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Combinatorics of Representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$

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Abstract. The $q = 0$ combinatorics for $U_q(\widehat{\mathfrak{sl}}(n))$ is studied in connection with solvable lattice models. Crystal bases of highest weight representations of $U_q(\widehat{\mathfrak{sl}}(n))$ are labelled by paths which were introduced as labels of corner transfer matrix eigenvectors at $q = 0$. It is shown that the crystal graphs for finite tensor products of l-th symmetric tensor representations of $U_q(\mathfrak{sl}(n))$ approximate the crystal graphs of level l representations of $U_q(\widehat{\mathfrak{sl}}(n))$. The identification is made between restricted paths for the RSOS models and highest weight vectors in the crystal graphs of tensor modules for $U_q(\widehat{\mathfrak{sl}}(n))$.

1. Introduction

1.1 R Matrices and Paths. The eminent role of the quantized enveloping algebras in solvable lattice models is widely known. The R matrices, which are the intertwiners of tensor product representations, give the Boltzmann weights of lattice models with commuting transfer matrices [1].

Consider $U_q(\widehat{\mathfrak{sl}}(n))$. Let (V, π) be the l-th symmetric tensor representation of $U_q(\mathfrak{sl}(n))$. We can extend this representations to a family of representations (V, π_x) of $U_q(\widehat{\mathfrak{sl}}(n))$ with an auxiliary parameter x. The R matrix $R(x, y)$ is an element of $\mathrm{End}(V \otimes V)$ which intertwines two representations $(V \otimes V, \pi_x \otimes \pi_y)$ and $(V \otimes V, \pi_y \otimes \pi_x)$. Set

$$\mathscr{A}_l^+ = \left\{ v = \sum_{i=0}^{n-1} v_i \epsilon_i \;\middle|\; v_i \in \mathbf{Z}_{\geqq 0}, \sum_{i=0}^{n-1} v_i = l \right\},$$

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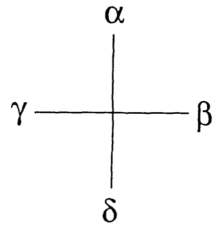


Fig. 1.1. Elementary configuration for vertex models

where $\epsilon_j = \left(0, \ldots, \overset{j\text{-th}}{1}, \ldots, 0\right)$ $(0 \leqq j < n)$. We choose a vector $v_\nu \in V$ with weight ν so that $\{v_\nu | \nu \in \mathscr{A}_l^+\}$ constitutes a base of V. The matrix elements of $R(x, y)$ with respect to this base give the Boltzmann weights of a solvable vertex model. The fluctuation variables of the model live on the bonds of the lattice, say $\mathscr{L}$, and they take values in the set $\{v_\nu | \nu \in \mathscr{A}_l^+\}$ which we identify with $\mathscr{A}_l^+$. (Fig. 1.1.) The simplest case $n = 2, l = 1$ is the 6 vertex model.

An interesting phenomenon was found in the study of the 1 point functions of solvable lattice models. The spectra of the logarithm of the corner transfer matrices in the infinite lattice limit $(N \to \infty)$ have an equally spaced distribution [2] and their generating functions often coincide with the characters or the branching functions of some affine Lie algebras (see e.g. [3]).

For the models corresponding to (V, π_x), the statement is as follows. Let Λ_i $(i = 0, \ldots, n-1)$ be the fundamental weights of $U_q(\widehat{\mathfrak{sl}}(n))$. Fix a dominant integral weight $\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_l}$. A ground state of the model is specified by this choice. A path of length N is a sequence $(\eta_0, \ldots, \eta_{N-1}) \in (\mathscr{A}_l^+)^N$. The corner transfer matrix is a matrix indexed by paths of length N. It depends on the choice of the ground state. A Λ-path is an infinite path $(\eta_0, \eta_1, \ldots)$ such that $\eta_k = \eta_{\Lambda,k}$ for $k \gg 1$, where $\eta_{\Lambda,k} \overset{\text{def}}{=} \epsilon_{\gamma_1 + k} + \cdots + \epsilon_{\gamma_l + k}$. We denote the set of Λ-paths by $\mathscr{P}(\Lambda)$. The following is proved in [4].

Theorem 1.1. *Let $M(\Lambda)$ be the irreducible highest weight representation of $\widehat{\mathfrak{sl}}(n)$ with highest weight Λ, and $M(\Lambda)_\mu$ the weight space of weight μ. Define the weight μ_η of a Λ-path η by*

$$\mu_\eta = \Lambda - \sum_{k \geqq 0} (\bar{\eta}_k - \bar{\eta}_{\Lambda,k}) - \omega(\eta)\delta \quad (\delta\text{: the null root}),$$

$$\omega(\eta) = \sum_{k \geqq 1} k(H(\eta_{k-1}, \eta_k) - H(\eta_{\Lambda,k-1}, \eta_{\Lambda,k})),$$

where

$$\bar{\epsilon}_i = \Lambda_{i+1} - \Lambda_i,$$

$$H(\epsilon_{i_1} + \cdots + \epsilon_{i_l}, \epsilon_{i'_1} + \cdots + \epsilon_{i'_l}) = \min_{\sigma \in \mathfrak{S}_l} \sum_{j=1}^{l} \theta(i_{\sigma(j)} - i'_j),$$

$$\theta(i) = 1 \quad \text{if} \quad i \geqq 0$$
$$= 0 \quad \text{otherwise}.$$

Then we have

$$\#\{\eta \in \mathscr{P}(\Lambda) | \mu_\eta = \mu\} = \dim M(\Lambda)_\mu \quad \text{for any } \mu. \tag{1.1}$$

The proof in [4] is to construct an explicit base of $M(\Lambda)$ labelled by Λ-paths. However, it is somewhat unnatural to consider the case $q=1$ in dealing with paths, because the corner transfer matrix method is based on the behaviour of the R matrix in the low temperature limit, i.e., $q=0$. In this paper we shall give a natural proof of (1.1) by constructing the crystal base of the irreducible $U_q(\widehat{\mathfrak{sl}}(n))$-module with highest weight Λ using Λ-paths as labels.

1.2. Crystal Base and Paths. Kashiwara [5] found certain bases of the integrable highest weight representations of the quantized enveloping algebras which exhibit a remarkably simple structure at $q=0$. He named them the crystal bases. Misra and Miwa [6] noticed that the paths ($l=1$) as explained in the previous section give appropriate labels to the crystal base in level 1 representation of $U_q(\widehat{\mathfrak{sl}}(n))$. It is apparent that the crystal base provides a powerful tool to attack the combinatorial problems related to the corner transfer matrix method. The aim of this paper is to establish the role of paths as the labels of crystal bases in arbitrary level representations of $U_q(\widehat{\mathfrak{sl}}(n))$.

Let e,f,t be the Chevalley generators of $U_q(\mathfrak{sl}(2))$, and (V_l,π_l) the $l+1$ dimensional irreducible representation with the distinguished base $(v_{lk})_{0\leqq k\leqq l}$ such that

$$\pi_l(e)v_{lk}=[k]v_{lk-1},$$
$$\pi_l(f)v_{lk}=[l-k]v_{lk+1},$$
$$\pi_l(t)v_{lk}=q^{l-2k}v_{lk}.$$

Let e_i,f_i,t_i ($i=0,\dots,n-1$) be the Chevalley generators of $U_q(\widehat{\mathfrak{sl}}(n))$, and $U_q(\mathfrak{sl}(2))_i$ the algebra generated by $e_i,f_i,t_i^{\pm1}$. Let $M(\Lambda)$ be the irreducible $U_q(\widehat{\mathfrak{sl}}(n))$-module with highest weight $\Lambda=\Lambda_{\gamma_1}+\cdots+\Lambda_{\gamma_l}$. Set $K=\mathbf{Q}(q)$ and $A=\{f\in K\,|\,f \text{ has no pole at } q=0\}$.

Kashiwara's result is rephrased in this case as

Theorem 1.2. *There exists a base $(u_b)_{b\in B}$ of $M(\Lambda)$ with the following properties. Set*

$$L=\bigoplus_{b\in B}Au_b,$$

and identify the subset $\{u_b \bmod qL\,|\,b\in B\}\subset L/qL$ with B. For each i there exists an isomorphism of $U_q(\mathfrak{sl}(2))_i$-modules

$$\phi_i:\bigoplus_{l=0}^{\infty}\bigoplus_{j\in J_l}V_l^{(j)}\xrightarrow{\sim}M(\Lambda),\quad \textit{where } V_l^{(j)} \textit{ is a copy of } V_l,$$

such that

$$L=\phi_i\left(\bigoplus_{l=0}^{\infty}\bigoplus_{j\in J_l}\bigoplus_{k=0}^{l}Av_{lk}^{(j)}\right),\quad \textit{where } v_{lk}^{(j)} \textit{ is a copy of } v_{lk},$$

and

$$B=\{\phi_i(v_{lk}^{(j)}) \bmod qL\,|\,0\leqq l<\infty,\ j\in J_l,\ 0\leqq k\leqq l\}.$$

The pair (L,B) is called the crystal base.

The set B is endowed with a structure of colored oriented graph [5]. Suppose that $b,b'\in B$, and set $\phi_i^{-1}(b)=v_{lk}^{(j)},\ \phi_i^{-1}(b')=v_{l'k'}^{(j')}$. We draw an arrow of color i from

b to b', if and only if $j = j'$ and $k + 1 = k'$. We write this as $b \xrightarrow{i} b'$. The graph B is called the crystal graph.

1.3. Main Results. We study the crystal base and the crystal graph of the $U_q(\widehat{\mathfrak{sl}}(n))$-module $M(\Lambda)$.

The first result is to make the crystal graph B out of the paths $\mathscr{P}(\Lambda)$. This gives an alternative proof of Theorem 1.1. We give a simple combinatorial criterion for two paths $\eta, \eta' \in \mathscr{P}(\Lambda)$ to be joined by an arrow of color $i: \eta \xrightarrow{i} \eta'$.

The second result is to show that the crystal graph of the finite tensor product $\underbrace{V \otimes \cdots \otimes V}_{N}$ approximates the crystal graph B.

Let $\mathscr{P}_N(\Lambda)$ denote the set of Λ-paths η such that $\eta_k = \eta_{\Lambda,k}$ for $k \geqq N$. We have a natural inclusion

$$\mathscr{P}_N(\Lambda) \subset (\mathscr{A}_l^+)^{N+1}.$$

The finite dimensional representations are excluded from the category of the integrable representations of $U_q(\widehat{\mathfrak{sl}}(n))$ in [7]. However, we dare to consider this case. The set $\mathscr{A}_l^+$ can be identified with the crystal graph of V. Then $(\mathscr{A}_l^+)^{N+1}$ is the crystal graph of $\underbrace{V \otimes \cdots \otimes V}_{N+1}$ (see [5]). Our assertion is that the graph structure of $\mathscr{P}_N(\Lambda)$ inherited from $(\mathscr{A}_l^+)^{N+1}$ and the one inherited from $\mathscr{P}(\Lambda)$ are the same but the direction of the arrows are all reversed. Namely, there is an arrow of color i from η to η' in $(\mathscr{A}_l^+)^{N+1}$ if and only if there is an arrow of color i from η' to η in $\mathscr{P}(\Lambda)$. We have no explanation of this inversion.

The third result is to prove certain combinatorial identities arising from the restricted solid-on-solid (RSOS) models.

Set

$$P_k^+ = \left\{ \Lambda = \sum_{j=0}^{n-1} m_j \Lambda_j \,\middle|\, m_j \in Z_{\geqq 0}, \sum_{j=0}^{n-1} m_j = k \right\}.$$

For the vertex model the fluctuation variables are located on the bonds of the lattice $\mathscr{L}$. Consider the dual lattice $\mathscr{L}^*$. The RSOS model [8] is given on $\mathscr{L}^*$. Fix two positive integers l and l'. The fluctuation variables now live on the vertices of $\mathscr{L}^*$ and take values in $P_{l+l'}^+$. The Boltzmann weights are attached to configurations round a face. (Fig. 1.2.)

