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An Algorithm for Detecting Abelian Monopoles in SU_2-Valued Lattice Gauge-Higgs Systems

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Abstract. We present an algorithm which calculates the monopole number of an SU_2-valued lattice gauge field, together with a lattice Higgs field, on a simplicial lattice of dimension $\geqq 3$. The calculation is gauge invariant. The expected value of the monopole density (for a fixed Higgs field) does not depend on the Higgs field.

Introduction

This paper addresses the problem of locating the abelian monopoles of an SU_2-valued lattice gauge-Higgs field system on a complex of dimension $\geqq 3$. In the smooth case, these phenomena have been extensively studied from the analytic and algebraic-geometric side (for example, [2–6, 9, 16]) but we believe a more local and topological analysis, besides being of intrinsic interest, will be useful in the study of monopoles as vacuum fluctuations, especially by lattice-theoretic methods as in [7]. A summary of this material appeared in [12].

We present an algorithm which, given a generic SU_2-valued lattice gauge field $\mathbf{u}$, and a lattice Higgs field $\mathbf{e}$, defined on a locally ordered simplicial lattice Λ, associates to each oriented 3-simplex $\Delta \in \Lambda$ an integer $\mu_{\mathbf{u},\mathbf{e}}(\Delta)$, the monopole number of the pair $\mathbf{u}, \mathbf{e}$ in Δ.

The lattice gauge field $\mathbf{u}$, as usual, assigns to every oriented 1-simplex $\langle ij \rangle \in \Lambda$ an SU_2-element u_{ij}, with $u_{ji} = u_{ij}^{-1}$, and the lattice Higgs field $\mathbf{e}$ assigns to each vertex $\langle i \rangle \in \Lambda$ a unit vector $e_i \in \mathbf{R}^3$. If we change gauge via a family $\mathbf{g} = \{g_i : \langle i \rangle \in \Lambda\}$ of elements of SU_2, then in the new gauge $\mathbf{u}$ becomes the lattice gauge field $\mathbf{gug}^{-1}$ which assigns $g_i u_{ij} g_j^{-1}$ to $\langle ij \rangle$, and $\mathbf{e}$ transforms under the adjoint action to $\mathbf{geg}^{-1}$, which assigns $g_i * e_i = g_i e_i g_i^{-1}$ to $\langle i \rangle$, identifying SU_2 with the unit quaternions and $\mathbf{R}^3 = \{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\}$ with the pure imaginary quaternions as usual.

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This monopole number has the following properties:

• *Coboundary zero* (*Local conservation law*). Suppose Σ is an oriented 4-simplex of Λ. Then each 3-simplex $\Delta \in \partial\Sigma$ inherits an orientation from Σ, and with respect to these orientations

$$\sum_{\Delta \in \partial\Sigma} \mu_{\mathbf{u},\mathbf{e}}(\Delta) = 0,$$

i.e. $\mu_{\mathbf{u},\mathbf{e}}$ is a 3-cocycle (see Sect. 1). It follows that on the boundary of a 4-simplex the positive monopoles must exactly balance the negative ones; if we imagine joining each of the positives to a negative by an arc in the interior of the 4-simplex, we will obtain a set of closed curves in Λ, which are lattice monopole world lines associated to **u** and **e**.

(If Λ is a triangulation of an oriented 4-manifold, then each oriented monopole world line can be given a sign: for example, call it positive if it goes through a positive monopole on its way *out* of a positively oriented 4-simplex.)

• *Gauge invariance.* If both **u** and **e** are modified as above, then μ remains unchanged:

$$\mu_{\mathbf{gug}^{-1},\mathbf{geg}^{-1}}(\Delta) = \mu_{\mathbf{u},\mathbf{e}}(\Delta).$$

(This will be proved in Sect. 3.)

• **u**-*expected value independent of* **e**. Fix the coupling constant β and consider the monopole density

$$M_{\mathbf{u},\mathbf{e}} = \frac{1}{N} \sum_{\Delta \in \Lambda} |\mu_{\mathbf{u},\mathbf{e}}(\Delta)|,$$

where N is the number of 3-simplexes in the lattice, and its expected value (with respect to **u**)

$$\langle M_{\mathbf{e}} \rangle_\beta = \frac{1}{Z} \int_{\mathbf{G}} M_{\mathbf{u},\mathbf{e}} e^{-\beta S(\mathbf{u})} d\mathbf{u},$$

where $S(\mathbf{u})$ is the Wilson action of the lattice gauge field **u**, or any other gauge-invariant action. Here **G** is the space of all SU_2-valued lattice gauge fields on Λ, $d\mathbf{u}$ is the product of the Haar measures, and $\mathbf{Z} = \int_{\mathbf{G}} e^{-\beta S(\mathbf{u})} d\mathbf{u}$, as usual. This expected value is independent of **e**. For suppose $\mathbf{e}' = \mathbf{geg}^{-1}$ is another unit lattice Higgs field. It follows from gauge invariance that

$$\mu_{\mathbf{u},\mathbf{geg}^{-1}}(\Delta) = \mu_{\mathbf{g}^{-1}\mathbf{ug},\mathbf{e}}(\Delta);$$

since $\mathbf{u} \to \mathbf{g}^{-1}\mathbf{ug}$ is an action-preserving isometry of **G**, this substitution will not change the integral. We may thus define

$$\bar{M}(\beta) = \langle M_{\mathbf{e}} \rangle_\beta$$

for any choice of **e**. In particular one may choose the "constant" Higgs field $e_i \equiv (1,0,0)$ which makes the calculation somewhat easier (see Sect. 4, *C*).

This independence still holds if the Higgs field is made dynamical (with suppression of its radial degrees of freedom [14]) by adding to the action the gauge-invariant term

$$\beta_{\mathrm{Higgs}} S'(\mathbf{u},\mathbf{e}) = \beta_{\mathrm{Higgs}} \sum_{ij} e_i^\dagger u_{ij} * e_j.$$

The general idea of the construction is as follows: generically the lattice gauge field $\mathbf{u}$ determines a principal SU_2-bundle ξ over Λ; on the boundary of each 3-simplex Δ, together $\mathbf{u}$ and the lattice Higgs field $\mathbf{e}$ pick out a most plausible reduction of the structural group of ξ from SU_2 to U_1. This defines a certain U_1-bundle over $\partial\Delta$; the monopole number $\mu_{\mathbf{u},\mathbf{e}}(\Delta)$ is defined to be the first Chern number of that bundle.

Here is the plan of the rest of this paper. We first discuss the general problem of reduction of structural group of a bundle from a Lie group G to a subgroup H, following Steenrod [15]. Then we explain how (in a local sense) this reduction can be *forced* by a connection, and how the lattice analogue of this concept requires additional data which, when $G=SU_2$, $H=U_1$, amount to the choice of a lattice Higgs field. We show in detail how to compute monopole numbers on the lattice in the SU_2, U_1 case, and in the last section we check that this algorithm gives the expected answer when applied to a 't Hooft–Polyakov monopole on $\mathbf{R}^3$.

1. Reduction of Structural Group

We review here some material from Steenrod [15] in a notation appropriate for our purposes.

