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Cyclic L-Operator Related with a 3-State R-Matrix

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Abstract. We consider the problem of constructing a cyclic L-operator associated with a 3-state R-matrix related to the $U_q(sl(3))$ algebra at $q^N = 1$. This problem is reduced to the construction of a cyclic (i.e. with no highest weight vector) representation of some twelve generating element algebra, which generalizes the $U_q(sl(3))$ algebra. We found such representation acting in $C^N \otimes C^N \otimes C^N$. The necessary conditions of the existence of the intertwining operator for two representations are also discussed.

0. Introduction

Recently, it was observed [1] that the chiral Potts model [2–4] can be considered as a part of some new algebraic structure related to the six-vertex R-matrix. In particular, the high genus algebraic relations between the Boltzmann weights of the chiral Potts model arise as a condition of the existence of an intertwining operator for two different representations of some quadratic Hopf algebra [5–7], which generalizes the $U_q(sl(2))$ algebra. This structure leads to various functional relations [1, 8], which completely determine the spectrum of the chiral Potts model transfer matrix. In fact, the largest eigenvalue was very recently calculated [9] using these functional relations.

It is natural to make an attempt to find new solvable lattice models whose Boltzmann weights obey high genus algebraic relations generalizing the results of [1] for the case of other R-matrices.

As a simplest possibility, one can replace the six-vertex R-matrix by the eight-vertex one. In this way one can discover [10] two cases of the integrable deformation of the chiral Potts model. The first case is, in fact, the deformation of Fateev-Zamolodchikov model [11] into the "broken Z_N-model" of [12]. The second case is an integrable deformation of the super-integrable chiral Potts model [13]. Incidentally, the former case was recently studied in [14].

In the present paper we consider the case of the three-state R-matrix of [15, 16, 20], which is related to the $U_q(sl(3))$ algebra with $q^N = 1$. As in the case of [1], the problem of the construction of a cyclic L-operator is reduced to the construction of the cyclic (i.e. with no highest weight vector) representation of some quadratic Hopf algebra containing twelve generating elements. We found an N^3-dimensional representation of this algebra parametrized by twelve complex parameters. The condition of the existence of the intertwining operator for two such representations leads to a set of high degree algebraic relations in the parameter space, which, however, leave two "spectral variable" degrees of freedom just as in the case of [1].

Up to the moment we have not yet generalized the whole program of [1] for our case. We hope to consider this in subsequent publications.

The organization of the paper is as follows. In Sect. 1 we start from the R-matrix (1.1) and the Yang-Baxter equation (1.5) for an L-operator of the form (1.6). This equation is reduced to the algebra (1.10) for the elements $L^{\pm}_{ij}$. Then we introduce an equivalent algebra (1.16). For this algebra we have two non-trivial Casimir elements given by (1.17). In Sect. 2 first we consider the subalgebra of (1.16) defined by Eqs. (2.1) because the above mentioned Casimir elements (1.17) are expressed entirely in terms of it. Using a special choice for the generating elements of this subalgebra [Eqs. (2.11), (2.12), (2.15)] we realize them by explicit expressions through simple matrices X_i, Z_i in (2.23), (2.24). In Sect. 3 we restore the rest of the algebra (1.16) by introducing three more elements L^-_{ii} having simple commutation relations (2.2) with other elements. Substituting these results into (1.15) we obtain the representation of (1.10). This ends the construction of the solution of the Yang-Baxter Eq. (1.5). In Sect. 4 we consider the specialization of our main algebra (1.10) to the $U_q(sl(3))$ algebra. In Sect. 5 we discuss the necessary conditions for the intertwining of two L-operators (1.5).

1. The Main Algebra

Define a trigonometric R-matrix acting in $C^3 \otimes C^3$ with the following matrix elements (the indices run over three values 1, 2, 3) [15, 16, 20]:

$$R(x)_{ij,kl} = \delta_{ij}\delta_{kl}\delta_{ik}(xq - x^{-1}q^{-1}) + \delta_{ij}\delta_{kl}\varrho_{ik}(x - x^{-1}) + \delta_{il}\delta_{jk}\sigma_{ij}\,, \tag{1.1}$$

where δ_{ij} is the Kroneker symbol,

$$\varrho_{ij} \equiv \begin{cases} 0, & i = j; \\ \lambda, & (ij) = (12), (23), (31); \\ \lambda^{-1} & (ij) = (21), (32), (13), \end{cases} \tag{1.2}$$

$$\sigma_{ij} \equiv \begin{cases} 0, & i = j; \\ (q - q^{-1})x, & i < j; \\ (q - q^{-1})x^{-1} & i > j. \end{cases} \tag{1.3}$$

Here x is a variable, while q, λ are considered as constants. The $R(x)$ satisfies the Yang-Baxter equation (Fig. 1)

$$\sum_{i'',j'',k''} R(x)_{ii'',jj''} R(xy)_{i''i',kk''} R(y)_{j''j',k''k'} = \sum_{i'',j'',k''} R(y)_{jj'',kk''} R(xy)_{ii'',k''k'} R(x)_{i''i',j''j'}\,. \tag{1.4}$$

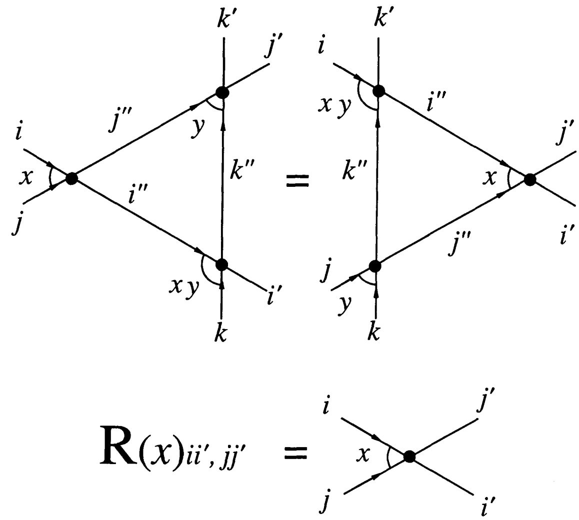

Fig. 1. Graphical representation of Eq. (1.4)

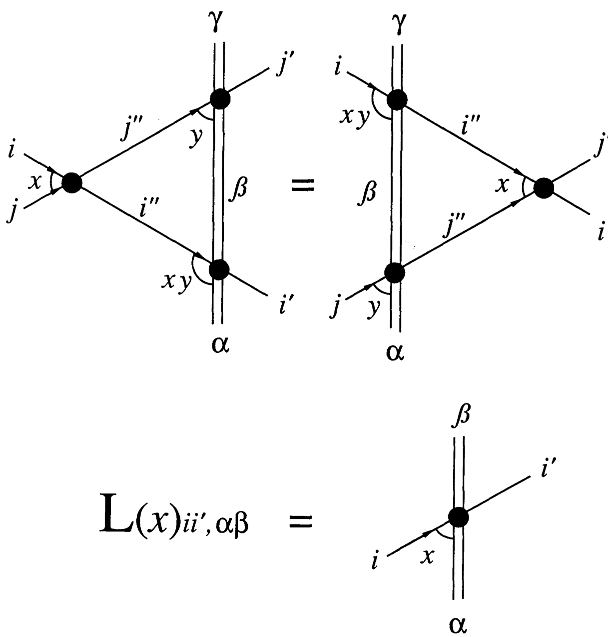

Fig. 2. Graphical representation of Eq. (1.5)

Let $L(x)$ be an operator in $C^3 \otimes C^M$, $M \geq 3$, satisfying the following equation (Fig. 2):

$$\sum_{i'',j'',\beta} R(x)_{ii'',jj''} L(xy)_{i''i',\alpha\beta} L(y)_{j''j',\beta\gamma} = \sum_{i'',j'',\beta} L(y)_{jj'',\alpha\beta} L(xy)_{ii'',\beta\gamma} R(x)_{i''i',j''j'} , \qquad (1.5)$$

where $L(x)_{ij,\alpha\beta}, i, j = 1,2,3; \alpha, \beta = 1, \ldots, M$ denote the matrix elements of $L(x)$. Such an operator is called a quantum L-operator related to a given R-matrix.

