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A Kinetic Equation with Kinetic Entropy Functions for Scalar Conservation Laws*

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Abstract. We construct a nonlinear kinetic equation and prove that it is well-adapted to describe general multidimensional scalar conservation laws. In particular we prove that it is well-posed uniformly in ε – the microscopic scale. We also show that the proposed kinetic equation is equipped with a family of kinetic entropy functions – analogous to Boltzmann's microscopic H-function, such that they recover Krushkov-type entropy inequality on the macroscopic scale. Finally, we prove by both – BV compactness arguments in the multidimensional case and by compensated compactness arguments in the one-dimensional case, that the local density of kinetic particles admits a "continuum" limit, as it converges strongly with $\varepsilon \downarrow 0$ to the unique entropy solution of the corresponding conservation law.

1. Introduction

Consider the scalar multi-dimensional conservation law

$$\frac{\partial}{\partial t}[u(x,t)] + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}[A_i(u(x,t))] = 0, \quad (x,t) \in R_x^d \times R_t^+, \quad A_i(\cdot) \in C^1, \tag{1.1}$$

with given initial conditions $u(x, t=0) = u_0(x)$. We are concerned here with a Boltzmann-like kinetic equation which describes (1.1), as its microscopic scale, $\varepsilon > 0$, tends to zero. To this end we introduce a scalar function, $f_\varepsilon(x, v, t)$, which can be viewed as a microscopic description for the density of particles located at

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$(x,t)\in R_x^d \times R_t^+$ with speed $v\in R$. Starting with given initial distribution, $f_\varepsilon(x,v,0)$, our kinetic model evolves according to

$$[\partial_t + a(v)\cdot\partial_x] f_\varepsilon(x,v,t) = \frac{1}{\varepsilon}[\chi_{u_\varepsilon(x,t)}(v) - f_\varepsilon(x,v,t)]. \tag{1.2a}$$

Equation (1.2a) tells us that the particles are transported along

$$a(v)\cdot\partial_x \equiv \sum_{i=1}^{d} a_i(v)\frac{\partial}{\partial x_i}, \quad a_i(\cdot)\equiv A_i'(\cdot),$$

and that their collisions are governed by the nonlinear kernel on the right. Here,

$$u_\varepsilon(x,t) = \int_v f_\varepsilon(x,v,t)dv, \tag{1.2b}$$

denotes the local density of particles at a given (x,t) location, and the "equilibrium function,' $\chi_{u_\varepsilon(x,t)}(v)$, is the signature of $u_\varepsilon(x,t)$, i.e.,

$$\chi_u(v) = \begin{cases} \operatorname{sgn} u, & \text{if } (u-v)v \geqq 0, \\ 0, & \text{if } (u-v)v < 0. \end{cases} \tag{1.2c}$$

The classical example of a kinetic model is of course the Boltzmann equation [1]. Equation (1.2) is closely related to the B.G.K. model of Boltzmann equation. Existence theory for Boltzmann equation and its simplified B.G.K. model can be found in [6, 11], respectively. In both cases, however, the question of convergence of the macroscopic moments to *weak* solutions of compressible Euler equations is still an open problem. (Consult [3] regarding an affirmative answer to this convergence question in the case of strong solutions.) In this paper we restrict our attention to the simpler scalar case, and we show that the proposed kinetic equation (1.2) is well adapted to describe strong as well as weak solutions of (1.1) as $\varepsilon\downarrow 0$.

The paper is organized as follows. In Sect. 2 we show that the kinetic equation (1.2) is well-posed in $L^\infty(R_t^+; L^1(R_x^d \times R_v))$. Next, we borrow our terminology from the framework of Boltzmann's kinetic equation. The microscopic scale, ε, in (1.2) can be viewed as the mean free path. In Sect. 3 we prove that the continuum or "fluid" limit of the local density of particles, $\lim_{\varepsilon\downarrow 0} u_\varepsilon(x,t)$ is the unique entropy solution of (1.1). A kinetic construction of conservative solutions was carried out by Giga and Miyakawa [7]. In fact their construction is nothing but a fractional splitting solution of our kinetic equation (1.2), namely, a kinetic approximation is constructed by a succession of small time steps, in which we first transport and then project the particles distribution according to (1.2). Here we improve on [7] by identifying the underlying kinetic equation which corresponds to (1.1). It is also shown here that this kinetic equation is equipped with (a family of) kinetic entropy functions which play an analogous role to Boltzmann's H-function. In particular, Krushkov entropy inequality [8, 9] is recovered in the "fluid" limit $\varepsilon\downarrow 0$.