Consider given a G-bundle ξ defined by coordinate patches $\{U_i\}$ and transition functions $\{v_{ij}: U_i \cap U_j \to G\}$. A reduction of the structural group of ξ to a subgroup $H \subset G$ is a choice of gauges $\{\lambda_i: U_i \to G\}$ such that with respect to the new gauges the transition functions take values in H, i.e.

$$\lambda_i v_{ij} \lambda_j^{-1}: U_i \cap U_j \to H \subset G.$$

The basic fact that allows a topological study of the existence and equivalence of structural group reductions is the following. *A reduction of the structural group of ξ to H gives a section in the associated bundle ξ/H with fiber G/H, and vice-versa.*

Let $[g]$ represent the right coset of g in G/H (thus $[hg]=[g]$ for $h \in H$), and let G act on G/H by $g \cdot [g'] = [g' g^{-1}]$. A section in ξ/H is then a collection $\{X_i: U_i \to G/H\}$ such that on $U_i \cap U_j$ we have $X_j = v_{ji} \cdot X_i$.

Now let $\{\lambda_i\}$ be the reducing family of gauges mentioned above, and set $X_i = [\lambda_i]$. Since at any point of $U_i \cap U_j$ we have $\lambda_i v_{ij} \lambda_j^{-1} \in H$, i.e. $[\lambda_i v_{ij}] = [\lambda_j]$, or $v_{ji} \cdot [\lambda_i] = [\lambda_j]$, it follows that the X_i define a section in ξ/H. The converse is proved in [15, Sect. 9.4]. It then follows immediately from [15, Theorem 14.4] that homotopic sections give equivalent reductions, where in particular *equivalent* means that the induced H-bundles are isomorphic.

The link with homotopy theory now comes naturally. Suppose that the base of ξ is triangulated as a simplicial complex Λ. The problem of reducing the structural group of ξ from G to H, i.e. of constructing a section in ξ/H, can be worked on stepwise over increasing skeleta of Λ and in the i-skeleton simplex by simplex, as follows.

Step 0. The section can be defined on the vertices of Λ by choosing a basepoint in the ξ/H-fiber over each vertex $\langle i \rangle$.

Step 1. If G/H is connected, then the section can be extended over the 1-skeleton: over each 1-simplex $\langle ij \rangle$ the bundle ξ/H must be trivial, i.e. isomorphic to

$G/H \times \langle ij \rangle$, so extending the section is equivalent to extending the map defined on the endpoints to a map of $\langle ij \rangle$ into G/H.

Step 2, *etc.* If G/H is simply connected, the same argument shows that the section can be extended over the 2-skeleton, and one can continue this procedure through the range of dimensions d for which $\pi_{d-1}(G/H) = 0$.

Suppose $\pi_d(G/H) \neq 0$ is the first nonzero homotopy group. Thus we may assume we have constructed a section X over the d-skeleton. The bundle must be trivial over each $(d+1)$-simplex Δ; so over the boundary $\partial\Delta$, a topological d-sphere, the section X appears as a map of $\partial\Delta$ into G/H, which represents an unambiguous element $c_X(\Delta)$ of $\pi_d(G/H)$ [15, Theorem 16.11], and X extends to Δ iff $c_X(\Delta) = 0$. The association $\Delta \to c_X(\Delta)$ is a $(d+1)$-cocycle [15, Theorem 32.4]. Furthermore it is clear that if $c_X(\Delta) = 0$, then the H-bundle η, defined on $\partial\Delta$ by the reduction over the d-skeleton, is trivial. The converse holds if $\pi_{d-1}G = 0$, for then a section in η, interpreted as a section in $\xi|_{\partial\Delta}$, extends to a section over all of Δ; passing to cosets gives a section in ξ/H extending X.

When ξ is a principal SU_2-bundle and $H = U_1$, we obtain an integer-valued 3-cochain. Since in this case $\pi_{d-1}G = \pi_2 SU_2 = 0$, the element $c_X(\Delta)$ is zero if and only if the reduced bundle on $\partial\Delta$ is trivial; in fact it turns out to be precisely the first Chern number of that bundle.

2. Forced Reductions, Higgs Fields

Suppose the G-bundle ξ has a connection ω. Then there is an induced connection ω_H on ξ/H (see below) and with respect to this connection it makes sense to ask how horizontal a section is. In particular (supposing G/H to be $(d-1)$-connected, as above) on the boundary $\partial\Delta$ of a $(d+1)$-simplex Δ of Λ, generically ω_H will pick out a family of "as horizontal as possible" sections of ξ/H, differing one from the other only by translation by an element of G. (Depending on what one minimizes, there is more than one way to define "as horizontal as possible;" we will not address this question here since the lattice has its own natural criteria for this concept.) Then we can say that ω has *forced* a certain reduction of the structural group over $\partial\Delta$. In particular, ω will have defined an H-bundle over $\partial\Delta$; this bundle is trivial if and only if the sections extend over Δ. The corresponding element of $\pi_d(G/H)$ is the *H-monopole "number"* of ω on Δ.

From now on we will focus on the case of interest in this paper, i.e. $G = SU_2$, $H = U_1$. Since $SU_2/U_1 = S^2$ is 1-connected and $\pi_2(S^2) = \mathbf{Z}$ the abelian monopole numbers will be integers associated to 3-simplexes.

The action of SU_2 on the 2-sphere SU_2/U_1 is equivalent to the adjoint action of SU_2 on $\mathbf{R}^3$, restricted to the unit $S^2 \subset \mathbf{R}^3$. To see the equivalence more explicitly, note that the map $SU_2/U_1 \to S^2$, going from the set of right U_1-cosets to the set of pure imaginary quaternions, and taking $[g] \to g^{-1} * \mathbf{i} = g^{-1}\mathbf{i}g$, is a well-defined bijection which commutes with the SU_2-actions.

So with $G = SU_2$, $H = U_!$, sections in ξ/H are *Higgs fields* [17], the abelian monopoles are winding numbers of Higgs fields [1], and our "as horizontal as possible" condition corresponds exactly to the inclusion in the Lagrangian [13, 14] of a term measuring the covariant derivative of the Higgs field.

Connections in associated bundles. For future reference, we give an explicit construction of ω_H.

Let $\xi = (\pi : E \to B)$. The connection ω is a smooth 1-form with values in the Lie Algebra $\mathbf{g}$ of G, defined on the total space E, and satisfying (1) $\omega|_p \circ (\rho_p)_* = \mathrm{id}_{\mathbf{g}}$ and (2) $\omega|_{gp} \circ (L_g)_* = \mathrm{Ad}(g)\omega|_p$ for every $p \in E$ and $g \in G$, where L_g is the left-action of g on E: $L_g(p) = gp$ and $\rho_p : G \to E$ takes g to gp.

Write ξ/H as $(\pi_H : E/H \to B)$, and let $\Pi : E \to E/H$ be the quotient map. The associated connection ω_H will take values in the tangent space to the fibers of E/H: at $q \in E/H$, with $\pi_H q = b$, we have $\omega_H|_q(w) : T_q(E/H) \to T_q(E/H|_b)$. It is defined by

$$\omega_H|_q(w) = (\Pi \circ (\rho_p))_*(\omega|_p(v)),$$

where p and v are chosen so that $\Pi(p) = q$ and $\Pi_*(v) = w$. One checks that this value is independent of the choice of p and v, and that ω_H is the identity on vectors tangent to the fibres.

