was first to associate – as in classical statistical mechanics – to a state of an infinite quantum system an infinite point process that can be interpreted as the position distribution of the state, i.e. the point process contains all information connected with (finite or infinite) position measurements. This step was done in [11]. The position distribution alone never will characterize the whole state of the quantum system. We relate to a given state a function describing the behaviour in bounded regions given a fixed configuration outside. This function which we called *conditional reduced density matrix* (c.r.d.m.) determines together with the position distribution the state completely, and allows to calculate all kinds of conditional intensities. The point process and the c.r.d.m. are defined in such a way that all troubles connected with infinity are "packed" into the probabilistic part (the point process) where one can use the results of classical statistical mechanics and the theory of infinite point processes. By integrating the c.r.d.m. with respect to the position distribution one gets the reduced density matrix of the state well-known in statistical physics. The advantage of such a description of locally normal states is that one obtains effective sufficient conditions on a point process Q and a function k ensuring that Q is the position distribution and k the c.r.d.m. of a locally normal state. This allows an explicit construction of states. Several examples of states of infinite boson systems are given, and an approach to equilibrium states is sketched.

First we will introduce a class of operators combining position measurements with other local measurements. This class is used for the characterization of locally normal states. However, independently of this application this class of operators may be of some interest, especially because of its connections with stochastic calculus and Maassen's kernel approach.

All our considerations are reduced to locally normal states of boson systems without spin. We further assume that the local algebras consist of all bounded linear operators on the symmetric Fock space over the bounded regions of the phase space G which is assumed to be a complete separable metric space equipped with a locally finite diffuse measure.

Let us sketch briefly the contents of the single sections of the present paper.

In Sect. 2 we introduce some basic notions and notations from point process theory. Further, we describe the symmetric Fock space in a manner adapted to the language of counting measures and point processes, i.e. this space is defined as the L_2-space over the set of all finite counting measures on the phase space. This space describes (by definition) indistinguishable systems. In our opinion this representation of the Fock space allows a more convenient description of boson systems because one achieves indistinguishibility by definition and not – as in the usual approach – by using operators of symmetrization acting on a space describing distinguishable particles. Further, we introduce in Sect. 2 the concept of quasilocal algebras, and we refer some results obtained in [11] concerning the position distribution of locally normal states.

In Sect. 3 we define the above mentioned type of operators on the Fock space representing combinations of counting procedures with other measurements.

In Sect. 4 it is shown that many operators describing physically meaningful measurements are operators of the above type, as for instance creation and annihilation operators and operators of second quantization.

Some further notions from point process theory, especially the important class of so-called Σ'_ν-point processes and conditional intensity measures are introduced in Sect. 5.

The next section deals with a description of normal states. We introduce the c.r.d.m. of a normal state and give a characterization of a normal state by its position distribution and its c.r.d.m. Section 7 contains the main results. We give sufficient conditions on a function k and a point process Q such that there exists a uniquely determined locally normal state with position distribution Q and c.r.d.m. k (Theorem 7.3). In Sect. 8 we give a method for an explicit construction of a state having a given point process as its position distribution. This method is based on an application of Theorem 7.3. Examples resulting from this approach are given in Sect. 9. In the subsequent section we will sketch a characterization of equilibrium states. Details of these investigations will appear in a forthcoming paper.

The proofs of all results are contained in the remaining sections.

The present paper represents an enlarged, generalized, and revised version of the reports [9, 10].

2. Basic Notions and Notations

2.1. Counting Measures and Point Processes

Let G be a complete separable metric space. By $\mathfrak{G}$ we denote the σ-algebra of Borel subsets of G, $\mathfrak{B}$ is the ring of bounded sets in $\mathfrak{G}$. G will represent the phase space of the considered boson systems (i.e. the space of the positions of the bosons). In applications, G usually will be an Euclidean space $\mathbb{R}^d$, $d \geqq 1$.

2.1. Definition. Let $\Lambda \in \mathfrak{G}$. A *counting measure* on Λ is an integer-valued locally finite measure on $[G, \mathfrak{G}]$ concentrated on Λ. By M_Λ we denote the set of all counting measures on Λ, i.e.

$$M_\Lambda := \{\varphi\colon \varphi \text{ is a measure on } [G, \mathfrak{G}],\ \varphi(\Lambda^c) = 0,\ \varphi(B) \in \mathbb{N} \text{ for } B \in \mathfrak{B}\}$$

where $\mathbb{N} := \{0, 1, \ldots\}$ and $\Lambda^c := G \setminus \Lambda$.

The elements of M_Λ may be interpreted as locally finite point configurations in Λ. Indeed, a measure φ on $[G, \mathfrak{G}]$ is a counting measure on Λ if and only if φ can be written in the form $\varphi = \sum_{j \in J} \delta_{x_j}$ with J an at most countable index set, $x_j \in \Lambda$ for all $j \in J$ and the sequence $(x_j)_{j \in J}$ having no accumulation points [each bounded subset of Λ contains only finitely many elements from $(x_j)_{j \in J}$]. By δ_x we denote the Dirac measure in x.

We equip M_Λ with the σ-algebra $\mathfrak{M}_\Lambda$ generated by all sets of the type

$$\{\varphi \in M_\Lambda : \varphi(B) = n\}, \qquad B \in \mathfrak{B},\ n \in \mathbb{N}.$$

The set $M_\Lambda^f := \{\varphi \in M_\Lambda : \varphi(\Lambda^c) < \infty\}$ of finite counting measures on Λ is obviously a measurable subset of M_Λ.

In the case $\Lambda = G$ we omit the index Λ in the above notations. For $\Lambda \in \mathfrak{G}$ we denote by v_Λ the restriction from M to M_Λ, i.e.

$$v_\Lambda(\varphi) := \varphi(\cdot \cap \Lambda) \quad (\varphi \in M). \tag{2.1}$$

We also write φ_Λ instead of $v_\Lambda(\varphi)$. Obviously, v_Λ is measurable.

Further, we set for arbitrary $\Lambda \in \mathfrak{G}$ ${}_\Lambda\mathfrak{M} := \{v_\Lambda^{-1}(Y) : Y \in \mathfrak{M}_\Lambda\}$. $({}_\Lambda\mathfrak{M})_{\Lambda \in \mathfrak{B}}$ is an increasing net of σ-subalgebras of $\mathfrak{M}$ ($\mathfrak{B}$ ordered with respect to inclusion), and ${}_G\mathfrak{M} = \mathfrak{M}$.

In the sequel we have to distinguish carefully between ${}_\Lambda\mathfrak{M}$ and $\mathfrak{M}_\Lambda$. Counting measures from a set belonging to $\mathfrak{M}_\Lambda$ have no mass points outside Λ while the sets from ${}_\Lambda\mathfrak{M}$ are determined by the behaviour of their elements inside Λ. More precisely, we have the following characterization of ${}_\Lambda\mathfrak{M}$ which is very easy to verify.

2.2. Lemma. *Let $\Lambda \in \mathfrak{G}$, $Y \in \mathfrak{M}$. The following conditions are equivalent:*
(i) $Y \in {}_\Lambda\mathfrak{M}$;
(ii) $Y = \{\varphi + \phi : \varphi \in v_\Lambda(Y),\ \phi \in M_{\Lambda^c}\}$;
(iii) *For all* $\varphi \in M$ $\chi_Y(\varphi) = \chi_{v_\Lambda(Y)}(\varphi_\Lambda)$,
where χ_X denotes the indicator function of a set X.

2.3. Definition. A *point process* is a probability measure on $[M, \mathfrak{M}]$. A point process P is called *finite* if $P(M^f) = 1$. P is called a point process on $\Lambda \in \mathfrak{G}$ if $P(M_\Lambda) = 1$.

According to the interpretation of counting measures a point process on Λ is the distribution of a random point system in the phase space Λ.

For details and further information about counting measures and point processes we recommend the monographs [21, 26].

2.2. The Fock Space Over Λ

The notion of the Fock space we want to introduce now is adapted to the language of counting measures.

Let ν be an arbitrary but in the sequel fixed locally finite diffuse measure on $[G, \mathfrak{G}]$ [i.e. $\nu(B) < \infty$ for all $B \in \mathfrak{B}$, $\nu(\{x\}) = 0$ for all $x \in G$].

For each $\Lambda \in \mathfrak{G}$ we define a σ-finite measure F_Λ on $[M, \mathfrak{M}]$ by

$$F_\Lambda(Y) := \chi_Y(\mathbb{O}) + \sum_{n \geqq 1} \frac{1}{n!} \int_{\Lambda^n} \nu^n(d[x_1, \ldots, x_n]) \chi_Y\left(\sum_{j=1}^n \delta_{x_j}\right) \quad (Y \in \mathfrak{M}), \tag{2.2}$$

where $\mathbb{O}$ denotes the empty configuration, i.e. $\mathbb{O} \in M$, $\mathbb{O}(G) = 0$.

Observe that F_Λ is concentrated on M_Λ^f and that for $\Lambda \in \mathfrak{B}$ the measure F_Λ is finite ($F_\Lambda(M) = \exp\{\nu(\Lambda)\}$. In the case $G = \mathbb{R}^d$, ν usually will be the d-dimensional Lebesgue measure.

The set of complex numbers is denoted by $\mathbb{C}$, and for a complex-valued function f by $\bar{f}$ its complex conjugate is denoted.

2.4. Definition. Let $\Lambda \in \mathfrak{G}$. The set

$$\mathcal{M}_\Lambda := \{\psi : M \to \mathbb{C},\ \Psi \text{ measurable},\ \operatorname{supp} \Psi \subseteqq M_\Lambda^f,\ \int F_\Lambda(d\varphi)\, |\Psi(\varphi)|^2 < \infty\}$$

endowed with the scalar product

$$(\Psi^1, \Psi^2)_\Lambda := \int F_\Lambda(d\varphi)\overline{\Psi^1(\varphi)}\Psi^2(\varphi)$$

we call the (symmetric) *Fock space over* Λ.

In the case $\Lambda = G$ we again omit the index Λ. A similar definition of the Fock space over $G = \mathbb{R}^1$ one can find in [19, 24, 25]. In [11], (Remark 2.5) Definition 2.4 is compared with the usual definition (cf. instance [15, 1, 2, 3]).

In the sequel we often will use the following property of the measures F_Λ:

2.5. Lemma. *Let* $\Lambda, \Lambda' \in \mathfrak{G}$, $\Lambda \cap \Lambda' = \emptyset$, *and let* $h: M \times M \to \mathbb{C}$ *be a* $F_\Lambda \times F_{\Lambda'}$-*integrable function. Then*

$$\int (F_\Lambda \times F_{\Lambda'})(d[\varphi_1, \varphi_2])h(\varphi_1, \varphi_2) = \int F_{\Lambda \cup \Lambda'}(d\varphi)h(\varphi_\Lambda, \varphi_{\Lambda'}). \tag{2.3}$$

Now, for $\Psi \in \mathcal{M}$ and $\varphi \in M$ we denote by $\Psi_\phi: M \to \mathbb{C}$ the mapping defined by

$$\Psi_\phi(\varphi) := \Psi(\varphi + \phi) \quad (\varphi \in M). \tag{2.4}$$

2.6. Remark. Since Ψ is concentrated on M^f we have $\Psi_\phi = 0$ for all $\phi \notin M^f$. Obviously, $\Psi_{\mathbb{O}} = \Psi$. However, $\Psi \in \mathcal{M}$ and $\phi \in M^f$ does not imply in general $\Psi_\phi \in \mathcal{M}$. One can construct easily $\Psi \in \mathcal{M}$ such that for all $\phi \in M^f$, $\phi \neq \mathbb{O}$ one has $\Psi_\phi \notin \mathcal{M}$. We will give now a class of functions Ψ with the property that for all $\phi \in M$, $\Psi_\phi \in \mathcal{M}$.

For arbitrary $\Lambda \in \mathfrak{G}$ we set

$$\mathcal{M}_\Lambda^f := \{\Psi \in \mathcal{M}_\Lambda : \operatorname{supp} \Psi \subseteqq M_\Lambda^m \textit{ for some } m \in \mathbb{N}\}, \tag{2.5}$$

where $M_\Lambda^m := \{\varphi \in M_\Lambda : \varphi(\Lambda) \leqq m\}$. Functions from $\mathcal{M}_\Lambda^f$ are called usually *finite-particle vectors.*

2.7. Lemma. *For all* $\Lambda \in \mathfrak{G}$, $\phi \in M$, *and* $\Psi \in \mathcal{M}_\Lambda^f$ *it holds* $\Psi_\phi \in \mathcal{M}_\Lambda^f$.

The set $\mathcal{M}_\Lambda^f$ is obviously dense in the Fock space $\mathcal{M}_\Lambda$ and it will represent the domain of definition for many operators considered in Sect. 3.

2.3. Quasilocal Algebras

For arbitrary measurable $f: M \to \mathbb{C}$ denote by O_f the operator of multiplication by f, i.e.

$$(O_f \Psi)(\varphi) = f(\varphi)\Psi(\varphi) \quad (\Psi \in \mathcal{M}, \varphi \in M). \tag{2.6}$$

If f is the indicator function of a set $Y \in \mathfrak{M}$, i.e. $f = \chi_Y$, we will denote O_{χ_Y} for brevity by O_Y.

Now, by $\mathcal{L}(\mathcal{M})$ we denote the algebra of all bounded linear operators on $\mathcal{M}$.

For each $\Lambda \in \mathfrak{G}$ we set

$$\mathcal{A}_\Lambda := \{A \in \mathcal{L}(\mathcal{M}) : A = O_{M_\Lambda} A O_{M_\Lambda}\}. \tag{2.7}$$

$\mathcal{A}_\Lambda$ may be identified with the algebra $\mathcal{L}(\mathcal{M}_\Lambda)$ of bounded linear operators on $\mathcal{M}_\Lambda$. $\mathcal{L}(\mathcal{M}_\Lambda)$ is a von Neumann-algebra. We have $\mathcal{A}_G = \mathcal{L}(\mathcal{M})$. For definition and properties of von Neumann-algebras cf. instance [28, 4, 7, Chap. 2, Sect. 1.5, 2, Chap. 2.4].

It is not difficult to show [11, Proposition 2.6] that for each $\Lambda \in \mathfrak{G}$ there exists a unique isomorphism I_Λ between $\mathcal{M}_\Lambda \otimes \mathcal{M}_{\Lambda^c}$ and such that for all $\Psi \in \mathcal{M}_\Lambda$, $\hat{\Psi} \in \mathcal{M}_{\Lambda^c}$, $\varphi \in M$,

$$I_\Lambda(\Psi \otimes \hat{\Psi})(\varphi) = \Psi(\varphi_\Lambda)\hat{\Psi}(\varphi_{\Lambda^c}) \tag{2.8}$$

($\otimes$ denotes the tensor product).

This allows a natural embedding of $\mathcal{A}_\Lambda$ into $\mathcal{L}(\mathcal{M})$. More precisely, for $\Lambda \in \mathfrak{B}$ we define a mapping $J_\Lambda : \mathcal{A}_\Lambda \to \mathcal{L}(\mathcal{M})$ by

$$J_\Lambda A := I_\Lambda(A \otimes \mathbb{1}_{\Lambda^c}) I_\Lambda^{-1} \quad (A \in \mathcal{A}_\Lambda), \tag{2.9}$$

where $\mathbb{1}_{\Lambda^c} = O_{M_{\Lambda^c}}$ is the identity in $\mathcal{A}_{\Lambda^c}$.

2.8. Definition. Let $\Lambda \in \mathfrak{G}$. The subalgebra ${}_\Lambda\mathcal{A}$ of $\mathcal{L}(\mathcal{M})$ defined by

$${}_\Lambda\mathcal{A} := \{J_\Lambda A : A \in \mathcal{A}_\Lambda\} \tag{2.10}$$

is called the *local algebra* on Λ.

Identifying isomorphic spaces we simply could write ${}_\Lambda\mathcal{A} = \mathcal{A}_\Lambda \otimes \mathbb{1}_{\Lambda^c}$.

2.9. Definition. The pair $[\mathcal{A}, ({}_\Lambda\mathcal{A})_{\Lambda \in \mathfrak{B}}]$, where $\mathcal{A}$ is the uniform closure of $\bigcup_{\Lambda \in \mathfrak{B}} {}_\Lambda\mathcal{A}$ we call the *quasilocal algebra* over the Fock space $\mathcal{M}$.

2.10. Remark. The uniform closure is the closure with respect to the operator norm in $\mathcal{L}(\mathcal{M})$. *If we assume that the phase space G is unbounded then $\mathcal{A}$ is a proper* subset of $\mathcal{L}(\mathcal{M})$. A more general definition of a quasilocal algebra is given e.g. in [2, Definition 2.6.3]. More information one can find in [33–36].

Finally, we want to make some remarks on multiplication operators. It is well-known that $\{O_f : f \in \mathcal{M}^b\}$ with

$$\mathcal{M}^b = \{f : M \to \mathbb{C},\ f \textit{ measurable and } F\textit{-a.e. bounded}\} \tag{2.11}$$

is the maximal set of multiplication operators contained in $\mathcal{L}(\mathcal{M})$ (cf. instance [22]). The relation between ${}_\Lambda\mathcal{A}$ and $\mathcal{A}_\Lambda$ corresponds to the relation between ${}_\Lambda\mathfrak{M}$ and $\mathfrak{M}_\Lambda$. Observe that for all $\Lambda \in \mathfrak{G}$ and $Y \in \mathfrak{M}_\Lambda$ one has $O_Y \in \mathcal{A}_\Lambda$.

2.11. Proposition. *Let $g \in \mathcal{M}^b$, $\Lambda \in \mathfrak{G}$. The following conditions are equivalent:*
(i) $O_g \in {}_\Lambda\mathcal{A}$.
(ii) *There exists a ${}_\Lambda\mathfrak{M}$-measurable function $\hat{g}$ such that $O_g = O_{\hat{g}}$.*
(iii) $O_g = J_\Lambda O_{g \circ v_\Lambda}$.

2.12. Remark. Observe that for $g \in \mathcal{M}^b$ we have $O_g = O_{g\chi_{M_f}}$. However, $g\chi_{M_f}$ is not ${}_\Lambda\mathfrak{M}$-measurable for any $\Lambda \in \mathfrak{B}$ (provided the whole space G is unbounded). Since ${}_\Lambda\mathcal{A}$ "does not feel" this ${}_\Lambda\mathfrak{M}$-measurability we cannot conclude from (i) that g is ${}_\Lambda\mathfrak{M}$-measurable but only (ii).

2.4. Locally Normal States and Their Position Distribution

Now, let $\tilde{\mathcal{A}}$ be a C^*-subalgebra of $\mathcal{L}(\mathcal{M})$. A positive normalized linear functional η on $\tilde{\mathcal{A}}$ is called a state on $\tilde{\mathcal{A}}$. A state η on a von Neumann subalgebra $\tilde{\mathcal{A}}$ of $\mathcal{L}(\mathcal{M})$ is called *normal* if there exists a density matrix ϱ on $\mathcal{M}$ (i.e. a positive trace-class operator with trace one) such that

$$\eta(A) = \mathrm{Tr}(\varrho A) \quad (A \in \tilde{\mathcal{A}}). \tag{2.12}$$

[By Tr we denote the trace in $\mathcal{L}(\mathcal{M})$].

2.13. Definition. A state ω on $\mathscr{A}$ is said to be *locally normal* if for all $\Lambda \in \mathfrak{B}$ the restriction ${}_\Lambda\omega$ of ω to ${}_\Lambda\mathscr{A}$ is a normal state, i.e. for all $\Lambda \in \mathfrak{B}$ there exists a density matrix ${}_\Lambda\varrho$ on $\mathscr{M}$ such that

$$\omega(A) = {}_\Lambda\omega(A) = \mathrm{Tr}({}_\Lambda\varrho A) \quad (A \in {}_\Lambda\mathscr{A}) \tag{2.13}$$

(cf. [2, Definitions 2.6.6, 2.4.20 and Theorem 2.4.21]).

2.14. Definition. Let ω be a locally normal state on $\mathscr{A}$ and Q a point process. Q is said to be the *position distribution of* ω if for all $Y \in \mathfrak{M}$ such that $O_Y \in \mathscr{A}$ we have

$$Q(Y) = \omega(O_Y). \tag{2.14}$$

Now we mention the main result of [11].

2.15. Theorem. *Let ω be a locally normal state on $\mathscr{A}$. There exists exactly one point process Q_ω being the position distribution of ω.*

2.16. Remark. The above result was shown in [11] in the case $G = \mathbb{R}^d$, ν the Lebesgue measure on $\mathbb{R}^d$. However, the proof of Theorem 2.15 for a more general phase space G requires only minor notational changes because all results from point process theory used in the proof are valid in this more general case.

Moreover, in [11] there is given a more detailed characterization of the class of point processes which may occur as position distributions of locally normal states of boson systems.

3. A Class of Operators on the Fock Space

In this section we will introduce operators $S(Y, A)$ corresponding to $A \in \mathscr{A}_\Lambda$ and sets $Y \in \mathfrak{M}$ that can be interpreted as a combination of A with the position measurement 0_Y. For the special case $Y = M$ and A being a self-adjoint operator concentrated on the L_2-space $\mathscr{M}_1$ over one-point configurations this operator coincides with the usual second quantization of A. Further, we will show that for A being the operator of a position measurement the expectation $\omega(S(Y, A))$ of $S(Y, A)$ in the state ω can be expressed with the aid of the compound Campbell measure of the position distribution Q_ω. Usually, the operator $S(Y, A)$ will be unbounded but we will give sufficient conditions on A and Y ensuring that $S(Y, A)$ will be bounded again. As examples of operators $S(Y, A)$ we will introduce (generalized) creation and annihilation operators. Especially, it will be shown that each element of ${}_\Lambda\mathscr{A}$ is an operator of the type $S(Y, A)$ for certain Y and A. Consequently, the whole state ω will be determined by the expectations $\omega(B)$ with B being operators of the type $S(Y, A)$.

The domain of an operator B on $\mathscr{M}$ will be denoted by $D(B)$.

3.1. Definition. Let $\Lambda \in \mathfrak{G}$, $Y \in \mathfrak{M}$, $A \in \mathscr{A}_\Lambda$. By $S_\Lambda(Y, A)$ we denote the set of all (possibly unbounded) operators B on $\mathscr{M}_\Lambda$ such that

(i) $$D(B) \supseteq \mathscr{M}_\Lambda^f,$$

(ii) $$(B\Psi)(\varphi) = \sum_{\hat\varphi \subseteq \varphi} \chi_Y(\hat\varphi)(A\Psi_{\hat\varphi})(\varphi - \hat\varphi) \quad (\Psi \in \mathscr{M}_\Lambda^f,\ F\text{-a.a. } \varphi),$$

(iii) $$B\Psi = \lim_{m\to\infty} B(O_{M_\Lambda^m}\Psi) \quad (\Psi \in D(B)).$$

In the case $\Lambda = G$ we write $S(Y, A)$ instead of $S_G(Y, A)$.

$\mathcal{M}^f_\Lambda$ was defined in (2.5). $\hat{\varphi} \subseteq \varphi$ means that $\hat{\varphi}$ is a subconfiguration of φ, i.e. $\varphi - \hat{\varphi} \in M$.