Let us search for an L-operator of the form

$$L(x) = xL^{+} + x^{-1}L^{-}, \tag{1.6}$$

where $L^{+}(L^{-})$ is independent of x and has an upper(lower) triangular form in $\mathbb{C}^3$. The most obvious non-trivial solution of this form for $M = 3$ is the R-matrix itself. From (1.1) it follows that

$$R(x) = xR^{+} + x^{-1}R^{-}, \tag{1.7}$$

where R^{+} and R^{-} satisfy the above requirements and

$$R_{12}^{+}P_{12}R_{12}^{-} = -P_{12}, \tag{1.8}$$

$$R_{12}^{+} + R_{12}^{-} = (q - q^{-1})P_{12} \tag{1.9}$$

with $P_{ij,kl} = \delta_{il}\delta_{jk}$ being the permutation matrix in $C^3 \otimes C^3$. By using of (1.6)–(1.9) Eq. (1.5) reduces to the following relations:

$$R_{12}^{-}L_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R_{12}^{-}, \tag{1.10a}$$

$$R_{12}^{-}L_1^{-}L_2^{+} = L_2^{+}L_1^{-}R_{12}^{-}. \tag{1.10b}$$

Explicitly we have

$$[L_{ii}^{\pm}, L_{jj}^{\pm}] = [L_{ii}^{+}, L_{jj}^{-}] = 0, \tag{1.11a}$$

$$L_{ii}^{\pm}L_{ij} = q^{\mp 1}\varrho_{ij}L_{ij}L_{ii}^{\pm}, \qquad i \neq j, \tag{1.11b}$$

$$L_{ii}^{\pm}L_{ji} = q^{\pm 1}\varrho_{ji}L_{ji}L_{ii}^{\pm}, \qquad i \neq j, \tag{1.11c}$$

$$L_{ii}^{\pm}L_{jk} = \lambda^{-2\varepsilon}L_{jk}L_{ii}^{\pm}, \tag{1.11d}$$

$$L_{ij}L_{ik} = \lambda^{\varepsilon}q^{-\varepsilon}L_{ik}L_{ij}, \tag{1.11e}$$

$$L_{ki}L_{ji} = \lambda^{\varepsilon}q^{\varepsilon}L_{ji}L_{ki}, \tag{1.11f}$$

$$[L_{ki}, L_{ij}] = -\varepsilon(q - q^{-1})\lambda^{-\varepsilon}L_{ii}^{\varepsilon}L_{kj}, \tag{1.11g}$$

$$L_{ij}L_{ji}\varrho_{ij} - L_{ji}L_{ij}\varrho_{ji} = (q - q^{-1})(L_{jj}^{-}L_{ii}^{+} - L_{jj}^{+}L_{ii}^{-}), \qquad i \neq j, \tag{1.11h}$$

where (i, j, k) in (1.11d)–(1.11g) is any permutation of $(1,2,3)$ and ε denotes its sign;

$$L_{ij} \equiv \begin{cases} L_{ij}^{+}, & i < j; \\ L_{ij}^{-}, & i > j. \end{cases} \tag{1.12}$$

These relations can be considered as the defining ones for some quadratic Hopf algebra [5–7] with twelve generating elements and co-multiplication $\Delta L_{ij}^{\pm} \equiv \sum_{k} L_{ik}^{\pm} \otimes L_{kj}^{\pm}$ which generalizes the $U_q(sl(3))$ algebra [6, 7].

We are interested in the most general irreducible finite dimensional representations of the algebra (1.10) satisfying the requirements

$$\det L_{ij}^{\pm} \neq 0, \qquad i, j = 1,2,3. \tag{1.13}$$

From the relations (1.11b)–(1.11f) it follows that this is possible provided that

$$q^{M}\lambda^{-M} = \lambda^{2M} = 1, \tag{1.14}$$

where M is the dimension of representation. It will be more convenient to deal with some other relations instead of (1.10). For this let us introduce a matrix S^+ in $C^3 \otimes C^M$ defined by

$$\sum_k S^+_{ik} L^+_{kj} = \sum_k L^+_{ik} S^+_{kj} = \delta_{ij}, \tag{1.15}$$

where in the right-hand side of (1.15) the identity matrix in C^M is implied. Solving Eq. (1.15) with respect fo L^+ we rewrite (1.10) as:

$$R^-_{12} L^-_1 L^-_2 = L^-_2 L^-_1 R^-_{12}, \tag{1.16a}$$

$$R^-_{12} S^+_2 S^+_1 = S^+_1 S^+_2 R^-_{12}, \tag{1.16b}$$

$$S^+_2 R^-_{12} L^-_1 = L^-_1 R^-_{12} S^+_2. \tag{1.16c}$$

One can show that the operators of the form

$$Q_N = \mathrm{tr}\{\Omega (L^- S^+)^N\}, \qquad N = 1, 2, \ldots, \tag{1.17}$$

where the trace is taken in C^3 and

$$\Omega \equiv \mathrm{diag}(q^2, 1, q^{-2}), \tag{1.18}$$

are Casimir elements of algebra (1.16). Only two of them Q_1 and Q_2 are independent.

2. The Subalgebra

There is a subalgebra of (1.16) being generated by seven elements $L^-_{21}, L^-_{32}, S^+_{12}, S^+_{23}$, $A_i \equiv L^-_{ii} S^+_{ii}$, $i = 1, 2, 3$ with the following defining relations:

$$\begin{aligned} [A_i, A_j] &= [L^-_{21}, S^+_{23}] = [L^-_{32}, S^+_{12}] = [L^-_{21}, A_3] \\ &= [L^-_{32}, A_1] = [S^+_{12}, A_3] = [S^+_{23}, A_1] = 0; \end{aligned} \tag{2.1a}$$

$$\begin{aligned} [L^-_{21}, S^+_{12}] &= (1 - \omega)(A_1 - A_2), \\ [L^-_{32}, S^+_{23}] &= (1 - \omega)(A_2 - A_3); \end{aligned} \tag{2.1b}$$

$$\begin{aligned} L^-_{32}(L^-_{21})^2 - (1+\omega) L^-_{21} L^-_{32} L^-_{21} + \omega (L^-_{21})^2 L^-_{32} &= 0, \\ (L^-_{32})^2 L^-_{21} - (1+\omega) L^-_{32} L^-_{21} L^-_{32} + \omega L^-_{21} (L^-_{32})^2 &= 0, \\ S^+_{12}(S^+_{23})^2 - (1+\omega) S^+_{23} S^+_{12} S^+_{23} + \omega (S^+_{23})^2 S^+_{12} &= 0, \\ (S^+_{12})^2 S^+_{23} - (1+\omega) S^+_{12} S^+_{23} S^+_{12} + \omega S^+_{23} (S^+_{12})^2 &= 0; \end{aligned} \tag{2.1c}$$