3. Lattice Definition of an "as Horizontal as Possible" Section

To go from a connection ω in a principal SU_2-bundle ξ over a manifold triangulated as a simplical complex Λ to a lattice gauge field $\mathbf{u}$ on Λ we need to choose a gauge at each vertex $\langle i \rangle$ of Λ, i.e. an identification of the fiber over $\langle i \rangle$ with the group SU_2. Then parallel transport by ω takes the identity element in the fiber over $\langle j \rangle$ to an element over the adjacent vertex $\langle i \rangle$ which we may label u_{ij}; assigning this element to the oriented 1-simplex $\langle ij \rangle$ defines $\mathbf{u}$. The same choice of gauges at each vertex identifies the fibers of ξ/H with $SU_2/U_1 = S^2$, and thus a (unit) lattice Higgs field, i.e. a collection $\mathbf{e} = \{e_i\}$ of elements of S^2, one for each vertex $\langle i \rangle$, becomes a family of basepoints in ξ/H.

Our lattice implementation of the scheme mentioned in the last section will involve extending a lattice Higgs field $\mathbf{e}$ to a section $X = X_{\mathbf{u},\mathbf{e}}$ in ξ/H defined over the entire 2-skeleton of Λ, which is "as horizontal as possible" with respect to $\mathbf{u}$, and then calculating the resulting winding number on the boundary of each 3-simplex.

For notational simplicity we assume that the local ordering of the vertices of Λ has been extended to a global ordering, with the i^{th} vertex labelled simply i. We will also abbreviate $u_{ij}u_{jk}$ to u_{ijk}, etc.

Note on general position: The definition [11] of an SU_2-bundle ξ from $\mathbf{u}$ involves the construction of a set of transition functions $v_{ij} : C_i \cap C_j \to SU_2$, where C_i is the cell dual to vertex i, etc. Suppose this has been done. Note that this procedure only works for $\mathbf{u}$ not belonging to a certain measure-zero set in the space of all SU_2-valued lattice gauge fields on Λ. The definition of an "as horizontal as possible" section X in ξ/H, using the lattice Higgs field $\mathbf{e}$ as a family of basepoints, will require eliminating in addition those lattice gauge fields that fail a set of measure-zero general-position conditions with respect to $\mathbf{e}$.

To define the section X we consider the 3-simplexes of Λ one by one. A typical 3-simplex is $\Delta = \langle i_0 i_1 i_2 i_3 \rangle$, with vertices so ordered; we relabel them $\langle 0123 \rangle$ to lighten the notation. We give Δ the orientation of the frame $(\overrightarrow{01}, \overrightarrow{02}, \overrightarrow{03})$.

For $i = 0, 1, 2, 3$ set $C_i^\Delta = \Delta \cap C_i$. A point in $x \in \Delta$ has barycentric coordinates t_0, t_1, t_2, t_3 which express it as a positive linear combination of the vertices. If

$x \in \Delta \cap C_i$, then $t_i \geqq \max_{j \neq i} t_j$ and x acquires modified barycentric coordinates [11] $s_j = t_j/t_i$, for $j \neq i$. These run from 0 to 1 and exhibit C_i^Δ as a 3-cube.

For $i = 3, 2, 1, 0$ in turn we define a map $X_i^\Delta : C_i^\Delta \to S^2$ in such a way that

$$X_i^\Delta(i) = e_i,$$

$$X_i^\Delta(x) = X_i^{\Delta'}(x) \quad \text{if} \quad x \in C_i \cap \Delta \cap \Delta',$$

$$X_i^\Delta(x) = v_{ij}(x) * X_j^\Delta(x) \quad \text{if} \quad x \in C_i \cap C_j \cap \Delta,$$

where as before $*$ represents the adjoint action of SU_2 on S^2, except that there may be an obstruction to extending X_0^Δ over the interior of C_0^Δ. These maps clearly form a section in ξ/H whenever they are defined.

The lattice implementation of "as horizontal as possible" is to keep each of the X_i^Δ as constant as possible, subject to these constraints, doing the necessary interpolations along geodesics in S^2 (and rejecting as non-generic any $\mathbf{u}$ for which unique interpolating geodesics do not exist). This involves a mix of geodesic interpolations in SU_2 (used in defining the v_{ij}) and in S^2.

On C_3^Δ, define $X_3^\Delta(s_0, s_1, s_2) \equiv e_3$.

Along $C_3^\Delta \cap C_2^\Delta$ the sections X_3^Δ and X_2^Δ must be related by the action of $v_{23} \equiv u_{23}$, i.e. $X_2^\Delta(s_0, s_1, s_3 = 1) = u_{23} * X_3^\Delta(s_0, s_1, s_2 = 1) = u_{23} * e_3$; to keep X_2^Δ as constant as possible, we let $\gamma_{23} : [0, 1] \to S^2$ be a parametrization of the shortest geodesic on S^2 from e_2 to $u_{23} * e_3$, proportionally to arclength, and let

$$X_2^\Delta(s_0, s_1, s_3) = \gamma_{23}(s_3).$$

This part of the construction does not work for the measure-zero set of lattice gauge fields for which $u_{23} * e_3 = -e_2$.

Along $C_3^\Delta \cap C_1^\Delta$ the sections X_3^Δ and X_1^Δ must be related by the action of the transition function v_{13} which is itself determined by geodesic interpolation (in SU_2) between u_{13} and $u_{12}u_{23}$, following [11]: $v_{13}(s_0, s_2) = \mathbf{g}_{123}(s_2)$. This gives $X_1^\Delta(s_0, s_2, s_3 = 1) = \mathbf{g}_{123}(s_2) * e_3$. Note that a geodesic in SU_2 acts on S^2 by rotations about some fixed axis, and applied to a point on S^2 will move it along the corresponding circle of latitude.

Along $C_2^\Delta \cap X_1^\Delta$, since $v_{12} \equiv u_{12}$, we must have $X_1^\Delta(s_0, s_2 = 1, s_3) = u_{12} * X_2^\Delta(s_0, s_1 = 1, s_3) = u_{12} * \gamma_{23}(s_3)$. Let $\eta_{123}(s_2, s_3)$ parametrize the geodesic cone in S^2 from e_1 to $\mathbf{g}_{123}(s_2) * e_3 \cup u_{12} * \gamma_{23}(s_3)$, and let

$$X_1^\Delta(s_0, s_2, s_3) = \eta_{123}(s_2, s_3).$$

This part of the construction does not work for the measure-zero set of lattice gauge fields for which $-e_1$ lies in the image of the two curves $\mathbf{g}_{123} * e_3$ and $u_{12} * \gamma_{23}$.

Finally we get to C_0^Δ. On $C_3^\Delta \cap C_0^\Delta$, we must have $X_0^\Delta(s_1, s_2, s_3 = 1) = v_{03} * e_3$, since $X_3 \equiv e_3$. Now, following [11], $v_{03}(s_1, s_2) = \mathbf{h}_{0123}(s_1, s_2)$ is the geodesic parametrization of two geodesic triangles in SU_2. As before, we have $S_0^\Delta(s_1, s_2, s_3 = 1) = v_{03}(s_1, s_2) * e_3$.

On $C_2^\Delta \cap C_0^\Delta$, we must have $X_0^\Delta(s_1, s_2 = 1, s_3) = v_{02}(s_1, s_3) * X_2^\Delta(s_0 = 1, s_1, s_3) = \mathbf{g}_{012}(s_1) * \gamma_{23}(s_3)$.