3.2. Remarks. a) Condition (i) is a very natural one because the finite particle vectors should belong to the domain of any reasonable operator on the Fock space. Further, (i) ensures that all operators from $S_\Lambda(Y, A)$ are densely defined.
b) Two operators from $S_\Lambda(Y, A)$ may differ only with respect to their domains of definition. Because of (ii) they coincide on $\mathcal{M}^f_\Lambda$. Since for each $m \in \mathbb{N}, O_{M^m_\Lambda}\Psi \in \mathcal{M}^f_\Lambda$ condition (iii) ensures that for $B_1, B_2 \in S_\Lambda(Y, A)$ it holds $B_1 = B_2$ if $D(B_1) = D(B_2)$.
c) If $B_1, B_2 \in S_\Lambda(Y, A)$ and B_1 is unbounded then B_2 is unbounded too. In what follows bounded operators are always assumed to be defined on the whole space. So if $B \in S_\Lambda(Y, A)$ and B is bounded then $S_\Lambda(Y, A) = \{B\}$, and we write simply $B = S_\Lambda(Y, A)$.
d) Also in the case that $S_\Lambda(Y, A)$ consists of unbounded operators for $\Psi \in \mathcal{M}^f_\Lambda$ we will write $S_\Lambda(Y, A)\Psi$ though $S_\Lambda(Y, A)$ denotes the set of operators. This will not lead to misunderstandings because they all coincide on $\mathcal{M}^f_\Lambda$ (ii)).
e) For each $\Psi \in \mathcal{M}^f_\Lambda$ the right side in (ii) is a well-defined function from M into $\mathbb{C}$. First observe that for $\varphi \notin M^f$ the right side is equal to zero because $\Psi_{\hat{\varphi}} \equiv 0$ (if $\hat{\varphi} \notin M^f$) or $(A\Psi_{\hat{\varphi}})(\varphi - \hat{\varphi}) = 0$ (if $\varphi - \hat{\varphi} \notin M^f$). Consequently, the sum over all $\hat{\varphi} \subseteq \varphi$ (the sum over all subconfiguration of φ) is a finite one. Further, from Lemma 2.7 we conclude that for all $\hat{\varphi}$ we have $\Psi_{\hat{\varphi}} \in \mathcal{M}_\Lambda$ (even $\Psi_{\hat{\varphi}} \in \mathcal{M}^f_\Lambda$). Thus $A\Psi_{\hat{\varphi}} \in \mathcal{M}_\Lambda$. However, there may exist $\Psi \in \mathcal{M}^f_\Lambda$ such that the function on the right side of (ii) is not an element of $\mathcal{M}$ (i.e. that it is not square integrable). In this case $S_\Lambda(Y, A)$ is the empty set.
f) Condition (ii) defining all operators from $S_\Lambda(Y, A)$ on $\mathcal{M}^f_\Lambda$ may be illustrated as follows: We "pick" out from the configuration φ one by one all subconfigurations $\hat{\varphi}$ and check whether $\hat{\varphi}$ belongs to Y or not. To the rest configuration $\varphi - \hat{\varphi}$ we apply the operator A holding $\hat{\varphi}$ fixed.

Operators concentrated on finite particle vectors will play an important role. For arbitrary $\Lambda \in \mathfrak{G}$ we set

$$\mathcal{A}^f_\Lambda := \{A \in \mathcal{A}_\Lambda : A = O_{M^m_\Lambda} A O_{M^m_\Lambda} \text{ for some } m \in \mathbb{N}\}, \tag{3.1}$$

and for $\Lambda \in \mathfrak{B}$

$$_\Lambda\mathcal{A}^f := \{J_\Lambda A : A \in \mathcal{A}^f_\Lambda\}. \tag{3.2}$$

Analogously, we set for arbitrary $\Lambda \in \mathfrak{G}$,

$$\mathfrak{M}^f_\Lambda := \{Y \in \mathfrak{M}_\Lambda : Y \subseteq M^m_\Lambda \text{ for some } m \in \mathbb{N}\}, \tag{3.3}$$

$$_\Lambda\mathfrak{M}^f := \{v_\Lambda^{-1}(Y) : Y \in \mathfrak{M}^f_\Lambda\}. \tag{3.4}$$

We will collect now some properties of the operator classes $S_\Lambda(Y, A)$.

3.3. Proposition. *Let $\Lambda \in \mathfrak{G}$, $Y \in \mathfrak{M}^f_\Lambda$, $A \in \mathcal{A}^f_\Lambda$. Then $S_\Lambda(Y, A) \in \mathcal{A}^f_\Lambda$. If A is self-adjoint then $S_\Lambda(Y, A)$ is self-adjoint.*

If A is positive then $S_\Lambda(Y, A)$ will be positive too.

The connection between S_Λ and S is given by the following proposition:

3.4. Proposition. *Let $\Lambda \in \mathfrak{G}$, $A \in \mathcal{A}^f_\Lambda$, $Y \in \mathfrak{M}^f_\Lambda$. Then*

$$J_\Lambda S_\Lambda(Y, A) = S(Y, J_\Lambda A) = S(v_\Lambda^{-1} Y, A).$$

Identifying isomorphic operators, Proposition 3.4 could be written simply in the form $S_\Lambda(Y, A) \otimes \mathbb{1}_{\Lambda^c} = S(Y, A \otimes \mathbb{1}_{\Lambda^c}) = S(Y + M_{\Lambda^c}, A)$. As an immediate consequence of 3.3 and 3.4 we obtain the following result:

3.5. Proposition. *Let $\Lambda \in \mathfrak{G}$, $Y \in \mathfrak{M}_\Lambda^f$, $A \in \mathscr{A}_\Lambda^f$. Then $S(Y, A) \in {}_\Lambda\mathscr{A}^f$. If A is self-adjoint (positive) $S(Y, A)$ will be self-adjoint (positive).*

Each operator from ${}_\Lambda\mathscr{A}$ itself is an operator of the type $S(Y, A)$. Indeed, we have the following result:

3.6. Proposition. *Let $\Lambda \in \mathfrak{B}$. For all $A \in {}_\Lambda\mathscr{A}$ we have*

$$A = S(M_{\Lambda^c}, O_{M_\Lambda} A O_{M_\Lambda}).$$

However, the above representation of A is not the only possible one. For all $\Lambda' \supseteqq \Lambda$, $\Lambda' \in \mathfrak{G}$ we get

$$A = S(M_{(\Lambda')^c}, O_{M_{\Lambda'}} A O_{M_{\Lambda'}}) \qquad (A \in {}_\Lambda\mathscr{A}).$$

Further, we get for all $A \in {}_\Lambda\mathscr{A}$ $A = S(\{\mathbf{O}\}, A)$ (taking $\Lambda' = G$). Proposition 3.6 allows another description of the isomorphism J_Λ between $\mathscr{A}_\Lambda$ and ${}_\Lambda\mathscr{A}$.

3.7. Proposition. *For all $\Lambda \in \mathfrak{G}$, $A \in \mathscr{A}_\Lambda$ we have $J_\Lambda A = S(M_{\Lambda^c}, A)$.*

Now we deal with the question what one can say about Y and A if $S(Y, A) \in {}_\Lambda\mathscr{A}$ for some $\Lambda \in \mathfrak{B}$.

3.8. Proposition. *Let $\Lambda \in \mathfrak{B}$, $A \in \mathscr{A}_\Lambda$, $Y \in \mathfrak{M}$. If $S(Y, A) \in {}_\Lambda\mathscr{A}$ then there exists an $\tilde{Y} \in {}_\Lambda\mathfrak{M}$ such that $O_{\tilde{Y}} = O_Y$.*

Proposition 3.8 is intuitively clear. Since each operator from ${}_\Lambda\mathscr{A}$ is "outside Λ" the identical operator there may occur "real" measurements only in the region Λ. Therefore if A is concentrated on $\mathscr{M}_\Lambda$ the set Y may contain only information about positions of points in Λ.

"Conversely" to 3.8 we have

3.9. Proposition. *Let $\Lambda \in \mathfrak{G}$, $A \in \mathscr{L}(\mathscr{M})$, $Y \in {}_\Lambda\mathfrak{M}$. If $S(Y, A) \in {}_\Lambda\mathscr{A}$ then there exists an operator $B \in \mathscr{A}_\Lambda$ such that $S(Y, A) = S(Y, B)$.*

However, we have to remark that $S(Y, A) \in {}_\Lambda\mathscr{A}$ for some $\Lambda \in \mathfrak{B}$ does not imply in general neither the conclusion of Proposition 3.8 nor of 3.9.

3.10. Proposition. *Let $\Lambda, \Lambda' \in \mathfrak{G}$, $\Lambda \cap \Lambda' = \emptyset$, $Y \in \mathfrak{M}_{\Lambda'}$, $A \in \mathscr{A}_\Lambda$. Then $S(Y, A) \in \mathscr{A}_{\Lambda \cup \Lambda'}$.*

So if A and Y (not necessarily from $\mathscr{A}_\Lambda^f$ respectively $\mathfrak{M}_{\Lambda'}^f$) "act" in disjoint regions $S(Y, A)$ will be bounded.

Finally, we want to discuss operators of the type $S(Y, A)$ if A corresponds to a position measurement. We remarked above that for general $Y \in \mathfrak{M}$ and $A \in \mathscr{L}(\mathscr{M})$ the class $S(Y, A)$ may be empty. However, for A being a multiplication operator with a bounded function on M $S(Y, A)$ always will be non-void.

3.11. Proposition. *Let $Y \in \mathfrak{M}$, $f \in \mathscr{M}^b$ (cf. (2.11)). Then $S(Y, O_f) \neq \emptyset$. Especially, for $Y_1, Y_2 \in \mathfrak{M}$ $S(Y_1, O_{Y_2}) \neq \emptyset$ and $S(Y_1, O_{Y_2})$ consists of positive (possibly unbounded) operators on $\mathscr{M}$.*

The expectation $\omega(S(Y, A))$ may be expressed with the aid of the so-called compound Campbell measure of the position distribution if A corresponds to a position measurement. This will be shown in Sect. 5.

3.12. Remark. There is close connection between the operator class introduced above and the stochastic calculus (cf. instance [39]). Each operator of the type $S(Y, A)$ can be expressed in the form $\mathscr{S}^c(O_Y \otimes A)\mathscr{D}^c$, where $\mathscr{S}^c$ and $\mathscr{D}^c$ are generalized Skorohod integrals and Malliavin derivatives. For details we refer to [12].

3.13. Remark. If $A \in \mathscr{L}(\mathscr{M})$ is an integral operator with kernel k then we get for all $Y \in \mathfrak{M}$, $\Psi \in \mathscr{M}$, $\varphi \in M$,

$$S(Y,A)\Psi(\varphi) = \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\varphi - \hat{\varphi}) \int F(d\tilde{\varphi}) k(\hat{\varphi}, \tilde{\varphi}) \Psi(\varphi - \hat{\varphi} + \tilde{\varphi}).$$

Especially, for $Y = M$ we get

$$S(M,A)\Psi(\varphi) = \sum_{\hat{\varphi} \subseteq \varphi} \int F(d\tilde{\varphi}) k(\hat{\varphi}, \tilde{\varphi}) \Psi(\varphi - \hat{\varphi} + \tilde{\varphi}). \tag{3.5}$$

Operators of the type (3.5) were considered by Maassen ([24, 25]) for the case that $G = \mathbb{R}^1$ and ν the Lebesgue measure.

4. Examples

In the sequel we denote for $\Lambda \in \mathfrak{G}$, $n \in \mathbb{N}$ by $M_{n,\Lambda} := \{\varphi \in M_\Lambda : \varphi(\Lambda) = n\}$ the set of n-particle configurations, and we put $M_n = M_{n,G}$.

4.1. The Number Operator

Let $A = O_{M_1}$. Obviously, we have $O_{M_1} \in \mathscr{A}^f$. For each $Y \in \mathfrak{M}$ we consider the class $S(Y, O_{M_1})$. For $\Psi \in \mathscr{M}^f$ and F-*a.a.* φ we obtain

$$\begin{aligned} S(Y, O_{M_1})\Psi(\varphi) &= \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi}) (O_{M_1} \Psi_{\hat{\varphi}})(\varphi - \hat{\varphi}) \\ &= \Psi(\varphi) \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi}) \chi_{M_1}(\varphi - \hat{\varphi}) \\ &= \Psi(\varphi) \int \varphi(dx) \chi_Y(\varphi - \delta_x). \end{aligned} \tag{4.1}$$

In the case $Y = M$ we get from (4.1) for all $\Psi \in \mathscr{M}^f$ and for F-*a.a.* φ

$$S(M, O_{M_1})\Psi(\varphi) = \varphi(G)\Psi(\varphi). \tag{4.2}$$

Consequently $S(M, O_{M_1})$ is the set of all number operators on $\mathscr{M}$ (which differ only with respect to their domains of definitions). For each $B \in S(M, O_{M_1})$ we have on $\mathscr{M}^f$ the representation (4.2). If $D(B) \supset \mathscr{M}^f$ we get $B\Psi$ for $\Psi \in D(B) \setminus \mathscr{M}^f$ from condition (iii) in Definition 3.1.

Analogously, the number operator (respectively the set of number operators) on $\mathscr{M}$ counting the particles in a region $\Lambda \in \mathfrak{B}$ (respectively $\Lambda \in \mathfrak{G}$) will be $S(M, O_{M_{1,\Lambda}})$. Indeed, for all $\Psi \in \mathscr{M}^f$ and F-*a.a.* φ

$$\begin{aligned} S(M, O_{M_{1,\Lambda}})\Psi(\varphi) &= \sum_{\hat{\varphi} \subseteq \varphi} \chi_M(\hat{\varphi}) \chi_{M_1}(\varphi - \hat{\varphi}) \chi_{M_\Lambda}(\varphi - \hat{\varphi}) \Psi(\varphi) \\ &= \Psi(\varphi) \int_\Lambda \varphi(dx) = \varphi(\Lambda)\Psi(\varphi). \end{aligned}$$

4.2. (Generalized) Creation Operators

Let $g \in \mathscr{M}_1$, and denote by $A^*(g)$ the operator on $\mathscr{M}$, defined by

$$(A^*(g)\Psi)(\varphi) = g(\varphi)\Psi(\mathbb{O}) \qquad (\Psi \in \mathscr{M}, \varphi \in M) \tag{4.3}$$

[$\mathbb{O}$ denotes the zero measure in M, i.e. $\mathbb{O}(G) = 0$].

Observe that $(A^*(g)\Psi)(\varphi) = 0$ for $\varphi \notin M_1$. Because of $A^*(g) = O_{M_1} A^*(g) O_{M_0}$ we have $A^*(g) \in \mathscr{A}^f$, and if $\operatorname{supp} g \subseteq M_\Lambda$ for some $\Lambda \in \mathfrak{G}$ then $A^*(g) \in \mathscr{A}^f_\Lambda$.

Let $Y \in \mathfrak{M}$. For each $\Psi \in \mathcal{M}^f$ and F-a.a. φ we obtain

$$S(Y, A^*(g))\Psi(\varphi) = \int \varphi(dx)\chi_Y(\varphi - \delta_x)g(\delta_x)\Psi(\varphi - \delta_x)$$
$$(S(Y, A^*(g))\Psi(\mathbf{0}) = 0). \tag{4.4}$$

For $Y \in \mathfrak{M}^f$ $S(Y, A^*(g))$ will be a bounded operator (Proposition 3.3). For general $Y \in \mathfrak{M}$ $S(Y, A^*(g))$ usually will be a set of unbounded operators on $\mathcal{M}$ having on $\mathcal{M}^f$ the representation (4.4). The "usual" creation operator we get by setting $Y = M$. Indeed, from (4.4) we conclude that for all $\Psi \in \mathcal{M}^f$ and F-a.a. φ we have

$$\begin{aligned} S(M, A^*(g))\Psi(\varphi) &= \int \varphi(dx)g(\delta_x)\Psi(\varphi - \delta_x) \\ &= \sum_{x:\varphi(\{x\})>0} g(\delta_x)\Psi(\varphi - \delta_x). \end{aligned} \tag{4.5}$$

The operator $B \in S(M, A^*(g))$ with $D(B) = \mathcal{M}^f$ we denote as usual by $a^*(g)$.

Let $\Lambda \in \mathfrak{G}$, $g \in \mathcal{M}_{1,\Lambda}$. For $\Psi \in \mathcal{M}_\Lambda^f$ and F-a.a. φ we get

$$\begin{aligned} S_\Lambda(M_\Lambda, A^*(g))\Psi(\varphi) &= \int \varphi(dx)\chi_{M_\Lambda}(\varphi - \delta_x)\chi_{M_\Lambda}(\delta_x)g(\delta_x)\Psi(\varphi - \delta_x) \\ &= \int \varphi(dx)g(\delta_x)\Psi(\varphi - \delta_x). \end{aligned}$$

Observe that because of $\Psi \in \mathcal{M}_\Lambda^f$ and $g \in \mathcal{M}_\Lambda$ the expression above will be equal to zero if $\varphi \notin M_\Lambda$. $S_\Lambda(M_\Lambda, A^*(g))$ represents the set of creation operators on $\mathcal{M}_\Lambda$ corresponding to g. The operator $B \in S_\Lambda(M_\Lambda, A^*(g))$ with $D(B) = \mathcal{M}_\Lambda^f$ we will denote by $a_\Lambda^*(g)$.

4.3. *(Generalized) Annihilation Operators*

Let $g \in \mathcal{M}_1$. We define an operator $A(g) \in \mathcal{L}(\mathcal{M})$ by setting

$$A(g)\Psi(\varphi) = \begin{cases} \int \nu(dx)\overline{g(\delta_x)}\Psi(\delta_x) & \Psi \in M,\ \varphi = \mathbf{0} \\ 0 & \Psi \in \mathcal{M},\ \varphi \neq \mathbf{0}. \end{cases} \tag{4.6}$$

(ν denotes again the locally finite diffuse measure on G – cf. Sect. 2.2). Observe that $A(g) = O_{M_0}A(g)O_{M_1}$. So we have $A(g) \in \mathcal{A}^f$, and if $g \in \mathcal{M}_{1,\Lambda}$ for some $\Lambda \in \mathfrak{G}$ then $A(g) \in \mathcal{A}_\Lambda^f$.

Let $Y \in \mathfrak{M}$. For all $\Psi \in \mathcal{M}^f$ and F-a.a. φ we get

$$S(Y, A(g))\Psi(\varphi) = \chi_Y(\varphi) \int \nu(dx)\overline{g(\delta_x)}\Psi(\varphi + \delta_x). \tag{4.7}$$

For $Y \in \mathfrak{M}^f$ $S(Y, A(g))$ is bounded. Analogously to Example 4.2 $S(M, A(g))$ represents the set of all annihilation operators on $\mathcal{M}$ with respect to g. In this case we get immediately from (4.7) for all $\Psi \in \mathcal{M}^f$ and F-a.a. φ

$$S(M, A(g))\Psi(\varphi) = \int \nu(dx)\overline{g(\delta_x)}\Psi(\varphi + \delta_x). \tag{4.8}$$

The operator $B \in S(M, A(g))$ with $D(B) = \mathcal{M}^f$ we denote by $a(g)$. By $a_\Lambda(g)$ we denote the operator from $S_\Lambda(M_\Lambda, a(g))$ with domain $\mathcal{M}_\Lambda^f$.

Let $Y \in \mathfrak{M}$, $g \in \mathcal{M}_1$. We still want to show that $B_1 \in S(Y, A^*(g))$ and $B_2 \in S(Y, A(g))$ are mutually adjoint. Indeed, for all $\Psi_1, \Psi_2 \in \mathcal{M}^f$ we obtain

$$\begin{aligned} (B_1\Psi_1, \Psi_2) &= \int F(d\varphi) \int \varphi(dx)\chi_Y(\varphi - \delta_x)\overline{g(\delta_x)\Psi_1(\varphi - \delta_x)}\Psi_2(\varphi) \\ &= \sum_{n \geq 1} \frac{1}{n!} \int \nu^n(d[x_1, \ldots, x_n]) \sum_{j=1}^n \chi_Y\left(\sum_{\substack{k=1 \\ k \neq j}}^n \delta_{x_k}\right) g(\delta_{x_j})\Psi_1\left(\sum_{\substack{k=1 \\ k \neq j}}^n \delta_{x_k}\right) \Psi_2\left(\sum_{k=1}^n \delta_{x_k}\right) \end{aligned}$$

$$= \sum_{n \geqq 1} \frac{1}{(n-1)!} \int \nu^{n-1}(d[x_1, \ldots, x_{n-1}]) \Psi_1 \left(\sum_{k=1}^{n-1} \delta_{x_k} \right) \chi_Y \left(\sum_{k=1}^{n-1} \delta_{x_k} \right)$$

$$\times \int \nu(dx) \Psi_2 \left(\sum_{k=1}^{n-1} \delta_{x_k} + \delta_x \right) \overline{g(\delta_x)}$$

$$= \int F(d\varphi) \Psi_1(\varphi) \chi_Y(\varphi) \int \nu(dx) \overline{g(\delta_x)} \Psi_2(\varphi + \delta_x) = (\Psi_1, B_2 \Psi_2).$$

Since norm-convergence in $\mathscr{M}$ implies weak convergence in $\mathscr{M}$ from Definition 3.1, (iii) follows that for all $\Psi_1 \in D(B_1)$, $\Psi_2 \in D(B_2)$ the equality $(B_1 \Psi_1, \Psi_2) = (\Psi_1, B_2 \Psi_2)$ holds. As one has to expect the canonical commutation relations hold for general $g \in \mathscr{M}_1$ only in the case $Y = M$. The proof of these CCR properties is rather straightforward and we will omit it. Thus $a(f)$, $a^*(g)$ represent "usual" creation and annihilation operators on $\mathscr{M}$ [respectively $a_\Lambda(f)$, $a^*_\Lambda(g)$ are the usual creation and annihilation operators on $\mathscr{M}_\Lambda$]. An approach to creation and annihilation operators similar to the above one (in the case $G = \mathbb{R}^1$) one can find for instance in [24, 25].

4.4. *Composition of Creation and Annihilation Operators*

In 4.2 and 4.3 we defined creation and annihilation operators with respect to functions from $\mathscr{M}_1$. Now we transfer this notion to functions from $\mathscr{M}_{m,\Lambda} := \{\Psi \in \mathscr{M}_\Lambda : \Psi = \Psi \chi_{M_m}\}$, $m \geqq 1$ $(M_{m,G} = \mathscr{M}_m)$.

Let m be a natural number, $m \geqq 1$, and $f^m \in \mathscr{M}_m$. We define operators $A^*(f^m)$, $A(f^m)$ from $\mathscr{L}(\mathscr{M})$ by

$$A^*(f^m)\Psi(\varphi) = m! f^m(\varphi) \Psi(\mathbb{O}) \qquad (\Psi \in \mathscr{M},\ \varphi \in M) \tag{4.9}$$

and

$$A(f^m)\Psi(\varphi) = \begin{cases} m! \int F(d\varphi) \overline{f^m(\varphi)} \Psi(\varphi) & \Psi \in \mathscr{M},\ \varphi = \mathbb{O} \\ 0 & \Psi \in \mathscr{M},\ \varphi \neq \mathbb{O}. \end{cases}$$

Observe that

$$A^*(f^m) = O_{M_m} A^*(f^m) O_{M_0}$$

and

$$A(f^m) = O_{M_0} A(f^m) O_{M_m}.$$

Thus $A^*(f^m)$, $A(f^m) \in \mathscr{A}^f$, and if $f^m \in \mathscr{M}_{m,\Lambda}$ for some $\Lambda \in \mathfrak{G}$ then $A^*(f^m)$, $A(f^m) \in \mathscr{A}^f_\Lambda$.