In Sect. 4 we revisit the question of the "fluid" limit in the case of one-dimensional kinetic model. Here we show that the compensated compactness theory of Murat–Tartar [10, 13] can be adapted as an alternative approach for providing an affirmative answer to the question of macroscopic convergence. The compensated compactness arguments allow us to pass to the continuum limit with minimal $L^1 \cap L^\infty$ information about the distribution function, f_ε, which may still oscillate around the "equilibrium function" χ_u. Finally, in Sect. 5, we indicate the extension of our results to the inhomogeneous case, in the presence of a (possibly stiff) source term.

2. The Kinetic Equation is Well-Posed

Let us rewrite (1.2) in the form

$$\frac{\partial}{\partial t} f_\varepsilon + a(v)\cdot\frac{\partial}{\partial x} f_\varepsilon + \frac{1}{\varepsilon} f_\varepsilon = \frac{1}{\varepsilon}\chi_{u_\varepsilon}, \tag{2.1}$$

separating between its linear part on the left and its nonlinear kernel on the right. By Duhammel's principle, (2.1) admits the following equivalent integral representation:

$$f_\varepsilon(x,v,t) = e^{-t/\varepsilon} f_\varepsilon(x - ta(v), v, 0) + \frac{1}{\varepsilon}\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}\chi_{u_\varepsilon(x-(t-\tau)a(v),\tau)}(v)\,d\tau. \tag{2.2}$$

The question of existence of a kinetic solution of (1.2) is now transformed into that of a fixed point solution for the right-hand side of (2.2). Fixing $T, T > 0$, we seek such a fixed point solution in $L^\infty([0,T]; L^1(R_x^d \times R_v))$. To this end, we let f_ε and g_ε be two different solutions of (1.2a) with corresponding densities $u_\varepsilon(x,t) = \int_v f_\varepsilon(x,v,t)\,dv$ and $w_\varepsilon(x,t) = \int_v g_\varepsilon(x,v,t)\,dv$. By (2.2), their difference does not exceed

$$\begin{aligned}\| f_\varepsilon(x,v,t) - g_\varepsilon(x,v,t)\|_{L^1(R_x^d\times R_v)} &\leqq e^{-t/\varepsilon}\| f_\varepsilon(x,v,0) - g_\varepsilon(x,v,0)\|_{L^1(R_x^d\times R_v)}\\ &\quad + \frac{1}{\varepsilon}\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}\|\chi_{u_\varepsilon(x,\tau)}(v) - \chi_{w_\varepsilon(x,\tau)}(v)\|_{L^1(R_x^d\times R_v)}\,d\tau.\end{aligned}$$

Using the properties of the signature function, χ, we therefore conclude

$$\begin{aligned}\| f_\varepsilon(x,v,t) - g_\varepsilon(x,v,t)\|_{L^1(R_x^d\times R_v)} &\leqq e^{-t/\varepsilon}\| f_\varepsilon(x,v,0) - g_\varepsilon(x,v,0)\|_{L^1(R_x^d\times R_v)}\\ &\quad + (1 - e^{-t/\varepsilon})\max_{0\leqq\tau\leqq t}\| f_\varepsilon(x,v,\tau) - g_\varepsilon(x,v,\tau)\|_{L^1(R_x^d\times R_v)}.\end{aligned} \tag{2.3}$$

The inequality (2.3) shows that the fixed point iterations

$$f_\varepsilon^{m+1}(x,v,t) = e^{-t/\varepsilon} f_\varepsilon(x - ta(v), v, 0) + \frac{1}{\varepsilon}\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}\chi_{u_\varepsilon^m(x-(t-\tau)a(v),\tau)}(v)\,d\tau, \tag{2.4}$$

are contracted (with a contraction factor of $1 - e^{-t/\varepsilon}$) to a fixed-solution solution of (2.2). Moreover, by (2.3) this kinetic solution is unique and continuously dependent on the initial data, for

$$\max_{0\leqq t\leqq T}\| f_\varepsilon(x,v,t) - g_\varepsilon(x,v,t)\|_{L^1(R_x^d\times R_v)} \leqq \| f_\varepsilon(x,v,0) - g_\varepsilon(x,v,0)\|_{L^1(R_x^d\times R_v)}. \tag{2.5}$$