On $C_1^\Delta \cap C_0^\Delta$, we must have $X_0^\Delta(s_1 = 1, s_2, s_3) = u_{01} * \eta_{123}(s_2, s_3)$.

The map X_0^Δ is thus determined on the "far sides" of C_0^Δ, i.e. those where at least one of the coordinates equals 1. The next step would be to set $X_0^\Delta(0, 0, 0) = e_0$;

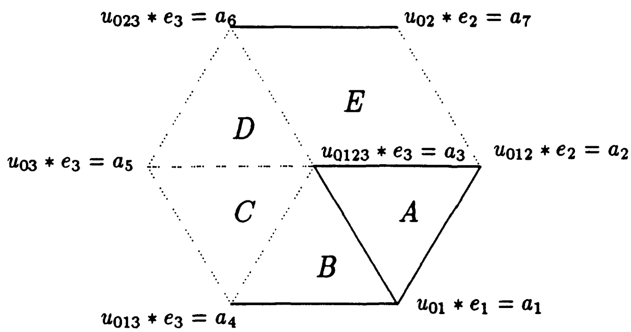

Fig. 1. The monopole number $\mu_{\mathrm{u,e}}(\Delta)$ of the oriented 3-simplex $\langle 0123 \rangle$ is the algebraic number of times this piece of surface on S^2 covers $-e_0$. Note that $u_{012} = u_{01}u_{12}$, etc.

since C_0^Δ is affinely a cone from $(0,0,0)$ onto the far sides, one would try to extend X_0^Δ to a continuous map defined on the rest of the cube by geodesic coning from e_0 onto the image of the far sides. *But this may not be possible*, because $-e_0$ may belong to that image. In fact it is easy to see that the obstruction to extending this map, (i.e. the monopole number $\mu_{\mathrm{u,e}}(\Delta)$ of the oriented simplex Δ), is generically equal to the algebraic number of times that $-e_0$ is covered by the piece of surface shown in Fig. 1. In this figure, solid lines are geodesics in S^2, dashed lines are circles of latitude with respect to an axis determined by the algorithm and the $*$ refers to the adjoint action of SU_2 on $S^2 \subset \mathbf{R}^3$.

The gauge invariance of $\mu_{\mathrm{u,e}}(\Delta)$ is now clear. For suppose we change gauge on the lattice using the family $\mathbf{g} = \{g_i\}$ of SU_2-elements. Then u_{ij} transforms to $u'_{ij} = g_i u_{ij} g_j^{-1}$ (and u_{ijk} to $g_i u_{ijk} g_k^{-1}$, etc.), while e_i transforms to $e'_i = g_i * e_i = g_i e_i g_i^{-1}$ in quaternionic notation. The triangle A, for example, transforms to the spherical triangle A' with vertices $u'_{01} * e'_1$, $u'_{012} * e'_2$, $u'_{0123} * e'_3$, i.e. $(g_0 u_{01} g_1^{-1}) * g_1 e_1 g_1^{-1} = g_0 * (u_{01} * e_1)$, etc., so the pair A', e'_0 differs from A, e_0 by the rotation $x \to g_0 * x$; and the intersection numbers will be the same.

4. Computation of Local Monopole Numbers

This section treats the computation of the number of times that $-e_0$ is covered by the piece of surface shown in Fig. 1; this is the monopole number $\mu_{\mathrm{u,e}}(\Delta)$ of the lattice gauge field $\mathbf{u}$ and the lattice Higgs field $\mathbf{e}$ in the oriented 3-simplex $\Delta = \langle 0123 \rangle$.

The piece of surface in question is naturally divided into five parts, labelled A through E in Fig. 1, and the computation is divided accordingly. The notation u_{012}, etc. is short for $u_{01}u_{12}$, etc.

A. This part is the simplest: a spherical triangle on S^2 with vertices $u_{01} * e_1 = a_1$, $u_{012} * e_2 = a_2$, $u_{0123} * e_3 = a_3$, listed in positive order around the boundary. The point $-e_0$ will belong to the interior of triangle A iff it can be written as a positive linear combination of the vertices: $-e_0 = t_1 a_1 + t_2 a_2 + t_3 a_3$, with all $t_i > 0$. So the three numbers $\det(-e_0, a_2, a_3)/\det(a_1, a_2, a_3)$, etc. must all be positive (or equivalently $\det(-e_0, a_2, a_3)\det(a_1, a_2, a_3) > 0$, etc.). Furthermore this intersection, if it exists, is counted positive if $\det(a_1, a_2, a_3) > 0$, and negative otherwise.

B. This part is a triangle with vertices a_1, a_3, and $u_{013} * e_3 = a_4$, listed in an order that gives the correct orientation. More precisely, the edge $a_3 a_4$ is an arc α of a circle of latitude, $\alpha = u_{01}\mathbf{g}_{123}(s_2) * e_3$, $0 \leqq s_2 \leqq 1$; and B consists of the union of the (unique) minimal geodesics in S^2 from a_1 to each point of α. We call B the *cone from* a_1 *on the oriented arc* $\vec{\alpha}$ and write $B = a_1 \wedge \vec{\alpha}$. Because of the doubling of angles corresponding to the projection $SU_2 \to SO_3$, the arc α will be greater than a semicircle in its circle of latitude if the length of $\mathbf{g}_{123}$ is greater than $\pi/2$ in the standard unit-sphere metric on SU_2. However, α can never be more than a circle of latitude.

It will be useful, here and in part E, to have an explicit form for the computation of the axis of a circle of latitude of the form $\mathbf{g} * e$, where $\mathbf{g}$ is a given geodesic in SU_2, and e a given point of S^2. In the typical case $\mathbf{g} = u_{01}\mathbf{g}_{123}$, where this would be the axis X_B of the circle bearing the arc α, it goes as follows. The geodesic $u_{01}\mathbf{g}_{123}$ in SU_2 is the image under left translation by $u_{01}u_{13}$ of the 1-parameter subgroup leading from the identity to $u = u_{3123}$. Let $R \in SO_3$ be the image of

$$u = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix}$$

(where $x^2 + y^2 + z^2 + w^2 = 1$) under the adjoint representation. The entries in the matrix R are the following well known quadratic functions of x, y, z, w:

$$\begin{pmatrix} x^2 + y^2 - z^2 - w^2 & 2(yz - xw) & 2(xz + yw) \\ 2(yz + xw) & x^2 - y^2 + z^2 - w^2 & 2(zw - xy) \\ 2(yw - xz) & 2(xy + zw) & x^2 - y^2 - z^2 + w^2 \end{pmatrix}.$$

The 1-parameter subgroup containing u gets mapped to the 1-parameter subgroup of SO_3 containing R. As do all such subgroups, this one consists of rotations about some fixed axis X. In particular R itself is such a rotation and maps the first basis vector ε_1 to its own first column R_1, and similarly ε_2 to R_2, ε_3 to R_3. The three vectors $R_1 - \varepsilon_1$, $R_2 - \varepsilon_2$, $R_3 - \varepsilon_3$, at least two of which must be nonzero if R is not the identity, are all perpendicular to the axis X; so we may take $(R_1 - \varepsilon_1) \times (R_2 - \varepsilon_2)$ (substituting the third vector if necessary) as X. Finally the translation by $u_{01}u_{13}$ will transform rotation about X to rotation about the axis $X_B = u_{01}u_{13} * X$.