In the sequel, for $\underline{x}^n := (x_1, \ldots, x_n)$, $x_j \in G$ for $j \in \{1, \ldots, n\}$ we write $\delta_{\underline{x}^n}$ instead of $\sum_{j=1}^n \delta_{x_j}$, and $\varphi(d\underline{x}^n)$ denotes the n^{th} *factorial measure of* $\varphi \in M$, $n \geqq 1$, i.e.

$$\varphi(d\underline{x}^n) := \varphi(dx_1)(\varphi - \delta_{x_1})(dx_2) \ldots (\varphi - \delta_{\underline{x}^{n-1}})(dx_n). \tag{4.10}$$

Now, let $Y \in \mathfrak{M}$, $f^m \in \mathscr{M}_m$, $m \geqq 1$. Using the above notations, we get for all $\Psi \in \mathscr{M}^f$ and F-*a.a.* φ

$$S(Y, A^*(f^m))\Psi(\varphi) = \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi}) A^*(f^m) \Psi_{\hat{\varphi}}(\varphi - \hat{\varphi})$$

$$= m! \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi}) f^m(\varphi - \hat{\varphi}) \Psi_{\hat{\varphi}}(\mathbb{O})$$

$$= \int \varphi(d\underline{x}^m) \chi_Y(\varphi - \delta_{\underline{x}^m}) f^m(\delta_{\underline{x}^m}) \Psi(\varphi - \delta_{\underline{x}^m}). \tag{4.11}$$

Hereby we have used the fact that for arbitrary function h on M we have

$$\sum_{\substack{\hat\varphi\subseteq\varphi\\ \hat\varphi\in M_m}} h(\hat\varphi)=\frac{1}{m!}\int\varphi(d\underline{x}^m)h(\delta_{\underline{x}^m}).$$

Similarly, one gets easily for each $\Psi\in\mathcal{M}^f$ and F-a.a. φ

$$S(Y,A(f^m))\Psi(\varphi)=m!\chi_Y(\varphi)\int F(d\hat\varphi)\overline{f^m(\hat\varphi)}\Psi(\varphi+\hat\varphi). \tag{4.12}$$

Obviously, in the case $m=1$ (4.12) coincides with (4.7).

For $Y\in\mathfrak{M}^f$ we get that $S(Y,A^*(f^m))$ and $S(Y,A(f^m))$ are bounded. For $Y\in\mathfrak{M}\setminus\mathfrak{M}^f$ all operators B from $S(Y,A^*(f^m))$ respectively $S(Y,A(f^m))$ have on $\mathcal{M}^f$ the representation (4.11) respectively (4.12). On $D(B)\setminus\mathcal{M}^f$ the operators are defined according to Definition 3.1, (iii).

Let $Y\in\mathfrak{M}$, $f^m\in\mathcal{M}_m$, $m\geqq 1$, $B_1\in S(Y,A^*(f^m))$, $B_2\in S(Y,A(f^m))$. For all $\Psi_1,\Psi_2\in\mathcal{M}^f$ we get

$$\begin{aligned}(B_1\Psi_1,\Psi_2)&=\int F(d\varphi)\int\varphi(d\underline{x}^m)\chi_Y(\varphi-\delta_{\underline{x}^m})\overline{f^m(\delta_{\underline{x}^m})}\,\overline{\Psi_1(\varphi-\delta_{\underline{x}^m})}\Psi_2(\varphi)\\
&=\sum_{n\geqq m}\frac{1}{n!}\int v^n(d\underline{x}^n)\frac{n!}{(n-m)!}\chi_Y\\
&\quad\times\chi_Y\left(\sum_{j=m+1}^n\delta_{x_j}\right)\overline{f^m\left(\sum_{j=1}^m\delta_{x_j}\right)}\,\overline{\Psi_1\left(\sum_{j=m+1}^n\delta_{x_j}\right)}\Psi_2(\delta_{\underline{x}^n})\\
&=\sum_{k\geqq 0}\frac{1}{k!}\int v^k(d\underline{x}^k)\overline{\Psi_1(\delta_{\underline{x}^k})}\chi_Y(\delta_{\underline{x}^k})\int v^m(d\underline{x}^m)\overline{f^m(\delta_{\underline{x}^m})}\Psi_2(\delta_{\underline{x}^m}+\delta_{\underline{x}^k})\\
&=\int F(d\varphi)\overline{\Psi_1(\varphi)}S(Y,A(f^m))\Psi_2(\varphi)=(\Psi_1,B_2\Psi_2).\end{aligned} \tag{4.13}$$

From (4.13) and Definition 3.1, (iii) we conclude that for all $\Psi_1\in D(B_1)$, $\Psi_2\in D(B_2)$ the equality $(B_1\Psi_1,\Psi_2)=(\Psi_1,B_2\Psi_2)$ is valid. Thus B_1 and B_2 are mutually adjoint.

Further, one easily checks that for $f^m\in\mathcal{M}_m$, $g^n\in\mathcal{M}_n$, $A:=A^*(f^m)A(g^n)$ is an integral operator on $\mathcal{M}$ ($A\in\mathcal{A}^f$) with kernel

$$k_A(\varphi_1,\varphi_2)=f^m(\varphi_1)g^n(\varphi_2)m!n! \tag{4.14}$$

and we get for all $Y\in\mathfrak{M}$, $\Psi\in\mathcal{M}^f$, and F-a.a. φ,

$$\begin{aligned}S(Y,A)\Psi(\varphi)&=\sum_{\hat\varphi\subseteq\varphi}\chi_Y(\hat\varphi)\int F(d\tilde\varphi)f^m(\varphi-\hat\varphi)\overline{g^n(\tilde\varphi)}\Psi(\hat\varphi+\tilde\varphi)m!n!\\
&=\int v^n(d\underline{x}^n)\int\varphi(d\underline{y}^m)f^m(\delta_{\underline{y}^m})\overline{g^n(\delta_{\underline{x}^n})}\\
&\quad\times\chi_Y(\varphi-\delta_{\underline{y}^m})\Psi(\varphi-\delta_{\underline{y}^m}+\delta_{\underline{x}^n}).\end{aligned} \tag{4.15}$$

Again from Proposition 3.3 we may conclude that $S(Y,A)$ is bounded for $Y\in\mathfrak{M}^f$.

Now, let $f^m\in\mathcal{M}_m$, $g^n\in\mathcal{M}_n$ be the symmetrized products of functions from $\mathcal{M}_1$, i.e.

$$f^m(\delta_{\underline{x}^m})=\frac{1}{m!}\sum_\sigma\prod_{j=1}^m f_{\sigma(j)}(\delta_{x_j})\qquad(\underline{x}^m=(x_1,\dots,x_m)\in G^m), \tag{4.16}$$

$$g^n(\delta_{\underline{x}^n})=\frac{1}{n!}\sum_\sigma\prod_{j=1}^n g_{\sigma(j)}(\delta_{x_j})\qquad(\underline{x}^n=(x_1,\dots,x_n)\in G^n) \tag{4.17}$$

with $f_j,g_j\in\mathcal{M}_1$ where the sum is taken over all permutations σ of $\{1,\dots,m\}$ respectively $\{1,\dots,n\}$.

For each $\Psi \in \mathscr{M}^f$ and F-a.a. φ we get

$$\begin{aligned} &a^*(f_1)\dots a^*(f_m)a(g_n)\dots a(g_1)\Psi(\varphi)\\ &=\int \varphi(dy_1)f_1(\delta_{y_1})a^*(f_2)\dots a(g_1)\Psi(\varphi-\delta_{y_1})\\ &=\int \varphi(dy_1)\int(\varphi-\delta_{y_1})(dy_2)f_1(\delta_{y_1})f_2(\delta_{y_2})a^*(f_3)\dots a(g_1)\Psi(\varphi-\delta_{y_1}-\delta_{y_2})\\ &=\int \varphi(d\underline{y}^m)f^m(\delta_{\underline{y}^m})a(g_n)\dots a(g_1)\Psi(\varphi-\delta_{\underline{y}^m})\\ &=\int \varphi(d\underline{y}^m)f^m(\delta_{\underline{y}^m})\int v^n(d\underline{x}^n)g^n(\delta_{\underline{x}^n})\Psi(\varphi-\delta_{\underline{y}^m}+\delta_{\underline{x}^n}).\end{aligned}$$

We thus may conclude [cf. (4.15)]

$$S(M,A^*(f^m)A(g^n))=\prod_{j=1}^{m} a^*(f_j)\prod_{k=1}^{m} a(g_j), \tag{4.18}$$

where f^m and g^n are the symmetrized products of $f_1,\dots,f_m$ and $g_1,\dots,g_n$ respectively. In this sense $S(M,A^*(f^m)A(g^n))$ are compositions of creation and annihilation operators, and the operator $B\in S(M,A^*(f^m)A(g^n))$ with $D(B)=\mathscr{M}^f$ we will denote by $a^*(f^m)a(g^n)$. Analogously, one may define corresponding operators on $\mathscr{M}_\Lambda$.

4.5. *Second Quantization of an Operator on $\mathscr{M}_1$*

Let A be an operator from $\mathscr{L}(\mathscr{M})$ concentrated on $\mathscr{M}_1$. Let $Y\in\mathfrak{M}$. For all $\Psi\in\mathscr{M}^f$ and F-a.a. $\varphi\in M$ we get

$$S(Y,A)\Psi(\varphi)=\int\varphi(dx)\chi_Y(\varphi-\delta_x)(A\Psi_{\varphi-\delta_x})(\delta_x). \tag{4.19}$$

For $Y\in\mathfrak{M}^f$ (4.19) defines a bounded operator on $\mathscr{L}(\mathscr{M})$. We will call $S(Y,A)$ second quantization of A. This is justified by the fact that in the case $Y=M$ we get on $\mathscr{M}^f$ for F-a.a. φ

$$S(M,A)\Psi(\varphi)=\int\varphi(dx)(A\Psi_{\varphi-\delta_x})(\delta_x)=\sum_{x:\varphi(\{x\})>0}(A\Psi_{\varphi-\delta_x})(\delta_x).$$

Thus $S(M,A)$ coincides with the usual second quantization of an operator A on $\mathscr{M}_1$. For historical and modern approaches to second quantization, cf. [15 and 3], but also for instance [2, Chap. 5.2.1], [7, Chap. 1, Sects. 1, 3], and [1, Part II].

5. Σ'_v-Point Processes – Conditional Intensity Measures

First we want to introduce the notions of so-called Σ'_v-point processes and Campbell measures.

5.1. Definition. Let Q be a point process (i.e. a probability measure on $[M,\mathfrak{M}]$) and n a positive integer. (i) The *n^{th} order reduced Campbell measure* $C_Q^{(n)}$ is the measure on $[G^n\times M,\mathfrak{G}^n\times\mathfrak{M}]$ given by

$$C_Q^{(n)}(B\times Y)=\int_M Q(d\varphi)\int_B\varphi(d\underline{x}^n)\chi_Y(\varphi-\delta_{\underline{x}^n})\qquad(B\in\mathfrak{G}^n,\ Y\in\mathfrak{M}). \tag{5.1}$$

(ii) The *compound Campbell measure* $C_Q^{(\infty)}$ is the measure on $[M\times M,\mathfrak{M}\times\mathfrak{M}]$ characterized by

$$C_Q^{(\infty)}(Y_1\times Y_2)=\int Q(d\varphi)\sum_{\substack{\hat\varphi\subseteq\varphi\\ \hat\varphi\in M^f}}\chi_{Y_1}(\hat\varphi)\chi_{Y_2}(\varphi-\hat\varphi)\qquad(Y_1,Y_2\in\mathfrak{M}). \tag{5.2}$$

Let us remark that $C_Q^{(n)}$ and $C_Q^{(\infty)}$ are σ-finite measures and $C_Q^{(\infty)}$ is concentrated on $M^f \times M$. We have the following relation between $C_Q^{(n)}$ and $C_Q^{(\infty)}$:

$$C_Q^{(n)}(B \times Y) = n!\, C_Q^{(\infty)}(\{\delta_{\underline{x}^n} : \underline{x}^n \in B\} \times Y\} \quad (B \in \mathfrak{G}^n,\ n \geq 1,\ Y \in \mathfrak{M}).$$

(For details see [21, Chap. 12.3] and [26].)

5.2. Definition. Let Q be a point process.
(i) Q is said to be a *Σ_ν^c-point process* if there exists a σ-finite measure S on $[M, \mathfrak{M}]$ (called a supporting measure of Q) such that

$$C_Q^{(1)} \ll \nu \times S. \tag{5.3}$$

(ii) Q is said to be a *Σ_ν'-point process* if

$$C_Q^{(1)} \ll \nu \times Q. \tag{5.4}$$

By $\ll$ we denote absolute continuity.

Observe that each Σ_ν'-point process is Σ_ν^c (put $S = Q$). The converse is not true.

For a more detailed discussion of Σ_ν'-point processes see e.g. [27, 29, 20, 26, 31, and 21, Chap. 13.2] Σ_ν^c-point processes are discussed in [38].

5.3. Lemma. *Let Q be a Σ_ν'-point process. Then*

$$C_Q^{(\infty)} \ll F \times Q, \tag{5.5}$$

and for Q-a.a. φ by

$$\eta_Q^\varphi(Y) := \int_Y F(d\phi) \frac{dC_Q^{(\infty)}}{d(F \times Q)}(\phi, \varphi) \quad (Y \in \mathfrak{M}) \tag{5.6}$$

there is defined a measure on $[M, \mathfrak{M}]$ with the following properties:
(i) *η_Q^φ is σ-finite and concentrated on $M^f := \{\varphi \in M : \varphi(G) < \infty\}$,*
(ii) *for all bounded Λ from $\mathfrak{G}$ $\eta_Q^{\varphi_{\Lambda^c}} \circ v_\Lambda$ is a finite measure on $[M, {}_\Lambda\mathfrak{M}]$,*
(iii) *for all bounded Λ from $\mathfrak{G}$ and $X \in {}_\Lambda\mathfrak{M}$*

$$Q(X \mid {}_{\Lambda^c}\mathfrak{M})(\varphi) = \eta_Q^{\varphi_{\Lambda^c}}(v_\Lambda X) / \eta_Q^{\varphi_{\Lambda^c}}(M_\Lambda). \tag{5.7}$$

$Q(X \mid {}_{\Lambda^c}\mathfrak{M})$ denotes the conditional probability of X with respect to the σ-algebra ${}_{\Lambda^c}\mathfrak{M}$.

A proof of the statements in Lemma 5.3 can be found in [21, Chap. 13.2] and the Russian edition of [26, Chap. 9.1].

5.4. Definition. The family $(\eta_Q^\varphi)_{\varphi \in M}$ of measures related to a Σ_ν'-point process Q by (5.6) we call the family of *conditional intensity measures of Q.*

5.5. Remark. The intensity measure I_Q of a point process Q is a measure on $[G, \mathfrak{G}]$ defined by $I_Q(\Lambda) = \int Q(d\varphi)\varphi(\Lambda)$, $\Lambda \in \mathfrak{G}$. Because of $I_Q = C_Q^{(1)}(\cdot \times M)$ an easy calculation shows

$$I_Q(\Lambda) = \int Q(d\varphi) \eta_Q^\varphi(\{\delta_x : x \in \Lambda\}) \quad (\Lambda \in \mathfrak{G}). \tag{5.8}$$

Equations (5.7) and (5.8) justify Definition 5.4.

5.6. Remark. Since the Radon-Nikodym-derivative $dC_Q^{(\infty)}/d(F \times Q)$ is only $F \times Q$-*a.e.* uniquely determined the family $(\eta_Q^\varphi)_{\varphi \in M}$ is only Q-*a.s.* unique, i.e. if $(\eta_Q^\varphi)_{\varphi \in M}$ and $(\hat{\eta}_Q^\varphi)_{\varphi \in M}$ are two versions of the family of conditional intensity measures then $Q(\{\varphi$: *η_Q^φ is a measure and $\eta_Q^\varphi(Y) = \hat{\eta}_Q^\varphi(Y)$ for all $Y \in \mathfrak{M}$*$\}) = 1$. Finite

Σ'_ν-point processes Q are determined by their conditional intensity. This is not true for general (infinite) Σ'_ν-point processes. We have the following statement which is an easy consequence of Lemma 5.3.

5.7. Lemma. *Let Q be a finite Σ'_ν-point process, and $(\eta_Q^\varphi)_{\varphi\in M}$ its family of conditional intensity measures. Then*

$$Q(X)=\eta_Q^{\emptyset}(X)/\eta_Q^{\emptyset}(M) \qquad (X\in\mathfrak{M}). \tag{5.9}$$

The expectation $\omega(S(Y,A))$ may be expressed by the position distribution Q_ω and the conpound Campbell measure or the conditional intensity measures of Q_ω if A corresponds to a position measurement.

5.8. Proposition. *Let ω be a locally normal state with position distribution Q_ω, and assume that Q_ω is a Σ'_ν-point process. Then*
(i) *for all $\Lambda\in\mathfrak{B}$, $Y_1\in\mathfrak{M}_\Lambda$, $Y_2\in{}_\Lambda\mathfrak{M}$,*

$$\omega(S(Y_1,O_{Y_2}))=\int_{Y_2} Q_\omega(d\varphi)\eta_{Q_\omega}^\varphi(Y_1)=C_{Q_\omega}^{(\infty)}(Y_1\times Y_2),$$

(ii) *for all $\Lambda\in\mathfrak{B}$, $g\in\mathscr{M}_\Lambda$, $Y\in_\Lambda\mathfrak{M}$ such that $S(Y,0_g)\in{}_\Lambda\mathscr{A}$,*

$$\omega(S(Y,0_g))=\int_Y Q_\omega(d\varphi)\int\eta_{Q_\omega}^\varphi(d\hat\varphi)g(\hat\varphi)=\int C_{Q_\omega}^{(\infty)}(d\hat\varphi,d\varphi)g(\hat\varphi)\chi_Y(\varphi).$$

We will see in the subsequent sections that an analogous representation holds for $\omega(S(Y,A))$ with A being an arbitrary local measurement. The conditional intensity measures only have to be replaced by the conditional reduced density matrix.

6. The Conditional Reduced Density Matrix of a Normal State

Let ω be a normal state on $\mathscr{L}(\mathscr{M})$. Thus there exists a density matrix ϱ on $\mathscr{M}$ such that

$$\omega(A)=\mathrm{Tr}(\varrho A) \qquad (A\in\mathscr{L}(\mathscr{M})). \tag{6.1}$$

6.1. Proposition. *The position distribution Q_ω of ω is a finite Σ_ν^c-point process.*

The proof of Proposition 6.1 is completely analogous to the proof of Proposition 3.1 in [11] (one only has to cancel the index Λ), and we will omit it.

In the sequel we will consider only states the position distributions of which are Σ'_ν-point processes. We already remarked that each Σ'_ν-point process is of the type Σ_ν^c.

6.2. Definition. A normal state ω on $\mathscr{L}(\mathscr{M})$ is called a *normal Σ'_ν-state* if Q_ω is a Σ'_ν-point process.

Before we will give the main characterization of normal Σ'_ν-states on $\mathscr{L}(\mathscr{M})$ let us still make a notational convention.

6.3. Definition. Let $T:M^2\to\mathbb{C}$, $k:M^3\to\mathbb{C}$ be measurable functions, $\Lambda\in\mathfrak{G}$. For all $\varphi_1,\varphi_2\in M$ we set

$$T\underset{\Lambda}{*}k(\varphi_1,\varphi_2):=\int F_\Lambda(d\varphi)T(\varphi_1,\varphi)k(\varphi,\varphi_1,\varphi_2) \tag{6.2}$$

provided the right side of (6.2) makes sense (in the case $\Lambda=G$ we omit the index Λ).

6.4. Theorem. *Let ω be a normal Σ'_ν-state. There exists a $F\times F\times Q_\omega$-a.e. uniquely determined measurable function $k_\omega: M^3\to\mathbb{C}$ with the following properties:*
(i) *For all $Y\in\mathfrak{M}$ and integral operators $A\in\mathcal{L}(\mathcal{M})$ such that $S(Y,A)\in\mathcal{L}(\mathcal{M})$ we have*

$$\omega(S(Y,A))=\int_Y Q_\omega(d\varphi)\int F(d\varphi_1)k_A * k_\omega(\varphi_1,\varphi), \tag{6.3}$$

where k_A is a kernel of A.
(ii) *For $F\times Q$-a.a. $(\hat\varphi,\varphi)$*

$$k_\omega(\hat\varphi,\hat\varphi,\varphi)=\frac{d\eta^{\varphi}_{Q_\omega}}{dF}(\hat\varphi). \tag{6.4}$$

6.5. Definition. Let ω be a normal Σ'_ν-state. The $F\times F\times Q_\omega$*-a.e.* uniquely determined function k_ω associated to ω by Theorem 6.4 is called the *conditional reduced density matrix (c.r.d.m.) of the normal state ω.*

6.6. Remark. In Sect. 3 there are given sufficient criteria on Y and A ensuring $S(Y,A)\in\mathcal{L}(\mathcal{M})$. Since integral operators are dense in $\mathcal{L}(\mathcal{M})$ and for each $A\in\mathcal{L}(\mathcal{M})$ we have $A=S(\{\mathbb{O}\},A)$ (6.3) enables us to calculate $\omega(A)$ for all $A\in\mathcal{L}(\mathcal{M})$.

6.7. Remark. In quantum statistical mechanics one is also interested in the expectation $\omega(A)$ for certain unbounded operators on $\mathcal{M}$. Since ω is a functional on $\mathcal{L}(\mathcal{M})$ it is not quite easy to give such expectations a precise mathematical meaning (cf. [2], part II). However, observe that the right side of (6.3) may make sense also for unbounded operators of the type $S(Y,A)$. So, without further assumptions about the state ω we can define $\omega(S(Y,A))$ by the right side of (6.3) for all operators of the type $S(Y,A)$ for which the integral on the right side of (6.3) exists (possibly equal $\pm\infty$). This allows to give the relation between the c.r.d.m. k_ω of ω and the reduced density matrix commonly used in statistical mechanics (cf. 2, 33, 34, 35]).

Let m be a positive integer and $f_j, g_j, j\in\{1,\dots,m\}$ be functions from $\mathcal{M}_1$. Further, let $A=a^*(f_1)\cdot\ldots\cdot a^*(f_m)a(g_m)\cdot\ldots\cdot a(g_1)$ be the symmetrized product of the creation and annihilation operators corresponding to the functions f_j, g_j [with $D(A)=\mathcal{M}^f$ – cf. Sect. 4.4]. A function $\lambda: G^{2m}\to\mathbb{C}$ is called the m^{th} *reduced density matrix of ω* if for all operators A of the above type

$$\omega(A)=\int \nu^m(d\underline{x}^m)\int \nu^m(d\underline{y}^m)\prod_{j=1}^m \overline{g_j(\delta_{y_j})}f_j(\delta_{x_j})\lambda(\underline{y}^m,\underline{x}^m) \tag{6.5}$$

(cf. instance [2, Chap. 6.3.3]).

In Sect. 4.4 we observed that $A=S(M,B)$, where B is an integral operator (from $\mathcal{A}^f$) with kernel

$$k_B(\delta_{\underline{x}^m},\delta_{\underline{y}^m})=\sum_\sigma\prod_{j=1}^m f_{\sigma(j)}(\delta_{x_j})\sum_\sigma\prod_{k=1}^m\overline{g_{\sigma(k)}(\delta_{y_k})},$$

where the sum is taken over all permutation $\{\sigma(1),\dots,\sigma(m)\}$ of $\{1,\dots,m\}$. A very easy calculation shows that if the right side of (6.3) exists it will be equal to

$$\int \nu^m(d\underline{x}^m)\int \nu^m(d\underline{y}^m)\prod_{j=1}^m\overline{g_j(\delta_{y_j})}f_j(\delta_{x_j})\int_M Q_\omega(d\varphi)k_\omega(\delta_{\underline{x}^m},\delta_{\underline{y}^m},\varphi).$$

Finally, we get

$$\lambda(\underline{x}^m,\underline{y}^m)=\int_M Q_\omega(d\varphi)k_\omega(\delta_{\underline{x}^m},\delta_{\underline{y}^m},\varphi)\qquad (\nu^{2m}\text{-a.a.}(\underline{x}^m,\underline{y}^m)). \tag{6.6}$$

Thus, $\lambda(\underline{x}^m, \underline{y}^m) = E_{Q_\omega} k_\omega(\delta_{\underline{x}^m}, \delta_{\underline{y}^m}, \cdot)$. This justifies to call k_ω the conditional reduced density matrix. We prefer to use k_ω instead of λ. There are several disadvantages of the reduced density matrices. First, there exist normal states for which the functions λ are not finite or do not exist at all. Moreover, even if for a state ω for each $m \geqq 1$ the m^{th} reduced density matrix exists and is finite *a.e.* the state possibly will not be determined by its reduced density matrices. Even the position distribution is in general not determined by them. This is caused by the fact that the reduced density matrices are connected only with "global" second quantizations ($Y = M$). The function k_ω however exists *a.e.* and determines together with the position distribution the whole state ω (Theorem 6.4).