We summarize this by stating

Theorem 2.1. *The kinetic model (1.2) is well-posed in $L^\infty(R_t^+; L^1(R_x^d \times R_v))$. Moreover, its solution operator is nonexpansive in this topology*, i.e., *(2.5) holds.*

We close this section with several remarks.

1. *L^∞-bound.* To see that the solution operator associated with the kinetic model (1.2) is uniformly bounded, we use (2.2), obtaining

$$\|f_\varepsilon(\cdot,v,t)\|_{L^\infty(R_x^d)} \leqq e^{-t/\varepsilon}\|f_\varepsilon(\cdot,v,0)\|_{L^\infty(R_x^d)} + (1-e^{-t/\varepsilon})\max_{0\leqq\tau\leqq t}\|\chi_{u_\varepsilon(\cdot,\tau)}(v)\|_{L^\infty(R_x^d)}$$

$$\leqq e^{-t/\varepsilon}\|f_\varepsilon(\cdot,v,0)\|_{L^\infty(R_x^d)} + 1 - e^{-t/\varepsilon}. \tag{2.6}$$

2. *Finite Speed of Propagation.* We assume that initially, $f_\varepsilon(x,\cdot,0)$ has a compact support in R_v. Let us first show that $f_\varepsilon(x,\cdot,t)$ remains compactly supported. Indeed, by (2.6), $f_\varepsilon(\cdot,v,t)$ and hence $u_\varepsilon(\cdot,t)$ are uniformly bounded, and therefore the contributions of $\chi_{u(\cdot,\cdot)}(v)$ on the right-hand side of (2.2) are supported by $v\in[-u_\infty,u_\infty]$, where $u_\infty = \|u_\varepsilon(x,t)\|_{L^\infty(R_x^d\times R_t^+)}$. Consequently, $f_\varepsilon(x,\cdot,t)$ given in (2.2) remains compactly supported for all $t>0$, with support contained in $\text{supp}_v f_\varepsilon(x,\cdot,0)\cup[-u_\infty,u_\infty]$. (Note that after an initial kinetic layer of order $O(\varepsilon)$, the contribution of the initial data in (2.2) decays exponentially fast. Thereafter, $f_\varepsilon(x,\cdot,t)$ is in fact "essentially" supported in $[-u_\infty,u_\infty]$).

With this in mind we now turn to prove the finite spatial speed of propagation. We shall need a refined version of estimate (2.6). To this end we first observe that according to (2.2) – which we rewrite as

$$f_\varepsilon(x,v,t) = e^{-t/\varepsilon}f(x-ta(v),v,0) + (1-e^{-t/\varepsilon})\left[\frac{\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}\chi_{u_\varepsilon(x-(t-\tau)a(v),\tau)}(v)\,d\tau}{\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}\,d\tau}\right],$$

$f_\varepsilon(x,v,t)$ is given by a convex combination of $f_\varepsilon(x-ta(v),v,0)$ and $\chi_{u_\varepsilon(x-(t-\tau)a(v),\tau)}(v)$. By Jensen's inequality, therefore, we have for *any* convex function, $U(f)$,

$$U(f_\varepsilon(x,v,t)) \leqq e^{-t/\varepsilon}U(f_\varepsilon(x-ta(v),v,0)) + \frac{1}{\varepsilon}\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}U(\chi_{u_\varepsilon(x-(t-\tau)a(v),\tau)}(v))\,d\tau. \tag{2.7}$$