$$\begin{aligned} L^-_{32} A_2 &= \omega A_2 L^-_{32}, & L^-_{21} A_2 &= \omega^{-1} A_2 L^-_{21}, \\ S^+_{23} A_2 &= \omega^{-1} A_2 S^+_{23}, & S^+_{12} A_2 &= \omega A_2 S^+_{12}, \\ L^-_{32} A_3 &= \omega^{-1} A_3 L^-_{32}, & L^-_{21} A_1 &= \omega A_1 L^-_{21}, \\ S^+_{23} A_3 &= \omega A_3 S^+_{23}, & S^+_{12} A_1 &= \omega^{-1} A_1 S^+_{12}, \end{aligned} \tag{2.1d}$$

where $\omega \equiv q^2$. Note that this algebra does not depend on λ. Moreover, if one adds three more elements, e.g. L^-_{ii}, $i = 1, 2, 3$ with the relations

$$[L^-_{ii}, L^-_{jj}] = [L^-_{ii}, A_j] = 0, \tag{2.2a}$$

$$\begin{aligned}
L^-_{32}L^-_{11} &= \lambda^{-2}L^-_{11}L^-_{32}, & L^-_{32}L^-_{33} &= q^{-1}\lambda L^-_{33}L^-_{32}\,,\\
S^+_{23}L^-_{11} &= \lambda^{2}L^-_{11}S^+_{23}, & S^+_{23}L^-_{33} &= q\lambda^{-1}L^-_{33}S^+_{23}\,,\\
L^-_{21}L^-_{11} &= q\lambda L^-_{11}L^-_{21} & L^-_{21}L^-_{33} &= \lambda^{-2}L^-_{33}L^-_{21}\,,\\
S^+_{12}L^-_{11} &= q^{-1}\lambda^{-1}L^-_{11}S^+_{12}, & S^+_{12}L^-_{33} &= \lambda^{2}L^-_{33}S^+_{12}\,,\\
L^-_{32}L^-_{22} &= q\lambda L^-_{22}L^-_{32}, & L^-_{21}L^-_{22} &= q^{-1}\lambda L^-_{22}L^-_{21}\,,\\
S^+_{23}L^-_{22} &= q^{-1}\lambda^{-1}L^-_{22}S^+_{23}, & S^+_{12}L^-_{22} &= q\lambda^{-1}L^-_{22}S^+_{12}\,,
\end{aligned} \tag{2.2b}$$

then the resulting algebra is equivalent to the whole one (1.16) with

$$S^+_{ii} = A_i(L^-_{ii})^{-1}, \tag{2.3a}$$

$$L^-_{31} = -\frac{\lambda}{(q-q^{-1})}\,[L^-_{21}, L^-_{32}](L^-_{22})^{-1}, \tag{2.3b}$$

$$S^+_{13} = \frac{\lambda}{(q-q^{-1})}\,[S^+_{12}, S^+_{23}](S^+_{22})^{-1}. \tag{2.3c}$$

First, we shall construct the representation of algebra (2.1). Let us introduce the new notations:

$$\begin{aligned}
H_1 &\equiv L^-_{32}, & G_1 &\equiv S^+_{23}\,,\\
H_3 &\equiv L^-_{21}, & G_3 &\equiv S^+_{12}\,.
\end{aligned} \tag{2.4}$$

After some tedious manipulations one can rewrite Q_1 and Q_2 from (1.17) in terms of generators of the algebra (2.1):

$$H_iN_jG_i = \varepsilon_{ji}\frac{(1-\omega)}{(1+\omega)^2}\,(\omega Q_1^2 - Q_2) + \frac{\omega^{\theta_{ji}}}{1+\omega}\,(N_1N_3 + N_3N_1) - \alpha_jN_j + \beta_j, \tag{2.5a}$$

$$G_iN_jH_i = \varepsilon_{ij}\frac{(1-\omega)}{(1+\omega)^2}\,(\omega Q_1^2 - Q_2) + \frac{\omega^{\theta_{ij}}}{1+\omega}\,(N_1N_3 + N_3N_1) - \alpha'_jN_j + \beta'_j, \tag{2.5b}$$

where

$$N_i \equiv H_iG_i + \omega^{\theta_{ij}}A_j + \omega^{\theta_{ji}}A_2 = G_iH_i + \omega^{\theta_{ji}}A_j + \omega^{\theta_{ij}}A_2; \tag{2.6}$$

$$\begin{aligned}
\alpha_i &\equiv \omega^{\theta_{ij}}(\omega^{\varepsilon_{ji}}A_i + \omega^{\varepsilon_{ij}}A_2)\,,\\
\alpha'_i &\equiv \omega^{\theta_{ji}}(\omega^{\varepsilon_{ij}}A_i + \omega^{\varepsilon_{ji}}A_2)\,;
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\beta_i &\equiv \varepsilon_{ij}\omega\frac{(1-\omega)}{1+\omega}\,A_2(A_i - \omega^{\varepsilon_{ij}}(A_j + Q_1))\,,\\
\beta'_i &\equiv \varepsilon_{ji}\omega\frac{(1-\omega)}{1+\omega}\,A_2(A_i - \omega^{\varepsilon_{ji}}(A_j + Q_1))\,.
\end{aligned} \tag{2.8}$$

In these formulae the indiuces i, j run over two values $1, 3$ and not coincide. The symbols θ_{ij}, ε_{ij} mean the following:

$$\theta_{ij} \equiv \begin{cases} 1, & i > j;\\ 0, & i < j,\end{cases} \tag{2.9}$$

$$\varepsilon_{ij} \equiv \theta_{ij} - \theta_{ji}\,. \tag{2.10}$$

Define four operators

$$\kappa_1 \equiv \frac{1}{(1-\omega)^2 \Delta^{1/3}} \left(\omega H_3 N_1 (H_3)^{-1} - N_1\right), \tag{2.11a}$$

$$\kappa_3 \equiv \frac{1}{(1-\omega)^2 \Delta^{1/3}} \left(\omega G_1 N_3 (G_1)^{-1} - N_3\right), \tag{2.11b}$$

$$\phi_i \equiv \frac{1}{\Delta^{1/3}} A_i \left(\kappa_i + \frac{1}{(1-\omega)\Delta^{1/3}} N_i\right), \qquad i = 1,3, \tag{2.12}$$

with

$$\Delta \equiv A_1 A_2 A_3 \tag{2.13}$$

being the Casimir element. From (2.1), (2.5) it follows that they form the closed algebra

$$[\kappa_1, \kappa_3] = \phi_1 - \phi_3, \tag{2.14a}$$

$$[\phi_1, \phi_3] = \kappa_1 - \kappa_3, \tag{2.14b}$$

$$\omega \phi_i \kappa_i - \kappa_i \phi_i = \frac{\omega}{1-\omega}, \qquad i = 1,3, \tag{2.14c}$$

$$\omega \kappa_i \phi_j - \phi_j \kappa_i = \frac{\omega}{1-\omega}, \qquad i \neq j. \tag{2.14d}$$