The calculation of the intersection number of region B with the point $-e_0$ is not as straightforward as in the case of A. Because of angle-doubling the region may fold back on itself, and in general the geometry depends subtly on the relative position of a_1 and α. To avoid some of this delicate geometry we will substitute for B a region B' which is the union of a geodesic-sided triangle and what we shall call a lens (so membership in B' will be easy to establish) and justify the substitution by an invariance principle for intersection numbers (see [8]), which for present purposes can be expressed thus: Let B' be an oriented 2-chain on S^2 such that $\partial B' = \partial B$ as oriented 1-cycles, and such that the 2-chain $B - B'$ is null-homologous. Then the intersection numbers are equal: $B' . (-e_0) = B \cdot (-e_0)$.

Before we can describe the 2-chain B' that we shall use, we need some notation (see Fig. 2).

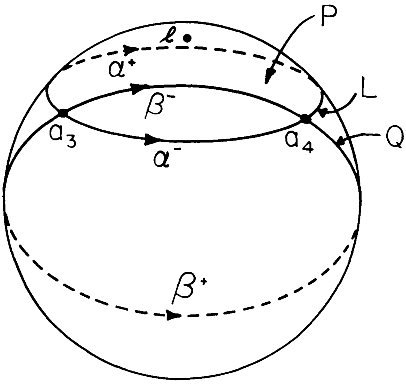

Fig. 2. Geometric set-up for part B of the calculation

Let Q be the great circle in S^2 through a_3 and a_4. Let L be the circle of latitude carrying α, and $l = \pm X_B$ the unique unit vector such that L has equation $(x, l) = r$ with $r > 0$, using $(\,,\,)$ for the euclidean inner product. (Note that generically L will not be a great circle.) Let P be the polar cap $P = \{x:(x, l) \geqq r\}$; also let $-L = \{x:(x, l) = -r\}$ and $-P = \{x:(x, l) \leqq -r\}$ be antipodal to L and P respectively. Generically, a_3 and a_4 are not antipodal in L, so there is a unique decomposition of Q into the union of a *major* arc β^+ and a *minor* arc β^-, each oriented from a_3 to a_4. Let β denote one of β^+ and β^-. Let $B'_1 = a_1 \wedge \vec{\beta}$ be the cone from a_1 on the arc β, with the orientation given by the order a_1, a_3, a_4 of its vertices. Let B'_2 be the unique portion of S^2 such that, in the standard orientation, $\partial B'_2 = \alpha - \beta$, and the interior of B'_2 is one component of $S^2 - (L \cup Q)$. We call B'_2 a *lens*, and write $B'_2 = \mathscr{L}(\alpha\beta)$. Finally set $B' = B'_1 + B'_2$ as an oriented 2-chain. Then

$$\begin{aligned}\partial B' &= \partial B'_1 + \partial B'_2 \\ &= \overrightarrow{a_1 a_3} + \beta + \overrightarrow{a_4 a_1} + \alpha - \beta \\ &= \partial B_1 .\end{aligned}$$

We shall show how to choose $\beta = \beta^+$ or β^- so that also

$$(*) \quad -a_1 \notin |B| \cup |B'_1| \cup |B'_2| ,$$

where as usual $|B|$ is the point-set underlying the chain B, etc. Then $B - B'$ is null-homologous, so, by the invariance principle, $B \cdot (-e_0) = B'_1 \cdot (-e_0) + B'_2 \cdot (-e_0)$ and it will remain only to show how to compute the two terms on the right.

Choice of β.

(a) $\beta = \beta^-$ if $a_1 \notin -P$

(b) $\beta = \beta^+$ if $a_1 \in -P$.

Note that the vertex a_1 is in $-P$ if $(a_1, l) < -r$.

Verification of $(*)$. (See Fig. 3, in which case (a) has been subdivided into (a′) if a_1 and α are on the same side of Q and (a'') if they are on opposite sides.)

Since $|B|$ and $|B'_1|$ are made up of shortest geodesics from a_1, we have that

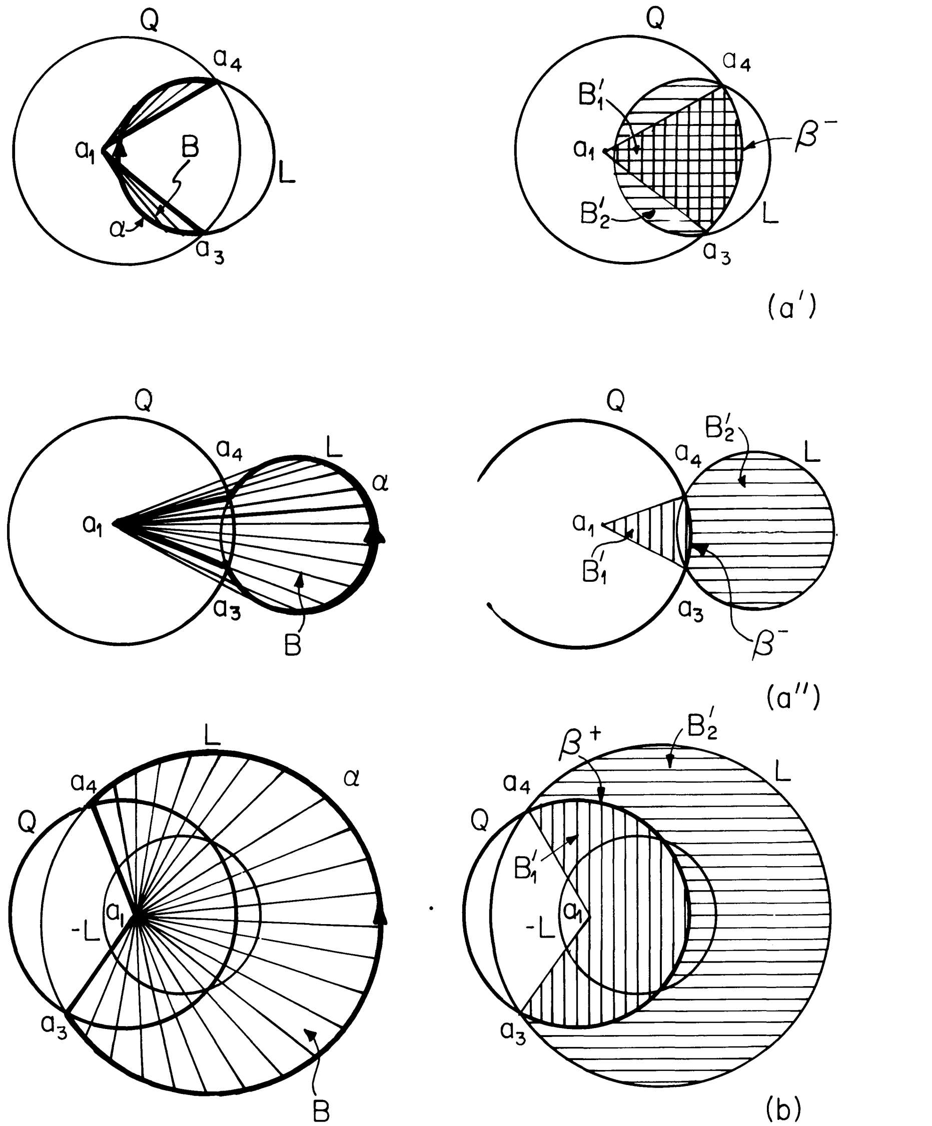

Fig. 3. B, B_1' and B_2' in cases (a'), (a''), and (b), as seen in stereographic projection from $-a_1$. In these pictures, α is drawn as the major arc of L

generically $-a_1 \notin |B| \cup |B_1'|$. So it remains to check that $-a_1 \notin |B_2'|$. Note that in general β^- lies inside the polar cap P, and β^+ lies outside it.