In particular, consider the case $U(f)=|f|^p$. If we let a_∞ denote the maximal speed of propagation,

$$a_\infty = \left\{\max_i |a_i(v)|,\quad v\in\text{supp}_v f_\varepsilon(x,\cdot,t)\right\}, \tag{2.8}$$

and $B[r]=[-r,r]^d\subset R_x^d$, is the ball of radius r, then (2.7) implies

$$\|f_\varepsilon(\cdot,v,t)\|^p_{L^\infty(B[r])} \leqq e^{-t/\varepsilon}\|f_\varepsilon(\cdot,v,0)\|^p_{L^\infty(B[r+ta_\infty])} + \frac{1}{\varepsilon}\int_{\tau=0}^{t} e^{(\tau-t)/\varepsilon}\|\chi_{u_\varepsilon(\cdot,\tau)}(v)\|_{L^\infty(B[r+(t-\tau)a_\infty])}\,d\tau.$$

Integrating the last inequality with respect to v we find

$$\int_v \|f_\varepsilon(\cdot,v,t)\|^p_{L^\infty(B[r])}dv \leqq e^{-t/\varepsilon}\int_v \|f_\varepsilon(\cdot,v,0)\|^p_{L^\infty(B[r+ta_\infty])}dv + (1-e^{-t/\varepsilon})\max_{0\leqq\tau\leqq t}\int_v \|f_\varepsilon(\cdot,v,\tau)\|_{L^\infty(B[r+(t-\tau)a_\infty])}dv. \tag{2.9}$$

If, on the one hand, we take the p-root of both sides and let $p\uparrow\infty$, we obtain

$$\| f_\varepsilon(x,v,t)\|_{L^\infty(B[r]\times R_v)} \leqq \max\{\| f_\varepsilon(x,v,0)\|_{L^\infty(B[r+ta_\infty]\times R_v)}, 1\}, \tag{2.10}$$

in agreement with what we had before, consult (2.6). If on the other hand, we set $p=1$ in (2.9), we find that the function $F(\tau)$,

$$F(\tau)\equiv\int_v \| f_\varepsilon(\cdot,v,\tau)\|_{L^\infty(B[r+(t-\tau)a_\infty])}dv, \quad 0\leqq\tau\leqq t,$$

satisfies

$$F(t)\leqq e^{-t/\varepsilon}F(0)+(1-e^{-t/\varepsilon})\max_{0\leqq\tau\leqq t}F(\tau),$$

and hence $F(t)\leqq F(0)$. Thus, we have a finite speed of propagation ($\leqq a_\infty$) of the uniform bound on the moments

$$\int_v \| f_\varepsilon(x,v,t)\|_{L^\infty(B[r])}dv \leqq \int_v \| f_\varepsilon(x,v,0)\|_{L^\infty(B[r+ta_\infty])}dv. \tag{2.11}$$

In particular, the local density is uniformly bounded by the initial data,

$$\| u_\varepsilon(x,t)\|_{L^\infty(R_x^d\times R_t^+)} \leqq \int_v \| f_\varepsilon(x,v,0)\|_{L^\infty(R_x^d)}dv. \tag{2.12}$$

In summary, we conclude that for initial data $f_\varepsilon(x,v,0)\in L^1(R_v;L^\infty(R_x^d))$ which are compactly supported in R_v, the corresponding kinetic solution $f_\varepsilon(x,v,t)$ remains compactly supported in R_v and is uniformly bounded in $L^1(R_v;L^\infty(R_x^d))$, due to a finite speed of propagation $\leqq a_\infty$, given by

$$a_\infty=\left\{\max_i |a_i(v)|, v\in[-v_\infty,v_\infty]\cup\mathrm{supp}_v f_\varepsilon(x,v,0)\right\}, \quad v_\infty\equiv\| f_\varepsilon(x,v,0)\|_{L^1(R_v;L^\infty(R_x^d))}.$$