Let us take them together with H_3, G_1, A_1, A_3 as a generating set of operators in (2.1). Apart from (2.14) we have

$$\begin{aligned}[H_3, G_1] &= [H_3, \kappa_i] = [H_3, \phi_i] = [H_3, A_3] \\ &= [G_1, \kappa_i] = [G_1, \phi_i] = [G_1, A_1] \\ &= [A_j, \kappa_i] = [A_j, \phi_i] = [A_1, A_3] = 0,\end{aligned} \tag{2.15a}$$

$$H_3 A_1 = \omega A_1 H_3, \qquad G_1 A_3 = \omega A_3 G_1. \tag{2.15b}$$

Then relations (2.14a) and (2.14b) can be replaced by their resolved form with two Casimir elements ϱ, σ,

$$\kappa_i \kappa_j = \varrho - \frac{\phi_j + \omega \phi_i}{1-\omega}, \qquad i \neq j, \tag{2.16a}$$

$$\phi_i \phi_j = \sigma - \frac{\kappa_i + \omega \kappa_j}{1-\omega}, \qquad i \neq j. \tag{2.16b}$$

To satisfy (2.5) the relations

$$\varrho = \frac{\omega Q_1^2 - Q_2}{(1+\omega)(1-\omega)^2 \Delta^{2/3}}, \qquad \sigma = \frac{\omega Q_1}{(1-\omega)^2 \Delta^{1/3}} \tag{2.17}$$

must be valid. So, if we know a representation of the algebra (2.14), then solving (2.12) with respect to N_i:

$$N_i = (1-\omega)\Delta^{1/3}\left(\Delta^{1/3}(A_i)^{-1}\phi_i - \kappa_i\right), \qquad i = 1,3, \tag{2.18}$$

we know the representation of (2.6).

To construct a representation of (2.14) let us choose any relation from (2.14c), (2.14d), which has the following form:

$$\omega AB - BA = \frac{\omega}{1-\omega}, \tag{2.19}$$

where A, B denote any pair of operators from (2.14c), (2.14d). Introduce an operator

$$C \equiv AB + \frac{\omega}{(1-\omega)^2} = \omega^{-1}BA + \frac{1}{(1-\omega)^2}\,. \tag{2.20}$$

One can show that it satisfies the simple commutation relations

$$AC = \omega^{-1}CA, \qquad BC = \omega CB\,, \tag{2.21}$$

which can be realized explicitly as (in the case of nongenerate A)

$$\begin{aligned} A &= aX_0\,, \\ C &= cZ_0\,, \\ B &= \frac{1}{a}\,X_0^{-1}\left(cZ_0 - \frac{\omega}{(1-\omega)^2}\right), \end{aligned} \tag{2.22}$$

where a, c are arbitrary parameters. Here the matrices Z_i, X_i have the following properties:

$$\begin{aligned} &[X_i, X_j] = [Z_i, Z_j] = 0\,, \\ &Z_i X_j = \omega^{\delta_{ij}} X_j Z_i, \qquad i,j = 0,1,3\,, \end{aligned} \tag{2.23}$$

and can be realized explicitly as

$$\langle n\,|X_i|\,m\rangle = \bar{\delta}_{n,m+\delta_i}\,, \tag{2.24a}$$

$$\langle n\,|Z_i|\,m\rangle = \omega^{n_i}\bar{\delta}_{n,m}\,. \tag{2.24b}$$

We use Dirac's notations for bra- and ket-vectors with three component indices ($n = (n_0, n_1, n_3)$) running over N^3 values, where N is a minimal number such that

$$\omega^N = 1\,; \tag{2.25}$$

$$\bar{\delta}_{n,m} \equiv \begin{cases} 1, & n = m \pmod N); \\ 0, & \text{otherwise}, \end{cases} \tag{2.26}$$

and δ_i means the addition of unity modulo N to the i-the component of the index. The two pairs of matrices Z_i, X_i with $i = 1,3$ will be used below. Let us choose

$$\begin{aligned} A &\equiv \kappa_3\,, \\ B &\equiv \phi_1\,. \end{aligned} \tag{2.27}$$

Then by means of (2.16) we have (the case $c \neq 0$):

$$\kappa_1 = \frac{1}{\omega c}\,Z_0^{-1}\left(\varrho\phi_1 - \frac{1}{1-\omega}\,\phi_1^2 - \frac{\omega}{1-\omega}\,\sigma - \frac{\omega^2}{(1-\omega)^2}\,\kappa_3\right), \tag{2.28a}$$

$$\phi_3 = \frac{1}{c}\,Z_0^{-1}\left(\sigma\kappa_3 + \frac{\omega}{1-\omega}\,\kappa_3^2 + \frac{1}{1-\omega}\,\varrho - \frac{1}{(1-\omega)^2}\,\phi_1\right). \tag{2.28b}$$

Taking into account (2.15) the operators H_3, G_1, A_1, A_3 can be realized in terms of Z_i, X_i by the formulae

$$\begin{aligned} H_3 &= h_3 Z_1, \qquad G_1 = g_1 Z_3\,, \\ A_1 &= a_1 X_1, \qquad A_3 = a_3 X_3\,, \end{aligned} \tag{2.29}$$

where h_3, g_1, a_1, a_3 are arbitrary parameters. For the remaining operators in (2.1) we have

$$A_2 = a_2 X_1^{-1} X_3^{-1} \tag{2.30a}$$

with the parameter a_2 being such that $\Delta = a_1 a_2 a_3$,

$$H_1 = \frac{1}{g_1}\left[(1-\omega)\frac{\Delta^{2/3}}{a_1} X_1^{-1}\phi_1 - (1-\omega)\Delta^{1/3}\kappa_1 - \omega a_2 X_1^{-1}X_3^{-1} - a_3 X_3\right] \times Z_3^{-1}, \tag{2.30b}$$

$$G_3 = \frac{1}{h_3}\left[(1-\omega)\frac{\Delta^{2/3}}{a_3} X_3^{-1}\phi_3 - (1-\omega)\Delta^{1/3}\kappa_3 - \omega a_2 X_1^{-1}X_3^{-1} - a_1 X_1\right] \times Z_1^{-1}. \tag{2.30c}$$

Thus, we have constructed the representation of the subalgebra (2.1). The expressions of the generating elements are given by Eqs. (2.4), (2.29), (2.30). This representation contains nine complex parameters a_1, a_2, a_3, g_1, h_3, a, c, ϱ, σ.

It is interesting to note that four-element algebra (2.14) at $\omega \neq -1$ contains a central extension of the algebra recently introduced in [19] as a new possible quantum deformation of the $sl(2)$ algebra. Define

$$\begin{aligned} e_i &\equiv \frac{1}{1+\omega}(\xi_i\phi_i - \xi_i^{-1}\kappa_i), \qquad i = 1,3, \\ e_2 &\equiv \frac{1}{1+\omega}(\xi_2\phi_3 - \xi_2^{-1}\kappa_1 + \xi_3\xi_1^{-1}[\kappa_1,\phi_3]) \end{aligned} \tag{2.31}$$

with ξ_i being arbitrary complex parameters with one constraint

$$\xi_1\xi_2\xi_3 = 1. \tag{2.32}$$

Then from (2.14) and (2.16) it follows that

$$\omega e_i e_j - e_j e_i = e_k + \zeta_k, \tag{2.33}$$

where (i,j,k) is any even permutation of $(1,2,3)$ and the central elements ζ_i have the following explicit form:

$$\zeta_i = \frac{\omega-1}{(\omega+1)^2}(\varrho\xi_i + \sigma\xi_i^{-1}) + \frac{\omega}{(\omega-1)(\omega+1)^2}(\xi_j\xi_k^{-1} + \xi_k\xi_j^{-1}), \qquad i \neq j \neq k \neq i. \tag{2.34}$$

Note that the parameters ξ_i can be chosen so that the elements e_i, $i = 1,3$ will be proportional to N_i. The algebra (2.33) with $\zeta_i = 0$ was introduced in [19], where some plausible arguments in favour of the existence of co-multiplication in this case were also given. There is a Casimir element generalizing that of ref. [19]:

$$(2+\omega^2)(e_1e_2e_3 + e_2e_3e_1 + e_3e_1e_2) - (2\omega+\omega^{-1})(e_2e_1e_3 + e_1e_3e_2 + e_3e_2e_1) + 3\sum_{i=1}^{3}\zeta_i e_i. \tag{2.35}$$

A question about the explicit formula for the co-multiplication law in this algebra is obscure until now.