(a) If $a_1 \notin -P$, then $-a_1 \notin P$. By our choice $B_2' = \mathscr{L}(\alpha, \beta^-) \subseteq P$, so $-a_1 \notin |B_2'|$.

(b) If $a_1 \in -P$, then $-a_1 \in P$. By our choice $B_2' = \mathscr{L}(\alpha, \beta^+)$ is in the complement of P. So $-a_1 \notin |B_2'|$.

Computation of $B_1' \cdot (-e_0)$. We think of vectors in $\mathbf{R}^3$ as column vectors; three of these can be grouped to make a 3×3 matrix (v_1, v_2, v_3). In case this matrix is

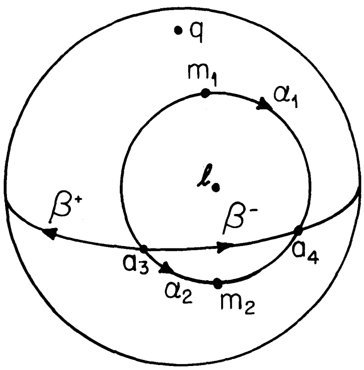

Fig. 4. The computation of $B_2' \cdot (-e_0)$

non-singular, set $\varepsilon(v_1, v_2, v_3) = +1$ or -1 according as the matrix has positive or negative determinant. If $\beta = \beta^-$, then B_1' is a triangle of minimal geodesics on S^2 and $B_1' \cdot (-e_0)$ is computed as in (A): if $\varepsilon(-e_0, a_i, a_j) = \varepsilon(a_k, a_i, a_j)$ for every cyclic permutation i, j, k of $1, 3, 4$ then

$$B_1' \cdot (-e_0) = \varepsilon(a_1, a_3, a_4);$$

otherwise, $B_1' \cdot (-e_0) = 0$.

If $\beta = \beta^+$, then $|B_1'|$ is in the hemisphere of S^2 determined by Q and $\vec{a}_1$, and is the complement in this hemisphere of the minimal geodesic triangle $\alpha_1 \wedge \beta^-$. Hence $B_1' \cdot (-e_0) = 0$ unless

(i) $\varepsilon(-e_0, a_3, a_4) = \varepsilon(a_1, a_3, a_4)$ and
(ii) at least one of these two equations *fails*:

$$\varepsilon(-e_0, a_1, a_4) = \varepsilon(a_3, a_1, a_4),$$
$$\varepsilon(-e_0, a_1, a_3) = \varepsilon(a_4, a_1, a_3).$$

In this case,

$$B_1' \cdot (-e_0) = -\varepsilon(a_1, a_3, a_4).$$

Computation of $B_2' \cdot (-e_0)$. The point $-e_0$ is in the lens $|B_2'|$ if it is on the same side of Q as α and on the same side of L as β. The calculation depends on the relative position of a_1 and α with respect to Q. (See Fig. 4, where α_1 and α_2 represent the two possibilities for α: being on the same or opposite sides of Q as q, respectively.)

1. Set $m = u_{01} g_{123}(\frac{1}{2}) * e_3$, so m is the midpoint of α. (In practice we may substitute $m' = (\frac{1}{2} u_{013} + \frac{1}{2} u_{1023}) * e_3$, which is a positive scalar multiple of m.)

2. The great circle Q has equation $(x, q) = 0$, where $q = a_3 \times a_4$. So a_1 and α are on the same or opposite sides of Q according as (a_1, q) and (m, q) have the same or opposite signs.

3. Finally, $e_0 \in |B_2'|$ if
(i) $(-e_0, q)$ and (m, q) have the same sign and
(ii) $(-e_0, l) > r$ or $< r$ according as $\beta = \beta^-$ or β^+.

If (i) and (ii) hold, then

$$B_2'\cdot(-e_0)=\varepsilon(m)\varepsilon(\beta),$$

where $\varepsilon(m)=\pm 1$ is the sign of (m,q) and where $\varepsilon(\beta)=1$ if $\beta=\beta^+$ and $=-1$ if $\beta=\beta^-$.
Otherwise, $B_2'\cdot(-e_0)=0$.

C. This part and the next are the images under the map $f:SU_2\to S^2$ taking u to $u*e_3$ of the two spherical triangles T_1 and T_2 making up the image of $\mathbf{h}_{0123}$, continuing with the notation of [11]. Since any or all of the sides of T_1 and T_2 may have length $>\pi/2$ it is simpler because of angle doubling to substitute for the calculation of the intersection number of $f(T_1)$ with the point $-e_0$ the equivalent calculation of the intersection of T_1 itself with the great circle $S=f^{-1}(-e_0)$, and similarly for T_2.

We first consider the special case $e_0=e_3=e=(1,0,0)$.

With this choice, S is the set S_e of SU_2 matrices whose SO_3 projections take e to $-e$, i.e. have the form

$$\begin{pmatrix}-1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & \sin\theta & -\cos\theta\end{pmatrix}$$

for some angle θ. Comparing with the explicit form of the adjoint representation given above shows that S_e is exactly the set of SU_2 matrices lying in the (z,w)-plane.

The vertices of triangle T_1 are u_{03}, u_{013}, u_{0123} with quaternionic coordinates we will call (p_1,p_2,p_3,p_4), (q_1,q_2,q_3,q_4), (r_1,r_2,r_3,r_4) respectively. This spherical triangle will intersect the (z,w)-plane if and only if the triangle in the (x,y)-plane, with vertices (p_1,p_2), (q_1,q_2), (r_1,r_2), contains the origin, i.e. if

$$\det\begin{pmatrix}p_1 & r_1\\ p_2 & r_2\end{pmatrix}\det\begin{pmatrix}q_1 & r_1\\ q_2 & r_2\end{pmatrix}<0$$

and

$$\det\begin{pmatrix}q_1 & p_1\\ q_2 & p_2\end{pmatrix}\det\begin{pmatrix}q_1 & r_1\\ q_2 & r_2\end{pmatrix}<0.$$

The sign of the intersection may be calculated as follows. The matrix $J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is mapped to $-e$ by f. The tangent vectors $\left.\frac{\partial}{\partial x}\right|_J$ and $\left.\frac{\partial}{\partial y}\right|_J$ give a basis for the normal space of S_e at J. The differential of f takes these vectors to the vectors $(0,0,-2)$ and $(0,2,0)$ in the tangent space to S^2 at e; so the basis is mapped to a negatively oriented basis on S^2. On the other hand, projection into the (x,y)-plane maps them to the standard positive basis. It follows that the sign of the intersection will be positive if the projected triangle wraps negatively around the origin, and vice-versa. The vertices were listed above in positive order, so a negative wrapping (and a positive sign) correspond, for example, to $\det\begin{pmatrix}p_1 & q_1\\ p_2 & q_2\end{pmatrix}<0$.