Formula (6.4) elucidates the connection between the c.r.d.m. and the conditional intensity measure of the position distribution. The c.r.d.m. cannot be chosen arbitrarily on the "diagonal" $k_\omega(\hat{\varphi}, \hat{\varphi}, \varphi)$ but in such a way that the trace formula is valid.

We will deal now with the converse question. What conditions a function $k: M^3 \to \mathbb{C}$ has to fulfill so that there would exist a normal Σ'_v-state ω with c.r.d.m. $k_\omega = k$?

6.8. Theorem. *Let $k: M^3 \to \mathbb{C}$ be a measurable function and Q a finite Σ'_v-point process with the following properties: There exists a positive trace-class operator K on $\mathcal{M}$ such that*

(i) *for all $Y \in \mathfrak{M}^f$*

$$\mathrm{Tr}(KO_Y) = \eta_Q^{\mathbb{O}}(Y) = \int_Y F(d\varphi) k(\varphi, \varphi, \mathbb{O}); \tag{6.7}$$

(ii) *for all integral operators $A \in \mathcal{L}(\mathcal{M})$*

$$\mathrm{Tr}(KA) = \int F(d\varphi) k_A * k(\varphi, \mathbb{O}), \tag{6.8}$$

where k_A is a kernel of A;

(iii) *for all $\varphi_1, \varphi_2, \varphi, \hat{\varphi} \in M$*

$$k(\varphi_1 + \varphi, \varphi_2 + \varphi, \hat{\varphi}) = k(\varphi, \varphi, \hat{\varphi}) k(\varphi_1, \varphi_2, \varphi + \hat{\varphi}). \tag{6.9}$$

Then there exists exactly one normal Σ'_v-state ω such that $Q_\omega = Q$ and $k_\omega = k$ a.e.

6.9. Remarks. (i) gives the connection between k and the point process Q.

(ii) guarantees normality of the state which has to be constructed.

(iii) represents a compatibility condition. Loosely speaking, one could interpret (6.9) in the following way:

Passing over from the configuration $\varphi_1 + \varphi$ to $\varphi_2 + \varphi$ having around the configuration $\hat{\varphi}$ is the same as adding first φ to $\hat{\varphi}$ and then passing from φ_1 to φ_2 having around the configuration $\varphi + \hat{\varphi}$.

Observe that (6.9) and (6.4) imply (on the "diagonal")

$$\kappa_Q(\varphi_1 + \varphi, \hat{\varphi}) = \kappa_Q(\varphi, \hat{\varphi}) \kappa_Q(\varphi_1, \varphi + \hat{\varphi}) \tag{6.10}$$

for $F^2 \times Q$-*a.a.* $(\varphi_1, \varphi, \hat{\varphi})$, where κ_Q denotes (a version of) the Radon-Nikodym derivative $\dfrac{dC_Q^{(\infty)}}{d(F \times Q)}$.

(6.10) is a well-known characteristic property of (not necessarily finite) Σ'_v-point processes (cf. instance [26, 31, and 38]).

Summarizing, we get the following result:

6.10. Theorem. *Let $k: M^3 \to \mathbb{C}$ be a measurable function and Q a finite Σ'_v-point process. The following conditions are equivalent:*
(I) *There exists a normal Σ'_v-state ω such that $Q_\omega = Q$ and $k_\omega = k$ a.e.*
(II) *k and Q fulfill assumptions* (i) *to* (iii) *of Theorem 6.8.*

6.11. Remark. All results of this section remain true if we consider normal states on $\mathcal{L}(\mathcal{M}_\Lambda)$, $\Lambda \in \mathfrak{G}$. We only have to replace M, $\mathcal{M}$, F by M_Λ, $\mathcal{M}_\Lambda$, and F_Λ respectively. Especially, all results are true for the restrictions ω_Λ of a locally normal state on $\mathcal{A}$ to $\mathcal{L}(\mathcal{M}_\Lambda)$, $\Lambda \in \mathfrak{B}$.

7. The Conditional Reduced Density Matrix of a Locally Normal State

We want to extend now Theorem 6.8 to the case of locally normal states.

7.1. Definition. Let ω be a locally normal state on $\mathcal{A}$ such that its position distribution Q_ω is a Σ'_v-point process. A measurable function $k: M^3 \to \mathbb{C}$ is called the *conditional reduced density matrix (c.r.d.m.) of* ω if k has the following properties:

(i) $k(\hat{\varphi}, \hat{\varphi}, \varphi_{\Lambda^c}) = \dfrac{d\eta_{Q_\omega}^{\varphi_{\Lambda^c}}}{dF}(\hat{\varphi})$ $\quad$ ($F_\Lambda \times Q_\omega$-*a.a.* $(\hat{\varphi}, \varphi)$, $\Lambda \in \mathfrak{B}$).

(ii) For all $\Lambda \in \mathfrak{B}$ and Q_ω-*a.a.* φ

$$k(\cdot, \cdot, \varphi_{\Lambda^c}) \chi_{M_\Lambda \times M_\Lambda}(\cdot, \cdot)$$

is the kernel of a positive trace-class operator on $\mathcal{M}_\Lambda$.

(iii) For all $\Lambda \in \mathfrak{B}$, $Y \in {}_\Lambda\mathfrak{M}$ and all integral operators $A \in \mathcal{A}_\Lambda$ such that $S(Y, A) \in {}_\Lambda\mathcal{A}$ we have

$$\omega(\mathrm{S}(\mathrm{Y}, \mathrm{A})) = \int_Y Q_\omega(d\varphi) \int F_\Lambda(d\hat{\varphi}) k_A \underset{\Lambda}{*} k(\hat{\varphi}, \varphi), \tag{7.1}$$

where k_A is a kernel of A.

7.2. Remarks. 1°. In [17] it is shown that the c.r.d.n. of a locally normal state is *a.e.* uniquely determined (provided it exists), i.e. if $k_1, k_2: M^3 \to \mathbb{C}$ are measurable functions satisfying conditions (i)$\to$(iii) of Definition 7.1 then for $F \times F \times Q_\omega$-*a.a.* $(\varphi_1, \varphi_2, \varphi)$ we have $k_1(\varphi_1, \varphi_2, \varphi) = k_2(\varphi_1, \varphi_2, \varphi)$. In this paper we will not make use of this fact.

2°. It is easy to check that the (a.e.-uniquely determined) c.r.d.m. of a normal Σ'_v-state is a c.r.d.m. in the sense Definition 7.1. So if the locally normal state is a normal one [i.e. may be extended to a normal state on $\mathcal{L}(\mathcal{M})$] both notions coincide.

3°. Condition (i) in Definition 7.1 gives the connection between the c.r.d.m. and the position distribution of ω. We see that the c.r.d.m. has to be chosen on the "diagonal" in each bounded region Λ for a fixed configuration φ outside in such a way that

$$k(\hat{\varphi}, \hat{\varphi}, \varphi) = \kappa_{Q_\omega}(\hat{\varphi}, \varphi) \qquad (\hat{\varphi} \in M_\Lambda),$$

where κ_{Q_ω} is the Radon-Mikodym derivative $dC_{Q_\omega}^{(\infty)}/d(F \times Q_\omega)$.

4°. Condition (ii) ensures the existence of the family of so-called *conditional states* $(\omega_\Lambda^\varphi)_{\varphi\in M_{\Lambda^c}}$ for each $\Lambda\in\mathfrak{B}$ which describe the behaviour of the system inside Λ having outside the configuration φ (cf. [17]).

5°. From (iii) follows that ω will be determined by Q_ω and the c.r.d.m. k. Indeed, for all $\Lambda\in\mathfrak{B}$ and all $A\in\mathscr{A}_\Lambda$ we have $J_\Lambda A=S(M_{\Lambda^c},A)$ (cf. Proposition 3.7). So we get from (7.1) for all integral operators $A\in\mathscr{A}_\Lambda$ (with kernel k_A)

$$\omega(J_\Lambda A)=\int_{M_{\Lambda^c}} Q_\omega(d\varphi)\int F_\Lambda(d\hat{\varphi})k_A \underset{\Lambda}{*} k(\hat{\varphi},\varphi).$$

Since the integral operators from $\mathscr{A}_\Lambda$ are dense in $\mathscr{A}_\Lambda$ the continuity of the state allows to calculate from the above formula the expectations of all local operators through which the whole state ω will be determined.

6°. The connections between the c.r.d.m. and the reduced matrices in the case of normal states given in Sect. 6 remain true in the locally normal case without any changes.

7.3. Theorem. *Let Q be Σ_v'-point process, and $k: M^3\to\mathbb{C}$ a measurable mapping satisfying the following conditions:*
(I) *For all $\Lambda\in\mathfrak{B}$ and Q-a.a. φ*

$$k(\cdot,\cdot,\varphi_{\Lambda^c})\chi_{M_\Lambda\times M_\Lambda}(\cdot,\cdot)$$

is the kernel of a positive trace-class operator $K_\Lambda^{\varphi_{\Lambda^c}}$ on $\mathscr{M}_\Lambda$ such that

$$\mathrm{Tr}(K_\Lambda^{\varphi_{\Lambda^c}}O_Y)=\int_Y F(d\hat{\varphi})k(\hat{\varphi},\hat{\varphi},\varphi_{\Lambda^c})=\eta_Q^{\varphi_{\Lambda^c}}(Y)\quad (Y\in\mathfrak{M}_\Lambda). \tag{7.2}$$

(II) *For all $\varphi_1,\varphi_2,\varphi,\hat{\varphi}\in M$,*

$$k(\varphi_1+\varphi,\varphi_2+\varphi,\hat{\varphi})=k(\varphi,\varphi,\hat{\varphi})k(\varphi_1,\varphi_2,\varphi+\hat{\varphi}). \tag{7.3}$$

Then there exists a unique locally normal state ω on $\mathscr{A}$ such that $Q_\omega=Q$ and k is the c.r.d.m. of ω.

7.4. Remark. The interpretation of the conditions k has to fulfill is analogous to the case of finite systems (cf. Remarks 6.9). However, while a finite point process is determined completely by $k(\hat{\varphi},\hat{\varphi},\varphi)$ this is not true for infinite point processes. In [17] we show that a locally normal state is a normal one if and only if the position distribution is a finite point process. Furthermore, in [17] we deal with the problem of the existence of the c.r.d.m. to a given locally normal state. However, the examples below show already that a wide and important class of locally normal states allow a characterization by their position distributions and their c.r.d.m.

8. On the Construction of Certain States

First we present a method of constructing states which is based on an application of Theorem 7.3 and will be applied in Sect. 9 to more specific examples.

Let Q be an arbitrary Σ_v'-point process. We denote by $\kappa_Q^{(1)}$ a version of $dC_Q^{(1)}/d(v\times Q)$. Further, let $\Phi: G\times M\to\mathbb{C}$ be a measurable mapping satisfying

$$|\Phi(x,\varphi)|^2=\kappa_Q^{(1)}(x,\varphi)\quad (C_Q^{(1)}\text{-a.a. } (x,\varphi)) \tag{8.1}$$

and

$$\Phi(x,\varphi)\Phi(y,\varphi+\delta_x)=\Phi(y,\varphi)\Phi(x,\varphi+\delta_y)\quad (C_Q^{(2)}\text{-}a.a.\ (x,y,\varphi)). \tag{8.2}$$

For instance, take

$$\Phi(x,\varphi)=e^{ic}\sqrt{\kappa_Q^{(1)}(x,\varphi)} \qquad (x\in G,\ \varphi\in M),$$

where c is an arbitrary real constant.

For all $n\geqq 1$, $\underline{x}^n=(x_1,\dots,x_n)\in G^n$ and $\varphi\in M$ we set

$$\Phi_n(\underline{x}^n,\varphi)=\prod_{j=1}^{n}\Phi\left(x_j,\varphi+\sum_{m=1}^{j-1}\delta_{x_m}\right) \tag{8.3}$$

$\left(\sum_{m=1}^{0}\delta_{x_m}:=0\right)$. Because of (8.2) $\Phi_n(\cdot,\varphi)$ is a symmetric function on G^n. Consequently, the function $\tilde{\Phi}: M\times M\to\mathbb{C}$ defined by

$$\tilde{\Phi}(\delta_{\underline{x}^n},\varphi)=\Phi_n(\underline{x}^n,\varphi) \qquad (n\geqq 1,\ \underline{x}^n\in G^n,\ \varphi\in M) \tag{8.4}$$

and

$$\tilde{\Phi}(\mathbb{O},\varphi)=1, \qquad \tilde{\Phi}(\hat{\varphi},\varphi)=0 \qquad (\hat{\varphi}\in M\setminus M^f,\ \varphi\in M) \tag{8.5}$$

is a well-defined measurable function.

Finally, we set

$$k(\varphi_1,\varphi_2,\varphi)=\tilde{\Phi}(\varphi_1,\varphi)\overline{\tilde{\Phi}(\varphi_2,\varphi)} \qquad (\varphi_1,\varphi_2,\varphi\in M). \tag{8.6}$$

This completes already the whole construction. We have the following result:

8.1. Proposition. *Let Q be a Σ'_v-point process, Φ a function satisfying* (8.1) *and* (8.2). *Then there exists exactly one locally normal state ω on $\mathscr{A}$ such that $Q_\omega=Q$ and k defined by* (8.6) *is the c.r.d.m. of ω. Moreover, ω is a normal state (i.e. may be extended to a normal one on $\mathscr{L}(\mathscr{M})$) if and only if Q is a finite point process.*

Observe that we get from Proposition 8.1 that each Σ'_v-point process (respectively finite Σ'_v-point process) Q gives rise to at least one locally normal state (respectively normal state) ω with position distribution Q_ω. This part of Proposition 8.1 was shown (for more general point processes) already in [11, Theorem 3.3].

9. Examples

9.1. Pure Normal States on $\mathscr{L}(\mathscr{M})$

First we want to illustrate the construction given above by dealing with a very simple example. Let Ψ be an arbitrary normalized wave function, i.e. $\Psi\in\mathscr{M}$, $\|\Psi\|=1$. We set

$$Q(Y)=\int_Y F(d\varphi)\,|\Psi(\varphi)|^2 \qquad (Y\in\mathfrak{M}). \tag{9.1}$$

Obviously, by (9.1) there is defined a finite point process. However, Q is not necessarily of the type Σ'_v.

9.1. Lemma (cf. [31]). *Let Q be the point process defined by* (9.1). *Q is a Σ'_v-point process if and only if the following implication holds:*

$$|\Psi(\varphi+\delta_x)|>0 \textit{ implies } |\Psi(\varphi)|>0 \qquad (v\times F\text{-a.a. } (x,\varphi)). \tag{9.2}$$

Now, we assume that Q defined by (9.1) is a Σ_ν'-point process. We define a function $\Phi: G \times M \to \mathbb{C}$ by

$$\Phi(x,\varphi) = \frac{\Psi(\varphi+\delta_x)}{\Psi(\varphi)} \qquad ((x,\varphi)\in G\times M), \tag{9.3}$$

where we make the convention $\frac{0}{0}=0$. Because of (9.2) Φ is a well-defined measurable function. An easy calculation shows that

$$\kappa_Q^{(1)}(x,\varphi) = |\Phi(x,\varphi)|^2 \qquad (\nu\times Q\text{-}a.a.\ (x,\varphi)). \tag{9.4}$$

Consequently, Φ satisfies (8.1). Condition (8.2) results immediately from (9.3). By the construction (8.3) to (8.6) we finally get the almost everywhere well-defined function

$$k(\varphi_1,\varphi_2,\varphi) = \frac{\Psi(\varphi_1+\varphi)\overline{\Psi(\varphi_2+\varphi)}}{|\Psi(\varphi)|^2} \qquad (\varphi_1,\varphi_2,\varphi\in M). \tag{9.5}$$

By Proposition 8.1 there exists exactly one normal state ω on $\mathcal{L}(\mathcal{M})$ such that $Q_\omega = Q$ and k is the c.r.d.m. of ω.

As one could expect ω is nothing else but the pure normal state on $\mathcal{L}(\mathcal{M})$ given by the wave function Ψ, i.e. we have the following result:

9.2. Proposition. *Let Ψ be from $\mathcal{M}$, $\|\Psi\|=1$ and assume that Ψ satisfies* (9.2). *Let Q be defined by* (9.1) *and k by* (9.5). *Further, let ω be the state obtained by Proposition 8.1. Then*

$$\omega(A) = (\Psi, A\Psi) \qquad (A\in\mathcal{L}(\mathcal{M})). \tag{9.6}$$

9.2. Coherent States

First we want to introduce the notion of a Poisson point process. Let I be a locally finite measure on $[G,\mathfrak{G}]$ [i.e. $I(\Lambda)<\infty$ for all $\Lambda\in\mathfrak{B}$].

9.3. Definition. A point process P is called a *Poisson point process with intensity measure I* if for all $m>0$, $n_1,\dots,n_m\in\mathbb{N}$ and $B_1,\dots,B_m\in\mathfrak{B}$, $B_i\cap B_j=\emptyset$ for $i\neq j$ we have

$$P(\{\varphi\in M:\varphi(B_1)=n_1,\dots,\varphi(B_m)=n_m\}) = \exp\left\{-I\left(\bigcup_{j=1}^m B_j\right)\right\}\prod_{j=1}^m \frac{(I(B_j))^{n_j}}{n_j!} \tag{9.7}$$

(cf. instance [26, 21]).

By $L_2^{\text{loc}}(G,\nu)$ we denote the space of all locally square integrable functions, i.e. $g\in L_2^{\text{loc}}(G,\nu)$ if $g: G\to\mathbb{C}$ and $\int_\Lambda \nu(dx)\,|g(x)|^2<\infty$ for all $\Lambda\in\mathfrak{B}$. For $g\in L_2^{\text{loc}}(G,\nu)$ we denote by I^g the locally finite measure on $[G,\mathfrak{G}]$ with ν-density $|g|^2$, i.e. $I^g(\Lambda)=\int_\Lambda \nu(dx)\,|g(x)|^2$.

So each $g\in L_2^{\text{loc}}(G,\nu)$ gives rise to a Poisson point process with intensity measure I^g. This point process we will denote by P^g. From (9.7) we conclude that P^g is uniquely determined by g.

We want to consider now states of infinite boson systems where all bosons are in the same "one-particle state."

Let g be an arbitrary function from $L_2^{\mathrm{loc}}(G,\nu)$. We define a function $\Phi: G\times M\to\mathbb{C}$ by

$$\Phi(x,\varphi)=g(x). \tag{9.8}$$

Φ fulfills condition (8.1) with respect to the Poisson point process P^g because

$$\kappa_{P^g}^{(1)}(x,\varphi)=|g(x)|^2 \quad (x\in G,\ \varphi\in M).$$

Condition (8.2) is trivially satisfied. By the construction (8.3)–(8.6) we obtain a function $k^g: M^3\to\mathbb{C}$,

$$k^g(\varphi_1,\varphi_2,\varphi)=\tilde{\Phi}(\varphi_1)\overline{\tilde{\Phi}(\varphi_2)} \quad (\varphi_1,\varphi_2,\varphi\in M) \tag{9.9}$$

with

$$\tilde{\Phi}(\varphi)=\begin{cases} 1 & \varphi=\mathbb{O} \\ \prod_{j=1}^{m} g(x_j) & \varphi=\sum_{j=1}^{m}\delta_{x_j},\ x_j\in G,\ m\geqq 1 \\ 0 & \varphi\in M\setminus M^f. \end{cases} \tag{9.10}$$

From Proposition 8.1 we thus obtain a uniquely determined locally normal state ω with $Q_\omega=P^g$ and $k_\omega=k^g$.

9.4. Definition. Let g be from $L_2^{\mathrm{loc}}(G,\nu)$. The locally normal state ω on $\mathscr{A}$ with $Q_\omega=P^g$ and the c.r.d.m. k^g is called the *coherent* (or Glauber) *state with respect to* g, and we will denote it by ω^g.

To call ω^g a coherent state is justified by the fact that in the case $g\in L_2(G,\nu)$ Definition 9.4 coincides with the usual definition of a coherent state (cf. instance [30]). Indeed, assume $g\in L_2(G,\nu)$. Then it is easy to observe that P^g is a finite Poisson point process. From Proposition 8.1 we conclude that ω^g is a normal state. $\tilde{\Phi}$ defined by (9.10) belongs to $\mathscr{M}$, and one has $\|\tilde{\Phi}\|^2=\exp\{\|g\|^2\}$, where $\|g\|^2=\int\nu(dx)\,|g(x)|^2$. Consequently, $\Psi: M\to\mathbb{C}$ defined by

$$\Psi(\varphi):=\exp\{-\tfrac{1}{2}\|g\|^2\}\tilde{\Phi}(\varphi) \quad (\varphi\in M) \tag{9.11}$$

has the properties $\Psi\in\mathscr{M}$, $\|\Psi\|=1$, and as in Example 9.1 one easily gets

$$\omega^g(A)=(\Psi,A\Psi) \quad (A\in\mathscr{L}(\mathscr{M})). \tag{9.12}$$

Thus ω^g is a pure normal state, and from the definition of Ψ we see that ω^g may be interpreted as a state of free bosons being all in the same "one-particle state g" (cf. instance [30]).

Coherent states were discussed in detail in [13] (in the case $G=\mathbb{R}^d$, ν the d-dimensional Lebesgue measure). Let us mention only some interesting properties of coherent states. Let ω be a locally normal state on $\mathscr{A}$. ω is coherent if and only if for all $\Lambda,\Lambda'\in\mathfrak{B}$, $\Lambda\cap\Lambda'=\emptyset$ and all $A\in{}_\Lambda\mathscr{A}$, $B\in{}_{\Lambda'}\mathscr{A}$ it holds

$$\omega(AB)=\omega(A)\omega(B) \tag{9.13}$$

(cf. [13], Sect. 2.4). This result is a generalization of a well-known characterization of Poisson point processes with diffuse intensity measure by local independence (cf. [26, Theorem 1.11.8]). A characterization of normal coherent states of photons by local independence was given already in [37] (with A,B being creation and annihilation operators).

Finally, we want to give still another interpretation of coherent states if g is only locally square integrable. Let g be from $L_2^{\mathrm{loc}}(G,\nu)$ and $A\in\mathscr{L}(\mathscr{M}_1)$ such that

(g, Ag) exists (for instance let $A \geqq 0$ or assume $A \in \mathscr{A}_\Lambda$ for some $\Lambda \in \mathfrak{B}$). An easy calculation shows

$$(g, Ag) = \omega^g(S(M, A)). \tag{9.14}$$

$S(M, A)$ is the operator of (global) second quantization of A (cf. 4.5). For $g \in L_2^{\mathrm{loc}}(G, \nu)$ [but $g \notin L_2(G, \nu)$] thus (9.14) coincides with the physical interpretation of a non-normalizable wave function as a "wave packet" of independent particles. These particles do not exist as single particles but only as part of the infinite system. According to the interpretation of the Poisson point process P^g, $|g|^2$ is not the density of the position distribution of a single particle but of the "wave packet."