3. *Monotonicity.* The signature function $\chi_u(v)$ is an increasing function of u. Consequently, the fixed-point iterations (2.4) show that

$$f_\varepsilon(x,v,0)\geqq g_\varepsilon(x,v,0)\Rightarrow f_\varepsilon(x,v,t)\geqq g_\varepsilon(x,v,t), \quad \text{for all } (x,v), \tag{2.13}$$

namely, the solution operator associated with the kinetic equation (1.2) is *monotone.* In particular, if we compare a given kinetic solution (compactly supported in $R_x^d\times R_v$) with the steady state solutions $\chi_{\mathrm{Const.}}(v)$, i.e., if initially we have

$$\chi_k(v)\leqq f_\varepsilon(x,v,0) \quad \text{or} \quad f_\varepsilon(x,v,0)\leqq\chi_K(v), \tag{2.14a}$$

we obtain

$$\chi_k(v)\leqq f_\varepsilon(x,v,t) \quad \text{or} \quad f_\varepsilon(x,v,t)\leqq\chi_K(v); \tag{2.14b}$$

in agreement with (2.6). And, since the kinetic solution operator is also conservative, the Crandall–Tartar lemma [5] implies the L^1-contraction stated in (2.5). In fact, at this point we can state a little more, namely,

4. *L^1-Contraction Revisited.* Taking into account the finite-speed of propagation, we can repeat – along the lines of Remark 2, a localized version of estimate (2.3) which sharpens the L^1-contraction estimate (2.5) into

$$\| f_\varepsilon(x,v,t)-g_\varepsilon(x,v,t)\|_{L^1(B[r]\times R_v)} \leqq \| f_\varepsilon(x,v,0)-g_\varepsilon(x,v,0)\|_{L^1(B[r+ta_\infty]\times R_v)}. \tag{2.15}$$

5. The various estimates quoted above indicate that after an initial layer of order

$O(\varepsilon)$, the kinetic solution asymptotes to the "equilibrium function," $\chi_{u(x,t)}(v)$, where – as will be shown in the next section, $u(x,t)$ is the unique entropy solution of (1.1).

3. Kinetic Entropy Functions

Our analysis of the kinetic model (1.2) hinges on the construction of certain kinetic entropy functions. A kinetic entropy function in this context is a function, $H(f)$, such that as in Boltzmann's H-Theorem, any solution of (1.2) obeys the additional entropy inequality

$$\int_v [\partial_t + a(v)\cdot\partial_x] H(f_\varepsilon)dv \leqq 0. \tag{3.1}$$

We shall construct a family of such kinetic entropy functions depending on extra fixed parameter k, k real. To this end, we integrate (1.2a) against $\operatorname{sgn}(f_\varepsilon - \chi_k)$ over the phase space. Invoking a standard regularization argument of the signum function we obtain

$$\int_v [\partial_t + a(v)\cdot\partial_x] |f_\varepsilon - \chi_k| dv = -\frac{1}{\varepsilon}\int_v \operatorname{sgn}(f_\varepsilon - \chi_k)(f_\varepsilon - \chi_{u_\varepsilon})dv. \tag{3.2}$$

Noting that the expression on the right is upper-bounded by

$$-\frac{1}{\varepsilon}\int_v \operatorname{sgn}(f_\varepsilon - \chi_k)(f_\varepsilon - \chi_{u_\varepsilon})dv = -\frac{1}{\varepsilon}\int_v [|f_\varepsilon - \chi_k| + \operatorname{sgn}(f_\varepsilon - \chi_k)(\chi_k - \chi_{u_\varepsilon})]dv$$
$$\leqq -\frac{1}{\varepsilon}\left[\int_v |f_\varepsilon - \chi_k| dv - |u_\varepsilon - k|\right],$$

we arrive at

Theorem 3.1. *For any solution* $f_\varepsilon \in L^\infty(R_t^+; L^1(R_x^d \times R_v))$ *of the kinetic model (1.2), the following inequality holds*:

$$\int_v [\partial_t + a(v)\cdot\partial_x] |f_\varepsilon - \chi_k| dv \leqq -\frac{1}{\varepsilon}\left[\int_v |f_\varepsilon - \chi_k| dv - |u_\varepsilon - k|\right]. \tag{3.3}$$

Now, the right-hand side of (3.3) is clearly nonpositive for

$$-\frac{1}{\varepsilon}\int_v |f_\varepsilon - \chi_k| dv \leqq -\frac{1}{\varepsilon}\left|\int_v (f_\varepsilon - \chi_k)dv\right| = -\frac{1}{\varepsilon}|u_\varepsilon - k|. \tag{3.4}$$