Now we turn to the case of a general e_0 and e_3. Suppose h is an SU_2 matrix such that $h*e=e_0$, and k is one with $k*e=e_3$. If a matrix g belongs to S_e, then the product hgk^{-1} will belong to S, and vice-versa, i.e. $S_e\to S$ under the isometry

$g \to hgk^{-1}$. So the intersection number of S with T_1, say, is the same as that of S_e with the triangle $h^{-1}T_1k$, and can be calculated as above. It remains to find the matrices h and k.

Let us find h. The point e has quaternionic coordinates $(0,1,0,0)$. Suppose $e_0 = (0,a,b,c)$ and $h = (x,y,z,w)$. Then writing $heh^{-1} = e_0$ leads to the three equations

$$x^2 + y^2 - z^2 - w^2 = a,$$
$$2(yz + xw) = b,$$
$$2(yw - xz) = c,$$

to which we may add

$$x^2 + y^2 + z^2 + w^2 = 1.$$

Considering $(x,y,0) = V_1$ and $(w,z,0) = V_2$ as vectors in $\mathbf{R}^3$, with standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$, this set of equations may be rewritten as

$$|V_1|^2 = (1+a)/2,$$
$$|V_2|^2 = (1-a)/2,$$
$$V_1 \cdot V_2 = \quad b/2 \quad = |V_1||V_2|\cos\theta,$$
$$(V_2 \times V_1)\cdot\mathbf{k} = \quad c/2 \quad = |V_1||V_2|\sin\theta.$$

As long as we avoid the degenerate cases $a = \pm 1$, this can be solved for $|V_1|, |V_2|$, and the angle θ from V_2 to V_1. One can then choose V_1 to lie along the x-axis, i.e. $x = \sqrt{(1+a)/2}$, $y = 0$, and then $w = \dfrac{b}{2x}$, $z = \dfrac{-c}{2x}$.

D. This is another triangle made up of circles of latitude, and can be analyzed in precisely the same way. Here the vertices in positive order are $u_{03} * e_3 = a_3$, $u_{0123} * e_3 = a_5$, and $u_{023} * e_3 = a_6$.

E. This part is the quadrilateral $X_0^4(C_2^4 \cap C_0^4)$. The vertices are a_6, a_3, a_2 and $u_{02} * e_2 = a_7$, listed positively. The map is defined by $X_0^4(s_1, s_3) = \mathbf{g}_{012}(s_1) * \gamma_{23}(s_3)$. This is a kind of ruled surface: when s_1 is fixed the image describes geodesics in S^2 (the sides a_3a_2 and a_7a_6 are of this type), whereas when s_3 is fixed the image describes circles of latitude about a common axis X_E.

To calculate $E\cdot(-e_0)$ we shall again use the invariance principle for intersection numbers that we used to compute $B\cdot(-e_0)$. That is, we shall find an oriented 2-chain E' such that $\partial E' = \partial E$ as oriented 1-cycles, and $E - E'$ is null-homologous in S^2. Let E_1' be the cone $a_2 \wedge \overrightarrow{a_6a_3}$ from a_2 on the oriented arc $\overrightarrow{a_6a_3}$ of ∂E; and let E_2' be the cone $a_6 \wedge \overrightarrow{a_2a_7}$, where again $\overrightarrow{a_2a_7}$ is part of ∂E. Set $E' = E_1' + E_2'$. Then $\partial E' = \partial(E_1' + E_2') = \partial E$.

It remains to show that $E - E'$ is null-homologous. The next lemma will show that, provided the circles of latitude L_1 and L_2, bearing a_6a_3 and a_2a_7 respectively, are not antipodal (this is a generic condition) then $|E| \cup |E'|$ fails to contain one or the other unit vector along the axis X_E. It follows that $E - E'$ is null-homologous. Hence $E\cdot(-e_0) = E'\cdot(-e_0) = E_1'\cdot(-e_0) + E_2'\cdot(-e_0)$. Since E_1' and E_2' are both geodesic cones on arcs of latitude, the last two values can be computed by the algorithm used to find $B\cdot(-e_0)$.

Lemma. *Let L_1 and L_2 be parallel circles of latitude which are not antipodal. Let l_1 and $l_2 = -l_1$ be the unit vectors on the axis X of symmetry of L_1 and L_2, chosen so that for $x_1 \in L_1$, $x_2 \in L_2$ we have $(l_1, x_1) > (l_1, x_2)$. Let $\alpha(x_1, x_2)$ be the (unique) minimal geodesic from x_1 to x_2. Then there is one of l_1 and l_2–call it l – such that for every pair $x_1 \in L_1$, $x_2 \in L_2$ we have $l \notin \alpha(a_1, x_2)$.*

Proof. Suppose, to the contrary, that $\alpha = \alpha(x_1, x_2)$ passes through l_1, and $\alpha' = \alpha(x_1', x_2')$ passes through l_2. Let R be a rotation about axis X that carries x_1' to x_1; set $x_3 = R(x_2')$. Since l_2 is invariant under R, it follows that $\alpha'' = \alpha(x_1, x_3)$ also passes through l_2. Now $\alpha \cup \alpha''$ is a broken geodesic that passes through antipodal points l_1 and l_2. Therefore $\alpha \cup \alpha''$ must in fact be unbroken, that is, part of a single great circle. But now, since $\alpha \cup \alpha''$ is more than a semi-circle, the only way for its endpoints x_2 and x_3 to be on L_2 (which is perpendicular to axis X) is if $x_2 = x_3$. But this would imply that α' and α'' are both minimal geodesics from x_1 to x_2; in other words, that x_1 and x_2 are antipodal, which is contrary to hypothesis.

5. An Example

There are various arbitrary elements in our algorithm. One is the local vertex ordering, another our interpretation of "as horizontal as possible." There is one global constraint on our monopole numbers: for any 3-cycle $\Sigma \subset \Lambda$, say $\Sigma = \sum \varepsilon_i \Delta_i$, with $\varepsilon_i = \pm 1$ so that $\partial \Sigma = 0$, the sum

$$\sum_{\Delta_i \in \Sigma} \varepsilon_i \mu_{\mathbf{u},\mathbf{e}}(\Delta_i)$$

must equal zero. This sum is in fact a characteristic invariant of ξ/U_1, where ξ is the SU_2-bundle corresponding to $\mathbf{u}$; since ξ, as an SU_2-bundle on a 3-complex, must be trivial, so must ξ/U_1.

In general the assignment of monopole numbers to individual simplexes will depend on the relative position of $\mathbf{u}, \mathbf{e}$, and the particulars of the algorithm. In this section, however, we show that for a particular, smooth configuration, a modified version of the Prasad–Sommerfield solution of the 't Hooft-Polyakov monopole on $\mathbf{R}^3$, our algorithm gives the expected individual monopole numbers: 1 for the simplex enclosing the origin, and 0 for all the others.

Following [13] we consider the gauge field A on $\mathbf{R}^3$ described as follows. This is a connection in the trivial SU_2-bundle, i.e. an $\mathbf{su}_2$-valued 1-form, so for each $\mathbf{x} \in \mathbf{R}^3$ it gives a linear map $A|_{\mathbf{x}}: T\mathbf{R}^3_{\mathbf{x}} \to \mathbf{su}_2$. Identifying both of these spaces with $\mathbf{R}^3$ as usual (in particular, using the imaginary quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as a basis for $\mathbf{su}_2$), we may write this map as

$$A|_{\mathbf{x}}(\mathbf{v}) = \frac{f(r)}{2r^2} \mathbf{x} \times \mathbf{v},$$

where $r = \|\mathbf{x}\|$, $f(r)$ is described just below, and $\times$ is the vector cross-product. In the notation of Sect. 2, this expression would give $\omega|_{(\mathbf{x},1)}(\mathbf{v})$ at the point $(\mathbf{x}, 1) \in \mathbf{R}^3 \times SU_2$. On vectors tangent to the fiber, $\omega|_{(\mathbf{x},1)}$ is the identity map.