3. Representation of the Main Algebra

To write down the L^-_{ii}, S^+_{ii}, L^-_{31}, S^+_{13} in terms of X_i and Z_i let us define the integer numbers s_1, s_2, s_3 by

$$(q\lambda)^{-1} = \omega^{s_1}, \qquad q\lambda^{-1} = \omega^{s_2}, \qquad \lambda^2 = \omega^{s_3}, \tag{3.1}$$

where $\omega = \exp(2\pi i/N)$. From (3.1) and (2.25) it follows that equations

$$\begin{gathered} s_1 + s_2 + s_3 = 0 \ (\mathrm{mod}\, N)\,, \\ s_2 - s_1 = 1 \ (\mathrm{mod}\, N) \end{gathered} \tag{3.2}$$

are valid. By use of (1.16), (3.1), (3.2), and (3.3) we come to

$$\begin{aligned} L^-_{11} &= b_1 X_1^{-s_1} X_3^{s_3}, \qquad S^+_{11} = a_1/b_1 X_1^{s_2} X_3^{-s_3}, \\ L^-_{22} &= b_2 X_1^{-s_2} X_3^{s_1}, \qquad S^+_{22} = a_2/b_2 X_1^{s_1} X_3^{-s_2}, \\ L^-_{33} &= b_3 X_1^{-s_3} X_3^{s_2}, \qquad S^+_{33} = a_3/b_3 X_1^{s_3} X_3^{-s_1}, \end{aligned} \tag{3.3}$$

$$S^+_{13} = \omega^{s_1} \frac{b_2 g_1}{a_2 h_3} X_1^{-s_2} X_3^{s_1} \left((1-\omega) \frac{\Delta^{2/3}}{a_3} X_1 \phi_3 - \omega a_2 \right) Z_3 Z_1^{-1}, \tag{3.4a}$$

$$L^-_{31} = \omega^{s_1} \frac{h_3}{b_2 g_1} X_1^{s_1} X_3^{-s_2} \left((1-\omega) \frac{\Delta^{2/3}}{a_1} X_3 \phi_1 - \omega a_2 \right) Z_1 Z_3^{-1}. \tag{3.4b}$$

That ends our construction of the representation for the algebra (1.16). The transition to that of (1.10) is straightforward. For completness we list the whole set of formulae:

$$\begin{aligned} L^+_{11} &= b_1/a_1 X_1^{-s_2} X_3^{s_3}, \qquad L^-_{11} = b_1 X_1^{-s_1} X_3^{s_3}, \\ L^+_{22} &= b_2/a_2 X_1^{-s_1} X_3^{s_2}, \qquad L^-_{22} = b_2 X_1^{-s_2} X_3^{s_1}, \\ L^+_{33} &= b_3/a_3 X_1^{-s_3} X_3^{s_1}, \qquad L^-_{33} = b_3 X_1^{-s_3} X_3^{s_2}; \end{aligned} \tag{3.5a}$$

$$\begin{aligned} L^+_{12} = -\omega^{s_1} \frac{b_1 b_2}{a_1 a_2 h_3} X_1^{s_3} X_3^{-s_1} \bigg[&(1-\omega) \frac{\Delta^{2/3}}{a_3} X_3^{-1} \phi_3 \\ &- (1-\omega)\Delta^{1/3}\kappa_3 - \omega a_2 X_1^{-1} X_3^{-1} - a_1 X_1 \bigg] Z_1^{-1}, \end{aligned} \tag{3.5b}$$

$$L^+_{13} = -\frac{b_1 b_2 b_3 g_1}{\omega \Delta h_3} ((1-\omega)\Delta^{1/3}\kappa_3 + a_1 X_1) Z_3 Z_1^{-1}, \tag{3.5c}$$

$$L^+_{23} = -\omega^{s_1} \frac{b_2 b_3 g_1}{a_2 a_3} X_1^{s_2} X_3^{-s_3} Z_3\,, \tag{3.5d}$$

$$L^-_{21} = h_3 Z_1\,, \tag{3.5e}$$

$$L^-_{31} = \omega^{s_1} \frac{h_3}{b_2 g_1} X_1^{s_1} X_3^{-s_2} \left((1-\omega) \frac{\Delta^{2/3}}{a_1} X_3 \phi_1 - \omega a_2 \right) Z_1 Z_3^{-1}, \tag{3.5f}$$

$$\begin{aligned} L^-_{32} = \frac{1}{g_1} \bigg[&(1-\omega) \frac{\Delta^{2/3}}{a_1} X_1^{-1} \phi_1 - (1-\omega)\Delta^{1/3}\kappa_1 - \omega a_2 X_1^{-1} X_3^{-1} - a_3 X_3 \bigg] \\ &\times Z_3^{-1}. \end{aligned} \tag{3.5g}$$

This representation is realized in $C^N \otimes C^N \otimes C^N$ and is defined by twelve complex parameters, namely. a_i, b_i, $i = 1, 2, 3$, g_1, h_3, a, c, ϱ, σ. For the meaning of the other symbols in (3.5) refer to (2.22)–(2.28), (3.1), (3.2).

Consider now a simplest one-dimensional realization of (2.14) when all the operators commute among themselves. In this case we have

$$\phi_i = \phi; \qquad \kappa_i = \kappa, \qquad \phi\kappa = -\frac{\omega}{(1-\omega)^2}\,; \tag{3.6}$$

$$\varrho = \kappa^2 + \frac{1+\omega}{1-\omega}\,\phi, \qquad \sigma = \phi^2 - \frac{1+\omega}{1-\omega}\,\kappa, \tag{3.7}$$

where ϕ and κ are some parameters. One sees that there is a relation between ϱ and σ owing to (3.6), (3.7) and therefore by (2.17) so between Q_1, Q_2 and Δ. Let us consider more closely the structure of formulae (3.5) in that case. It is not difficult to see that the long expressions in square brackets in (3.5b), (3.5g) factorize leading to the formulae

$$L^+_{12} = \omega^{s_1}\,\frac{b_1 b_2}{a_1 a_2 h_3} X_1^{s_3} X_3^{-s_1} \left(1 + \frac{\omega a_2 X_1^{-1} X_3^{-1}}{(1-\omega)\Delta^{1/3}\kappa}\right) ((1-\omega)\Delta^{1/3}\kappa + a_1 X_1) Z_1^{-1}, \tag{3.8}$$

$$L^-_{32} = -\frac{(1-\omega)\Delta^{1/3}\kappa}{a_2 g_1} \left(1 + \frac{\omega a_2 X_1^{-1} X_3^{-1}}{(1-\omega)\Delta^{1/3}\kappa}\right) \left(\frac{\Delta^{2/3}}{(1-\omega)\kappa a_1} X_3 + a_2\right) Z_3^{-1}. \tag{3.9}$$