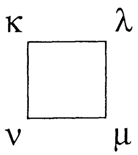


Fig. 1.2. Elementary configuration of RSOS models

We impose two restrictions on $\kappa, \lambda, \mu, \nu \in P_{l+l'}^+$. The first condition is that

$$\lambda - \kappa, \mu - \lambda, \nu - \kappa, \mu - \nu \in \{\bar{\xi} \in P_0 \,|\, \xi \in \mathscr{A}_l^+\}.$$

Note that $\xi \in \mathscr{A}_l^+$ is uniquely written as

$$\xi = \sigma(\nu) - \nu, \quad \nu \in P_l^+.$$

Here σ is a $\mathbf{Z}$-linear map such that $\sigma(\Lambda_i) = \Lambda_{i+1}$. Suppose that $\mu, \mu' \in P^+_{l+l'}$ and $\mu' - \mu = \sigma(\nu) - \nu \in \mathscr{A}^+_l$ ($\nu \in P^+_l$). The pair (μ, μ') is called *admissible* if and only if $\mu - \nu = \mu' - \sigma(\nu)$ belongs to $P^+_{l'}$. The second condition is that the pairs $(\kappa, \lambda), (\lambda, \mu)$, $(\kappa, \nu), (\nu, \mu)$ are admissible. We omit the expression of the Boltzmann weights. See [8]. The simplest case $n = 2, l = l' = 1$ is equivalent to the Ising model.

For this model the definition of the paths is slightly modified. (We call them restricted paths.) A path is a pair of sequences (μ, η) such that $\mu = (\mu_k)_{k \geqq 0}, \mu_k \in P^+_{l+l'}$, and $\eta = (\eta_k)_{k \geqq 0}, \eta_k \in \mathscr{A}^+_l$ with the restriction

$$\mu_{k+1} - \mu_k = \bar{\eta}_k \quad \text{for} \quad k \geqq 0.$$

Let Λ and Λ' be dominant integral weights of level l and l', respectively. A (Λ', Λ)-path (μ, η) is such that η is a Λ-path, μ is admissible, i.e., (μ_k, μ_{k+1}) is admissible for any $k \geqq 0$, and for $k \gg 1$,

$$\mu_k - \nu_k = \mu_{k+1} - \sigma(\nu_k) = \Lambda',$$

where

$$\mu_{k+1} - \mu_k = \sigma(\nu_k) - \nu_k, \quad \nu_k \in P^+_l.$$

Suppose that (μ, η) and (μ', η') are (Λ', Λ)-paths. If $\eta = \eta'$ then $\mu = \mu'$. Therefore we can say η is a (Λ', Λ)-path (if ever μ exists).

Consider the $U_q(\widehat{\mathfrak{sl}}(n))$-modules $M(\Lambda)$ and $M(\Lambda')$. Let $B = \mathscr{P}(\Lambda)$ and $B' = \mathscr{P}(\Lambda')$ be the crystal graphs. The crystal graph of $M(\Lambda') \otimes M(\Lambda)$ is $B' \times B$. Our assertion is that $(b', b) \in B' \times B$ is highest, i.e., there is no arrow in $B' \times B$ pointing to (b', b), if and only if b' is highest and b is a (Λ', Λ)-path. To put it in a different way, we obtain a combinatorial way of labelling the highest weight vectors in the tensor product $M(\Lambda') \otimes M(\Lambda)$.

The plan of this paper is as follows. In Sect. 2 we review the basic facts about the crystal base. In Sect. 3 we make the crystal base for $M(\Lambda)$ in terms of $\mathscr{P}(\Lambda)$. In Sect. 4 the finite size approximation to the crystal base is discussed. In Sect. 5 the restricted paths are identified with the highest weight vectors in the crystal base of $M(\Lambda') \otimes M(\Lambda)$.

2. Crystal Base

The purpose of this section is to give a brief review of the crystal base following [5, 7].

2.1. $U_q(\widehat{\mathfrak{sl}}(n))$. Let us first fix notations concerning the affine Lie algebras [9]. We shall consider the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$ over the field $\mathbf{Q}$. Let $C = (c_{ij})_{i,j=0}^{n-1}$ denote the associated generalized Cartan matrix: $c_{ij} = 2\delta^{(n)}_{ij} - \delta^{(n)}_{ij-1} - \delta^{(n)}_{ij+1}$, where $\delta^{(n)}_{ij} = 1$ if $i \equiv j \bmod n$, $\delta^{(n)}_{ij} = 0$ otherwise. Let $\mathfrak{h}$ be a $\mathbf{Q}$-vector space and $\mathfrak{h}^*$ its dual, with distinguished bases such that

$$\mathfrak{h} = \left(\bigoplus_{i=0}^{n-1} \mathbf{Q} h_i \right) \oplus \mathbf{Q} D, \quad \mathfrak{h}^* = \left(\bigoplus_{i=0}^{n-1} \mathbf{Q} \Lambda_i \right) \oplus \mathbf{Q} \delta,$$

$$\langle \Lambda_i, h_j \rangle = \delta_{ij}, \quad \langle \Lambda_i, D \rangle = 0, \quad \langle \delta, h_i \rangle = 0, \quad \langle \delta, D \rangle = 1.$$

Let further $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta^{(n)}_{i0} \delta$. Here and in what follows we extend the suffixes of Λ_i to $i \in \mathbf{Z}$ by $\Lambda_i = \Lambda_{i'}$ for $i \equiv i' \bmod n$. We define the weight lattice $P=\left(\bigoplus_{i=0}^{n-1}\mathbf{Z}\Lambda_i\right)\oplus\mathbf{Z}\delta$, its dual $P^\vee=\left(\bigoplus_{i=0}^{n-1}\mathbf{Z}h_i\right)\oplus\mathbf{Z}D$ and the root lattice $Q=\bigoplus_{i=0}^{n-1}\mathbf{Z}\alpha_i$.

Throughout this paper we set

$$K=\mathbf{Q}(q),\quad A=\{f\in K\,|\,f \text{ has no pole at } q=0\}.$$

The algebra $U_q=U_q(\widehat{\mathfrak{sl}}(n))$ is an associative algebra over K with 1, generated by the symbols $\{e_i,f_i|0\leqq i\leqq n-1\}$ and q^h $(h\in P^\vee)$. The defining relations [10, 11] are as follows (we set $t_i=q^{h_i}$):

(i) $$q^hq^{h'}=q^{h+h'}\quad(h,h'\in P^\vee),\quad q^0=1,$$

(ii) $$q^he_jq^{-h}=q^{\langle\alpha_j,h\rangle}e_j,\quad q^hf_jq^{-h}=q^{-\langle\alpha_j,h\rangle}f_j,$$

(iii) $$[e_i,f_j]=\delta_{ij}\frac{t_i-t_i^{-1}}{q-q^{-1}},$$

(iv) $$\sum_{k=0}^{1-c_{ij}}(-)^k\begin{bmatrix}1-c_{ij}\\k\end{bmatrix}e_i^{1-c_{ij}-k}e_je_i^k=0\quad(i\neq j),$$

$$\sum_{k=0}^{1-c_{ij}}(-)^k\begin{bmatrix}1-c_{ij}\\k\end{bmatrix}f_i^{1-c_{ij}-k}f_jf_i^k=0\quad(i\neq j).$$

Here

$$\begin{bmatrix}m\\k\end{bmatrix}=\frac{[m]!}{[k]![m-k]!},\quad [m]!=\prod_{1\leqq l\leqq m}[l],\quad [m]=\frac{q^m-q^{-m}}{q-q^{-1}}.$$

Note that the algebra U_q has a Hopf algebra structure with comultiplication $\Delta\colon U_q\to U_q\otimes U_q$ given by:

$$\Delta(q^h)=q^h\otimes q^h\quad h\in P^\vee,$$
$$\Delta(e_i)=e_i\otimes 1+t_i\otimes e_i,$$
$$\Delta(f_i)=f_i\otimes t_i^{-1}+1\otimes f_i.$$

The tensor product of U_q-modules becomes a U_q-module via Δ.

2.2. Crystal Base. Let M be a U_q-module. The weight space M_λ $(\lambda\in P)$ is defined by

$$M_\lambda=\{u\in M\,|\,q^hu=q^{\langle\lambda,h\rangle}u \text{ for all } h\in P^\vee\}.$$

For each i, let $U_{qi}=U_q(\mathfrak{sl}(2))_i$ denote the subalgebra of U_q generated by e_i,f_i,t_i and t_i^{-1}. A U_q-module M is called *integrable* if

(i) $M=\bigoplus_{\lambda\in P}M_\lambda$,
(ii) $\dim M_\lambda<\infty$ for each $\lambda\in P$,
(iii) for each i, M is a union of finite-dimensional representations over U_{qi}.

In [5] Kashiwara defines the following operators on M: for $0\leqq i\leqq n-1$,

$$\tilde{e}_i=(qt_i\Delta_i)^{-1/2}e_i,\quad \tilde{f}_i=t_i(qt_i\Delta_i)^{-1/2}f_i,$$

where

$$\Delta_i=q^{-1}t_i+qt_i^{-1}+(q-q^{-1})^2e_if_i-2.$$

Definition 2.1. [5] A pair (L, B) is called a crystal base of M if it satisfies the following conditions:

(i) L is a free A-module such that $K \otimes_A L \cong M$,
(ii) B is a base of the $\mathbf{Q}$-vector space L/qL,
(iii) $L = \bigoplus_{\lambda \in P} L_\lambda$ and $B = \bigcup_{\lambda \in P} B_\lambda$, where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,
(iv) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all i,
(v) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all i,
(vi) for any i and $u, v \in B$, $u = \tilde{e}_i v$ if and only if $v = \tilde{f}_i u$.

As noted in [5] B has a structure of colored oriented graph (the colors are labelled by $i (0 \leqq i \leqq n-1)$): For $u, v \in B$, we draw an arrow of color i $u \xrightarrow{i} v$ if and only if $v = \tilde{f}_i u$. The set B endowed with this structure is called the crystal graph of M.

Let Λ be a dominant integral weight. Let $M(\Lambda)$ denote the irreducible highest weight U_q-module with highest weight Λ and highest weight vector u_Λ. Set

$$L(\Lambda) = \sum A \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\Lambda \subset M(\Lambda)$$

and

$$B(\Lambda) = \{ v = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\Lambda \bmod qL \mid v \neq 0 \} \subset L/qL.$$

The following states the existence and uniqueness of a crystal base.