For simplicity's sake we have replaced the cut-off function $1\text{-}vr/\sinh(vr)$ of [13] by a C^∞ positive function $f(r)$ which is identically zero for $r \leqq v^{-1}$ and identically one for $r \geqq 2v^{-1}$.

Together with A we consider the radial Higgs field defined on the complement of $\mathbf{0}$ by $\varphi(\mathbf{x}) = \mathbf{x}/r$ (continuing with the identification of $\mathbf{R}^3$ with $\mathbf{su}_2$. Here we omit the cutoff function which would make φ well defined at $\mathbf{0}$.)

We make this simplified continuum monopole into a lattice gauge-Higgs system as follows. We triangulate $\mathbf{R}^3$ as a simplicial complex Λ such that the origin is contained in the interior of a simplex Δ_0 and such that the simplexes intersecting the spherical shell $v^{-1} \leqq r \leqq 2v^{-1}$ have their edge lengths bounded by a number L we shall discuss presently. To apply our algorithm, we order the vertices of Λ; we define an SU_2-valued lattice gauge field $\mathbf{u}$ on Λ by setting u_{ij} to be the path-ordered integral of A along the edge $\langle ij \rangle$; and we extract from φ the lattice Higgs field $\mathbf{e}$ with $e_i = \varphi(i)$ for each vertex $\langle i \rangle \in \Lambda$.

We will show that for this combination of $\mathbf{u}$ and $\mathbf{e}$ the monopole number $\mu_{\mathbf{u},\mathbf{e}}(\Delta)$ is zero for every simplex $\Delta \neq \Delta_0$, while $\mu_{\mathbf{u},\mathbf{e}}(\Delta_0) = 1$.

There are four possibilities for Δ, which we will consider separately:

1. Δ entirely contained in the region $r \leqq v^{-1}$, but $\Delta \neq \Delta_0$;
2. $\Delta = \Delta_0$;
3. Δ intersecting the shell $v^{-1} < r < 2v^{-1}$;
4. Δ entirely contained in the region $r \geqq 2v^{-1}$.

In the first two cases the gauge field is zero, so $\mathbf{u}$ is the identity lattice gauge field; this makes Fig. 1 collapse down to the spherical triangle A with vertices e_1, e_2, e_3.

1. Any linear simplex $\Delta = \langle 0123 \rangle$ which does not contain the origin must lie entirely in a half-space, and the same must be true for the four Higgs field values e_0, e_1, e_2 and e_3. So the triangle A lies entirely in the same hemisphere as e_0, and $\mu_{\mathbf{u},\mathbf{e}}(\Delta) = 0$.

2. Since Δ_0 contains the origin $\mathbf{0}$, we can write $\mathbf{0}$ as a positive linear combination of the vertices $0, 1, 2, 3$; and therefore also as a positive linear combination of e_0, e_1, e_2, e_3. But this means that $-e_0$ can be written as a positive linear combination of e_1, e_2, e_3, i.e. $-e_0$ is covered exactly once by the triangle A. If Δ_0 is given a positive orientation by the vertex ordering, i.e. if the three vectors $\overrightarrow{01}$, $\overrightarrow{02}$, $\overrightarrow{03}$ form a positively oriented frame, the same must hold for the three vertices $1, 2, 3$ thought of as vectors; then according to the algorithm $\mu_{\mathbf{u},\mathbf{e}}(\Delta_0) = +1$.

3. For $\Delta = \langle 0123 \rangle$ in this case we will show that if L is sufficiently small all the pieces of Fig. 1 lie in the open hemisphere about e_0, so again the monopole number must be zero. Here we can argue by continuity starting from case 1. The transporters are path-ordered integrals of the connection from A along segments of length $< L$; we first take $L < v^{-1}$; then these segments lie in the compact region $v^{-1} - L \leqq r \leqq 2v^{-1} + L$, on which the coefficients of A are bounded, so by further controlling L we can make Fig. 1, for all Δ's in this case, uniformly as close as we please to the figure in case 1; in particular we can make each of these figures lie in the open hemisphere about its e_0. A calculation shows $L < .34v^{-1}$ to be sufficient.

4. In this region, where $A|_{\mathbf{x}}(\mathbf{v}) = (1/2r^2)\mathbf{x} \times \mathbf{v}$, we use the fact that φ *is parallel with respect to this connection.* Very briefly, this is established as follows; we use notation from the end of Sect. 2. The associated bundle is here $E/U_1 = \mathbf{R}^3 \times S^2$, with $\Pi: E/U_1$ taking $(\mathbf{x}, g) \in \mathbf{R}^3 \times SU_2$ to $(\mathbf{x}, g^{-1}\mathbf{i}g)$, and induced connection ω_1. The tangent space

to E/U_1 at $(\mathbf{x}, \mathbf{h})$ is $T\mathbf{R}^3|_{\mathbf{x}} \oplus TS^2|_{\mathbf{h}}$. If $\mathbf{v} \in TS^2|_{\mathbf{h}}$ then by definition $\omega_1(\mathbf{v}) = \mathbf{v}$, while if $\mathbf{v} \in T\mathbf{R}^3|_{\mathbf{x}}$, a straightforward calculation leads to

$$\omega_1|_{(\mathbf{x},\mathbf{h})}(\mathbf{v}) = \frac{1}{r^2}\mathbf{h} \times (\mathbf{x} \times \mathbf{v}),$$

identifying $TS^2|_{\mathbf{h}}$ with the set of vectors in $\mathbf{R}^3$ perpendicular to $\mathbf{h}$, as usual. In this context, to say that φ is parallel means that the tangent 3-plane at $(\mathbf{x}, \varphi(\mathbf{x}))$ to the graph of φ in E/U_1 coincides with the kernel of $\omega_1|_{(\mathbf{x},\varphi(\mathbf{x}))}$, i.e. that $\omega_1|_{(\mathbf{x},(\mathbf{x}/r))}(\mathbf{v}) = -\varphi_*\mathbf{v}$, for any $\mathbf{v} \in T\mathbf{R}^3|_{\mathbf{x}}$.

Now

$$\omega_1|_{(\mathbf{x},(\mathbf{x}/r))}(\mathbf{v}) = \frac{1}{r^2}\frac{\mathbf{x}}{r} \times (\mathbf{x} \times \mathbf{v}),$$

using the expression for ω_1 above, whereas one can easily check that

$$\varphi_*\mathbf{v} = \frac{-1}{r^3}\mathbf{x} \times (\mathbf{x} \times \mathbf{v}),$$

so φ is indeed parallel.

This fact means for us that $u_{ij} * e_j$, which is what we get when we parallel-translate e_j along $\langle ji \rangle$, is precisely e_i. So if $\Delta = \langle 0123 \rangle$ is in this region, the vertices of Fig. 1, along with the surface they span, all coalesce at e_0, and this surface has no chance of covering $-e_0$.

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