Consequently, Theorem 3.1 yields

Corollary 3.2. *For any* k, k *real, the following functions*

$$H(f_\varepsilon) \equiv H_k(f_\varepsilon) = |f_\varepsilon - \chi_k| \tag{3.5}$$

are kinetic entropy functions, i.e., *we have*

$$\int_v [\partial_t + a(v)\cdot\partial_x] |f_\varepsilon - \chi_k| dv \leqq 0. \tag{3.6}$$

Let us point out that our kinetic entropy functions, $H_k(f_\varepsilon)$, are intimately related to the entropy functions used by Krushkov in [8]. Indeed, as $\varepsilon \downarrow 0$ we expect (and later on prove) that f_ε approaches χ_u. With this in mind, the inequality (3.6) turns into Krushkov's entropy inequality [8]

$$\frac{\partial}{\partial t}|u-k| + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}[\operatorname{sgn}(u-k)(A_i(u)-A_i(k))] \leqq 0, \quad \text{for any real } k. \tag{3.7}$$

To make this last point more precise, we shall need several lemmata. We start with

Lemma 3.3. *Let $f_\varepsilon \in L^\infty(R_t^+; L^1(R_v; L^\infty(R_x^d)))$ be the solution of the kinetic equation (1.2), subject to given initial data $f_\varepsilon(x,v,0)$ which are compactly supported in R_v. Assume that $u_\varepsilon(x,t) \equiv \int_v f_\varepsilon(x,v,t)dv$ satisfies*

$$u_\varepsilon(x,0) \to u_0(x) \quad \text{in } L^1(R_x^d), \tag{3.8}$$

and

$$\text{a subsequence of } u_\varepsilon(x,t) \to u(x,t) \quad \text{in } L^\infty_{\text{loc}}(R_t^+; L^1(R_x^d)). \tag{3.9}$$

Then the sequence $u_\varepsilon(x,t)$ converges strongly in $L^\infty_{\text{loc}}(R_t^+; L^1(R_x^d))$ to $u(x,t)$, which is the unique entropy solution of the conservation law (1.1), i.e., *(3.7) holds.*

Note. If we take $k > \|u\|_{L^\infty(R_x^d \times [0,T])}$, then the entropy inequality (3.7) yields

$$\frac{\partial}{\partial t}u + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} A_i(u) \geqq 0,$$

i.e., $u(x,t)$ is a supersolution of (1.1); similarly, taking $k < -\|u\|_{L^\infty(R_x^d \times [0,T])}$ shows that $u(x,t)$ is a subsolution of (1.1). Hence, (3.7) implies that $u(x,t)$ solves the conservation law (1.1).

To prove Lemma 3.3 we first prepare

Lemma 3.4. *Let $f_\varepsilon \in L^\infty(R_t^+; L^1(R_v; L^\infty(R_x^d)))$ be the solution of the kinetic equation (1.2), subject to given initial data $f_\varepsilon(x,v,0)$ which are compactly supported in R_v. Then for any k, k real, we have*

$$\int_v |f_\varepsilon - \chi_k| dv - |u_\varepsilon - k| \xrightarrow[\varepsilon\downarrow 0]{} 0 \quad \text{in } L^1_{\text{loc}}(R_t^+ \times R_x^d), \tag{3.10}$$

and for any $b(\cdot) \in L^\infty(R_v)$,

$$\int_v b(v)|f_\varepsilon - \chi_k| dv - \operatorname{sgn}(u_\varepsilon - k)\int_v b(v)(f_\varepsilon - \chi_k)dv \xrightarrow[\varepsilon\downarrow 0]{} 0 \quad \text{in } L^1_{\text{loc}}(R_t^+ \times R_x^d). \tag{3.11}$$

Proof. The vanishing limit in (3.10) follows from the inequality (3.3), for

$$\begin{aligned} 0 &\leqq \int_0^T \int_x \left[\int_v |f_\varepsilon - \chi_k| dv - |u_\varepsilon - k| \right] dx\,dt \\ &\leqq -\varepsilon \int_0^T \int_x \int_v [\partial_t + a(v)\partial_x] |f_\varepsilon - \chi_k| dv\,dx\,dt \xrightarrow[\varepsilon\downarrow 0]{} 0. \end{aligned} \tag{3.12}$$