9.3. States Preserving the Number of Particles

Let Q be an arbitrary Σ'_ν-point process, and $\Phi: G \times M \to \mathbb{C}$ be a measurable function satisfying (8.1) and (8.2). Let $\tilde{\Phi}$ be defined by (8.4) and (8.5). We define a function $\hat{k}$ by

$$\hat{k}(\varphi_1, \varphi_2, \varphi) = \begin{cases} 0 & \text{if} \quad \varphi_1(G) \neq \varphi_2(G) \\ k(\varphi_1, \varphi_2, \varphi) & \text{otherwise}, \end{cases} \tag{9.15}$$

where k is the function defined by (8.6). Exactly as in the proof of Theorem 8.1 one easily shows that the pair $[Q, k]$ fulfills the assumptions of Theorem 7.3. Thus there exists a locally normal state ω on $\mathscr{A}$ such that $Q_\omega = Q$ and $k_\omega = \hat{k}$ a.e. Because of the definition (9.15) of the c.r.d.m. the state ω is concentrated on operators preserving the number of particles [i.e. if $A \in \mathscr{A}$ and for all $n \in \mathbb{N}$ $O_{M_n} A O_{M_n} \equiv 0$ then $\omega(A) = 0$].

In [8] we considered the time evolution and the question of invariance of such states with respect to a given potential (in the case $G = \mathbb{R}^d$, ν the Lebesgue measure). We introduced a certain function $W_\omega: \mathbb{R}^d\ M \to \mathbb{R}^d$ that can be interpreted as an "average velocity field." Under certain differentiability conditions (cf. [8]) this function W_ω is equal to

$$W_\omega(x, \varphi) = \frac{1}{\kappa_Q^{(1)}(x, \varphi)} (\mathrm{Re}\, h(x, \varphi) - \mathrm{Im}\, h(x, \varphi)),$$

where

$$h(x, \varphi) = \mathrm{grad}_y \hat{k}(\delta_x, \delta_y, \varphi)|_{y=x}.$$

If A^s denotes the velocity operator of the s^{th} component of the particles, $s \in \{1, \ldots, d\}$ we get for all $Y \in \mathfrak{M}$,

$$\omega(S(Y, A^s)) = \int_{\mathbb{R}^d \times Y} C_Q^{(1)}(d[x, \varphi]) W_\omega^s(x, \varphi) \tag{9.16}$$

provided this expectation exists (for convenience we set the particle's mass and Planck's constant equal to one).

For pairs $[Q_t, \hat{k}_t]$ (or equivalently $[Q_t, W_t]$) we considered in [8] equations of motion being equivalent to the usual Schrödinger equation if the states are normal ones, i.e. if the point processes Q_t are finite. Especially, one may conclude from the results in [8] that states the c.r.d.m. of which we have the property

$$\hat{k}(\delta_x, \delta_y, \varphi) = \sqrt{\kappa_Q^{(1)}(x, \varphi) \kappa_Q^{(1)}(y, \varphi)}$$

may be interpreted as equilibrium states. For details we refer to [8].

9.4. An Infinite Linear Chain of Coupled Harmonic Oscillators

In this example we consider one-dimensional systems, i.e. $G=\mathbb{R}$, ν the Lebesgue measure on $\mathbb{R}$. For a configuration $\varphi\in M$ and a point $x\in\mathbb{R}$ denote by $x_+(\varphi)$ [respectively $x_-(\varphi)$] the smallest mass point of φ greater than x (respectively the greatest one less or equal x).

The local (or conditional) potential $U:\mathbb{R}\times M\to\mathbb{R}$ defined by

$$U(x,\varphi)=\frac{a^2}{2}\left(\frac{x_+(\varphi)+x_-(\varphi)}{2}-x\right)^2, \tag{9.17}$$

where a is a positive constant describes harmonic oscillations of x around the centre of the two neighboured points in φ.

Let Q be a stationary simple recurrent point process with continuous density f_Q of the spacing distribution function

$$F_Q(x):=Q_0(\{\varphi:0_+(\varphi)\leqq x\}),$$

where Q_0 is the Palm distribution of Q (cf. [26, Chap. 9.5] or [8, Sects. 5 and 10]). For what follows we only use the fact that a point process Q of this type is determined completely by the density f_Q and that $\kappa_Q^{(1)}$ has the form

$$\kappa_Q^{(1)}(x,\varphi)=\frac{f_Q(x-x_-(\varphi))f_Q(x_+(\varphi)-x)}{f_Q(x_+(\varphi)-x_-(\varphi))}\qquad (C_Q^{(1)}\text{-}a.a.\ (x,\varphi)) \tag{9.18}$$

(cf. [8, Sect. 10]).

Setting

$$\Phi(x,\varphi)=\sqrt{\kappa_Q^{(1)}(x,\varphi)}\qquad (x\in\mathbb{R},\ \varphi\in M)$$

we obtain from the construction as in Example 9.3 [cf. (8.15)] a state ω with $Q_\omega=Q$ and the c.r.d.m. $\hat{k}$, where

$$\hat{k}(\varphi_1,\varphi_2,\varphi)=\tilde{\Phi}(\varphi_1,\varphi)\tilde{\Phi}(\varphi_2,\varphi)\delta_{\varphi_1(\mathbb{R})}(\varphi_2(\mathbb{R}))\,.$$

In [8, Sect. 10] we proved that the state ω described by Q and $\hat{k}$ is invariant with respect to the time evolution according to the potential U given by (9.17) if the density f_Q of the spacing distribution of Q satisfies

$$f_Q(x)=\sqrt{\frac{2a}{\pi}}\exp\left\{-\frac{ax^2}{2}\right\}\qquad (x\geqq 0),$$

i.e. if f_Q corresponds to a (one-sided) normal distribution.

10. Some Remarks on Quantum Equilibrium States

In a forthcoming paper we will consider conditional intensities of equilibrium states. In this section we will sketch only some basic ideas. For that reason we restrict here our considerations to the case of a free Bose gas with phase space $G=\mathbb{R}^1$.

We start with free bosons moving in a finite box $\Lambda:=[r,s]$ with natural boundary conditions (cf. [2]). H denotes the corresponding Hamilton operator and N denotes the number operator in $\mathcal{M}_\Lambda$. Then for all $\beta>0$ and $\alpha<0$

$\exp\{-\beta(H-\alpha N)\}$ is a positive trace-class operator (cf. [2]). By ω_Λ we denote the normal state on $\mathscr{A}_\Lambda$ related to the density matrix

$$\varrho_\Lambda = \frac{\exp\{-\beta(H-\alpha N)\}}{\operatorname{Tr}\exp\{-\beta(H-\alpha N)\}}. \tag{10.1}$$

It is easy to verify that the position distribution Q_{ω_Λ} of ω_Λ is a finite Σ'_v-point process.

Using the Feynman-Kac formula (cf. [2]) and the representation of k_{ω_Λ} (see the proof of Theorem 6.4) the c.r.d.m. can be calculated explicitly. The result can be formulated as follows:

First we set

$$\sigma = \frac{\beta\hbar^2}{2m} \tag{10.2}$$

(m is the mass of the boson, $\hbar$ Planck's constant).

Denote by $\pi(\varphi_1, \varphi_2)$ the set of all one-to-one mappings from the support of $\varphi_1 \in M$ onto the support of $\varphi_2 \in M$.

$\pi(\varphi) := \pi(\varphi, \varphi)$ is the set of all permutations of the mass-points of φ.

One obtains for all $\varphi_1, \varphi_2, \varphi \in M$, $\varphi_1(\Lambda) \neq 0$,

$$k_{\omega_\Lambda}(\varphi_1, \varphi_2, \varphi) = \frac{\exp\{\alpha\beta\varphi_1(\Lambda)\}}{\sqrt{(2\pi\sigma)^{\varphi_1(\Lambda)}}} \times \frac{\sum\limits_{g\in\pi(\varphi_1+\varphi,\varphi_2+\varphi)} \exp\left\{-\frac{1}{2\sigma}\int(\varphi+\varphi_1)(dx)(x-g(x))^2\right\}}{\sum\limits_{g\in\pi(\varphi,\varphi)} \exp\left\{-\frac{1}{2\sigma}\int\varphi(dx)(x-g(x))^2\right\}}. \tag{10.3}$$

If $\pi(\varphi_1, \varphi_2) = \emptyset$ the sum $\sum\limits_{g\in\pi(\varphi_1,\varphi_2)}$ will be set equal to zero. Observe that for all $\varphi \in M$ we have $k_{\omega_\Lambda}(\mathbb{O}, \mathbb{O}, \varphi) = 1$. Since $\pi(\varphi_1, \varphi_2) = \emptyset$ if $\varphi_1(\mathbb{R}) \neq \varphi_2(\mathbb{R})$ we further have $k_{\omega_\Lambda}(\varphi_1, \varphi_2, \varphi) = 0$ if $\varphi_1(\mathbb{R}) \neq \varphi_2(\mathbb{R})$.

Now, for $\varphi \in M_\Lambda$, $\varphi(\Lambda) \neq 0$ we denote by P_φ the probability measure on $\pi(\varphi)$ (i.e. the distribution of a random permutation of the points in φ) given by

$$P_\varphi(\{g\}) := \frac{1}{Z}\exp\left\{-\frac{1}{2\sigma}\int\varphi(dx)(x-g(x))^2\right\} \quad (g \in \pi(\varphi)), \tag{10.4}$$

where Z is the normalization factor. With this notation we get from (10.3)

$$\kappa_{Q_{\omega_\Lambda}}(\hat{\varphi}, \varphi) = k_{\omega_\Lambda}(\hat{\varphi}, \hat{\varphi}, \varphi) = \frac{\exp\{\alpha\beta\cdot\hat{\varphi}(\Lambda)\}}{\sqrt{(2\pi\sigma)^{\hat{\varphi}(\Lambda)}}} \cdot \frac{1}{P_{\varphi+\hat{\varphi}}(\{g : g(x) = x \text{ for } x \in \hat{\varphi}\})}. \tag{10.5}$$

If for a fixed β we take $\sigma \to 0$ (i.e. the mass m will be very large compared with $\hbar$) one gets easily from (10.4) and (10.5)

$$\kappa_{Q_{\omega_\Lambda}}(\varphi_1, \varphi) \xrightarrow[\sigma\to 0]{} \exp\{\alpha\beta\varphi_1(\Lambda)\}.$$

So we get in the limit that the position distribution of ω will be a Poisson point process with intensity $\exp\{\alpha\beta\}$, i.e. the position distribution of a free gas in classical mechanics. It is also possible to express k_{ω_Λ} explicitly in terms of the probability measures P_φ what gives hints for a characterization of the state ω of the infinite volume ideal Bose gas (cf. [2]) corresponding to the same inverse temperature $\beta>0$ and chemical potential $\alpha<0$. The position distribution Q_ω of this infinite ideal Bose gas is discussed in detail in [14]. It is shown there that Q_ω is an infinitely divisible point process on $\mathbb{R}$. Especially, clustering representations of this distribution are considered. Using Q_ω one may define for Q_ω-*a.a.* φ a distribution $\hat{P}_\varphi$ of a random permutation of the infinite point configuration φ. Now, if $(\Lambda_n)_{n\geqq 0}$ is an increase sequence from $\mathfrak{B}$ with $\lim_{n\to\infty} \Lambda_n=\mathbb{R}$ we get that $Q_{\omega_{\Lambda_n}}$ converges weakly toward Q_ω, and for Q_ω-*a.a.* φ the sequence $P_{\varphi_{\Lambda_n}}$ converges weakly to $\hat{P}_\varphi$. Further, one can show that for $F\times F\times Q_\omega$-*a.a.* $(\varphi_1,\varphi_2,\varphi)$ $\lim_{n\to\infty} k_{\omega_{\Lambda_n}}(\varphi_1,\varphi_2,\varphi_{\Lambda_n})$ exists and defines a function $k: M^3\to\mathbb{C}$ given by

$$k(\varphi_1,\varphi_2,\varphi)=\frac{\exp\{\alpha\beta\varphi_1(\Lambda)\}}{\sqrt{(2\pi\sigma)^{\varphi_1(\Lambda)}}}\times\frac{\int\hat{P}_{\varphi+\varphi_1}(dg)\exp\left\{-\frac{1}{2\sigma}\int\varphi_1(dx)(h(x)-x)(h(x)+x-2g(x))\right\}}{\hat{P}_{\varphi+\varphi_1}(\{g:g(x)=x \text{ for } x\in\varphi_1\})}. \tag{10.6}$$

It turns out that k given by (10.6) is just the c.r.d.m. of ω. It would be interesting to calculate the c.r.d.m. k of infinite equilibrium states corresponding to more general types of potentials. One could follow the ideas of Dobrushin, Lanford, and Ruelle from classical statistical mechanics and consider infinite equilibrium states as states corresponding to a given c.r.d.m. of a certain type. This concept can be realized for the ideal Bose gas and should be investigated for more general models in the future.

Proofs

11. Proofs from Section 2. In the sequel we will use again the abbreviation $\delta_{\underline{x}^n}$ for $\sum_{j=1}^{n}\delta_{x_j}$, $\underline{x}^n=(x_1,\dots,x_n)$. Further, we write $\nu^0(d\underline{x}^0)$ instead of $\delta_{\mathbb{O}}$, i.e. for arbitrary $\Lambda\in\mathfrak{G}$ and $g:M\to\mathbb{C}$ we set

$$\int_{\Lambda^0}\nu^0(d\underline{x}^0)g(\delta_{\underline{x}^0})=g(\mathbb{O}). \tag{11.1}$$

11.1. Proof of Lemma 2.5

Using the above notations we get

$$\begin{aligned}\int F_{\Lambda\cup\Lambda'}(d\varphi)h(\varphi_\Lambda,\varphi_{\Lambda'})&=\sum_{n\in\mathbb{N}}\frac{1}{n!}\int_{(\Lambda\cup\Lambda')^n}\nu^n(d\underline{x}^n)h((\delta_{\underline{x}^n})_\Lambda,(\delta_{\underline{x}^n})_{\Lambda'})\\&=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\int_{\Lambda^k}\nu^k(d\underline{x}^k)\int_{(\Lambda')^{n-k}}\nu^{n-k}(d\underline{y}^{n-k})h(\delta_{\underline{x}^k},\delta_{\underline{y}^{n-k}})\\&=\sum_{n\in\mathbb{N}}\sum_{n\geqq k}\frac{1}{k!(n-k)!}\int_{\Lambda^k}\nu^k(d\underline{x}^k)\int_{(\Lambda')^{n-k}}\nu^{n-k}(d\underline{y}^{n-k})h(\delta_{\underline{x}^k},\delta_{\underline{y}^{n-k}})\\&=\sum_{k\in\mathbb{N}}\frac{1}{k!}\int\nu^k(d\underline{x})\sum_{n\in\mathbb{N}}\frac{1}{n!}\int\nu^n(d\underline{y}^n)h(\delta_{\underline{x}^k},\delta_{\underline{y}^n})\\&=\int(F_\Lambda\times F_{\Lambda'})(d[\varphi_1,\varphi_2])h(\varphi_1,\varphi_2).\quad\square\end{aligned}$$

11.2. Proof of Lemma 2.7

Let Ψ be from $\mathcal{M}_\Lambda^f$. There exists a natural number m_0 such that $\operatorname{supp}\Psi \subseteqq M_\Lambda^{m_0}$ [cf. (2.5)].

For $\hat{\varphi} \notin M_\Lambda^{m_0}$ we have $\Psi_{\hat{\varphi}} \equiv 0$ and thus $\Psi_{\hat{\varphi}} \in \mathcal{M}_\Lambda^f$. We obtain the following estimations [where we again use the notation (11.1)]:

$$\begin{aligned}\int F_\Lambda(d\hat{\varphi}) \int F_\Lambda(d\varphi)\,|\Psi_{\hat{\varphi}}(\varphi)|^2 &= \int F_\Lambda(d\hat{\varphi}) \int F_\Lambda(d\varphi)\,|\Psi(\varphi+\hat{\varphi})|^2\\ &= \sum_{n=0}^{m_0} \frac{1}{n!} \int_{\Lambda^n} v^n(d\underline{x}^n) \sum_{k=0}^{m_0-n} \frac{1}{k!} \int_{\Lambda^k} v^n(d\underline{y}^k)\,|\Psi(\delta_{\underline{x}^n}+\delta_{\underline{y}^k})|^2\\ &= \sum_{n=0}^{m_0} \sum_{k=0}^{m_0-n} \binom{n+k}{n} \int_{M_{n+k,\Lambda}} F_\Lambda(d\varphi)\,|\Psi(\varphi)|^2 \leqq C\|\Psi\|^2,\end{aligned}$$

where C is a constant depending only on m_0. Consequently, we get

$$\int_{M^\Lambda} F_\Lambda(d\hat{\varphi})\,\|\Psi_{\hat{\varphi}}\|^2 < \infty\,.$$

Thus, for F_Λ-*a.a.* $\hat{\varphi}$ we have $\Psi_{\hat{\varphi}} \in \mathcal{M}$. Moreover, because of $\Psi_{\hat{\varphi}}(\varphi) = 0$ for $\varphi \notin M_\Lambda^{m_0}$ we have for F_Λ-*a.a.* $\hat{\varphi}$ $\Psi_{\hat{\varphi}} \in \mathcal{M}_\Lambda^f$. It is easy to deduce from this fact that for F_Λ-*a.a.* $\hat{\varphi}$ we have that for all $\tilde{\varphi} \subseteqq \hat{\varphi}$ $\Psi_{\tilde{\varphi}} \in \mathcal{M}_\Lambda^f$. We denote this set by $\hat{M}$, i.e.

$$\hat{M} = \{\hat{\varphi} \in M_\Lambda : \Psi_{\tilde{\varphi}} \in \mathcal{M}_\Lambda^f \text{ for all } \tilde{\varphi} \subseteqq \hat{\varphi}\}\,.$$

Since $F_\Lambda(\hat{M}^c) = 0$ and we identify two functions from $\mathcal{M}_\Lambda$ if they are equal F_Λ-*a.e.* we have $\Psi = \Psi_{\chi_{\hat{M}}}$.

For $\hat{\varphi} \in \hat{M}$ we have $\Psi_{\hat{\varphi}} \in \mathcal{M}_\Lambda^f$. For $\hat{\varphi} \notin \hat{M}$ we get

$$(\Psi_{\chi_{\hat{m}}})_{\hat{\varphi}}(\varphi) = \chi_{\hat{M}}(\hat{\varphi}+\varphi)\Psi(\hat{\varphi}+\varphi)\,.$$

$\hat{M}$ was so defined that $\hat{\varphi} \notin \hat{M}$ implies $\hat{\varphi}+\varphi \notin M$ for all $\varphi \in M$. Consequently, for $\hat{\varphi} \notin \hat{M}$ we get $(\Psi_{\chi_{\hat{M}}})_{\hat{\varphi}} \equiv 0 \in \mathcal{M}_\Lambda^f$. This proves Lemma 2.7. □

11.3. Proof of Proposition 2.11

It is enough to show 2.11 for indicator functions, i.e. for functions g of the type χ_Y, $Y \in \mathfrak{M}$.

Let $\Lambda \in \mathfrak{B}$, $Y \in \mathfrak{M}$. We will show that (i) implies (ii). For all $\varphi \in M$ we have

$$\chi_Y(\varphi) = \chi_{v_\Lambda(Y)}(\varphi_\Lambda)\chi_{v_{\Lambda^c}(Y)}(\varphi_{\Lambda^c})\,. \tag{11.2}$$

Consequently, we get $O_Y = I_\Lambda(O_{v_\Lambda(Y)} \otimes O_{v_{\Lambda^c}(Y)})I_\Lambda^{-1}$. From the assumption $O_Y \in {}_\Lambda\mathcal{A}$ we conclude that there exists an $A \in \mathcal{A}_\Lambda$ such that

$$O_{v_\Lambda(Y)} \otimes O_{v_{\Lambda^c}(Y)} = A \otimes \mathbb{1}_{\Lambda^c}\,.$$

Thus we have $A = O_{v_\Lambda(Y)}$ and $O_{v_{\Lambda^c}(Y)} = \mathbb{1}_{\Lambda^c}$. Since $\mathbb{1}_{\Lambda^c} = O_{M_{\Lambda^c}}$ we get $O_{v_{\Lambda^c}(Y)} = O_{M_{\Lambda^c}}$. Setting

$$\hat{Y} := \{\varphi+\hat{\varphi} : \varphi \in v_\Lambda(Y),\ \hat{\varphi} \in M_{\Lambda^c}\}$$

we thus get $O_{\hat{Y}} = O_Y$. Because of $v_\Lambda(\hat{Y}) = v_\Lambda(Y)$ we conclude from Lemma 2.2 $\hat{Y} \in {}_\Lambda\mathfrak{M}$.

Now, assume (ii) holds. From Lemma 2.2 we get for all $\Psi \in \mathcal{M}$, $\varphi \in M$

$$\begin{aligned}O_Y\Psi(\varphi) &= O_{\hat{Y}}\Psi(\varphi) = \chi_{v_\Lambda(\hat{Y})}(\varphi_\Lambda)\Psi(\varphi) = \chi_{v_\Lambda(Y)}(\varphi_\Lambda)\Psi(\varphi)\\ &= I_\Lambda(O_{v_\Lambda(Y)} \otimes \mathbb{1}_{\Lambda^c})I_\Lambda^{-1}\Psi(\varphi) = (J_\Lambda O_{v_\Lambda(Y)})\Psi(\varphi)\,.\end{aligned}$$

Thus (iii) is valid.

Finally, suppose (iii) holds. For arbitrary $X \in \mathfrak{M}$ we have $O_{v_\Lambda(X)} \in \mathscr{A}_\Lambda$. Thus for all $X \in \mathfrak{M}$ we get $J_\Lambda O_{v_\Lambda(X)} \in {}_\Lambda\mathscr{A}$. From the assumption $O_Y = J_\Lambda O_{v_\Lambda(Y)}$ we thus get $O_Y \in {}_\Lambda\mathscr{A}$. □

12. Proofs from Section 3

We start with proving several lemmas.

12.1. Lemma. *Let $\Lambda \in \mathfrak{G}$ and $h: M \times M \to \mathbb{C}$ be a measurable mapping. Then for all natural numbers n, m the following equalities hold (provided the integrals exist):*

$$\int F_\Lambda(d\varphi) \sum_{\hat\varphi \subseteq \varphi} h(\hat\varphi, \varphi - \hat\varphi) = \int F_\Lambda(d\varphi) \int F_\Lambda(d\hat\varphi) h(\hat\varphi, \varphi) \tag{12.1}$$

and

$$\int_{M_{n+m}} F_\Lambda(d\varphi) \sum_{\substack{\hat\varphi \subseteq \varphi \\ \hat\varphi(\Lambda) = m}} h(\hat\varphi, \varphi - \hat\varphi) = \int_{M_n} F_\Lambda(d\varphi) \int_{M_m} F_\Lambda(d\hat\varphi) h(\hat\varphi, \varphi). \tag{12.2}$$

Proof. In the sequel we often will use the following abbreviation

$$\delta_{\underline{x}^n_{i_1, \ldots, i_k}} := \sum_{j=1}^{n} \delta_{x_j} - \sum_{l=1}^{k} \delta_{x_{i_l}} \qquad (\underline{x}^n \in \Lambda^n,\ k \leqq n,\ \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}) \tag{12.3}$$

(i.e. from the configuration $\delta_{\underline{x}^n}$ we take out the points $x_{i_1}, \ldots, x_{i_k}$).