Introducing new matrices W_i instead of Z_i,

$$W_i^{-1} \equiv \frac{1}{\left(1 - \left(\frac{a_2}{(\omega-1)\Delta^{1/3}\kappa}\right)^N\right)^{1/N}} \left(1 + \frac{\omega a_2 X_1^{-1} X_3^{-1}}{(1-\omega)\Delta^{1/3}\kappa}\right) Z_i^{-1}, \quad i = 1, 3, \tag{3.10}$$

for which we have the same algebraic relations (2.23) where all Z_i are replaced by the W_i, we rewrite the (3.5b)–(3.5g) in form

$$L^+_{12} = \omega^{s_1}\,\frac{b_1 b_2}{a_1 a_2 h'_3} X_1^{s_3} X_3^{-s_1} ((1-\omega)\Delta^{1/3}\kappa + a_1 X_1) W_1^{-1}, \tag{3.11a}$$

$$L^+_{23} = -\omega^{s_1}\,\frac{b_2 b_3 g'_1}{a_2 a_3} X_1^{s_2} X_3^{-s_3} \left(1 + \frac{a_2 X_1^{-1} X_3^{-1}}{(1-\omega)\Delta^{1/3}\kappa}\right) W_3, \tag{3.11b}$$

$$L^+_{13} = -\frac{b_1 b_2 b_3 g'_1}{\omega \Delta h'_3} ((1-\omega)\Delta^{1/3}\kappa + a_1 X_1) W_3 W_1^{-1}, \tag{3.11c}$$

$$L^-_{21} = h'_3 \left(1 + \frac{a_2 X_1^{-1} X_3^{-1}}{(1-\omega)\Delta^{1/3}\kappa}\right) W_1, \tag{3.11d}$$

$$L^-_{32} = -\frac{(1-\omega)\Delta^{1/3}\kappa}{a_2 g'_1} \left(\frac{\Delta^{2/3}}{(1-\omega)\kappa a_1} X_3 + a_2\right) W_3^{-1}, \tag{3.11e}$$

$$L^-_{31} = -\omega^{s_1}\,\frac{\omega h'_3}{b_2 g'_1} X_1^{s_1} X_3^{-s_2} \left(\frac{\Delta^{2/3}}{(1-\omega)\kappa a_1} X_3 + a_2\right) W_1 W_3^{-1}, \tag{3.11f}$$

where the symbols g'_1 and h'_3 mean the following:

$$\frac{g_1}{g'_1} = \frac{h_3}{h'_3} \equiv \left(1 - \left(\frac{a_2}{(\omega-1)\Delta^{1/3}\kappa}\right)^N\right)^{1/N}. \tag{3.12}$$

Let us define the new parameters by formulae:

$$\xi \equiv (\omega-1)^{1/2}\Delta^{1/6}\kappa^{1/2}, \qquad u_i^- \equiv b_i, \qquad u_i^+ \equiv b_i/a_i, \qquad i=1,2,3\,; \tag{3.13}$$

$$c_{12} = -\frac{1}{c_{21}} \equiv \xi\,\frac{u_2^+}{h_3'}, \quad c_{23} = -\frac{1}{c_{32}} \equiv \xi^{-1}u_3^+ g_1', \quad c_{13} = -\frac{1}{c_{31}} \equiv -\xi\,\frac{u_2^+u_3^+g_1'}{\omega h_3'}. \tag{3.14}$$

Then relations (3.11) take the following compact form:

$$L_{ij} = c_{ij}(u_i^-\xi^{-1}X_i - u_i^+\xi)X_i^{-\theta s_2}W_{ij}X_j^{\theta s_1}, \qquad i \neq j, \tag{3.15}$$

where L_{ij} is defined in (1.12),

$$W_{12} \equiv W_1^{-1}, \qquad W_{23} \equiv W_3, \qquad W_{13} \equiv W_1^{-1}W_3, \qquad X_2 \equiv X_1^{-1}X_3^{-1} \tag{3.16}$$

and

$$\theta \equiv \begin{cases} 1, & \text{if } (i,j,k) \text{ is even permutation of } (1,2,3); \\ 0, & \text{otherwise}. \end{cases} \tag{3.17}$$

Here the matrices X_i, W_{ij} satisfy the closed algebraic relations

$$[X_i, X_{i'}] = [W_{ij}, W_{i'j'}] = 0, \qquad W_{ij}X_p = \omega^{\delta_{jp}-\delta_{ip}}X_pW_{ij}, \qquad i \neq j, \qquad i' \neq j' \tag{3.18}$$

with additional constraints:

$$W_{ij}W_{ji} = W_{ij}W_{jk}W_{ki} = X_1X_2X_3 = 1, \qquad i \neq j \neq k \neq i\,. \tag{3.19}$$

At last note that the parameters in (3.15) can be considered independently of their definition (3.13), (3.14) if we impose on them the following constraints:

$$c_{ij}c_{ji} = -1, \qquad c_{ij}c_{jk}c_{ki} = \varepsilon\omega^{\varepsilon}\xi^{-\varepsilon}, \qquad i \neq j \neq k \neq i, \qquad \varepsilon = 2\theta - 1\,. \tag{3.20}$$

4. Specialization to the $U_q(sl(3))$ Algebra

Let us make more transparent the connection of the algebra (1.10) with the $U_q(sl(3))$ announced in Sect. 1. Impose the following constraints:

$$\lambda = 1\,, \tag{4.1a}$$

$$L_{ii}^+L_{ii}^- = L_{11}^{\pm}L_{22}^{\pm}L_{33}^{\pm} = 1\,. \tag{4.1b}$$

Then the algebra (1.10) as a Hopf algebra with co-multiplication $\bar{\Delta}L_{ij}^{\pm} \equiv \sum_k L_{kj}^{\pm} \otimes L_{ik}^{\pm}$ is equivalent to $U_q(sl(3))$ by the following identification:

$$\begin{aligned} L_{11}^+ &= k_1^{-4/3}k_2^{-2/3}, & L_{11}^- &= k_1^{4/3}k_2^{2/3}, \\ L_{22}^+ &= k_1^{2/3}k_2^{-2/3}, & L_{22}^- &= k_1^{-2/3}k_2^{2/3}, \\ L_{33}^+ &= k_1^{2/3}k_2^{4/3}, & L_{33}^- &= k_1^{-2/3}k_2^{-4/3}, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} L_{12}^+ &= (q-q^{-1})k_1^{-1/3}k_2^{-2/3}e_1, & L_{21}^- &= -(q-q^{-1})k_1^{1/3}k_2^{2/3}f_1, \\ L_{23}^+ &= (q-q^{-1})k_1^{2/3}k_2^{1/3}e_2, & L_{32}^- &= -(q-q^{-1})k_1^{-2/3}k_2^{-1/3}f_2. \end{aligned} \tag{4.2b}$$

Here we omitted the corresponding expressions for operators L_{13}^+ and L_{31}^- since they are dependent ones by (1.11g). Hereafter, the commutation relations (1.11) lead to the standard commutation relations of the $U_q(sl(3))$ algebra [7]:

$$[k_i, k_j] = 0, \tag{4.3a}$$

$$k_i e_j = q^{a_{ij}/2} e_j k_i, \qquad k_i f_j = q^{-a_{ij}/2} f_j k_i, \tag{4.3b}$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, \tag{4.3c}$$

$$e_i e_j^2 - (q + q^{-1}) e_j e_i e_j + e_j^2 e_i = 0, \qquad i \neq j, \tag{4.3d}$$

$$f_i f_j^2 - (q + q^{-1}) f_j f_i f_j + f_j^2 f_i = 0, \qquad i \neq j, \tag{4.3e}$$

where

$$a_{ii} = 2, \qquad a_{ij} = -1, \qquad i \neq j, \tag{4.4}$$

and the following co-multiplication law:

$$\begin{aligned} \Delta(k_i) &= k_i \otimes k_i, \\ \Delta(e_i) &= k_i \otimes e_i + e_i \otimes k_i^{-1}, \\ \Delta(f_i) &= k_i \otimes f_i + f_i \otimes k_i^{-1}. \end{aligned} \tag{4.5}$$