Theorem 2.2. [7]
(i) *Let $M(\Lambda)$ be as above. Then the pair $(L(\Lambda), B(\Lambda))$ is a crystal base for $M(\Lambda)$.*
(ii) *Let M be an integrable module isomorphic to $\bigoplus_j M(\lambda_j)$, and let (L, B) be its crystal base. Then there is an isomorphism $M \xrightarrow{\sim} \bigoplus_j M(\lambda_j)$ which sends (L, B) to $\bigoplus_j (L(\lambda_j), B(\lambda_j))$.*

The crystal base of tensor product modules is given by

Theorem 2.3. [5] *Let (L_j, B_j) be crystal bases of M_j $(j = 1, 2)$. Then $(L_1 \otimes L_2, B_1 \times B_2)$ is a crystal base of $M_1 \otimes M_2$. Here $B_1 \times B_2 \hookrightarrow (L_1 \otimes L_2)/q(L_1 \otimes L_2) \cong (L_1/qL_1) \otimes (L_2/qL_2)$ is given by $(u, v) \mapsto u \otimes v$.*

The graph structure of $B_1 \times B_2$ is described as follows [5]. For a crystal base (L, B) and $b \in B$, we define $l_i^{(\pm)}(b) \in \mathbf{Z}_{\geqq 0}$ to be

$$l_i^{(\pm)}(b) = \text{the length of the } i \text{ string above/below } b \text{ in the graph } B. \tag{2.1}$$

This means that there exists a sequence $b^{(j)} \in B (-l_i^{(+)}(b) \leqq j \leqq l_i^{(-)}(b))$ satisfying $b^{(0)} = b$, $b^{(j)} \xrightarrow{i} b^{(j+1)}$, such that if $b' \in B$ then neither $b' \xrightarrow{i} b^{(l_-)}$ $(l_- = -l_i^{(+)}(b))$ nor $b^{(l_+)} \xrightarrow{i} b'$ $(l_+ = l_i^{(-)}(b))$ is valid. If $b \in B_\lambda$ then

$$l_i^{(-)}(b) - l_i^{(+)}(b) = \langle \lambda, h_i \rangle. \tag{2.2}$$

Using these notations we have, for $u \in B_1$ and $v \in B_2$,

$$\begin{aligned} \tilde{f}_i(u \otimes v) &= \tilde{f}_i u \otimes v \quad \text{if} \quad l_i^{(-)}(u) > l_i^{(+)}(v), \\ &= u \otimes \tilde{f}_i v \quad \text{otherwise}, \end{aligned} \tag{2.3a}$$

$$\begin{aligned} \tilde{e}_i(u \otimes v) &= \tilde{e}_i u \otimes v \quad \text{if} \quad l_i^{(-)}(u) \geqq l_i^{(+)}(v), \\ &= u \otimes \tilde{e}_i v \quad \text{otherwise}. \end{aligned} \tag{2.3b}$$

3. Crystal Graphs for Integrable Representations

3.1 Fock Representation of $U_q(\widehat{\mathfrak{sl}}(n))$. The aim of this section is to determine the crystal graph for highest weight modules $M(\Lambda)$ with dominant integral highest weight Λ. We begin with some combinatorial objects which will play a role in the description of the graph.

Definition 3.1. An *extended Young diagram* Y is a sequence $(y_k)_{k\geqq 0}$ such that
(i) $y_k \in \mathbf{Z}$, $y_k \leqq y_{k+1}$ for all k,
(ii) there exists fixed $y_\infty \in \mathbf{Z}$ such that $y_k = y_\infty$ for $k \gg 0$.

The integer y_∞ is called the *charge* of Y:
For example, pictorially,

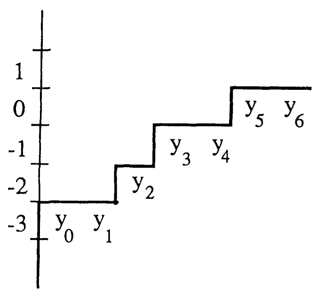


Fig. 3.1. Extended Young diagram

where $Y=(-2,-2,-1,0,0,1,1,1,\ldots)$ is an extended Young diagram of charge $y_\infty = 1$. Thus an extended Young diagram $Y=(y_k)_{k\geqq 0}$ is an infinite Young diagram (see [4]) drawn on the lattice in the right half plane with sites $\{(i,j)\in \mathbf{Z}\times\mathbf{Z} \mid i \geqq 0\}$, where y_k denotes the "depth" of the k-th column. Note that if $y_k \neq y_{k+1}$ for some k, then we will have corners in the extended Young diagram. For instance, in the above example $y_1 \neq y_2$. So we have a *convex corner* at site $(2,-2)$ and a *concave corner* at $(2,-1)$. If a corner is located at site (i,j), it is called a *d diagonal corner* where $d=i+j$.

Definition 3.2. We define a *pattern* to be a map

$$t: \mathbf{Z}\times\mathbf{Z}_{\geqq 0} \to \mathbf{Z}$$
$$(j,k) \mapsto t_{jk}$$

such that
(i) for all j, $(t_{jk})_{k\geqq 0}$ is an extended Young diagram,
(ii) $t_{jk} \leqq t_{j+1\,k}$ for all j and k,
(iii) $t_{j+l\,k} = t_{jk} + n$ for all j and k.

We say the pattern t is *normalized* if $0 \leqq \gamma_1 \leqq \cdots \leqq \gamma_l < n$, where $\gamma_j = t_{j\infty}$ is the charge of $(t_{jk})_{k\geqq 0}$. We call $\gamma = (\gamma_1, \ldots, \gamma_l)$ the charge of t. We identify the pattern t with a sequence $Y=(Y_j)_{j\in\mathbf{Z}}$ of extended Young diagrams $Y_j = (t_{jk})_{k\geqq 0}$. If $t_{jk} < t_{jk+1}$ then Y_j has a convex corner and concave corner at the k-th column. (Fig. 3.2.) Let $\mathscr{T}$ denote the set of all patterns.

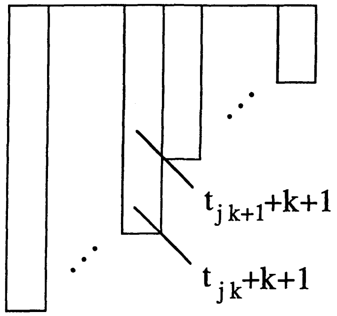


Fig. 3.2. Convex corner on the $t_{jk}+k+1$ diagonal and concave corner on the $t_{jk+1}+k+1$ diagonal

In the sequel we fix a positive integer l and set

$$P_l=\left\{\Lambda=\sum_{j=0}^{n-1} m_j\Lambda_j \mid m_j\in\mathbf{Z}, \sum_{j=0}^{n-1} m_j=l\right\},$$

$$P_l^+=\left\{\Lambda=\sum_{j=0}^{n-1} m_j\Lambda_j\in P_l \mid m_j\geqq 0\right\}.$$

Let $\epsilon_j=(0,\ldots,\overset{j\text{-th}}{1},\ldots,0)$ $(0\leqq j<n)$ denote the standard base vectors of $\mathbf{Z}^n$. We extend the suffixes of ϵ_i to $i\in\mathbf{Z}$ by $\epsilon_{i'}=\epsilon_i$ for $i'\equiv i \bmod n$. We set

$$\mathscr{A}=\bigoplus_{j=0}^{n-1}\mathbf{Z}\epsilon_j,\quad \mathscr{A}_l^+=\left\{\nu=\sum_{i=0}^{n-1}\nu_i\epsilon_i \mid \nu_i\in\mathbf{Z}_{\geqq 0}, \sum_{i=0}^{n-1}\nu_i=l\right\}.$$

Definition 3.3. A *path* is a pair (μ,η) such that
(i) $\mu=(\mu_k)_{k\geqq 0}$, $\mu_k\in P_l$,
(ii) $\eta=(\eta_k)_{k\geqq 0}$, $\eta_k\in\mathscr{A}_l^+$,
(iii) $\mu_{k+1}-\mu_k=\bar{\eta}_k$ for all k,
where $-:\mathscr{A}\to P_0$ is the $\mathbf{Z}$-linear map given by $\bar{\epsilon}_j=\Lambda_{j+1}-\Lambda_j$.

Let $\Lambda\in P_l^+$, A Λ-*path* (μ,η) is a path such that $\mu_k=\sigma^k(\Lambda)$ for $k\gg 0$, where $\sigma(\Lambda_j)=\Lambda_{j+1}$ for all j. Note that in this case μ is uniquely determined from Λ and η. Hence we will call η a Λ-path. Let $\mathscr{P}(\Lambda)$ denote the set of all Λ-paths.

We have a map

$$\begin{aligned}\pi:\quad \mathscr{T} &\to \bigcup_{\Lambda\in P_l^+}\mathscr{P}(\Lambda)\\ t=(t_{jk}) &\mapsto \eta=(\eta_k),\end{aligned}$$

where $\eta_k=\epsilon_{t_{1k}+k}+\cdots+\epsilon_{t_{lk}+k}$. For a Λ-path $\eta\in\mathscr{P}(\Lambda)$, we say t is a *lift* of η if $t\in\pi^{-1}(\eta)$. The following proposition is analogous to Proposition 5.2 in [4].

Proposition 3.4. *For any $\eta\in\mathscr{P}(\Lambda)$, there exists a unique normalized lift $t=(t_{jk})$ such that $t_{jk}\geqq t'_{jk}$ for all j,k for any $t'=(t'_{jk})\in\pi^{-1}(\eta)$. This t is called the highest lift of η. Furthermore, a normalized pattern t is a highest lift if and only if for each $k\geqq 0$ there exists some j such that $t_{j+1\,k}>t_{jk+1}$.*

For fixed $\Lambda=\Lambda_{\gamma_1}+\cdots+\Lambda_{\gamma_l}$ with $0\leqq\gamma_1\leqq\cdots\leqq\gamma_l<n$, define

$$\mathscr{Y}(\Lambda)=\{\mathbf{Y}=(Y_j)_{j\in\mathbf{Z}}\in\mathscr{T}\mid\mathbf{Y}\text{ has charge }\gamma=(\gamma_1,\ldots,\gamma_l)\}.$$

Note that $\mathbf{Y}\in\mathscr{Y}(\Lambda)$ is completely determined by $(Y_1,\ldots,Y_l)$ using periodicity with respect to j with period l. So we will identify $\mathbf{Y}$ with $(Y_1,\ldots,Y_l)$. The Fock space

$$\mathscr{F}(\Lambda)=\bigoplus_{\mathbf{Y}\in\mathscr{Y}(\Lambda)}K\mathbf{Y}$$

is the vector space over the field $K=\mathbf{Q}(q)$ having all $\mathbf{Y}\in\mathscr{Y}(\Lambda)$ as base vectors.

For $d\in\mathbf{Z}$, $j=1,2,\ldots,l$ define symbols $e_{dj}^{\infty}, f_{dj}^{\infty}, t_{dj}^{\infty}$ and let these act on $\mathscr{F}(\Lambda)$ as follows. Let $\mathbf{Y}=(Y_1,\ldots,Y_l)\in\mathscr{Y}(\Lambda)$. If Y_j has a d diagonal convex corner, then

$$e_{dj}^{\infty}\mathbf{Y}=(Y_1,\ldots,Y_j',\ldots,Y_l), \tag{3.1}$$

wherein Y_j' is the same as Y_j except the convex corner is replaced by a concave corner,

$$e_{dj}^{\infty}\mathbf{Y}=0\quad\text{otherwise.}$$

If Y_j has a d diagonal concave corner, then

$$f_{dj}^{\infty}\mathbf{Y}=(Y_1,\ldots,Y_j'',\ldots,Y_l), \tag{3.2}$$

wherein Y_j'' is the same as Y_j except the concave corner is replaced by a convex corner.

$$\begin{aligned} f_{dj}^{\infty}\mathbf{Y}&=0 && \text{otherwise,}\\ t_{dj}^{\infty}\mathbf{Y}&=q\mathbf{Y}, && \text{if } Y_j \text{ has a } d \text{ diagonal concave corner,}\\ &=q^{-1}\mathbf{Y}, && \text{if } Y_j \text{ has a } d \text{ diagonal convex corner,}\\ &=\mathbf{Y}, && \text{otherwise.}\end{aligned} \tag{3.3}$$

Define also the operator s_{jk} by

$$s_{jk}\mathbf{Y}=q^a\mathbf{Y},\quad a=\#\{p\in\mathbf{Z}\mid t_{jk}<p\leqq\gamma_j,\ p+k\equiv\operatorname{mod}(n)\}.$$

For (d,j), $(d',j')\in\mathbf{Z}\times\{1,2,\ldots,l\}$ we say

$$(d,j)<(d',j')\quad\text{if and only if}\quad d<d',\quad\text{or}\quad d=d'\quad\text{and}\quad j>j'.$$

The following proposition can be proved by an argument similar to Theorem 6.1 in [12].