For arbitrary $n, m \geqq 0$ we obtain the following chain:

$$\begin{aligned}
&\int_{M_{n+m}} F_\Lambda(d\varphi) \sum_{\substack{\hat\varphi \subseteq \varphi \\ \hat\varphi(\Lambda) = m}} h(\hat\varphi, \varphi - \hat\varphi) = \frac{1}{(n+m)!} \int_{\Lambda^{n+m}} \nu^{n+m}(d\underline{x}^{n+m}) \\
&\quad \times \sum_{\substack{\{i_1, \ldots, i_m\} \\ \subseteq \{1, \ldots, n+m\}}} h\left(\sum_{j=1}^{m} \delta_{x_{i_j}}, \delta_{\underline{x}^{n+m}_{i_1, \ldots, i_m}} \right) \\
&= \frac{1}{(n+m)!} \binom{n+m}{m} \int_{\Lambda^{n+m}} \nu^{n+m}(d\underline{x}^{n+m}) h\left(\sum_{j=1}^{m} \delta_{x_j}, \sum_{l=m+1}^{m+n} \delta_{x_l} \right) \\
&= \frac{1}{n!} \int_{\Lambda^n} \nu^n(d\underline{x}^n) \frac{1}{m!} \int_{\Lambda^m} \nu^m(d\underline{y}^m) h(\delta_{\underline{y}^m}, \delta_{\underline{x}^n}) = \int_{M_n} F_\Lambda(d\varphi) \int_{M_m} F_\Lambda(d\hat\varphi) h(\hat\varphi, \varphi).
\end{aligned}$$

This proves (12.2). From (12.2) we get

$$\begin{aligned}
&\int F_\Lambda(d\varphi) \int F_\Lambda(d\hat\varphi) h(\hat\varphi, \varphi) = \sum_{n, m \in \mathbb{N}} \int_{M_n} F_\Lambda(d\varphi) \int_{M_m} F_\Lambda(d\hat\varphi) h(\hat\varphi, \varphi) \\
&= \sum_{n, m \in \mathbb{N}} \int_{M_{n+m}} F_\Lambda(d\varphi) \sum_{\substack{\hat\varphi \subseteq \varphi \\ \hat\varphi(\Lambda) = m}} h(\hat\varphi, \varphi - \hat\varphi) \\
&= \sum_{n \in \mathbb{N}} \sum_{l=0}^{n} \int_{M_n} F_\Lambda(d\varphi) \sum_{\substack{\hat\varphi \subseteq \varphi \\ \hat\varphi(\Lambda) = l}} h(\hat\varphi, \varphi - \hat\varphi) \\
&= \sum_{n \in \mathbb{N}} \int_{M_n} F_\Lambda(d\varphi) \sum_{\hat\varphi \subseteq \varphi} h(\hat\varphi, \varphi - \varphi) = \int F_\Lambda(d\varphi) \sum_{\hat\varphi \subseteq \varphi} h(\hat\varphi, \varphi - \hat\varphi).
\end{aligned}$$

Thus (12.1) holds. □

12.2 Lemma. *Let $h: M \to \mathbb{C}$ be a measurable function. Then for all $\Lambda \in \mathfrak{G}$ the following equality holds (provided the integrals exist):*

$$\int F_\Lambda(d\varphi) \int F_\Lambda(d\hat\varphi) h(\varphi + \hat\varphi) = \int F_\Lambda(d\varphi) h(\varphi) 2^{\varphi(\Lambda)}. \tag{12.4}$$

Proof. We set $\hat{h}(\varphi_1, \varphi_2) = h(\varphi_1 + \varphi_2)$. From (12.1) we get

$$\int F_A(d\varphi) \int F_A(d\hat{\varphi}) h(\varphi + \hat{\varphi}) = \int F_A(d\varphi) \sum_{\hat{\varphi} \subseteq \varphi} \hat{h}(\hat{\varphi}, \varphi - \hat{\varphi})$$
$$= \int F_A(d\varphi) h(\varphi) \sum_{\hat{\varphi} \subseteq \varphi} 1 = \int F_A(d\varphi) h(\varphi) 2^{\varphi(A)}. \quad \square$$

12.3. Proof of Proposition 3.3

1°. Assume $\Lambda \in \mathfrak{G}$, $A \in \mathscr{A}_\Lambda^f$, $Y \in \mathfrak{M}_\Lambda^f$. Let B be an operator from $S_\Lambda(Y, A)$. For $\Psi \in \mathscr{M}_\Lambda^f$ and F-*a.a.* φ we get from the definition of $S_\Lambda(Y, A)$,

$$B\Psi(\varphi) = \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi}) A\psi_{\hat{\varphi}})(\varphi - \hat{\varphi}). \tag{12.5}$$

Because of Lemma 2.7 the right side of (12.5) is a well-defined function from M into $\mathbb{C}$. We will show that B is bounded. For $\Psi \in \mathscr{M}_\Lambda^f$ we get

$$\int F_A(d\varphi) |B\Psi(\varphi)|^2 = \int F_A(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \sum_{\varphi_2 \subseteq \varphi} \chi_Y(\varphi_1) \chi_Y(\varphi_2) \overline{A\Psi_{\varphi_1}(\varphi - \varphi_1)} A\Psi_{\varphi_2}(\varphi - \varphi_2)$$
$$= \int F_A(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \sum_{\varphi_2 \subseteq \varphi - \varphi_1} \sum_{\varphi_3 \subseteq \varphi - \varphi_1 - \varphi_2} \chi_Y(\varphi_1 + \varphi_2) \chi_Y(\varphi_1 + \varphi_3)$$
$$\times \overline{A\Psi_{\varphi_1 + \varphi_2}(\varphi - \varphi_1 - \varphi_2)} A\Psi_{\varphi_1 + \varphi_3}(\varphi - \varphi_1 - \varphi_3)$$
$$= \int F_A(d\varphi) \int F_A(d\varphi_1) \sum_{\varphi_2 \subseteq \varphi} \sum_{\varphi_3 \subseteq \varphi - \varphi_2} \chi_Y(\varphi_1 + \varphi_2) \chi_Y(\varphi_1 + \varphi_3)$$
$$\times \overline{A\Psi_{\varphi_1 + \varphi_2}(\varphi - \varphi_2)} A\Psi_{\varphi_1 + \varphi_3}(\varphi - \varphi_3)$$
$$= \int F_A(d\varphi) \int F_A(d\varphi_1) \int F_A(d\varphi_2) \int F_A(d\varphi_3) \chi_Y(\varphi_1 + \varphi_2) \chi_Y(\varphi_1 + \varphi_3)$$
$$\times \overline{A\Psi_{\varphi_1 + \varphi_2}(\varphi + \varphi_3)} A\Psi_{\varphi_1 + \varphi_3}(\varphi + \varphi_2)$$
$$\leqq \int F_A(d\varphi) \int F_A(d\varphi_1) \int F_A(d\varphi_2)$$
$$\times \int F_A(d\varphi_3) \chi_Y(\varphi_1 + \varphi_2) \chi_Y(\varphi_1 + \varphi_3) |A\Psi_{\varphi_1 + \varphi_2}(\varphi + \varphi_3)|^2$$
$$= \int F_A(d\varphi) \int F_A(d\varphi_1) \chi_Y(\varphi_1) |A\Psi_{\varphi_1}(\varphi)|^2 2^{\varphi_1(A)} 2^{\varphi(A)}. \tag{12.6}$$

In the last step we applied Lemma 12.2. From the assumptions about Y and A we get that there exists an $m \in \mathbb{N}$ such that $Y \subseteq M_\Lambda^m$, $A = O_{M_\Lambda^m} A O_{M_\Lambda^m}$. We thus get applying once more Lemma 12.2,

$$\int F_A(d\varphi) |B\Psi(\varphi)|^2 \leqq 2^{2m} \|A\|^2 \int_{M^{2m}} F_A(d\varphi) |\Psi(\varphi)|^2 \cdot 2^{2m} \leqq 2^{4m} \|A\|^2 \|\Psi\|^2.$$

Consequently, there exists a constant C depending on A and Y but not on $\Psi \in \mathscr{M}_\Lambda^f$ such that $\|B\Psi\| \leqq C \|\Psi\|$ for all $\Psi \in \mathscr{M}_\Lambda^f$. B may be extended to the whole space $\mathscr{M}_\Lambda$ and $B \in \mathscr{A}_\Lambda$. From $B = O_{M^{2m}} B O_{M^{2m}}$ we finally get $B \in \mathscr{A}_\Lambda^f$.

2°. Now, let $A \in \mathscr{A}_\Lambda^f$ be self-adjoint. Because of $S_\Lambda(Y, A) \in \mathscr{A}_\Lambda$ we only have to prove that $S_\Lambda(Y, A)$ is symmetric. For $\Psi^1, \Psi^2 \in \mathscr{M}_\Lambda^f$ we get using Lemma 12.1 and Lemma 2.7

$$(\Psi^1, S_\Lambda(Y, A)\Psi^2)_\Lambda = \int F_A(d\varphi) \sum_{\hat{\varphi} \subseteq \varphi} \Psi_{\hat{\varphi}}^1(\varphi - \hat{\varphi}) \chi_Y(\hat{\varphi}) A\Psi_{\hat{\varphi}}^2(\varphi - \hat{\varphi})$$
$$= \int F_A(d\hat{\varphi}) \chi_Y(\hat{\varphi}) (\Psi_{\hat{\varphi}}^1, A\Psi_{\hat{\varphi}}^2)_\Lambda = \int F_A(d\hat{\varphi}) \chi_Y(\hat{\varphi}) (A\Psi_{\hat{\varphi}}^1, \Psi_{\hat{\varphi}}^2)_\Lambda$$
$$= \int F_A(d\varphi) \int F_A(d\hat{\varphi}) \chi_Y(\hat{\varphi}) \overline{A\Psi_{\hat{\varphi}}^1(\varphi)} \Psi_{\hat{\varphi}}^2(\varphi)$$
$$= \int F_A(\varphi) \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi}) \overline{A\Psi_{\hat{\varphi}}^1(\varphi - \hat{\varphi})} \Psi^2(\varphi)$$
$$= (S_\Lambda(Y, A)\Psi^1, \Psi^2)_\Lambda.$$

From Definition 3.1, (iii) we conclude that the above equality holds for all $\Psi^1, \Psi^2 \in \mathcal{M}_\Lambda$. Thus $S_\Lambda(Y, A)$ is self-adjoint.

3°. Now let $A \in \mathscr{A}^f_\Lambda$ be a positive operator, $Y \in \mathfrak{M}^f_\Lambda$. For all $\Psi \in \mathcal{M}^f_\Lambda$ we get applying Lemma 12.1 and Lemma 2.7,

$$\begin{aligned}(\Psi, S_\Lambda(Y, A)\Psi)_\Lambda &= \int F_\Lambda(d\varphi) \sum_{\hat\varphi \subseteqq \varphi} \chi_Y(\hat\varphi)\overline{\Psi(\varphi)} A \Psi_{\hat\varphi}(\varphi - \hat\varphi)\\ &= \int_Y F_\Lambda(d\hat\varphi) \int F_\Lambda(d\varphi)\overline{\Psi_{\hat\varphi}(\varphi)} A \Psi_{\hat\varphi}(\varphi) = \int_Y F_\Lambda(d\hat\varphi)(\Psi_{\hat\varphi}, A\Psi_{\hat\varphi})_\Lambda \geqq 0.\end{aligned}$$

From Definition 3.1, (iii) and because of $S_\Lambda(Y, A) \in \mathscr{A}_\Lambda$ we get

$$(\Psi, S_\Lambda(Y, A)\Psi)_\Lambda \geqq 0 \quad \text{for all} \quad \Psi \in \mathcal{M}_\Lambda. \quad \square$$

12.4. Proof of Proposition 3.4

Let $\Lambda \in \mathfrak{B}$, $A \in \mathscr{A}^f_\Lambda$, $Y \in \mathfrak{M}^f_\Lambda$. From Proposition 3.3 we know that $S_\Lambda(Y, A) \in \mathscr{A}^f_\Lambda$. Hence $J_\Lambda S_\Lambda(Y, A) \in {}_\Lambda\mathscr{A}^f$. Observe that for each $B \in \mathscr{A}_\Lambda$ we have

$$(J_\Lambda B\Psi)(\varphi) = (BO_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})(\varphi_\Lambda) \quad (\Psi \in \mathcal{M}, \varphi \in M). \tag{12.7}$$

(Observe that $O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}} \in \mathcal{M}_\Lambda$.)

For all $\Psi \in \mathcal{M}^f$, $\varphi \in M$ we get using (12.7),

$$\begin{aligned}(J_\Lambda S_\Lambda(Y, A)\Psi)(\varphi) &= (S_\Lambda(Y, A)O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})(\varphi_\Lambda)\\ &= \sum_{\hat\varphi \subseteqq \varphi_\Lambda} \chi_Y(\hat\varphi)(A(O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})_{\hat\varphi})(\varphi_\Lambda - \hat\varphi)\\ &= \sum_{\hat\varphi \subseteqq \varphi_\Lambda} \chi_Y(\hat\varphi)(AO_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}+\hat\varphi})(\varphi_\Lambda - \hat\varphi)\\ &= \sum_{\hat\varphi \subseteqq \varphi_\Lambda} \chi_Y(\hat\varphi)(AO_{M_\Lambda}(\Psi_{\hat\varphi})_{\varphi_{\Lambda^c}})(\varphi_\Lambda - \hat\varphi)\\ &= \sum_{\hat\varphi \subseteqq \varphi_\Lambda} \chi_Y(\hat\varphi)(J_\Lambda A\Psi_{\hat\varphi})(\varphi_{\Lambda_c} + \varphi_\Lambda - \hat\varphi)\\ &= \sum_{\hat\varphi \subseteqq \varphi_\Lambda} \chi_Y(\hat\varphi)(J_\Lambda A\Psi_{\hat\varphi})(\varphi - \hat\varphi) = \sum_{\hat\varphi \subseteqq \varphi} \chi_Y(\hat\varphi)(J_\Lambda A\Psi_{\hat\varphi})(\varphi - \hat\varphi)\\ &= (S(Y, J_\Lambda A)\Psi)(\varphi).\end{aligned} \tag{12.8}$$

Analogously, one shows $J_\Lambda S_\Lambda(Y, A) = S(v_\Lambda^{-1} Y, A)$. $\square$

12.5. Proof of Proposition 3.6

From the assumption $A \in {}_\Lambda\mathscr{A}$ we get $A = J_\Lambda O_{M_\Lambda} A O_{M_\Lambda}$. Using (12.7) we get for all $\Psi \in \mathcal{M}^f$, $\varphi \in M$,

$$\begin{aligned}(A\Psi)(\varphi) &= (O_{M_\Lambda} A O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})(\varphi_\Lambda)\\ &= \sum_{\hat\varphi \subseteqq \varphi} \chi_{M_{\Lambda^c}}(\hat\varphi)(O_{M_\Lambda} A O_{M_\Lambda}\Psi_{\hat\varphi})(\varphi - \hat\varphi)\\ &= (S(M_{\Lambda^c}, O_{M_\Lambda} A O_{M_\Lambda})\Psi)(\varphi).\end{aligned}$$

Since A was assumed to be bounded we get from Definition 3.1, (iii)

$$A\Psi = S(M_{\Lambda^c}, O_{M_\Lambda} A O_{M_\Lambda}) \quad \text{for all} \quad \Psi \in \mathcal{M}.$$

12.6. Proof of Proposition 3.7

Assume $A \in \mathscr{A}_\Lambda$, $\Lambda \in \mathfrak{B}$. We have $A = O_{M_\Lambda}(J_\Lambda A)O_{M_\Lambda}$. So from Proposition 3.6 we conclude

$$J_\Lambda A = S(M_{\Lambda^c}, O_{M_\Lambda}(J_\Lambda A)O_{M_\Lambda}) = S(M_{\Lambda^c}, A). \quad \square$$

12.7. Proof of Proposition 3.8

Assume $Y \in \mathfrak{M}$, $A \in \mathscr{A}_\Lambda$, $S(Y, A) \in {}_\Lambda\mathscr{A}$ for some $\Lambda \in \mathfrak{B}$. From the identity (11.2) we obtain especially

$$\chi_{M_\Lambda}(\varphi - \hat{\varphi}) = \chi_{M_\Lambda}(\varphi_\Lambda - \hat{\varphi}_\Lambda)\chi_{\{\mathbf{O}\}}(\varphi_{\Lambda^c} - \hat{\varphi}_{\Lambda^c}) \qquad (\varphi, \hat{\varphi} \in M,\ \hat{\varphi} \subseteq \varphi). \tag{12.9}$$

From (11.2) and (12.9) we get for all $\Psi \in \mathscr{M}^f$ and F-*a.a.* φ,

$$\begin{aligned} S(Y, A)\Psi(\varphi) &= \sum_{\hat{\varphi} \subseteq \varphi} \chi_{v_{\Lambda^c}(Y)}(\hat{\varphi}_{\Lambda^c})\chi_{v_\Lambda(Y)}(\hat{\varphi}_\Lambda)A\Psi_{\hat{\varphi}}(\varphi - \hat{\varphi}) \\ &= \sum_{\hat{\varphi} \subseteq \varphi_\Lambda} \chi_{v_\Lambda(Y)}(\hat{\varphi})\chi_{v_{\Lambda^c}(Y)}(\varphi_{\Lambda^c})A\Psi_{\hat{\varphi} + \varphi_{\Lambda^c}}(\varphi_\Lambda - \hat{\varphi}) \\ &= I_\Lambda(S_\Lambda(v_\Lambda(Y), A) \otimes O_{v_{\Lambda^c}(Y)})I_\Lambda^{-1}\Psi(\varphi). \end{aligned} \tag{12.10}$$

From the assumption $S(Y, A) \in {}_\Lambda\mathscr{A}$ we get for all $\Psi \in \mathscr{M}^f$ and F-*a.a.* φ,

$$\begin{aligned} S(Y, A)\Psi(\varphi) &= (O_{M_\Lambda}S(Y, A)O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})(\varphi_\Lambda) \\ &= \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi})A(O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})_{\hat{\varphi}}(\varphi_\Lambda - \hat{\varphi}) \\ &= \sum_{\hat{\varphi} \subseteq \varphi_\Lambda} \chi_{v_\Lambda(Y)}(\hat{\varphi})AO_{M_\Lambda}\Psi_{\varphi_{\Lambda^c} + \hat{\varphi}}(\varphi_\Lambda - \hat{\varphi})\chi_{v_{\Lambda^c}(Y)}(\mathbf{O}). \end{aligned} \tag{12.11}$$

$\mathbf{O} \in v_{\Lambda^c}(Y)$ because otherwise we would have $S(Y, A) \equiv 0$.

But since we assumed $S(Y, A) \in {}_\Lambda\mathscr{A}$ this cannot happen (each operator from ${}_\Lambda\mathscr{A}$ is "outside Λ" the identical operator and thus not identically zero).

Consequently, from (12.11) we get

$$S(Y, A) = I_\Lambda(S_\Lambda(v_\Lambda(Y), A) \otimes \mathbb{1}_{\Lambda^c})I_\Lambda^{-1}. \tag{12.12}$$

Combining (12.12) and (12.10) we get

$$O_{v_{\Lambda^c}(Y)} = \mathbb{1}_{\Lambda^c} = O_{M_{\Lambda^c}}.$$

Thus, for $\hat{Y} \in \mathfrak{M}$ such that $v_\Lambda(\hat{Y}) = v_\Lambda(Y)$, $v_{\Lambda^c}(Y) = M_{\Lambda^c}$ we get $O_{\hat{Y}} = O_Y$. But from Lemma 2.2 we know that $\hat{Y} \in {}_\Lambda\mathfrak{M}$. $\square$

12.8. Proof of Proposition 3.9

Let $\Lambda \in \mathfrak{B}$, $A \in \mathscr{L}(\mathscr{M})$, $Y \in {}_\Lambda\mathfrak{M}$, $S(Y, A) \in {}_\Lambda\mathscr{A}$. From Lemma 2.2 we know that for all $\varphi \in M$ we have $\chi_Y(\varphi) = \chi_{v_\Lambda(Y)}(\varphi_\Lambda)$. Further, we have $S(Y, A) = J_\Lambda(O_{M_\Lambda}S(Y, A)O_{M_\Lambda})$. Using the above observations we get for all $\Psi \in \mathscr{M}^f$ and F-*a.a.* φ,

$$\begin{aligned} S(Y, A)\Psi(\varphi) &= \sum_{\hat{\varphi} \subseteq \varphi_1} \chi_Y(\hat{\varphi})(A(O_{M_\Lambda}\Psi_{\varphi_{\Lambda^c}})_{\hat{\varphi}})(\varphi_\Lambda - \hat{\varphi}) \\ &= \sum_{\hat{\varphi} \subseteq \varphi_\Lambda} \chi_Y(\hat{\varphi})\chi_{M_\Lambda}(\varphi_\Lambda - \hat{\varphi})(AO_{M_\Lambda}\Psi_{\varphi_{\Lambda^c} + \hat{\varphi}})(\varphi_\Lambda - \hat{\varphi}) \\ &= \sum_{\hat{\varphi} \subseteq \varphi} \chi_Y(\hat{\varphi})(O_{M_\Lambda}AO_{M_\Lambda}\Psi_{\hat{\varphi}})(\varphi - \hat{\varphi}) = S(Y, O_{M_\Lambda}AO_{M_\Lambda})\Psi(\varphi). \end{aligned}$$

But $0_{M_\Lambda}A0_{M_\Lambda} \in \mathscr{A}_\Lambda$. $\square$

12.9. Proof of Proposition 3.10

Let $\Lambda, \Lambda' \in \mathfrak{B}$, $\Lambda \cap \Lambda' = \emptyset$, $Y \in \mathfrak{M}_{\Lambda'}$, $A \in \mathscr{A}_\Lambda$.

For all $\Psi \in \mathscr{M}$, $\varphi \in M$ the sum

$$\sum_{\hat{\varphi} \subseteqq \varphi} \chi_Y(\hat{\varphi})(A\Psi_{\hat{\varphi}})(\varphi - \hat{\varphi})$$

exists ($\varphi \in Y \subseteqq M_{\Lambda'}$ and $\hat{\varphi} \subseteqq \varphi$ imply $\varphi \subseteqq \varphi_{\Lambda'} \subseteqq \varphi_{\Lambda^c}$). Further, if $\hat{\varphi} \neq \varphi_{\Lambda'}$ then $\varphi - \hat{\varphi} \notin M_\Lambda$ because $\Lambda \cap \Lambda' = \emptyset$. Consequently,

$$S(Y, A)\Psi(\varphi) = \chi_Y(\varphi_{\Lambda'})(A\Psi_{\varphi_{\Lambda'}})(\varphi - \varphi_{\Lambda'}).$$

From $A \in \mathscr{A}_\Lambda$ we get that $S(Y, A)\Psi(\varphi) = 0$ for $\varphi \notin M_{\Lambda \cup \Lambda'}$.