Hence, any representation of (1.10) obeying the constraints (4.1) becomes the representation of $U_q(sl(3))$. For example, let us rewrite (3.11) as a representation of $U_q(sl(3))$. The relations (4.1b) give

$$a_i = b_i^2, \qquad i = 1,2,3, \qquad b_1 b_2 b_3 = 1. \tag{4.6}$$

Expressing now k_i, e_i, f_i from (4.2) and using (3.5a), (3.11) and (4.6) we obtain:

$$\begin{aligned} k_1 &= \sqrt{b_1/b_2} X_1^{1/2} X_3^{1/4}, \qquad k_2 = \sqrt{b_2/b_3} X_1^{-1/4} X_3^{-1/2} \\ e_1 &= \frac{\sqrt{b_1/b_2}\,\xi}{(q - q^{-1}) h_3'} \left[\frac{b_1}{\xi} X_1^{1/2} - \frac{\xi}{b_1} X_1^{-1/2} \right] \tilde{W}_1^{-1}, \\ e_2 &= \frac{\sqrt{b_2/b_3}\, g_1'}{(q - q^{-1}) \xi} \left[\frac{b_2}{\xi} X_2^{1/2} - \frac{\xi}{b_2} X_2^{-1/2} \right] \tilde{W}_3, \\ f_1 &= \frac{\sqrt{b_2/b_1}\, h_3'}{(q - q^{-1}) \xi} \left[\frac{b_2}{\xi} X_2^{1/2} - \frac{\xi}{b_2} X_2^{-1/2} \right] \tilde{W}_1, \\ f_2 &= \frac{\sqrt{b_3/b_2}\,\xi}{(q - q^{-1}) g_1'} \left[\frac{b_3}{\xi} X_3^{1/2} - \frac{\xi}{b_3} X_3^{-1/2} \right] \tilde{W}_3^{-1}, \\ & X_2 \equiv X_1^{-1} X_3^{-1}, \qquad \xi \equiv (\omega - 1)^{1/2} \kappa^{1/2}, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \tilde{W}_1 &\equiv X_1^{-1/2} X_3^{-1/4} W_1, \\ \tilde{W}_3 &\equiv q^{-1} X_1^{-1/4} X_3^{-1/2} W_3. \end{aligned} \tag{4.8}$$

Note that Eqs. (4.7) contain five independent parameters.

5. Necessary Conditions for S-Matrix Existence

Let $L(x)$ and $L(\tilde{x})$ be two solutions of Eq. (1.5) just constructed with two different sets of parameters. Below the argument $x(\tilde{x})$ will be omitted since by redefinition

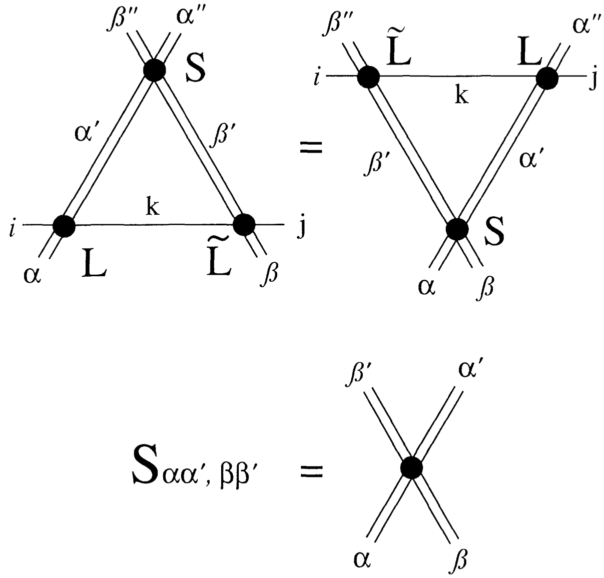

Fig. 3. Graphical representation of Eq. (5.1)

of the parameters of the representation it can be absorbed into other parameters. Let us find necessary conditions for the existence of an intertwining matrix S, which satisfies the equation (Fig. 3)

$$\Delta L_{ij} S = S \bar{\Delta} L_{ij} , \tag{5.1}$$

where

$$\Delta L_{ij} \equiv \sum_k L_{ik} \otimes \tilde{L}_{kj} , \tag{5.2a}$$

$$\bar{\Delta} L_{ij} \equiv \sum_k L_{kj} \otimes \tilde{L}_{ik} . \tag{5.2b}$$

From (5.1) it follows that the equations

$$(\Delta L_{ij})^n S = S(\bar{\Delta} L_{ij})^n \tag{5.3}$$

with n being an arbitrary positive integer must be valid too. From (5.3) it follows that (if S^{-1} does exist)

$$\mathrm{tr}(\Delta L_{ij})^n = \mathrm{tr}(\bar{\Delta} L_{ij})^n , \tag{5.4}$$

where the trace is taken in C^M ($M = N^3$). Let us expand the operators ΔL_{ij} in the sum

$$\Delta L_{ij} = \Delta L_{ij}^{+} + \Delta L_{ij}^{0} + \Delta L_{ij}^{-} , \tag{5.5}$$

where

$$\begin{aligned} \Delta L_{ij}^{\pm} &\equiv \sum_k L_{ik}^{\pm} \otimes \tilde{L}_{kj}^{\pm} , \\ \Delta L_{ij}^{0} &\equiv \sum_k L_{ik}^{+} \otimes \tilde{L}_{kj}^{-} + L_{ik}^{-} \otimes \tilde{L}_{kj}^{+} . \end{aligned} \tag{5.6}$$

By using (1.10) one can show that the individual terms in (5.5) are commutative with each other (a similar expansion is valid for the $\bar{\Delta}L_{ij}$). This means that they are intertwined independently, i.e. the equations

$$\begin{aligned} \mathrm{tr}(\Delta L_{ij}^{\pm})^n &= \mathrm{tr}(\bar{\Delta} L_{ij}^{\pm})^n, \\ \mathrm{tr}(\Delta L_{ij}^{0})^n &= \mathrm{tr}(\bar{\Delta} L_{ij}^{0})^n \end{aligned} \tag{5.7}$$

should be valid. For the cyclic representations with $\omega^N = 1$, where N is a prime number Eqs. (5.7) at $n < N$ are trivial (of the form $0 = 0$). Calculating the traces in the case when $n = N$ and discarding the common factors N^3 we have the following equations:

$$\sum_k (L_{ik}^{\pm})^N (\tilde{L}_{kj}^{\pm})^N = \sum_k (L_{kj}^{\pm})^N (\tilde{L}_{ik}^{\pm})^N, \tag{5.8a}$$

$$\sum_k (L_{ik}^{+})^N (\tilde{L}_{kj}^{-})^N + (L_{ik}^{-})^N (\tilde{L}_{kj}^{+})^N = \sum_k (L_{kj}^{+})^N (\tilde{L}_{ik}^{-})^N + (L_{kj}^{-})^N (\tilde{L}_{ik}^{+})^N. \tag{5.8b}$$