Proposition 3.5. *The algebra $U_q(\widehat{\mathfrak{sl}}(n))$ acts on $\mathscr{F}(\Lambda)$ by the following equations:*

$$e_i=\sum_{\substack{d\equiv i\operatorname{mod}(n)\\ 1\leqq j\leqq l}}\Bigg(\prod_{\substack{(d',j')>(d,j)\\ d'\equiv i\operatorname{mod}(n),1\leqq j'\leqq l}}t_{d'j'}^{\infty}\Bigg)e_{dj}^{\infty} \tag{3.4}$$

$$f_i=\sum_{\substack{d\equiv i\operatorname{mod}(n)\\ 1\leqq j\leqq l}}f_{dj}^{\infty}\Bigg(\prod_{\substack{(d',j')<(d,j)\\ d'\equiv i\operatorname{mod}(n),1\leqq j'\leqq l}}(t_{d'j'}^{\infty})^{-1}\Bigg) \tag{3.5}$$

and

$$t_i=\prod_{\substack{d\equiv i\operatorname{mod}(n)\\ 1\leqq j\leqq l}}t_{dj}^{\infty},\quad q^D=\prod_{\substack{1\leqq j\leqq l\\ k\geqq 0}}s_{jk}. \tag{3.6}$$

Under the above action $\mathscr{F}(\Lambda)$ is an integrable $U_q(\widehat{\mathfrak{sl}}(n))$-module.

Set $\Phi=(\phi_1,\ldots,\phi_l)\in\mathscr{Y}(\Lambda)$, where ϕ_j $(1\leqq j\leqq l)$ denotes the empty extended Young diagram of charge γ_j (i.e., $t_{jk}=\gamma_j$ for all $k\geqq 0$). Observe that $\Phi\in\mathscr{F}(\Lambda)$ is a highest weight vector with highest weight Λ. The space $M(\Lambda)=U_q(\widehat{\mathfrak{sl}}(n))\Phi$ is the irreducible integrable highest weight $U_q(\mathfrak{sl}(n))$-module with highest weight Λ.

3.2. $U_q(\mathfrak{sl}(2))$ *Decomposition of the Fock Space.* Let Y be any extended Young diagram. We color the corners in Y as follows. If d is the diagonal number of any corner and $d\equiv i \bmod n$, then we say it is a corner of color i. A convex (respectively concave) corner of color i is called i-convex (respectively i-concave) corner.

Fix some color i. For an extended Young diagram Y we denote by $\bar{Y}$ the Young diagram obtained from Y by removing all the i-convex corners. Let η be a Λ path and t be the highest lift of η. We construct a sequence $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_m)$ in such a way that the following hold:

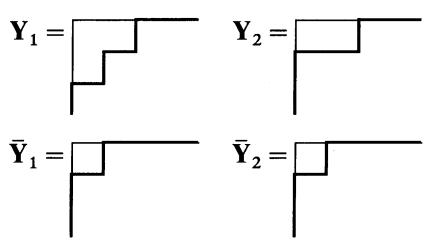
(i) $\sum_{j=1}^{l}\#\{i\text{-concave corner of }\bar{Y}_j\}=m$.
(ii) Each ε_r is either 0 or 1.
(iii) We can define $j(r)(1\leqq j(r)\leqq l)$ and $d(r)$ in such a way that $\bar{Y}_{j(r)}$ has a $d(r)$ diagonal i-concave corner.
(iv) If $\varepsilon_r=0$ (respectively $\varepsilon_r=1$) then $Y_{j(r)}$ has a $d(r)$ diagonal i-concave (respectively i-convex) corner.
(v) If $r_1<r_2$ then $(d(r_1),j(r_1))>(d(r_2),j(r_2))$.

With these conditions ε is uniquely determined from η and i. Fixing i we call ε the *signature* of η (or $\mathbf{Y}$). Set $\bar{\mathbf{Y}}=(\bar{Y}_1,\bar{Y}_2,\ldots,\bar{Y}_l)$. Note that $\mathbf{Y}$ is uniquely determined by $\bar{\mathbf{Y}}$ and ε. So we write $\mathbf{Y}=(\bar{\mathbf{Y}},\varepsilon)$.

Example. Let $n=2$, $l=2$, $i=1$ and

$$\eta=(\epsilon_0+\epsilon_1,2\epsilon_0,2\epsilon_0,2\epsilon_1,\ldots)\in\mathscr{P}(2\Lambda_0).$$

Then $\mathbf{Y}=(Y_1,Y_2)$, $\bar{\mathbf{Y}}=(\bar{Y}_1,\bar{Y}_2)$ where



Note that as an ordered set

$$\{(d,j)\mid 1\leqq j\leqq 2,\ d\equiv 1 \bmod 2,\ \bar{Y}_j \text{ has a } d \text{ diagonal 1-concave corner}\}$$
$$=\{(1,1),(1,2),(-1,1),(-1,2)\}.$$

Hence $\varepsilon=(1,1,1,0)$ and $\mathbf{Y}=(\bar{\mathbf{Y}},(1,1,1,0))$.

For fixed $\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m)$ we partition the set

$$\{1,2,\ldots,m\}=J\sqcup K_1\sqcup\cdots\sqcup K_t$$

into disjoint subsets by the following inductive procedure (see [6]):
(i) If there is no j such that $(\varepsilon_j,\varepsilon_{j+1})=(0,1)$, define $J=\{1,2,\ldots,m\}$.
(ii) If there is some j such that $(\varepsilon_j,\varepsilon_{j+1})=(0,1)$ define $K_1=\{j,j+1\}$.

(iii) Apply (i) and (ii) to $\{1,2,\dots,m\}\setminus K_1$ to choose J or K_2 and repeat this as necessary to choose J and $K_1, K_2, \dots, K_t$.
Let $\varepsilon_J = (\varepsilon_{i_1}, \dots, \varepsilon_{i_r})$, where $J = \{i_1, \dots, i_r\}$ and $i_1 < \cdots < i_r$. We call 0 or 1 in the signature ε *relevant* if and only if it is in ε_J.

For i, $\mathbf{Y}$, $\bar{\mathbf{Y}}$, ε, J, $K_1, \dots, K_t$ as above, define an element of $\mathscr{F}(\Lambda)$ by (see [6])

$$[\mathbf{Y}]_i = \sum_{\substack{J = J_0 \sqcup J_1 \\ |J_1| = n_1}} \sum_{S \subset \{1,2,\dots,t\}} q^{\#(J_0,J_1)}(-q)^{|S|}(\bar{\mathbf{Y}}, \varepsilon(J_0, J_1, S)), \tag{3.7}$$

where

$$n_1 = \#\{j \in J \mid \varepsilon_j = 1\},$$
$$\#(J_0, J_1) = \#\{(j,j') \mid j < j', j \in J_0, j' \in J_1\},$$

and $\varepsilon(J_0, J_1, S) = (\tau_1, \tau_2, \dots, \tau_r)$ is determined by
(i) $\tau_j = 0$ if $j \in J_0$
(ii) $\tau_j = 1$ if $j \in J_1$
(iii) $(\tau_j, \tau_{j'}) = (1, 0)$ if $j < j'$ and $\{j,j'\} = K_s$, $s \in S$
(iv) $(\tau_j, \tau_{j'}) = (0, 1)$ if $j < j'$ and $\{j,j'\} = K_s$, $s \notin S$.

The following theorems are analogous to Theorems 3.1 and 3.2 in [6] and follow similarly.

Theorem 3.6. *Let Y, $\bar{Y}$, ε, J, $K_1, \dots, K_t$ be as above. For each $k = 0, 1, \dots, r$ there is a unique vector $\mathbf{Y}_k = (\bar{\mathbf{Y}}, \tau) \in \mathscr{Y}(\Lambda)$ such that the partition $\{1,2,\dots,m\} = J \sqcup K_1 \sqcup \cdots \sqcup K_t$ is the same with Y and $\#\{j \in J \mid \tau_j = 1\} = k$. Furthermore, $V_r = \bigoplus_{k=0}^{r} K[\mathbf{Y}_k]_i$ is an $(r+1)$-dimensional irreducible integrable U_{qi}-module with highest weight vector $[\mathbf{Y}_0]_i$. Set $L_i = \bigoplus_{k=0}^{r} A[\mathbf{Y}_k]_i$ and $B_i = \{[\mathbf{Y}_k]_i \mid 0 \leqq k \leqq r\}$. Then (L_i, B_i) is the crystal base for the U_{qi}-module V_r.*

Theorem 3.7. *Let $L(\mathscr{F}(\Lambda)) = \bigoplus_{\mathbf{Y} \in \mathscr{Y}(\Lambda)} A\mathbf{Y}$ and $B(\mathscr{F}(\Lambda)) = \mathscr{Y}(\Lambda)$. Then the pair $(L(\mathscr{F}(\Lambda)), B(\mathscr{F}(\Lambda)))$ is the crystal base for the integrable $U_q(\widehat{\mathfrak{sl}}(n))$-module $\mathscr{F}(\Lambda)$.*

The next theorem is an immediate consequence of Theorems 3.6, 3.7 and the definition of crystal graph (see [5, 7]).

Theorem 3.8. *Let $\mathbf{Y}, \mathbf{Y}' \in B(\mathscr{F}(\Lambda))$. In the crystal graph $B(\mathscr{F}(\Lambda))$, $\mathbf{Y} \xrightarrow{i} \mathbf{Y}'$ if and only if the following hold*
(i) $\mathbf{Y} = (\bar{\mathbf{Y}}, (\varepsilon_1, \dots, \varepsilon_m))$, $\mathbf{Y}' = (\bar{\mathbf{Y}}, (\varepsilon'_1, \dots, \varepsilon'_m))$.
(ii) *The partition $\{1,2,\dots,m\} = J \sqcup K_1 \sqcup \cdots \sqcup K_t$ is the same for both $\mathbf{Y}$ and $\mathbf{Y}'$.*
(iii) *There exists $k \in J$ such that $\varepsilon_k = 0$, $\varepsilon'_k = 1$, $\varepsilon_j = \varepsilon'_j = 1$ if $j \in J$ and $j < k$, $\varepsilon_j = \varepsilon'_j = 0$ if $j \in J$ and $j > k$.*

Suppose that η, η' are Λ-paths and $\mathbf{Y}, \mathbf{Y}' \in \mathscr{Y}(\Lambda)$ are their highest lifts. Then for any $i = 0, 1, \dots, n-1$, we write $\eta \xrightarrow{i} \eta'$ if and only if $\mathbf{Y} \xrightarrow{i} \mathbf{Y}'$ in $B(\mathscr{F}(\Lambda))$.

3.3. Crystal Base for $M(\Lambda)$. Recall that $M(\Lambda) = U_q(\widehat{\mathfrak{sl}}(n))\Phi$, where $\Phi = (\phi_1, \dots, \phi_l)$, and $\phi_j (1 \leqq j \leqq l)$ is the empty extended Young diagram of charge γ_j. Let $B(\mathscr{F}(\Lambda))_\Phi$ denote the Φ-connected component in the crystal graph $B(\mathscr{F}(\Lambda))$. By Theorem 2.2, $B(\mathscr{F}(\Lambda))_\Phi$ is the crystal graph of $M(\Lambda)$. Let $\mathscr{H}(\Lambda)$ denote the set of $\mathbf{Y} \in B(\mathscr{F}(\Lambda)) = \mathscr{Y}(\Lambda)$

such that $\mathbf{Y}$ is the highest lift of some $\eta \in \mathscr{P}(\Lambda)$. Suppose that $t = (t_{jk})_{k \geqq 0}$ is the pattern of $\mathbf{Y} = (Y_1, \ldots, Y_l) \in B(\mathscr{F}(\Lambda))$. Recall that by definition Y_{l+j} denotes the extended Young diagram of charge $(\gamma_j + n)$ which is obtained by giving an upward vertical shift of n units to Y_j. It follows from definition and Proposition 3.4, that $\mathbf{Y} = (Y_1, \ldots, Y_l) \in \mathscr{H}(\Lambda)$ if and only if the following conditions hold:

$$Y_1 \supset Y_2 \supset \cdots \supset Y_l. \tag{3.8}$$

$$Y_l \supset Y_{l+1}. \tag{3.9}$$

$$\text{For each } k \geqq 0 \text{ there exists some } j \text{ such that } t_{j+1,k} > t_{j,k+1}. \tag{3.10}$$

The following lemma is an immediate consequence of Theorem 3.8.