It is easy to check that $S(Y, A)$ is bounded. Thus $S(Y, A) \in \mathscr{A}_{\Lambda \cup \Lambda'}$. □

12.10. Proof of Proposition 3.11

Let Y_1, Y_2 be from $\mathfrak{M}$. Using several times Lemma 12.1 and Lemma 12.2 we get for $\Psi \in \mathscr{M}^f$,

$$\begin{aligned}
\|S(Y_1, O_{Y_2})\Psi\|^2 &= \int F(d\varphi_1) \sum_{\varphi_2 \subseteqq \varphi_1} \sum_{\varphi_3 \subseteqq \varphi_1} \chi_{Y_1}(\varphi_2)\chi_{Y_2}(\varphi_1 - \varphi_2) \\
&\quad \times \chi_{Y_1}(\varphi_3)\chi_{Y_2}(\varphi_1 - \varphi_3)|\Psi(\varphi_1)|^2 \\
&= \int F(d\varphi_1) \sum_{\varphi_2 \subseteqq \varphi_1} \sum_{\varphi_3 \subseteqq \varphi_1 - \varphi_2} \sum_{\varphi_4 \subseteqq \varphi_1 - \varphi_2 - \varphi_3} \chi_{Y_1 \times Y_1}(\varphi_2 + \varphi_3, \varphi_2 + \varphi_4) \\
&\quad \times \chi_{Y_2 \times Y_2}(\varphi_1 - \varphi_2 - \varphi_3, \varphi_1 - \varphi_2 - \varphi_4)|\Psi(\varphi_1)|^2 \\
&= \int F^4(d[\varphi_1, \varphi_2, \varphi_3, \varphi_4])\chi_{Y_1 \times Y_1}(\varphi_2 + \varphi_3, \varphi_2 + \varphi_4) \\
&\quad \times \chi_{Y_2 \times Y_2}(\varphi_1 + \varphi_4, \varphi_1 + \varphi_3)|\Psi(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)|^2 \\
&\leqq \int F^4(d[\varphi_1, \varphi_2, \varphi_3, \varphi_4])|\Psi(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)|^2 \\
&\leqq \int F(d\varphi)2^{2\varphi(G)}|\Psi(\varphi)|^2 \leqq C\|\Psi\|^2,
\end{aligned} \tag{12.13}$$

where C is a constant depending on $\Psi \in \mathscr{M}^f$ ($\Psi = \Psi_{\chi_{M^m}}$ for some $m \in \mathbb{N}$). Thus $S(Y_1, O_{Y_2})\Psi \in \mathscr{M}$ for all $\Psi \in \mathscr{M}^f$. Consequently, $S(Y_1, O_{Y_2}) \neq \emptyset$. Further, for all $\Psi \in \mathscr{M}^f$ we get

$$\begin{aligned}
(\Psi, S(Y_1, O_{Y_2})\Psi) &= \int F(d\varphi)\overline{\Psi(\varphi)} \sum_{\hat{\varphi} \subseteqq \varphi} \chi_{Y_1}(\hat{\varphi})\chi_{Y_2}(\varphi - \hat{\varphi})\Psi(\varphi) \\
&= \int F(d\varphi) \int F(d\hat{\varphi})\chi_{Y_1}(\hat{\varphi})\chi_{Y_2}(\varphi)|\Psi(\varphi + \hat{\varphi})|^2 \geqq 0.
\end{aligned}$$

We thus get that all operators from $S(Y_1, O_{Y_2})$ are positive. Analogous estimations as in (12.13) prove that for $Y \in \mathfrak{M}$, $f \in \mathscr{M}^b$, $|f| \leqq a$ *F-a.e.* we get

$$\|S(Y, O_f)\Psi\|^2 \leqq a^2 C\|\Psi\|^2$$

for all $\Psi \in \mathscr{M}^f$, where C is a constant depending on Ψ. This proves 3.11.

13. Proofs from Sections 5 and 6

13.1. Proof of Theorem 6.4

1°. Let ω be a normal state on $\mathscr{L}(\mathscr{M})$. Thus there exists a density matrix ϱ on $\mathscr{M}$ such that $\omega(A) = \mathrm{Tr}(\varrho A)$ for all $A \in \mathscr{L}(\mathscr{M})$. Consequently, there exists an orthonormal system $(\Psi_s)_{s\in S}$ of elements from $\mathcal{M}$ (with S a finite or countable index set) and a sequence $(\alpha_s)_{s\in S}$, $\alpha_s>0$, $\sum_{s\in S}\alpha_s=1$ such that

$$\varrho=\sum_{s\in S}\alpha_s(\Psi_s,\cdot)\Psi_s. \tag{13.1}$$

Having this representation of ϱ we get for all $Y\in\mathfrak{M}$,

$$\begin{aligned}Q_\omega(Y)&=\mathrm{Tr}(\varrho O_Y)=\sum_{s\in S}\alpha_s(\Psi_s,O_Y\Psi_s)\\&=\int_Y F(d\varphi)\sum_{s\in S}\alpha_s|\Psi_s(\varphi)|^2=\int_Y F(d\varphi)D(\varphi),\end{aligned} \tag{13.2}$$

where we have set for all $\varphi\in M$,

$$D(\varphi):=\sum_{s\in S}\alpha_s|\Psi_s(\varphi)|^2.$$

Thus $Q_\omega\ll F$, and $D(\varphi)$ is a version of dQ_ω/dF.

2°. Observe that the density $D(\varphi)$ has the following property:

$$\kappa(\hat\varphi,\varphi)D(\varphi)=D(\hat\varphi+\varphi)\qquad(F\times F\text{-a.a.}\,(\hat\varphi,\varphi)), \tag{13.3}$$

where $\kappa:=dC_{Q_\omega}^{(\infty)}/d(F\times Q_\omega)$. Indeed, for all $Y_1,Y_2\in\mathfrak{M}$ we get from (13.2) and Lemma 12.1,

$$\begin{aligned}C_{Q_\omega}^{(\infty)}(Y_1\times Y_2)&=\int F(d\varphi)\sum_{\hat\varphi\subseteq\varphi}D(\varphi)\chi_{Y_1}(\hat\varphi)\chi_{Y_2}(\varphi-\hat\varphi)\\&=\int F(d\varphi)\int F(d\hat\varphi)\chi_{Y_1}(\hat\varphi)\chi_{Y_2}(\varphi)D(\varphi+\hat\varphi)\\&=\int_{Y_2}F(d\varphi)\int_{Y_1}F(d\hat\varphi)D(\varphi+\hat\varphi).\end{aligned} \tag{13.4}$$

On the other hand, from (5.5) and (13.2) we obtain

$$C_{Q_\omega}^{(\infty)}(Y_1\times Y_2)=\int_{Y_2}F(d\varphi)\int_{Y_1}F(d\hat\varphi)D(\varphi)\kappa(\hat\varphi,\varphi). \tag{13.5}$$

Combining (13.4) and (13.5) we obtain (13.3).

(13.3) implies that for $F\times F$-a.a. $(\hat\varphi,\varphi)$ we have the implication:

$$D(\varphi+\hat\varphi)>0\Rightarrow D(\varphi)>0.$$

Especially, the function $k_\omega:M^3\to\mathbb{C}$ given by

$$k_\omega(\varphi_1,\varphi_2,\varphi)=\frac{\sum_{s\in S}\alpha_s\Psi_s(\varphi_1+\varphi)\overline{\Psi_s(\varphi_2+\varphi)}}{D(\varphi)}\qquad(\varphi_1,\varphi_2,\varphi\in M) \tag{13.6}$$

is a well-defined measurable function (we set $\frac{0}{0}=0$).

3°. Now, let Y be from $\mathfrak{M}$, A an integral operator from $\mathscr{L}(\mathcal{M})$ such that $S(Y,A)\in\mathscr{L}(\mathcal{M})$. First we additionally assume that $S(Y,A)$ is concentrated on $\mathcal{M}^f$, i.e. there exists an $m\geqq 1$ such that $S(Y,A)O_{M^m}=S(Y,A)$ ($M^m=\{\varphi\in M:\varphi(G)\leqq m\}$). Observe that for all $\hat\varphi\in M$ we have $(O_{M^m}\Psi)_{\hat\varphi}\in\mathcal{M}$ (Lemma 2.7). Denoting by k_A a kernel of A we get for arbitrary $\Psi\in\{\Psi_s:s\in S\}$ the following chain of equalities:

$$\begin{aligned}&(\Psi,S(Y,A)\Psi)\\&\quad=\int F(d\varphi)\overline{\Psi(\varphi)}\sum_{\varphi_1\subseteq\varphi}\chi_Y(\varphi-\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\Psi(\varphi-\varphi_1+\varphi_2)\\&\quad=\int Q_\omega(d\varphi)\frac{1}{D(\varphi)}\sum_{\varphi_1\subseteq\varphi}\chi_Y(\varphi-\varphi_1)\overline{\Psi(\varphi)}\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\Psi(\varphi-\varphi_1+\varphi_2)\end{aligned}$$

$$=\int_Y C^{(\infty)}_{Q_\omega}(d[\varphi_1,\varphi])\frac{1}{D(\varphi+\varphi_1)}\overline{\Psi(\varphi+\varphi_1)}\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\Psi(\varphi+\varphi_2)$$

$$=\int_Y Q_\omega(d\varphi)\int F(d\varphi_1)\frac{\kappa(\varphi_1,\varphi)}{D(\varphi+\varphi_1)}\overline{\Psi(\varphi+\varphi_1)}\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\Psi(\varphi+\varphi_2)$$

$$=\int_Y Q_\omega(d\varphi)\int F(d\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\frac{\overline{\Psi(\varphi_1+\varphi)}\Psi(\varphi_2+\varphi)}{D(\varphi)}. \tag{13.7}$$

Consequently, by Definition (13.6) we get

$$\omega(S(Y,A))=\sum_{s\in S}\alpha_s(\Psi_s,S(Y,A)\Psi_s)$$
$$=\int_Y Q_\omega(d\varphi)\int F(d\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)k_\omega(\varphi_2,\varphi_1,\varphi)$$
$$=\int_Y Q_\omega(d\varphi)\int F(d\varphi_1)k_A * k_\omega(\varphi_1,\varphi). \tag{13.8}$$

4°. Now, let $S(Y,A)$ be from $\mathscr{L}(\mathscr{M})$. From the definition of $S(Y,A)$ [Definition 3.1, (iii)] and the continuity of ω we deduce that (13.8) holds for arbitrary $Y\in\mathfrak{M}$ and integral operators $A\in\mathscr{L}(\mathscr{M})$ such that $S(Y,A)\in\mathscr{L}(\mathscr{M})$. This proves (i).

5°. From Lemma 5.3, (5.6) we know that for Q_ω-*a.a.* φ $\eta^\varphi_Q\ll F$ and κ is a version of $d\eta^\varphi_Q/dF$. From step 2° of the above proof we immediately get (6.4).

6°. We still have to prove that k_ω is *a.e.* uniquely determined. For arbitrary $Y_0, Y_1, Y_2\in\mathfrak{M}^f$ the function $\chi_{Y_1}(\varphi_1)\chi_{Y_2}(\varphi_2)$, $\varphi_1,\varphi_2\in M$ represents the kernel of a Hilbert-Schmidt operator A from $\mathscr{A}$. Consequently, $S(Y_0,A)\in\mathscr{A}^f$ (cf. Proposition 3.3).

So if $\tilde{k}:M^3\to\mathbb{C}$ is a measurable function satisfying (6.3) and (6.4) we get

$$\int_{Y_0} Q_\omega(d\varphi)\int_{Y_1} F(d\varphi_1)\int_{Y_2} F(d\varphi_2)(k_\omega(\varphi_1,\varphi_2,\varphi)-\tilde{k}(\varphi_1,\varphi_2,\varphi))=0. \tag{13.9}$$

Since F and Q_ω are concentrated on M^f from (13.9) we easily conclude that $k_\omega=\tilde{k}$ $F\times F\times Q_\omega$-*a.e.* □

13.2. Proof of Theorem 6.8

Since K is assumed to be a positive trace-class operator on $\mathscr{M}$ there exists an orthonormal sequence $(\Psi_s)_{s\in S}$ from $\mathscr{M}$ with S an at most countable index set and a sequence $(\alpha_s)_{s\in S}$, $\alpha_s>0$, $\sum_{s\in S}\alpha_s<\infty$ such that

$$K=\sum_{s\in S}\alpha_s(\Psi_s,\cdot)\Psi_s. \tag{13.10}$$

Consequently, $\hat{k}:M^2\to\mathbb{C}$ defined by

$$\hat{k}(\varphi_1,\varphi_2):=\sum_{s\in S}\alpha_s\Psi_s(\varphi_1)\overline{\Psi_s(\varphi_2)}\qquad(\varphi_1,\varphi_2\in M) \tag{13.11}$$

is a kernel of K.

We will show that $\omega(\cdot):=\mathrm{Tr}(\varrho\,\cdot)$ with

$$\varrho=Q(\{\mathbb{O}\})K \tag{13.12}$$

is the normal Σ'_ν-state with $Q_\omega=Q$ and $k_\omega=k$.

First observe that

$$\mathrm{Tr}\,\varrho=Q(\{\mathbb{O}\})\mathrm{Tr}\,K=Q(\{\mathbb{O}\})\eta^{\mathbb{O}}_Q(M). \tag{13.13}$$

Because of

$$Q(Y)=C_Q^{(\infty)}(Y\times\{\mathbb{O}\})=\int_{\{\mathbb{O}\}} Q(d\varphi)\eta_Q^{\varphi}(Y)=Q(\{\mathbb{O}\})\eta_Q^{\mathbb{O}}(Y) \qquad (Y\in\mathfrak{M})$$

from (13.13) we get $\operatorname{Tr}\varrho=1$. Thus, ω is a normal state. For all $Y\in\mathfrak{M}$ we get from (13.14) and assumption (6.7)

$$Q_\omega(Y)=\operatorname{Tr}(\varrho O_Y)=Q(\{\mathbb{O}\})\operatorname{Tr}(KO_Y)=Q(\{\mathbb{O}\})\eta_Q^{\mathbb{O}}(Y)=Q(Y).$$

Thus $Q_\omega=Q$.

From (13.10), (13.11), and (6.8) we conclude

$$k((\varphi_1,\varphi_2),\mathbb{O})=\hat{k}(\varphi_1,\varphi_2) \qquad (F\times F\text{-}a.a.\,(\varphi_1,\varphi_2)). \tag{13.15}$$

Now, let Y be from $\mathfrak{M}$, A an integral operator from $\mathscr{L}(\mathscr{M})$ such that $S(Y,A)\in\mathscr{L}(\mathscr{M})$. Let k_A denote a kernel of A. Then we get from (13.12), (13.14), (13.15), (6.9) and Lemma 12.1,

$$\begin{aligned}
\omega(S(Y,A))&=Q(\{\mathbb{O}\})\sum_{s\in S}\alpha_s(\Psi_s,S(Y,A)\Psi_s)\\
&=Q(\{\mathbb{O}\})\sum_{s\in S}\alpha_s\int F(d\varphi_1)\overline{\Psi_s(\varphi_1)}\sum_{\varphi\subseteq\varphi_1}\chi_Y(\varphi)(A(\Psi_s)_\varphi)(\varphi_1-\varphi)\\
&=Q(\{\mathbb{O}\})\sum_{s\in S}\alpha_s\int_Y F(d\varphi)\int F(d\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\\
&\quad\times\overline{\Psi_s(\varphi+\varphi_1)}\Psi_s(\varphi+\varphi_s)\\
&=Q(\{\mathbb{O}\})\int_Y F(d\varphi)\int F(d\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)\hat{k}(\varphi+\varphi_2,\varphi+\varphi_1)\\
&=Q(\{\mathbb{O}\})\int_Y F(d\varphi)\int F(d\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)k(\varphi+\varphi_2,\varphi+\varphi_1,\mathbb{O})\\
&=Q(\{\mathbb{O}\})\int_Y F(d\varphi)k(\varphi,\varphi,\mathbb{O})\int F(\varphi_1)\int F(d\varphi_2)k_A(\varphi_1,\varphi_2)k(\varphi_2,\varphi_1,\varphi)\\
&=Q(\{\mathbb{O}\})\int_Y \eta_Q^{\mathbb{O}}(d\varphi)\int F(d\varphi_1)k_A*k(\varphi_1,\varphi)\\
&=\int_Y Q(d\varphi)F(d\varphi_1)k_A*k(\varphi_1,\varphi).
\end{aligned}$$

Thus k fulfills condition (6.3) of Theorem 6.4.

From (6.7) and (6.9) we get for $F\times F\text{-}a.a.\,(\varphi_1,\varphi)$

$$\kappa_Q(\varphi_1+\varphi,\mathbb{O})=\kappa_Q(\varphi,\mathbb{O})k(\varphi_1,\varphi_1,\varphi),$$

where $\kappa_Q=dC_Q^{(\infty)}/d(F\times Q)$. One easily checks that for $F\times F\text{-}a.a.\,(\varphi_1,\varphi)$,

$$\kappa_Q(\varphi_1,\varphi)\kappa_Q(\varphi,\mathbb{O})=\kappa_Q(\varphi_1+\varphi,\mathbb{O}). \tag{13.16}$$

Since $\kappa_Q(\varphi_1+\varphi,\mathbb{O})>0$ implies $\kappa_Q(\varphi,\mathbb{O})>0$ we get for $F\times Q\text{-}a.a.\,(\varphi_1,\varphi)$ (6.4). From Theorem 6.4 we thus conclude $k_\omega=k$ *a.e.* what ends the proof. □

13.3. Proof of Theorem 6.10

The implication (II) ⇒ (I) is the contents of Theorem 6.8. Assume (I) is fulfilled.

In the proof of Theorem 6.4 we obtained that the *a.e.* uniquely determined function k_ω can be written in the form (13.6), i.e.

$$k_\omega(\varphi_1,\varphi_2,\varphi)=\frac{1}{D(\varphi)}\sum_{s\in S}\alpha_s\Psi_s(\varphi_1+\varphi)\overline{\Psi_s(\varphi_2+\varphi)} \qquad (\varphi_1,\varphi_2,\varphi\in M), \tag{13.17}$$

where we use the notations from step 1° in the proof of Theorem 6.4.

Since $0 < Q_\omega(\{\mathbb{O}\}) = D(\mathbb{O})$ we have that $\hat{k}(\varphi_1, \varphi_2) := k_\omega(\varphi_1, \varphi_2, \mathbb{O})$ is the kernel of a positive trace-class operator K with

$$\begin{aligned}\mathrm{Tr}(KO_Y) &= \int_Y F(d\varphi)\hat{k}(\varphi,\varphi) \qquad (Y \in \mathfrak{M}^f)\\ &= \int_Y F(d\varphi)\frac{D(\varphi)}{D(\mathbb{O})} = \frac{1}{Q_\omega(\{\mathbb{O}\})} Q_\omega(Y) = \eta^{\mathbb{O}}_{Q_\omega}(Y),\end{aligned}$$

where we used formula (13.14). Thus condition (i) is fulfilled. Condition (ii) follows directly from the definition of K. Finally, for all $\varphi_1, \varphi_2, \varphi, \hat{\varphi} \in M$ we get from (13.17),

$$\begin{aligned}k_\omega(\varphi_1+\varphi, \varphi_2+\varphi, \hat{\varphi}) &= \frac{D(\varphi+\hat{\varphi})}{D(\hat{\varphi})} \cdot \frac{1}{D(\varphi+\hat{\varphi})} \sum_{s\in S} \alpha_s \Psi_s(\varphi_1+\varphi+\hat{\varphi})\overline{\Psi_s(\varphi_2+\varphi+\hat{\varphi})}\\ &= k_\omega(\varphi,\varphi,\hat{\varphi})k_\omega(\varphi_1,\varphi_2,\varphi+\hat{\varphi})\end{aligned}$$

which proves (6.9). □

13.4. Proof of Proposition 5.8

From Proposition 3.4 we obtain $S(Y_1, O_{Y_2}) = J_\Lambda S_\Lambda(Y_1, O_{v_\Lambda Y_2})$. Denoting by ω_Λ the normal state on $\mathscr{L}(\mathscr{M}_\Lambda)$ given by $\omega_\Lambda(A) = \omega(J_\Lambda A)$ for $A \in \mathscr{L}(\mathscr{M}_\Lambda)$ we thus get

$$\omega(S(Y_1, O_{Y_2})) = \omega_\Lambda(S_\Lambda(Y_1, O_{v_\Lambda Y_2})).$$

Obviously, the position distribution Q_Λ of ω_Λ is given by

$$Q_\Lambda(Y) = Q_\omega(v_\Lambda^{-1} Y) \qquad (Y \in \mathfrak{M}_\Lambda). \tag{13.18}$$

There exists a density matrix ϱ on $\mathscr{M}_\Lambda$ such that $\omega_\Lambda = \mathrm{Tr}(\varrho\,\cdot)$. Let ϱ be written in the form (13.1) (with $\Psi_s \in \mathscr{M}_\Lambda$). Using the notations from step 1° of the proof of Theorem 6.4 and (13.18) we get

$$\begin{aligned}\omega(S(Y_1, O_{Y_2})) &= \mathrm{Tr}(\varrho S_\Lambda(Y_1, O_{v_\Lambda Y_2}))\\ &= \sum_{s\in S} \alpha_s \int F_\Lambda(d\varphi)\overline{\Psi_s(\varphi)} \sum_{\hat{\varphi} \subseteq \varphi} \chi_{Y_1}(\hat{\varphi})\chi_{v_\Lambda Y_2}(\varphi-\hat{\varphi})\Psi_s(\varphi)\\ &= \int F_\Lambda(d\varphi) D(\varphi) \sum_{\hat{\varphi}\subseteq\varphi} \chi_{Y_1}(\hat{\varphi})\chi_{v_\Lambda Y_2}(\varphi-\hat{\varphi})\\ &= C^{(\infty)}_{Q_\Lambda}(Y_1 \times v_\Lambda Y_2)\\ &= C^{(\infty)}_{Q_\omega}(Y_1 \times Y_2).\end{aligned} \tag{13.19}$$

From Lemma 5.3 ((5.6)) we obtain $C^{(\infty)}_{Q_\omega}(Y_1 \times Y_2) = \int_{Y_2} Q_\omega(d\varphi)\eta^{\varphi}_{Q_\omega}(Y_1)$. This proves (i). By the usual approximation procedure of measurable functions by step functions we obtain (ii). □

14. Proof of Theorem 7.3

Let $\Lambda \in \mathfrak{B}$ be fixed. For all $\varphi \in M_{\Lambda^c}$ we set

$$K^\varphi_\Lambda \Psi(\varphi_1) = \int F_\Lambda(d\varphi_2)k(\varphi_1, \varphi_2, \varphi)\Psi(\varphi_2) \qquad (\Psi \in \mathscr{M}_\Lambda,\ \varphi_1 \in M_\Lambda). \tag{14.1}$$

Denote by Q_{Λ^c} the measure $Q \circ v_{\Lambda^c}^{-1}$ (on $[M_{\Lambda^c}, \mathfrak{M}_{\Lambda^c}]$). From assumption (I) we conclude that for Q_{Λ^c}-*a.a.* φ K^φ_Λ is a positive trace-class operator on $\mathscr{M}_\Lambda$. Further, it is well-known (cf. the Russian edition of [26, Proposition 9.1.10]) that for Q_{Λ^c}-a.a. φ $0<\eta_Q^\varphi(M_\Lambda)<\infty$. Now we fix a $\varphi\in M_{\Lambda^c}$ such that K_Λ^φ is a positive trace-class operator and $0<\eta_Q^\varphi(M_\Lambda)<\infty$. From (7.2) we conclude

$$0<\operatorname{Tr}(K_\Lambda^\varphi)=\int F_\Lambda(d\hat{\varphi})k(\hat{\varphi},\hat{\varphi},\varphi)=\eta_Q^\varphi(M_\Lambda)<\infty\,. \tag{14.2}$$

There exists an orthonormal sequence $(\tilde{\Psi}_s)_{s\in S}$ from $\mathscr{M}_\Lambda$ and a sequence $(\tilde{\alpha}_s)_{s\in S'}$, $\tilde{\alpha}_s>0$, $\sum_{s\in S}\tilde{\alpha}_s<\infty$ with S an at most countable index set such that

$$\hat{k}(\varphi_1,\varphi_2)=\sum_{s\in S}\tilde{\alpha}_s\tilde{\Psi}_s(\varphi_1)\overline{\tilde{\Psi}_s(\varphi_2)}\qquad(\varphi_1,\varphi_2\in M_\Lambda) \tag{14.3}$$

is a kernel of K_Λ^φ ($\tilde{\Psi}_s$ and $\tilde{\alpha}_s$ depend on the fixed counting measure φ). For the kernel defined this way we get

$$\operatorname{Tr}(K_\Lambda^\varphi O_Y)=\int_Y F(d\hat{\varphi})\hat{k}(\hat{\varphi},\hat{\varphi})\qquad(Y\in\mathfrak{M}_\Lambda)\,.$$

So from (14.1) and (14.2) we may conclude

$$k(\varphi_1,\varphi_2,\varphi)=\hat{k}(\varphi_1,\varphi_2)\qquad(F_\Lambda\times F_\Lambda\text{-a.a.}\,(\varphi_1,\varphi_2)) \tag{14.4}$$

and

$$k(\hat{\varphi},\hat{\varphi},\varphi)=\hat{k}(\hat{\varphi},\hat{\varphi})=\kappa_Q(\hat{\varphi},\varphi)\qquad(F_\Lambda\text{-a.a.}\ \hat{\varphi})\,. \tag{14.5}$$

Consequently, for Q_{Λ^c}-a.a. $\varphi\in M_{\Lambda^c}$ by

$${}_\Lambda\tilde{\omega}^\varphi(A):=(\eta_Q^\varphi(M_\Lambda))^{-1}\operatorname{Tr}(K_\Lambda^\varphi O_{M_\Lambda}AO_{M_\Lambda})\qquad(A\in{}_\Lambda\mathscr{A}) \tag{14.6}$$

there is defined a normal state on ${}_\Lambda\mathscr{A}$.