Equations (5.8a) are equivalent to

$$\frac{(L_{11}^{+})^N - (L_{22}^{+})^N}{(L_{12}^{+})^N} = c_1^{+}, \qquad \frac{(L_{11}^{-})^N - (L_{22}^{-})^N}{(L_{21}^{-})^N} = c_1^{-}, \tag{5.9a}$$

$$\frac{(L_{33}^{+})^N - (L_{22}^{+})^N}{(L_{23}^{+})^N} = c_2^{+}, \qquad \frac{(L_{33}^{-})^N - (L_{22}^{-})^N}{(L_{32}^{-})^N} = c_2^{-}, \tag{5.9b}$$

$$\frac{c_1^{+}(L_{13}^{+})^N + (L_{23}^{+})^N}{c_2^{+}(L_{13}^{+})^N + (L_{12}^{+})^N} = c_3^{+}, \qquad \frac{c_1^{-}(L_{31}^{-})^N + (L_{32}^{-})^N}{c_2^{-}(L_{31}^{-})^N + (L_{21}^{-})^N} = c_3^{-}, \tag{5.9c}$$

where $c_i^{\pm}$, $i = 1,2,3$ are invariants (in the sense that they should be the same for L_{ij} and $\tilde{L}_{ij}$). For the generic case, i.e., when there are no special relations between $(L_{ij}^{\pm})^N$, Eqs. (5.8) are equivalent to

$$\begin{aligned} \frac{(L_{ii}^{\pm})^N - (L_{jj}^{\pm})^N}{(L_{12}^{+})^N} &= \frac{(\tilde{L}_{ii}^{\pm})^N - (\tilde{L}_{jj}^{\pm})^N}{(\tilde{L}_{12}^{+})^N}, \qquad i \neq j, \\ \frac{(L_{ij})^N}{(L_{12}^{+})^N} &= \frac{(\tilde{L}_{ij})^N}{(\tilde{L}_{12}^{+})^N}, \qquad i \neq j, \end{aligned} \tag{5.10}$$

where the symbol L_{ij} means as it stands in (1.12). Note that we have the eleven free parameters excluding the common normalization factor listed after Eqs. (3.5) and the nine equations (5.10), which define the two-dimensional spectral parameter surface just as in the case of [1]. However, we don't know if is it possible to factorize the two-dimensional complex surface defined by Eqs. (5.10) into a product of two complex curves.

A complete analysis of different particular cases with special relations between $(L_{ij}^{\pm})^N$ is too cumbersome and will not be presented here. Consider only one special case, when

$$(L_{11}^{\pm})^N = (L_{33}^{\pm})^N, \tag{5.11}$$

$$c_3^{+}c_3^{-} = 1, \qquad c_4^{+}c_4^{-} = c_2^{+}c_2^{-} + 1, \tag{5.12}$$

where

$$c_4^+ \equiv \frac{(L_{13}^+)^N - c_2^-(L_{12}^+)^N}{(L_{21}^-)^N}, \qquad c_4^- \equiv \frac{(L_{31}^-)^N - c_2^+(L_{21}^-)^N}{(L_{12}^+)^N}. \tag{5.13}$$

In this case Eqs. (5.8b) require only that $c_4^\pm$ should be the invariants.

For example, consider the representation in (3.11) with additional constraints

$$(L_{ii}^\pm)^N = (L_{jj}^\pm)^N, \qquad i,j = 1,2,3. \tag{5.14}$$

In this case Eqs. (5.12) are valid, and the invariants have the following explicit form:

$$c_1^\pm = c_2^\pm = 0, \tag{5.15}$$

$$c_3^\pm = \left(\frac{g_1' h_3'}{a_1(\omega-1)\kappa}\right)^{\pm N}, \qquad c_4^\pm = \left(\frac{b_1^3 (g_1')^2}{a_1^3 h_3'}\right)^{\pm N} c_3^\mp, \tag{5.16}$$

where we have used $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$.

To end this section let us list the formulae for $(L_{ij}^\pm)^N$ entering (5.8) for the general case of our L-operator given by (3.5),

$$(L_{ii}^+)^N = (b_i/a_i)^N, \qquad (L_{ii}^-)^N = b_i^N, \tag{5.17a}$$

$$(L_{12}^+)^N = \left(\frac{b_1 b_2}{a_1 a_2 h_3}\right)^N \left[-(\omega-1)^N \Delta^{N/3}\kappa_3^N - (1-\omega)^N \frac{\Delta^{2N/3}}{a_3^N}\phi_3^N + a_2^N + a_1^N\right], \tag{5.17b}$$

$$(L_{13}^+)^N = \left(\frac{b_1 b_2 b_3 g_1}{\Delta h_3}\right)^N [(\omega-1)^N \Delta^{N/3}\kappa_3^N - a_1^N], \tag{5.17c}$$

$$(L_{23}^+)^N = -\left(\frac{b_2 b_3 g_1}{a_2 a_3}\right)^N, \tag{5.17d}$$

$$(L_{21}^-)^N = h_3^N, \tag{5.17e}$$

$$(L_{31}^-)^N = \left(\frac{h_3}{b_2 g_1}\right)^N \left[(1-\omega)^N \frac{\Delta^{2N/3}}{a_1^N}\phi_1^N - a_2^N\right], \tag{5.17f}$$

$$(L_{32}^-)^N = \frac{1}{g_1^N}\left[(1-\omega)^N \frac{\Delta^{2N/3}}{a_1^N}\phi_1^N + (\omega-1)^N \Delta^{N/3}\kappa_1^N - a_2^N - a_3^N\right]; \tag{5.17g}$$

$$\kappa_3^N = a^N, \qquad \phi_1^N = \frac{(-1)^N}{a^N}\left(\frac{1}{(1-\omega)^{2N}} - c^N\right), \tag{5.18a}$$

$$\kappa_1^N = \frac{(-1)^N}{\phi_1^N}\left(\frac{1}{(1-\omega)^{2N}} - C_1^N\right), \tag{5.18b}$$

$$\phi_3^N = \frac{(-1)^N}{\kappa_3^N}\left(\frac{1}{(1-\omega)^{2N}} - C_2^N\right); \tag{5.18c}$$

$$C_1^N = \frac{1}{(1-\omega)^N c^N}(\nu_1^N - \phi_1^N)(\nu_2^N - \phi_1^N)(\nu_3^N - \phi_1^N), \tag{5.19a}$$

$$C_2^N = \frac{1}{(1-\omega)^N c^N}(\mu_1^N - \kappa_3^N)(\mu_2^N - \kappa_3^N)(\mu_3^N - \kappa_3^N), \tag{5.19b}$$

where the v_i are defined by the following system of equations:

$$\begin{aligned} v_1 + v_2 + v_3 &= (\omega^{-1} - 1)\varrho\,, \\ v_1 v_2 + v_2 v_3 + v_3 v_1 &= \omega^{-1}\sigma\,, \\ v_1 v_2 v_3 &= \frac{1}{(1-\omega)^3}\,, \end{aligned} \tag{5.20}$$

while the μ_i are given by formulae:

$$\mu_i = -\frac{\omega}{(1-\omega)^2 v_i}\,, \qquad i = 1,2,3\,. \tag{5.21}$$

Note, that expressions (5.17–5.19) are valid for any prime $N \geq 2$ and for any choice of s_1, s_2, s_3 satisfying (3.2).

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