Lemma 3.9. *Let $\mathbf{Y}, \mathbf{Y}' \in B(\mathscr{F}(\Lambda))$ and $\mathbf{Y} \xrightarrow{i} \mathbf{Y}'$ in the crystal graph $B(\mathscr{F}(\Lambda))$ for some i. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ (respectively $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_m)$) be the signature of $\mathbf{Y}$ (respectively $\mathbf{Y}'$) with respect to this color i. If $\varepsilon_a = 0$ and $\varepsilon'_a = 1$ for some $1 \leqq a \leqq m$, then $\varepsilon_{a-1} = \varepsilon'_{a-1} = 1$.*

Proposition 3.10. *Suppose $\mathbf{Y} \in \mathscr{H}(\Lambda)$, $\mathbf{Y}' \in B(\mathscr{F}(\Lambda))$ and $\mathbf{Y} \xrightarrow{i} \mathbf{Y}'$ for some color i. Then $\mathbf{Y}' \in \mathscr{H}(\Lambda)$.*

Proof. Suppose that $t = (t_{jk})$ and $t' = (t'_{jk})$ are the patterns of $\mathbf{Y}$ and $\mathbf{Y}'$, respectively. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ and $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_m)$ denote the signatures of $\mathbf{Y}$ and $\mathbf{Y}'$ respectively with respect to color i. Suppose $\mathbf{Y}'$ does not satisfy condition (3.8). Then there exists $d \equiv i \bmod n$ and j $(1 \leqq j \leqq l)$ such that Y_{j-1}, Y'_{j-1} and Y_j have d diagonal concave corners and Y'_j has a d diagonal convex corner. Therefore, for some a $(1 < a \leqq m)$, we have $\varepsilon_{a-1} = 0$, $\varepsilon_a = 0$, but $\varepsilon'_{a-1} = 0$, $\varepsilon'_a = 1$ which is a contradiction by Lemma 3.9.

Now suppose $\mathbf{Y}'$ does not satisfy condition (3.9). Then there exists $d \equiv i \bmod n$ such that Y_l and Y'_l have d diagonal concave corners, Y_1 has a $d-n$ diagonal concave corner, but Y'_1 has a $d-n$ diagonal convex corner. Again, this implies that $\varepsilon_{a-1} = 0$, $\varepsilon_a = 0$, $\varepsilon'_{a-1} = 0$ and $\varepsilon_a = 1$ for some $1 < a \leqq m$, which is a contradiction by Lemma 3.9.

Finally, suppose that $\mathbf{Y}'$ satisfies conditions (3.8) and (3.9), but does not satisfy condition (3.10). Then there exists $k_0 \geqq 0$ such that $t'_{j+1\,k_0} \leqq t'_{j\,k_0+1}$ for all $j \in \mathbf{Z}$. Since $\mathbf{Y} \in \mathscr{H}(\Lambda)$, there exists j_0 $(1 \leqq j_0 \leqq l)$ such that $t_{j_0+1\,k_0} > t_{j_0\,k_0+1}$. This implies $t_{j_0+1\,k_0} = t'_{j_0+1\,k_0} + 1$, $t_{j_0 k_0} = t'_{j_0 k_0} = t'_{j_0+1\,k_0}$. Note also that Y_{j_0+1} has an i-concave corner and Y'_{j_0+1} has an i-convex corner. They have the same diagonal number, say, d. For $j = j_0$ there are two cases.

(i) $t_{jk_0} < t_{jk_0+1}$ and $Y_j (= Y'_j)$ has a d diagonal concave corner.

(ii) $t_{jk_0} = t_{jk_0+1}$ and $Y_j (= Y'_j)$ has no d diagonal corner, and $t_{jk_0} = t_{j-1\,k_0+1}$.

The case (i) implies that for some a $(1 < a \leqq m)$ we have $\varepsilon_{a-1} = \varepsilon_a = \varepsilon'_a = 0$ and $\varepsilon'_{a-1} = 1$, which is a contradiction by Lemma 3.9. In the case (ii), we argue similarly, replacing j by $j-1$. Because of the periodicity in j, we will come to the case (i) in finite steps. This is a contradiction. □

Proposition 3.11. *Suppose $\mathbf{Y} \in \mathscr{H}(\Lambda)$ and $\mathbf{Y} \neq \Phi$. Then there exists a color i $(0 \leqq i < n)$ such that the signature ε of $\mathbf{Y}$ with respect to i contains 1 which is relevant.*

Proof. Let $t = (t_{jk})$ be the pattern of $\mathbf{Y}$. For $1 \leqq j \leqq l$, set $m_j = \max\{k \mid t_{jk} < \gamma_j\}$. Set $m = \max_{1 \leqq j \leqq l} \{m_j\}$. By (3.10) there exists an integer j_0 $(1 \leqq j_0 \leqq l)$ such that $t_{j_0 m} >$

$t_{j_0-1\,m+1}$. Choose the minimal integer j_1 that satisfies $j_1 \geqq j_0$ and $t_{j_1 m} < t_{j_1 m+1} = t_{j_1 \infty}$. Then Y_{j_1} has a $t_{j_1 m} + m + 1$ diagonal convex corner. Let i be the color of this corner. We shall show that the signature ε of $\mathbf{Y}$ contains $\varepsilon_a = 1$, which is corresponding to this corner (or, if $j_1 > l$, corresponding to the shifted corner in Y_{j_1-l}), as a relevant element. If it is not so, there is a' such that $a' < a$ and $\varepsilon_{a'} = 0$. This means that there exists j $(1 \leqq j \leqq l)$ such that $t_{jm} < t_{jm+1}$, $t_{jm+1} + m + 1 \equiv i \bmod n$. This is contradictory to the fact $t_{j_0 m} > t_{j_0-1\,m+1}$. □

Proposition 3.12. $B(\mathscr{F}(\Lambda))_\Phi = \mathscr{H}(\Lambda)$.

Proof. By Proposition 3.10, $B(\mathscr{F}(\Lambda))_\Phi \subseteqq \mathscr{H}(\Lambda)$. If $\mathbf{Y} \in \mathscr{H}(\Lambda)$ and $\mathbf{Y} \neq \Phi$, then by Proposition 3.11 there exists $\mathbf{Y}_1$ such that $\mathbf{Y}_1 \xrightarrow{i} \mathbf{Y}$. Hence using induction we get $\mathbf{Y} \in B(\mathscr{F}(\Lambda))_\Phi$. So $B(\mathscr{F}(\Lambda))_\Phi = \mathscr{H}(\Lambda)$. □

To sum up we have shown

Theorem 3.13. *There is a one to one correspondence between the set of Λ-paths $\mathscr{P}(\Lambda)$, the set of their highest lifts $\mathscr{H}(\Lambda)$ and the crystal graph $B(\mathscr{F}(\Lambda))_\Phi \subset B(\mathscr{F}(\Lambda))$ of $M(\Lambda)$. Their graph structure is described in Theorem 3.8.*

4. Finite Size Approximation

4.1. Symmetric Tensors and the Crystal Graph for its Tensor Power. Let U'_q denote the K-subalgebra of U_q generated by e_i, f_i and $t_i^{\pm 1}$ $(0 \leqq i \leqq n-1)$. In this section we shall consider finite dimensional representations of U'_q and the crystal graph for their tensor powers.

Let $V = \bigoplus_{\nu \in \mathscr{A}_l^+} K v_\nu$ be the vector space spanned by basis elements v_ν $(\nu \in \mathscr{A}_l^+)$ over K. We define the action of U'_q on V as follows:

$$e_i v_\nu = [\nu_i] v_{\nu + \epsilon_{i-1} - \epsilon_i},$$
$$f_i v_\nu = [\nu_{i-1}] v_{\nu - \epsilon_{i-1} + \epsilon_i},$$
$$t_i v_\nu = q^{\nu_{i-1} - \nu_i} v_\nu.$$

Here in the right-hand side v_ν with $\nu \notin \mathscr{A}_l^+$ is to be understood as 0. With respect to the subalgebra $U_q(\mathfrak{sl}(n)) \subset U'_q$, V is a highest weight module with highest weight vector $v_{l\epsilon_0}$ and the highest weight $l\Lambda_1$. Note however that it is *not* a highest weight module over U'_q.

The notion of a crystal base as given in Definition 2.1 carries over to U'_q-modules by replacing P with $P' = \bigoplus_{i=0}^{n-1} \mathbf{Z}\Lambda_i$. Setting $L = \bigoplus_{\nu \in \mathscr{A}_l^+} A v_\nu$ and $B = \{v_\nu \bmod qL \mid \nu \in \mathscr{A}_l^+\}$ one finds easily that (L, B) is the crystal base of the U'_q-module V. We shall identify B with $\mathscr{A}_l^+$. The crystal graph structure of $\mathscr{A}_l^+$ is given as follows: A vertex of the graph is represented by an element $\nu \in \mathscr{A}_l^+$, and for $\nu, \nu' \in \mathscr{A}_l^+$ an arrow from ν to ν' of color i is drawn if and only if $\nu' = \nu - \epsilon_{i-1} + \epsilon_i$. If $\nu \in \mathscr{A}_l^+$, then $l_i^{(+)}(\nu) = \nu_i$ and $l_i^{(-)}(\nu) = \nu_{i-1}$.

Remark. For U'_q-modules, existence of a crystal base is not always guaranteed; a simple example is the two dimensional module (in the case $n=2$) defined by

$$\pi(e_0)=\begin{pmatrix}0 & 0\\ q & 0\end{pmatrix},\quad \pi(f_0)=\begin{pmatrix}0 & q^{-1}\\ 0 & 0\end{pmatrix},$$
$$\pi(e_1)=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad \pi(f_1)=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

However Theorem 2.3 for the tensor product modules is valid without change.

Let now N be a positive integer. The crystal base for the tensor representation $V^{\otimes N}$ is given by $(L^{\otimes N}, B^N=(\mathscr{A}_l^+)^N)$. The crystal graph structure is described inductively as follows.

Suppose that the graph structure on $(\mathscr{A}_l^+)^N$ is already given. Let $v, v'\in\mathscr{A}_l^+$ and $b', b'\in(\mathscr{A}_l^+)^N$. Then $(v,b)\xrightarrow{i}(v',b')$ in $(\mathscr{A}_l^+)^{N+1}$ if and only if one of the following is valid.

Case A:

$$l_i^{(-)}(v)>l_i^{(+)}(b),\ v\xrightarrow{i}v' \quad\text{and}\quad b=b'.$$

Case B:

$$l_i^{(-)}(v)\leqq l_i^{(+)}(b),\ v=v' \quad\text{and}\quad b\xrightarrow{i}b'.$$

In Case A we have

$$l_i^{(+)}((v,b))=l_i^{(+)}(v),$$
$$l_i^{(-)}((v,b))=l_i^{(-)}(v)-l_i^{(+)}(b)+l_i^{(-)}(b).$$

In Case B we have

$$l_i^{(+)}((v,b))=l_i^{(+)}(v)-l_i^{(-)}(v)+l_i^{(+)}(b).$$
$$l_i^{(-)}((v,b))=l_i^{(-)}(b).$$

Example. Let $n=2$. Then the graph $(\mathscr{A}_l^+)^N$ for the cases $(l,N)=(1,2),(1,3),(2,2)$ is given as in Fig. 4.1(a), (b), (c), respectively.