Now we set

$${}_\Lambda\tilde{\omega}(A)=\int_{M_{\Lambda^c}}Q_{\Lambda^c}(d\varphi)\,{}_\Lambda\tilde{\omega}^\varphi(A)\qquad(A\in{}_\Lambda\mathscr{A})\,. \tag{14.7}$$

Obviously, ${}_\Lambda\tilde{\omega}$ is a state on ${}_\Lambda\mathscr{A}$. Applying Theorem 2.6.14 in [2] and Lebesgue's dominated convergence theorem we get that ${}_\Lambda\tilde{\omega}$ is again a normal state on ${}_\Lambda\mathscr{A}$.

Consequently, for all $\Lambda\in\mathfrak{B}$ by (14.6) there is defined a normal state on ${}_\Lambda\mathscr{A}$. We will prove now compatibility of this family of states, i.e. we fix two arbitrary sets Λ, $\Lambda^1\in\mathfrak{B}$, $\Lambda\subseteqq\Lambda^1$ and show

$${}_\Lambda\tilde{\omega}(A)={}_{\Lambda^1}\tilde{\omega}(A)\qquad(A\in{}_\Lambda\mathscr{A})\,.$$

Let $A\in\mathscr{A}_\Lambda$ be an integral operator with kernel k_A, and let $\varphi\in M_{(\Lambda^1)^c}$ be such that $K_{\Lambda^1}^\varphi$ is a positive trace-class operator on $\mathscr{M}_{\Lambda^1}$ the kernel of which is of the form

$$k(\varphi_1,\varphi_2,\varphi)=\sum_{s\in S}\alpha_s\Psi_s(\varphi_1)\overline{\Psi_s(\varphi_2)}\qquad(\varphi_1,\varphi_2\in M_{\Lambda^1}) \tag{14.8}$$

with $(\Psi_s)_{s\in S}$ an orthonormal sequence in $\mathscr{M}_{\Lambda^1}$, $\alpha_s>0$, $\sum_{s\in S}\alpha_s<\infty$.

Using several times assumption (7.3), Lemma 2.5 and (14.5) we get

$$\begin{aligned}
&\operatorname{Tr}(K_{\Lambda^1}^\varphi O_{M_{\Lambda^1}}J_\Lambda AO_{M_{\Lambda^1}})\\
&=\sum_{s\in S}\alpha_s\int F_{\Lambda^1}(d\varphi_1)\overline{\Psi_s(\varphi_1)}\int F_\Lambda(d\varphi_2)k_A(\varphi_{1_\Lambda},\varphi_2)\Psi_s(\varphi_2+\varphi_{1_{\Lambda^1\setminus\Lambda}})\\
&=\int_{M_{\Lambda^1\setminus\Lambda}}F(d\varphi_0)\int_{M_\Lambda}F(d\varphi_1)\int_{M_\Lambda}F(d\varphi_2)k_A(\varphi_1,\varphi_2)k(\varphi_2+\varphi_0,\varphi_1+\varphi_0,\varphi)\\
&=\int_{M_{\Lambda^1\setminus\Lambda}}F(d\varphi_0)k(\varphi_0,\varphi_0,\varphi)\int_{M_\Lambda}F(d\varphi_1)\int_{M_\Lambda}F(d\varphi_2)k_A(\varphi_1,\varphi_2)k(\varphi_2,\varphi_1,\varphi+\varphi_0)\\
&=\int_{M_{\Lambda^1\setminus\Lambda}}\eta_Q^\varphi(d\varphi_0)\int_{M_\Lambda}F(d\varphi_1)k_A\underset{\Lambda}{*}k(\varphi_1,\varphi_0+\varphi)\,.
\end{aligned} \tag{14.9}$$

Thus, we get from (14.7), (14.6), (14.9), Lemma 2.5, (6.10) and (5.7)

$$
\begin{aligned}
{}_{\Lambda^1}\tilde{\omega}(J_\Lambda A) &= \int_{M_{(\Lambda^1)^c}} Q_{(\Lambda^1)^c}(d\varphi) \int_{M_{\Lambda^1\setminus\Lambda}} \eta_Q^\varphi(d\varphi_0) \frac{1}{\eta_Q^\varphi(M_{\Lambda^1})} \\
&\quad \times \int_{M_\Lambda} F(d\varphi_2) k_\Lambda \underset{\Lambda}{*} k(\varphi_2, \varphi_0 + \varphi) \\
&= \int_{M_{(\Lambda^1)^c}} Q_{(\Lambda^1)^c}(d\varphi) \int_{M_{\Lambda^1\setminus\Lambda}} F(d\varphi_0) \int_{M_\Lambda} F(d\varphi_1) \kappa_Q(\varphi_0, \varphi) \kappa_Q(\varphi_1, \varphi + \varphi_0) \\
&\quad \times \frac{1}{\eta_Q^\varphi(M_{\Lambda^1}) \eta_Q^{\varphi+\varphi_0}(M_\Lambda)} \int_{M_\Lambda} F(d\varphi_2) k_\Lambda \underset{\Lambda}{*} k(\varphi_2, \varphi_0 + \varphi) \\
&= \int_{M_{(\Lambda^1)^c}} Q_{(\Lambda^1)^c}(d\varphi) \int_{M_{\Lambda^1}} F(d\varphi_0) \kappa_Q(\varphi_0, \varphi) \frac{1}{\eta_Q^\varphi(M_{\Lambda^1}) \eta_Q^{\varphi+\varphi_{0(\Lambda^1\setminus\Lambda)}}(M_\Lambda)} \\
&\quad \times \int_{M_\Lambda} F(d\varphi_2) k_\Lambda \underset{\Lambda}{*} k(\varphi_2, \varphi_{0(\Lambda^1\setminus\Lambda)} + \varphi) \\
&= \int_{M_{(\Lambda^1)^c}} Q_{(\Lambda^1)^c}(d\varphi) \frac{1}{\eta_Q^\varphi(M_{\Lambda^1})} \int_{M_{\Lambda^1}} \eta_Q^\varphi(d\varphi_0) \frac{1}{\eta_Q^{\varphi+\varphi_{0\Lambda^c}}(M_\Lambda)} \\
&\quad \times \int_{M_\Lambda} F(d\varphi_2) k_\Lambda \underset{\Lambda}{*} k(\varphi_2, \varphi_{\Lambda^1\setminus\Lambda} + \varphi) \\
&= \int_{M_{\Lambda^c}} Q_{\Lambda^c}(d\varphi) \frac{1}{\eta_Q^\varphi(M_\Lambda)} \int F(d\hat{\varphi}) k_\Lambda \underset{\Lambda}{*} k(\hat{\varphi}, \varphi) = {}_\Lambda\tilde{\omega}(J_\Lambda A)\,. \qquad (14.10)
\end{aligned}
$$

Thus we obtained that for all $\Lambda, \Lambda^1 \in \mathfrak{B}$, $\Lambda \subseteqq \Lambda^1$ ${}_{\Lambda^1}\tilde{\omega}(A) = {}_\Lambda\tilde{\omega}(A)$ for all integral operators from ${}_\Lambda\mathscr{A}$, and consequently, we have equality for all $A \in {}_\Lambda\mathscr{A}$. This proves that there exists a locally normal state ω on $\mathscr{A}$ such that ${}_\Lambda\omega = {}_\Lambda\tilde{\omega}$ for all $\Lambda \in \mathfrak{B}$.

We have to prove that $Q_\omega = Q$. For all $\Lambda \in \mathfrak{B}$ and $Y \in {}_\Lambda\mathfrak{M}$ we get using (14.7), (14.6), and (14.5) and Lemma 5.3 (5.7)

$$
\begin{aligned}
\omega(O_Y) = {}_\Lambda\omega(O_Y) = {}_\Lambda\tilde{\omega}(O_Y) &= \int_{M_{\Lambda^c}} Q_{\Lambda^c}(d\varphi)\, {}_\Lambda\tilde{\omega}^\varphi(O_Y) \\
&= \int_{M_{\Lambda^c}} Q_{\Lambda^c}(d\varphi) \frac{1}{\eta_Q^\varphi(M_\Lambda)} \int_{Y \cap M_\Lambda} F(d\hat{\varphi}) \kappa_Q(\hat{\varphi}, \varphi) \\
&= \int Q(d\varphi) \frac{\eta_Q^{\varphi_{\Lambda^c}}(v_\Lambda Y)}{\eta_Q^{\varphi_{\Lambda^c}}(M_\Lambda)} = \int Q(d\varphi) Q(Y|_{\Lambda^c}\mathfrak{M})(d\varphi) = Q(Y)\,.
\end{aligned}
$$

Since Q is determined by all sets from $\bigcup_{\Lambda \in \mathfrak{B}} {}_\Lambda\mathfrak{M}$ this implies $Q_\omega = Q$.

Finally, we have to show that the function k has the properties (i)–(iii) of Definition 7.1. Properties (i) and (ii) follow immediately from assumption (I).

We fix $\Lambda \in \mathfrak{B}$ and a $\varphi \in M_{\Lambda^c}$ such that K_Λ^φ is a positive trace-class operator with a kernel of the form (14.3) (where Ψ_s and α_s depend on Λ and the chosen counting measure φ). Let $Y \in {}_\Lambda\mathfrak{M}$, A an integral operator from $\mathscr{A}_\Lambda$ with kernel k_A such that $S(Y, A) \in {}_\Lambda\mathscr{A}$. Using the fact that $S(Y, A) = J_\Lambda S_\Lambda(v_\Lambda Y, A)$ (Proposition 3.4) we get using (14.3), (14.4), Lemma 12.1, (7.3), (14.5) and Lemma 5.3

$$
\begin{aligned}
\operatorname{Tr}(K_\Lambda^\varphi S_\Lambda(v_\Lambda Y, A)) &= \sum_{s \in S} \alpha_s \int F_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) \overline{\Psi_s(\varphi_1 + \varphi_2)} \chi_{v_\Lambda Y}(\varphi_2) A(\Psi_s)_{\varphi_2}(\varphi_1) \\
&= \int F_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A(\varphi_1, \varphi_2) \\
&\quad \times \int_{v_\Lambda Y} F_\Lambda(d\varphi_3) k(\varphi_2 + \varphi_3, \varphi_1 + \varphi_3, \varphi) \\
&= \int_{v_\Lambda Y} \eta_Q^\varphi(d\hat{\varphi}) \int F_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A(\varphi_1, \varphi_2) k(\varphi_2, \varphi_1, \varphi + \hat{\varphi})\,.
\end{aligned}
$$

Using (14.6) and (14.7) we finally get

$$\begin{aligned}\omega(S(Y,\Lambda)) &= \omega(J_\Lambda S_\Lambda(v_\Lambda Y,\Lambda))\\ &= \int_{M_{\Lambda^c}} Q_{\Lambda^c}(d\varphi)(\eta_Q^\varphi(M_\Lambda))^{-1}\operatorname{Tr}(K_\Lambda^\varphi S_\Lambda(v_\Lambda Y,\Lambda))\\ &= \int_{M_{\Lambda^c}} Q_{\Lambda^c}(d\varphi)\int_{v_\Lambda Y}\eta_Q^\varphi(d\hat\varphi)(\eta_Q^\varphi(M_\Lambda))^{-1}\int F_\Lambda(d\varphi_1)k_\Lambda \underset{\Lambda}{*} k(\varphi_1,\varphi+\hat\varphi)\\ &= \int_Y Q(d\varphi)\int F_\Lambda(d\varphi_1)k_\Lambda \underset{\Lambda}{*} k(\varphi_1,\varphi).\end{aligned}$$

This ends the proof of Theorem 7.3.

15. Proof of Proposition 8.1

We will show that the function k defined by (8.6) satisfies the assumptions of Theorem 7.3. It is easy to check (cf. also [26]) that for all $\delta_{\underline{x}^n}\in M$, $\underline{x}^n\in G^n$, $n\geqq 1$, and $\varphi\in M$,

$$\kappa_Q(\delta_{\underline{x}^n},\varphi)=\prod_{j=1}^{n}\kappa_Q^{(1)}\left(x_j,\varphi+\sum_{l=1}^{j-1}\delta_{x_l}\right), \tag{15.1}$$

where $\kappa_Q=dC_Q^{(\infty)}/d(F\times Q)$. Consequently, directly from the definition of $\tilde{\Phi}$ (5.1)–(5.5) we get

$$|\tilde\Phi(\hat\varphi,\varphi)|^2=\kappa_Q(\hat\varphi,\varphi)\qquad(\hat\varphi,\varphi\in M). \tag{15.2}$$

Now, for arbitrary $\Lambda\in\mathfrak{B}$ we set

$$\Psi_\Lambda^\varphi(\hat\varphi):=\tilde\Phi(\hat\varphi,\varphi)\chi_{M_\Lambda}(\hat\varphi)\qquad(\hat\varphi\in M,\ \varphi\in M_{\Lambda^c}).$$

Directly from the definition of $C_Q^{(\infty)}$ and κ_Q we get

$$\begin{aligned}\int_{M_{\Lambda^c}} Q(d\varphi)\int_M F(d\hat\varphi)|\Psi_\Lambda^\varphi(\hat\varphi)|^2 &= \int_{M_{\Lambda^c}} Q(d\varphi)\int_{M_\Lambda} F(d\hat\varphi)\kappa_Q(\hat\varphi,\varphi)\\ &= C_Q^{(\infty)}(M_\Lambda\times M_{\Lambda^c})=\int Q(d\varphi)\sum_{\substack{\hat\varphi\subseteq\varphi\\ \hat\varphi\in M^f}}\chi_{M_\Lambda}(\hat\varphi)\chi_{M_{\Lambda^c}}(\varphi-\hat\varphi)\\ &= \int Q(d\varphi)\chi_{M_\Lambda}(\varphi_\Lambda)\chi_{M_{\Lambda^c}}(\varphi_{\Lambda^c})=Q(M)=1.\end{aligned}$$

Consequently, for all $\Lambda\in\mathfrak{B}$ and Q-a.a. φ $\|\Psi_\Lambda^{\varphi_{\Lambda^c}}\|<\infty$ and thus $\Psi_\Lambda^{\varphi_{\Lambda^c}}\in\mathscr{M}_\Lambda$. This implies that for all $\Lambda\in\mathfrak{B}$ and Q-a.a. $\varphi\in M_{\Lambda^c}$ by

$$K_\Lambda^\varphi:=(\Psi_\Lambda^\varphi,\cdot)\Psi_\Lambda^\varphi \tag{15.3}$$

there is defined a positive trace-class operator on $\mathscr{M}_\Lambda$ with kernel

$$\Psi_\Lambda^\varphi(\varphi_1)\overline{\Psi_\Lambda^\varphi(\varphi_2)}=k(\varphi_1,\varphi_2,\varphi)\chi_{M_\Lambda\times M_\Lambda}(\varphi_1,\varphi_2)\qquad(\varphi_1,\varphi_2\in M).$$

Because of (15.1) condition (7.2) is satisfied. This proves (I).

From (8.4) and (8.3) we get for all $n,m\geqq 1$, $\underline{x}^n\in G^n$, $\underline{y}^m\in G^m$, and $\varphi\in M$,

$$\begin{aligned}\tilde\Phi(\delta_{\underline{x}^n}+\delta_{\underline{y}^m},\varphi) &= \prod_{j=1}^{n}\Phi\left(x_j,\varphi+\sum_{r=1}^{j-1}\delta_{x_r}\right)\prod_{l=1}^{m}\Phi\left(y_l,\varphi+\delta_{\underline{x}^n}+\sum_{s=1}^{l-1}\delta_{y_s}\right)\\ &= \tilde\Phi(\delta_{\underline{x}^n},\varphi)\Phi(\delta_{\underline{y}^m},\varphi+\delta_{\underline{x}^n}).\end{aligned} \tag{15.4}$$

(15.4) and part (8.5) of the definition of $\tilde{\Phi}$ imply

$$\tilde{\Phi}(\varphi_1+\varphi_2,\varphi)=\tilde{\Phi}(\varphi_1,\varphi)\tilde{\Phi}(\varphi_2,\varphi+\varphi_1) \qquad (\varphi_1,\varphi_2,\varphi\in M). \tag{15.5}$$

From (15.5) we immediately get that the function k defined by (8.6) fulfills condition (II) of Theorem 7.3.

Consequently, there exists a unique locally normal state ω on $\mathscr{A}$ such that $Q_\omega=Q$ and k is the c.r.d.m. of ω.

Now, if ω is a normal state then Q is a finite point process (Proposition 6.1). We prove the converse. If Q is a finite Σ'_v-point process then $Q(\{\mathbb{O}\})>0$ and $Q(X)=Q(\{\mathbb{O}\})\eta_Q^{\mathbb{O}}(X)$ for all $X\in\mathfrak{M}$ [cf. (13.14)].

Observe that $\tilde{\Phi}(\cdot,\mathbb{O})\in\mathscr{M}$. Indeed

$$\int F(d\varphi)\,|\Phi(\varphi,\mathbb{O})|^2=\int F(d\varphi)\kappa(\varphi,\mathbb{O})=\eta_Q^{\mathbb{O}}(M)=\frac{1}{Q(\{\mathbb{O}\})}.$$

Consequently, the operator K on $\mathscr{M}$ with kernel $k(\varphi_1,\varphi_2,\mathbb{O})$ is a positive trace class operator, and

$$\omega:=Q(\{\mathbb{O}\})\,\mathrm{Tr}(K\,\cdot) \tag{15.6}$$

defines a normal state on $\mathscr{L}(\mathscr{M})$ with position distribution $Q_\omega=Q$. Since it was already shown that k is the c.r.d.m. of a certain state $\tilde{\omega}$ and $Q_{\tilde{\omega}}=Q_\omega=Q$ we only have to show that k has the property (iii) of Definition 7.1. For that reason, let Λ be from $\mathfrak{B}$, $A\in\mathscr{A}_\Lambda$ an integral operator with kernel k_A, $Y\in{}_\Lambda\mathfrak{M}$ and assume $S(Y,A)\in{}_\Lambda\mathscr{A}$. From Definition (15.6) we get applying Lemma 12.1

$$\begin{aligned}
\omega(S(Y,A))&=Q(\{\mathbb{O}\})\,\mathrm{Tr}(KS(Y,A))\\
&=Q(\{\mathbb{O}\})\int F(d\varphi)\overline{\tilde{\Phi}(\varphi,\mathbb{O})}\sum_{\varphi_1\subseteq\varphi}\chi_Y(\varphi_1)\\
&\quad\times\int F_\Lambda(d\varphi_2)k_A(\varphi-\varphi_1,\varphi_2)\tilde{\Phi}(\varphi_1+\varphi_2,\mathbb{O})\\
&=Q(\{\mathbb{O}\})\int F_\Lambda(d\varphi)\int_Y F(d\varphi_1)\int F_\Lambda(d\varphi_2)k_A(\varphi,\varphi_2)k(\varphi_2+\varphi_1,\varphi+\varphi_1,\mathbb{O})\\
&=Q(\{\mathbb{O}\})\int_Y F(d\varphi_1)k(\varphi_1,\varphi_1,\mathbb{O})\int F_\Lambda(d\varphi)\\
&\quad\times\int F_\Lambda(d\varphi_2)k_A(\varphi,\varphi_2)k(\varphi_2,\varphi,\varphi_1)\\
&=Q(\{\mathbb{O}\})\int_Y \eta_Q^{\mathbb{O}}(d\varphi_1)\int F_\Lambda(d\varphi)k_A\underset{\Lambda}{*}k(\varphi,\varphi_1)\\
&=\int_Y Q(d\varphi_1)\int F_\Lambda(d\varphi)k_A\underset{\Lambda}{*}k(\varphi,\varphi_1).
\end{aligned}$$

Finally, we thus get that the normal Σ'_v-state ω given by (15.6) is the uniquely determined state ω on $\mathscr{A}$ such that $Q_\omega=Q$ and k is the c.r.d.m. of ω. □

16. Proof of Proposition 9.2

Let ω be the state obtained by Proposition 8.1, and denote by $\tilde{\omega}$ the normal state given by $\tilde{\omega}=(\Psi,\cdot\,\Psi)$.

Obviously, $Q_\omega=Q_{\tilde{\omega}}=Q$. For all $\Lambda\in\mathfrak{B}$ and $A\in\mathscr{A}_\Lambda$ we have $J_\Lambda A=S(M_{\Lambda^c},A)$ (Proposition 3.7). So, from Theorem 6.4 we conclude that for all integral operators

$A \in \mathscr{A}_\Lambda$ (with kernel k_A) we have applying Lemma 2.5,

$$\begin{aligned}\omega(J_\Lambda A) &= \int_{M_{\Lambda^c}} Q(d\varphi) \int F_\Lambda(d\varphi_1) k_A \underset{\Lambda}{*} k(\varphi_1, \varphi) \\ &= \int_{M_{\Lambda^c}} F(d\varphi) \int F_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A(\varphi_1, \varphi_2) k(\varphi_2, \varphi_1, \varphi) |\Psi(\varphi)|^2 \\ &= \int F_{\Lambda^c}(d\varphi) \int F_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A(\varphi_1, \varphi_2) \Psi(\varphi_2 + \varphi) \overline{\Psi(\varphi_1 + \varphi)} \\ &= \int F(d\varphi) \int F_\Lambda(d\varphi_2) k_A(\varphi_\Lambda, \varphi_2) \Psi(\varphi_2 + \varphi_{\Lambda^c}) \overline{\Psi(\varphi)} \\ &= \int F(d\varphi) \overline{\Psi(\varphi)} A \Psi_{\varphi_{\Lambda^c}}(\varphi_\Lambda) = \int F(d\varphi) \overline{\Psi(\varphi)} J_\Lambda A \Psi(\varphi) \\ &= (\Psi, J_\Lambda A \Psi) = \tilde{\omega}(J_\Lambda A).\end{aligned}$$

Since Λ was chosen arbitrarily and integral operators are dense in $\mathscr{L}(\mathscr{M})$ this implies $\omega = \tilde{\omega}$. □

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