Set

$$\mathscr{P}_N(\Lambda)=\{\eta=(\eta_0,\eta_1,\dots)\in\mathscr{P}(\Lambda)\,|\,\bar{\eta}_k=\sigma^{k+1}(\Lambda)-\sigma^k(\Lambda)\text{ if }k\geqq N\}.$$

We identify $\mathscr{P}_N(\Lambda)$ with a subset of $(\mathscr{A}_l^+)^{N+1}$ by

$$\iota_N:\mathscr{P}_N(\Lambda)\to(\mathscr{A}_l^+)^{N+1},\quad \eta\mapsto\iota_N(\eta)=(\eta_0,\dots,\eta_N).$$

For $\eta,\eta'\in\mathscr{P}_N(\Lambda)$ we write

$$\eta\xrightarrow[N]{i}\eta'$$

if and only if $\iota_N(\eta)\xrightarrow{i}\iota_N(\eta')$ in $(\mathscr{A}_l^+)^{N+1}$.

Our goal in Sect. 4 is the following theorem.

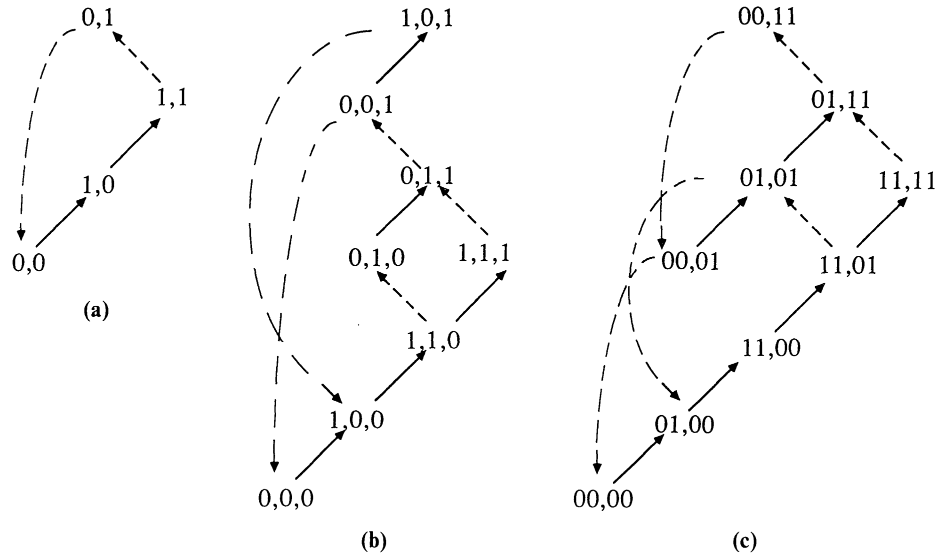


Fig. 4.1. Crystal graphs of $(\mathscr{A}_l^+)^N$ for $U_q(\widehat{\mathfrak{sl}}(2, \mathbf{C}))$. The color $i = 0, 1$ corresponds to the dashed and ordinary arrows, respectively. The symbols 0, 1 represent ϵ_0, ϵ_1, so that 0, 1 means (ϵ_0, ϵ_1), 00, 01 means $(2\epsilon_0, \epsilon_0 + \epsilon_1)$, and so on

Theorem 4.1. *Suppose that* $\eta, \eta' \in \mathscr{P}_N(\Lambda)$. *Then*

$$\eta \xrightarrow[N]{i} \eta' \quad \textit{if and only if} \quad \eta' \xrightarrow{i} \eta \quad \textit{in } \mathscr{P}(\Lambda).$$

The proof will be given in 4.3. Here we note only the following fact.

Lemma 4.2. *Suppose that* $b^{(1)}, b^{(2)}, b^{(3)} \in (\mathscr{A}_l^+)^{N+1}$ *satisfy*

$$b^{(1)} \xrightarrow{i} b^{(2)} \xrightarrow{i} b^{(3)}$$

and suppose also that $b_N^{(2)} = b_N^{(3)}$. *Then we have* $b_N^{(1)} = b_N^{(2)}$.

Proof. We use induction on N. If $N = 0$, $b^{(2)} \xrightarrow{i} b^{(3)}$ and $b^{(2)} = b^{(3)}$ are contradictory. Therefore the assertion is true. Suppose that the lemma is proved for $(\mathscr{A}_l^+)^N$. For $v^{(1)}, v^{(2)}, v^{(3)} \in \mathscr{A}_l^+$ and $b^{(1)}, b^{(2)}, b^{(3)} \in (\mathscr{A}_l^+)^N$ assume that

$$(v^{(1)}, b^{(1)}) \xrightarrow{i} (v^{(2)}, b^{(2)}) \xrightarrow{i} (v^{(3)}, b^{(3)})$$

is valid in $(\mathscr{A}_l^+)^{N+1}$. We shall show that if $b_{N-1}^{(2)} = b_{N-1}^{(3)}$ then $b_{N-1}^{(1)} = b_{N-1}^{(2)}$. If Case A is valid for $(v^{(1)}, b^{(1)}) \xrightarrow{i} (v^{(2)}, b^{(2)})$, then $b^{(1)} = b^{(2)}$, and therefore we have $b_{N-1}^{(1)} = b_{N-1}^{(2)}$. If

$$\text{Case B is valid for} \quad (v^{(1)}, b^{(1)}) \xrightarrow{i} (v^{(2)}, b^{(2)})$$

and

$$\text{Case A is valid for} \quad (\nu^{(2)}, b^{(2)}) \xrightarrow{i} (\nu^{(3)}, b^{(3)})$$

then we have

$$l_i^{(+)}(b^{(2)}) < l_i^{(-)}(\nu^{(2)}) = l_i^{(-)}(\nu^{(1)}) \leqq l_i^{(+)}(b^{(1)}) = l_i^{(+)}(b^{(2)}) - 1.$$

This is a contradiction.

Finally assume that Case B is valid for both

$$(\nu^{(1)}, b^{(1)}) \xrightarrow{i} (\nu^{(2)}, b^{(2)}) \quad \text{and} \quad (\nu^{(2)}, b^{(2)}) \xrightarrow{i} (\nu^{(3)}, b^{(3)}).$$

Then $b^{(1)} \xrightarrow{i} b^{(2)} \xrightarrow{i} b^{(3)}$ in $(\mathscr{A}_l^+)^N$. Therefore, by the induction hypothesis, if $b_{N-1}^{(2)} = b_{N-1}^{(3)}$ then $b_{N-1}^{(1)} = b_{N-1}^{(2)}$. □

4.2. Signature of a Path. Fix color i. In 3.2 we defined the signature ε of an extended Young diagram. Take $\eta \in \mathscr{P}(\Lambda)$ and let t be the highest lift of η. We shall give another description of the signature ε of η in terms of the pattern t. For $k = -1, 0, 1, \ldots$, we define

$$\begin{aligned} \alpha(k) &= \#\{j \mid 1 \leqq j \leqq l, t_{jk} < t_{jk+1}, t_{jk+1} + k + 1 \equiv i \bmod n\}, \\ \beta(k) &= \#\{j \mid 1 \leqq j \leqq l, t_{jk} < t_{jk+1}, t_{jk} + k + 1 \equiv i \bmod n\}. \end{aligned} \tag{4.1}$$

Here by convention we set $t_{j-1} = -\infty$ and $\beta(-1) = 0$ Set

$$\varepsilon^{(k)} = (\underbrace{0, \ldots, 0}_{\alpha(k)}, \underbrace{1, \ldots, 1}_{\beta(k)}).$$

Lemma 4.3.

$$\varepsilon = (\ldots, \varepsilon^{(k)}, \varepsilon^{(k-1)}, \ldots, \varepsilon^{(-1)}).$$

Proof. Suppose that $t_{jk} < t_{jk+1}$. Then the extended Young diagram Y_j corresponding to $(t_{jk})_{k \geqq 0}$ has a concave corner at the $(t_{jk+1} + k + 1)$-th diagonal and a convex corner at the $(t_{jk} + k + 1)$-th diagonal (see Fig. 3.2).

Therefore $\sum_{k=-1}^{\infty} \alpha(k)$ is the number of i-concave corners in $(Y_1, \ldots, Y_l)$, and $\sum_{k=-1}^{\infty} \beta(k)$ is the number of i-convex corners in $(Y_1, \ldots, Y_l)$. Hence ε and $(\ldots, \varepsilon^{(k)}, \varepsilon^{(k-1)}, \ldots, \varepsilon^{(-1)})$ have the same numbers of 0's and 1's.

Now we shall prove that the ordering of 0's and 1's are also the same in ε and in $(\ldots, \varepsilon^{(k)}, \varepsilon^{(k-1)}, \ldots, \varepsilon^{(-1)})$. First we consider the ordering within a single $\varepsilon^{(k)}$. Suppose that $1 \leqq j_1, j_2 \leqq l$ and

$$\begin{aligned} &t_{j_1 k} < t_{j_1 k+1}, \quad d_1 = t_{j_1 k} + k + 1 \equiv i \bmod n, \\ &t_{j_2 k} < t_{j_2 k+1}, \quad d_2 = t_{j_2 k+1} + k + 1 \equiv i \bmod n. \end{aligned}$$

There are four cases.

(i) $d_1 > d_2$.
(ii) $d_1 = d_2$, $j_1 \leqq j_2$.
(iii) $d_1 = d_2$, $j_1 > j_2$.
(iv) $d_1 < d_2$.

We shall show that (i), (ii) are contradictory. The cases (iii), (iv) mean the 1 corresponding to the i-convex corner at the d_1-th diagonal of Y_{j_1} is located to the right of the 0 corresponding to the i-concave corner at the d_2-th diagonal of Y_{j_2} in ε. This is consistent with the definition of $\varepsilon^{(k)} = (0, \ldots, 0, 1, \ldots, 1)$.

Consider the case (i). We have

$$t_{j_1 k} + k + 1 = t_{j_2 k+1} + k + 1 + rn \quad (r \geqq 1).$$

Therefore $t_{j_1 k} \geqq t_{j_2 + lk+1} > t_{j_2 + lk}$. On the other hand we have $j_2 + l > j_1$. This is a contradiction.

Next consider the case (ii). We have $t_{j_1 k} + k + 1 = t_{j_2 k+1} + k + 1$. Therefore $t_{j_1 k} = t_{j_2 k+1} > t_{j_2 k}$. On the other hand $j_1 \leqq j_2$. This is a contradiction. Thus we have checked the ordering in $\varepsilon^{(k)}$ is consistent with the ordering in ε.

Now we will show that the ordering of $\varepsilon^{(k)}$'s is also consistent with the ordering in ε. Suppose that $k_1 < k_2, t_{j_1 k_1} < t_{j_1 k_1 + 1}$ and $t_{j_2 k_2} < t_{j_2 k_2 + 1}$. Suppose also that $t_{j_1 k_1} + k_1 + 1$ or $t_{j_1 k_1 + 1} + k_1 + 1$ is equal to $i \bmod n$, and denote it by d_1. Since $t_{j_1 k_1} + n > t_{j_1 k_1 + 1}$ the d_1 is uniquely determined. Similarly we define d_2. This means that Y_{j_1} has a convex or concave corner at the d_1-th diagonal and Y_{j_2} has a concave or convex corner at the d_2-th diagonal. Again, there are four cases (i)–(iv) for (d_1, j_1) and (d_2, j_2), and a similar argument shows that neither (i) nor (ii) occurs. Therefore, in ε, 0 or 1 corresponding to (d_1, j_1) is located to the right of 0 or 1 corresponding to (d_2, j_2). This is consistent with the ordering $(\ldots, \varepsilon^{(k)}, \varepsilon^{(k-1)}, \ldots, \varepsilon^{(-1)})$. □

Remark. In the first definition of ε, we imposed the condition that $1 \leqq j(k) \leqq l$. Replacing this condition by $l_1 \leqq j(r) \leqq l_2$ such that $l_2 - l_1 = l$, we can define ε similarly. In fact, the equivalence to the second definition is valid for any choice of l_1, l_2 $(l_2 - l_1 = l)$. Therefore the definition of ε does not depend on this choice.

4.3. *Proof of Theorem 4.1.*

Proposition 4.4. *Let* $\eta, \eta' \in \mathscr{P}(\Lambda)$. *We have an arrow*

$$\eta' \xrightarrow{i} \eta$$

if and only if we can take highest lifts t, t' *corresponding to* η, η' *in such a way that the following are satisfied.*

(i) *For some* (j_0, k_0) *we have*

$$\begin{aligned} t'_{jk} &= t_{jk} + 1 \quad \text{if} \quad j \equiv j_0 \bmod l \quad \text{and} \quad k = k_0, \\ &= t_{jk} \qquad \text{otherwise.} \end{aligned}$$

(ii) $t_{j_0 k_0} + k_0 + 1 = t'_{j_0 k_0} + k_0 \equiv i \bmod n$.

(iii) $t_{j_0 k_0} < t_{j_0 k_0 + 1}$ *and the corresponding* 1 *in the signature of* η *is relevant.*

(iv) $t'_{j_0 k_0 - 1} < t'_{j_0 k_0}$ *and the corresponding* 0 *in the signature of* η' *is relevant.*

Now we come to the proof of Theorem 4.7. The assertion is true for $N = 0$, since the set $\mathscr{P}_0(\Lambda)$ consists of one vertex. Supposing the assertion is valid for N, we prove it for $N + 1$.

Take $v, v' \in \mathscr{A}_l^+$ and $\tilde{\eta}, \tilde{\eta}' \in \mathscr{P}_N(\sigma(\Lambda))$. Set $\eta = (v, \tilde{\eta})$ and $\eta' = (v', \tilde{\eta}')$. They belong to $\mathscr{P}_N(\Lambda)$. Assume that

$$(v,b)\xrightarrow{i}(v',b') \quad \text{in } (\mathscr{A}_l^+)^{N+2},$$

where $b=\iota_N(\tilde{\eta}), b'=\iota_N(\tilde{\eta}')$. We shall show that $\eta'\xrightarrow{i}\eta$.

Let t be a highest lift of η. Recall the definitions of $\alpha(k)$ and $\beta(k)$ for t. We also define

$$\gamma(k)=\#\{j\,|\,1\leqq j\leqq l, t_{jk}=t_{jk+1}, t_{jk}+k+1\equiv i \bmod n\}.$$

Then we have $v_{i-1}=\beta(0)+\gamma(0)$. Set $\tau=(\ldots,\varepsilon^{(2)},\varepsilon^{(1)})$. Here ε is the signature of η. The signature $\tilde{\varepsilon}$ corresponding to $\tilde{\eta}$ is

$$(\tau, \underbrace{0\cdots0}_{\alpha(0)+\gamma(0)}), \tag{4.2}$$

and ε reads as

$$\varepsilon=(\tau,\underbrace{0\cdots0}_{\alpha(0)}, \underbrace{1\cdots1}_{v_{i-1}-\gamma(0)}, \underbrace{0\cdots0}_{\alpha(-1)}). \tag{4.3}$$

Case A: We have $v'=v-\epsilon_{i-1}+\epsilon_i, b=b'$ and $l_i^{(-)}(v)>l_i^{(+)}(b)$. Note that $l_i^{(-)}(v)=v_{i-1}$. By the induction hypothesis and Proposition we also have $l_i^{(+)}(b)=l_i^{(-)}(\tilde{\eta})$, where the latter signifies the length of the i string below $\tilde{\eta}$ in $\mathscr{P}(\Lambda)$ defined similarly as in (2.1). Therefore we have $v_{i-1}>l_i^{(-)}(\tilde{\eta})$. This implies that the rightmost 1 of the block $\underbrace{1\cdots1}_{v_{i-1}-\gamma(0)}$ in (4.3) is relevant. Now consider η'. Set

$$j_0=\max\{j\,|\,1\leqq j\leqq l, t_{j0}<t_{j1}, t_{j0}\equiv i-1\}.$$

Since $v_{i-1}-\gamma(0)>0$, the set of j in the right-hand side is not void. Define a parttern t' by

$$\begin{aligned} t'_{jk}&=t_{jk}+1 \quad \text{if} \quad j\equiv j_0 \bmod l \quad \text{and} \quad k=0,\\ &=t_{jk} \qquad\quad \text{otherwise.}\end{aligned}$$

This pattern is a highest lift of η' and the corresponding signature is

$$\varepsilon'=(\tau,\underbrace{0\cdots0}_{\alpha(0)}, \underbrace{1\cdots1}_{v_{i-1}-\gamma(0)-1}, \underbrace{0\cdots0}_{\alpha(-1)+1}). \tag{4.4}$$

By Proposition 4.4 we have $\eta'\xrightarrow{i}\eta$.

Case B: Define ε,ε' and τ as in Case A. The change from ε to ε' is that the 1 in ε which is the rightmost in the 1 block of ε_J changes to 0 in ε'. It can be checked that this 1 is in τ. We have $v=v', \tilde{\eta}'\xrightarrow{i}\tilde{\eta}$ and $l_i^{(-)}(v)\leqq l_i^{(+)}(b)$. Therefore

$$v_{i-1}=l_i^{(-)}(v)\leqq l^{(+)}(b)=l_i^{(-)}(\tilde{\eta}).$$

From this we can check the conditions (iii) and (iv) in Proposition 4.4 and prove that $\eta'\xrightarrow{i}\eta$.

Finally we show that the arrow $\eta'\xrightarrow{i}\eta$ implies $(v,b)\xrightarrow{i}(v',b')$ in $(\mathscr{A}_l^+)^{N+2}$. From Proposition 4.4 the (j_0,k_0) is determined. First assume that $k_0=0$. In this case $b=b'$. The signatures ε,ε' and $\tilde{\varepsilon}$ are given by (4.3), (4.4) and (4.2), respectively.

Because of the condition (iii) of Proposition 4.4 we have $v_{i-1} > l_i^{(-)}(v)$. This means $l_i^{(-)}(v) > l_i^{(+)}(b)$. Therefore we have $(v,b) \xrightarrow{i} (v',b)$ in $(\mathscr{A}_l^+)^{N+2}$. Next assume that $k_0 \geqq 1$. In this case we have $v = v'$ and $\tilde{\eta}' \xrightarrow{i} \tilde{\eta}$. The signature ε is given by (4.3) but η' is given by (4.3) with one 1 in τ replaced by 0. The condition (iv) of Proposition 4.4 implies that $l_i^{(-)}(v) = v_{i-1} \leqq l_i^{(-)}(\tilde{\eta}) = l_i^{(+)}(b)$. Therefore we have $(v,b) \xrightarrow{i} (v,b')$ in $(\mathscr{A}_l^+)^{N+2}$. $\square$

5. Restricted Paths

5.1. Tensor Product of Integrable Modules. In this section we return to integrable U_q-modules in the sense of Sect. 2. Let (L,B) be the crystal base of an integrable module. We shall call an element $v \in B$ a *highest weight vector* if $v \in B_\mu$ with some $\mu \in P$ and $\tilde{e}_i v = 0$ for any $i = 0,\ldots,n-1$. The latter is equivalent to the condition that there is no arrow pointing to v.

Let Λ, Λ' be dominant integral weights of level l, l', and (L,B), (L',B') be the crystal bases of $M(\Lambda), M(\Lambda')$, respectively. In view of Theorem 3.13, we shall make an identification $B = \mathscr{P}(\Lambda), B' = \mathscr{P}(\Lambda')$. Let $\eta^{(\Lambda')}$ denote the highest weight vector of B'.

Let us consider the tensor product module $M(\Lambda') \otimes M(\Lambda)$. Take two paths $\eta \in \mathscr{P}(\Lambda), \eta' \in \mathscr{P}(\Lambda')$. In this section, we study the condition that $\eta' \otimes \eta$ is a highest weight vector in the crystal graph $B' \times B$ of $M(\Lambda') \otimes M(\Lambda)$, i.e.,

$$\tilde{e}_i(\eta' \otimes \eta) = 0 \quad \text{for all } i.$$

Lemma 5.1. *The vector $\eta' \otimes \eta \in B' \times B$ is a highest weight vector in the above sense if and only if*

(i) $\eta' = \eta^{(\Lambda')}$,
(ii) $\tilde{e}_i^{\langle \Lambda', h_i \rangle + 1} \eta = 0$ *for all i.*

Proof. From (2.2) and (2.3b) it follows that

$$\tilde{e}_i(\eta' \otimes \eta) = 0 \Leftrightarrow \tilde{e}_i \eta' = 0, \quad \tilde{e}_i^{\langle \lambda', h_i \rangle + 1} \eta = 0,$$

where λ' is the weight of η'. This implies $\eta' = \eta^{(\Lambda')}$, and therefore $\lambda' = \Lambda'$. $\square$

Let $\mu = (\mu_0, \mu_1, \ldots)$ be a sequence of integral weights, and let Λ be a dominant integral weight. We write $\mu - \Lambda$ to mean $(\mu_0 - \Lambda, \mu_1 - \Lambda, \ldots)$. Define the **Z**-linear map ${}^\vee : \mathscr{A} \to P$ by $\check{\epsilon}_j = \Lambda_j$. Note that

$$\bar{\eta}_k = \sigma(\check{\eta}_k) - \check{\eta}_k.$$

Definition 5.2. We call (μ, η) a (Λ', Λ)-path if and only if the following conditions are satisfied.

(i) $(\mu - \Lambda', \eta) \in \mathscr{P}(\Lambda)$,
(ii) $\mu_k - \check{\eta}_k \in P_{l'}^+$ for all $k \geqq 0$.

Given $\eta \in \mathscr{P}(\Lambda)$, we can define a sequence of integral weights $\mu = (\mu_0, \mu_1, \ldots)$ uniquely from the condition $\mu_k = \Lambda' + \sigma^k(\Lambda)$ $k \gg 0$ so that (i) is satisfied. The condition (ii) then restricts η from being arbitrary. In this sense we call η a restricted path if it is a (Λ', Λ)-path for some $\Lambda' \in P$.

Example. ($n=2$.)

$$\mu^{(1)}=(2\Lambda_1,\Lambda_0+\Lambda_1,2\Lambda_0,\Lambda_0+\Lambda_1,2\Lambda_0,\dots),$$
$$\eta^{(1)}=(\epsilon_1,\epsilon_1,\epsilon_0,\epsilon_1,\dots),$$
$$\mu^{(2)}=(2\Lambda_0,3\Lambda_0-\Lambda_1,2\Lambda_0,\Lambda_0+\Lambda_1,2\Lambda_0,\dots),$$
$$\eta^{(2)}=(\epsilon_1,\epsilon_0,\epsilon_0,\epsilon_1,\dots).$$

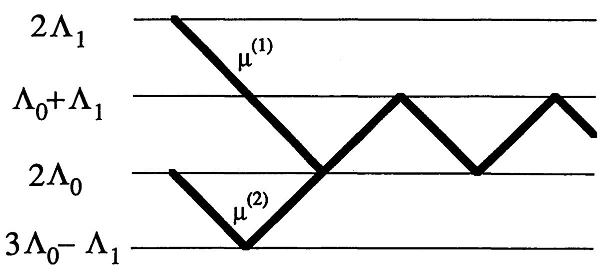


Fig. 5.1. Level 1 paths. $(\mu^{(1)},\eta^{(1)})$ is a (Λ_0,Λ_0)-path, but $(\mu^{(2)},\eta^{(2)})$ is not.

$$\mu^{(3)}=(2\Lambda_0+\Lambda_1,3\Lambda_1,2\Lambda_0+\Lambda_1,2\Lambda_0+\Lambda_1,2\Lambda_0+\Lambda_1,\dots),$$
$$\eta^{(3)}=(2\epsilon_0,2\epsilon_1,\epsilon_0+\epsilon_1,\epsilon_0+\epsilon_1,\dots),$$
$$\mu^{(4)}=(2\Lambda_0+\Lambda_1,3\Lambda_1,3\Lambda_1,2\Lambda_0+\Lambda_1,2\Lambda_0+\Lambda_1,\dots),$$
$$\eta^{(4)}=(2\epsilon_0,\epsilon_0+\epsilon_1,2\epsilon_1,\epsilon_0+\epsilon_1,\dots).$$

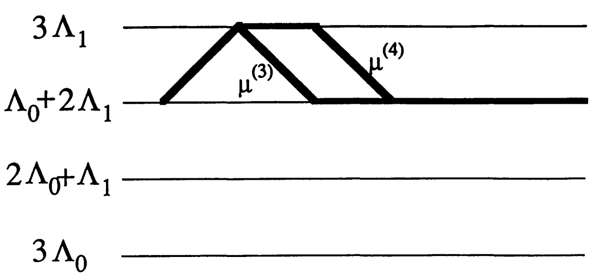


Fig. 5.2. Level 2 paths. $(\mu^{(3)},\eta^{(3)})$ is a $(\Lambda_0,\Lambda_0+\Lambda_1)$-path but $(\mu^{(4)},\eta^{(4)})$ is not.

Our goal is to prove the following theorem.

Theorem 5.3. *$\eta'\otimes\eta\in B'\times B$ is a highest weight vector if and only if $\eta'=\eta^{(\Lambda')}$ and η is a (Λ',Λ)-path.*

Before proving the theorem, we prepare

Lemma 5.4. *Consider a sequence of* 0 *or* 1 *of the form*

$$\varepsilon=(\dots,\varepsilon^{(k)},\varepsilon^{(k-1)},\dots,\varepsilon^{(-1)}),$$
$$\varepsilon^{(k)}=(\underbrace{0,\dots,0}_{\alpha(k)},\underbrace{1,\dots,1}_{\beta(k)}).$$

We assume that $\alpha(k) = \beta(k) = 0$ *if* $k \gg 0$. *We set* $\gamma(m) = \beta(m) - \alpha(m)$. *Let* n_s $(s = 0, 1)$ *be the number of s in* ε_J *(see 3.2 for the definition of* ε_J*). Then we have*

$$n_1 = \max_{k \geqq -1} \sum_{m \geqq k} \gamma(m), \tag{5.1}$$

$$n_1 - n_0 = \sum_{m \geqq -1} \gamma(m). \tag{5.2}$$

Proof. By the definition of ε_J (5.2) is obvious. Put

$$\tau = (\dots, \varepsilon^{(k)}, \varepsilon^{(k-1)}, \dots, \varepsilon^{(0)}).$$

If $\gamma(-1) = 0$ the proof of (5.1) is reduced to proving the same statement for τ. Therefore, without loss of generality we can assume that $\gamma(-1) \neq 0$. We apply the induction on the number of m $(\geqq -1)$ such that $\gamma(m) \neq 0$. Let $\bar{n}_s$ $(s = 0, 1)$ be the number of s in τ_J. Since $\gamma(-1) \neq 0$, by the induction hypothesis we can assume that

$$\bar{n}_1 = \max_{k \geqq 0} \sum_{m \geqq k} \gamma(m), \quad \bar{n}_1 = \bar{n}_0 = \sum_{m \geqq 0} \gamma(m). \tag{5.3}$$

Clearly, we have

$$\varepsilon_J = (\tau_J, \varepsilon^{(-1)})_J = (\underbrace{1, \dots, 1}_{\bar{n}_1}, \underbrace{0, \dots, 0}_{\bar{n}_0}, \varepsilon^{(-1)})_J.$$

We devide the proof into two cases.

(i) $\gamma(-1) > 0$.

In this case, $\varepsilon_J = (\underbrace{1, \dots, 1}_{\bar{n}_1}, \underbrace{0, \dots, 0}_{\bar{n}_0}, \underbrace{1, \dots, 1}_{\gamma(-1)})_J$. If $\bar{n}_0 \leqq \gamma(-1)$, $n_1 = \bar{n}_1 + \gamma(-1) - \bar{n}_0$ and $n_0 = 0$. From (5.3) and the inequality $\gamma(-1) - \bar{n}_0 \geqq 0$, we have (5.1). If $\bar{n}_0 > \gamma(-1)$, $n_1 = \bar{n}_1, n_0 = \bar{n}_0 - \gamma(-1)$. Since $n_0 > 0, \bar{n}_1 > \bar{n}_1 - \bar{n}_0 + \gamma(-1) = \sum_{m \geqq -1} \gamma(m)$. Therefore we get (5.1).

(ii) $\gamma(-1) < 0$.

In this case, $\varepsilon_J = (\underbrace{1, \dots, 1}_{\bar{n}_1}, \underbrace{0, \dots, 0}_{\bar{n}_0}, \underbrace{0, \dots, 0}_{-\gamma(-1)})_J$. Therefore, $n_1 = \bar{n}_1 = \max_{k \geqq -1} \sum_{m \geqq k} \gamma(m)$. □

Remark. If ε is the signature of some $\mathbf{Y} \in \mathscr{P}(\Lambda)$ then $\beta(-1) = 0$. Therefore we have

$$n_1 = \max_{k \geqq 0} \sum_{m \geqq k} \gamma(m).$$

Proof of Theorem 5.3. We define μ in such a way that Definition 5.2 (i) holds. From Lemma 5.1, it suffices to show that

$$\tilde{e}_i^{\langle \Lambda', h_i \rangle + 1} \eta = 0 \quad \text{for all} \quad i \Leftrightarrow (\mu, \eta) \text{ satisfies Definition 5.2 (ii).}$$

Fix i. Let ε be the signature of the highest lift of η with respect to the color i. Let $n_1^{(i)}$ be the number of 1 in ε_J. From Theorem 3.6, the following conditions are equivalent.

$$\tilde{e}_i^{\langle \Lambda', h_i \rangle + 1} \eta = 0 \Leftrightarrow n_1^{(i)} \leqq \langle \Lambda', h_i \rangle. \tag{5.4}$$

On the other hand, we have

$$(\mu_k-\check{\eta}_k)-(\mu_{k+1}-\check{\eta}_{k+1})=\check{\eta}_{k+1}-\sigma(\check{\eta}_k)$$
$$=\sum_{1\leqq j\leqq l}(\Lambda_{t_{jk+1}+k+1}-\Lambda_{t_{jk}+k+1}).$$

From (4.1), it turns out that

$$(\mu_k-\check{\eta}_k)-(\mu_{k+1}-\check{\eta}_{k+1})=\sum_{0\leqq i<n}(\alpha_i(k)-\beta_i(k))\Lambda_i.$$

Here we have exhibited the i-dependence of α, β explicitly. Recalling that $\mu_k-\check{\eta}_k=\Lambda'$ if $k\gg 0$, we get

$$\mu_k-\check{\eta}_k=\sum_{m\geqq k}\sum_{0\leqq i<n}(\alpha_i(m)-\beta_i(m))\Lambda_i+\Lambda'.$$

Therefore

$$\sum_{m\geqq k}(\beta_i(m)-\alpha_i(m))=\langle\Lambda'-\mu_k+\check{\eta}_k,h_i\rangle.$$

Using Lemma 5.4, we have

$$n_1^{(i)}=\max_{k\geqq 0}\langle\Lambda'-\mu_k+\check{\eta}_k,h_i\rangle. \tag{5.5}$$

From (5.4–5),

$$\tilde{e}_i^{\langle\Lambda',h_i\rangle+1}\eta=0\quad\text{for all}\quad i\Leftrightarrow\langle\mu_k-\check{\eta}_k,h_i\rangle\geqq 0\quad\text{for all } i \text{ and } k.$$

These are equivalent to the condition that $\mu_k-\check{\eta}_k$ is a dominant integral weight for all k. This completes the proof of the theorem. □

5.2. String Functions and Branching Coefficients. Let us consider a Λ-path (μ,η) (or simply η) and calculate its weight $\lambda(\eta)$. For $\alpha=\epsilon_{\mu_1}+\cdots+\epsilon_{\mu_l}$, $\beta=\epsilon_{\nu_1}+\cdots+\epsilon_{\nu_l}\in\mathscr{A}_l^+$ we define

$$H(\alpha,\beta)=\min_{\sigma\in\mathfrak{S}_l}\sum_{j=1}^{l}\theta(\mu_j-\nu_{\sigma(j)}),$$
$$\theta(r)=1\quad\text{if}\quad r\geqq 0$$
$$=0\quad\text{otherwise.}$$

Then we have

Theorem 5.5.

$$\lambda(\eta)=\mu_0-\omega(\eta)\delta,$$
$$\omega(\eta)=\sum_{k\geqq 1}k(H(\eta_{k-1},\eta_k)-H(\eta_{k-1}^{(\Lambda)},\eta_k^{(\Lambda)})).$$

Let $\mathbf{Y}=(Y_1,\ldots,Y_l)$ be the highest lift of η. Recalling the Fock representation, we see that the proof of Theorem 5.5 reduces to counting the number of nodes on d-th diagonal with $d\equiv i \bmod (n)$ in each extended Young diagram Y_j. The same problem was solved in a different setting ([4] Theorem 5.7). We omit the proof here.

From Theorem 5.5 we obtain

Corollary 5.6. *For $\Lambda' \in P_l$ we have*

$$\sum_n \dim M(\Lambda)_{\Lambda' - n\delta} q^n = \sum_{\substack{(\mu,\eta)\in\mathscr{P}(\Lambda) \\ \mu_0 = \Lambda'}} q^{\omega(\eta)}.$$

Next let Λ, Λ' be dominant integral weights, and consider the following tensor product decomposition:

$$M(\Lambda') \otimes M(\Lambda) = \sum_{\Lambda''} \Omega_{\Lambda'\Lambda\Lambda''} \otimes M(\Lambda'').$$

Here $\Omega_{\Lambda'\Lambda\Lambda''}$ is the space of the highest weight vectors in $M(\Lambda') \otimes M(\Lambda)$ whose weight is equal to Λ'' modulo $\mathbf{Z}\delta$. Combining Definition 5.2 and Theorem 5.5 we obtain

Corollary 5.7.

$$\sum_n \dim (\Omega_{\Lambda'\Lambda\Lambda''})_{\Lambda'' - n\delta} q^n = \sum_{\substack{(\mu,\eta)\in\mathscr{P}(\Lambda',\Lambda) \\ \mu_0 = \Lambda''}} q^{\omega(\eta)},$$

where $(\Omega_{\Lambda'\Lambda\Lambda''})_\lambda = \Omega_{\Lambda'\Lambda\Lambda''} \cap (M(\Lambda') \otimes M(\Lambda))_\lambda$ and $\mathscr{P}(\Lambda', \Lambda)$ denotes the set of (Λ', Λ)-paths.

The quantities appearing in Corollary 5.6 and 5.7 are called the string functions [13] and the branching coefficients, respectively. We have thus obtained a neat expression for both of them using paths.

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