To prove (3.11) we write

$$\begin{aligned} \int_v |f_\varepsilon - \chi_k| dv - |u_\varepsilon - k| &= \int_v \operatorname{sgn}(f_\varepsilon - \chi_k)(f_\varepsilon - \chi_k)dv - \operatorname{sgn}(u_\varepsilon - k)\int_v (f_\varepsilon - \chi_k)dv \\ &= \int_v (f_\varepsilon - \chi_k)s(v)dv. \end{aligned} \tag{3.13}$$

Here, $s(v) \equiv s(v; x, t)$ is the characteristic function given by

$$s(v) = \operatorname{sgn}(f_\varepsilon(x, v, t) - \chi_k(v)) - \operatorname{sgn}(u_\varepsilon(x, t) - k).$$

Now, since $s(v)$ is supported on the set

$$V = \{v \mid \operatorname{sgn}(f_\varepsilon - \chi_k) \neq \operatorname{sgn}(u_\varepsilon - k)\},$$

and since

$$\operatorname{sgn}(f_\varepsilon - \chi_k) \cdot s(v) \equiv 2, \quad \text{for} \quad v \in V,$$

we can rewrite (3.13) in the following form,

$$\int_v |f_\varepsilon - \chi_k| dv - |u_\varepsilon - k| = \int_{v \in V} |f_\varepsilon - \chi_k| \operatorname{sgn}(f_\varepsilon - \chi_k) s(v) dv = 2 \int_{v \in V} |f_\varepsilon - \chi_k| dv. \quad (3.14)$$

In view of (3.10), the identity (3.14) implies

$$\int_{v \in V} |f_\varepsilon - \chi_k| dv \xrightarrow[\varepsilon \downarrow 0]{} 0 \quad \text{in } L^1_{\text{loc}}(R_t^+ \times R_x^d). \quad (3.15)$$

We conclude by noting that for any $b \in L^\infty(R_v)$,

$$\int_v b(v)|f_\varepsilon - \chi_k| dv - \operatorname{sgn}(u_\varepsilon - k) \int_v b(v)(f_\varepsilon - \chi_k) dv$$

$$= \int_v b(v)(f_\varepsilon - \chi_k) s(v) dv = 2 \int_{v \in V} b(v)|f_\varepsilon - \chi_k| dv \leqq 2 \sup_v b(v) \cdot \int_{v \in V} |f_\varepsilon - \chi_k| dv, \quad (3.16)$$

and (3.11) follows from (3.16) together with (3.15). □

Equipped with Lemma 3.4 we turn to the

Proof (*of Lemma 3.3*). By our assumption (3.9), there is a strongly convergent subsequence (still denoted by) $u_\varepsilon(x, t) \to u(x, t)$. Utilizing (3.10) we obtain

$$\overline{\int_v |f_\varepsilon - \chi_k| dv} = \overline{|u_\varepsilon - k|} = |u - k|. \quad (3.17)$$

Here the overbar denotes the weak* L^∞-limit of the indicated quantities after extraction of appropriate subsequences, if necessary. (We note that the existence of the weak* L^∞ limits here and below are justified, since in view of (2.11), $f_\varepsilon(x, v, t)$ remains compactly supported in R_v and uniformly bounded with respect to ε in $L^1(R_v; L^\infty(R_x^d))$.)

By (1.2) we have

$$f_\varepsilon - \chi_{u_\varepsilon} = -\varepsilon[\partial_t + a(v) \cdot \partial_x] f_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} 0 \quad \text{in } \mathscr{D}',$$

and hence by (3.9)

$$\overline{\int_v a_i(v) f_\varepsilon dv} = \overline{\int_v a_i(v) \chi_{u_\varepsilon} dv} = \overline{A_i(u_\varepsilon)} = A_i(u). \quad (3.18)$$

This together with (3.11) gives

$$\overline{\int_v a_i(v)|f_\varepsilon - \chi_k| dv} = \overline{\operatorname{sgn}(u_\varepsilon - k) \int_v a_i(v)(f_\varepsilon - \chi_k) dv}$$

$$= \operatorname{sgn}(u - k) \cdot \left(\overline{\int_v a_i(v) f_\varepsilon dv} - \int_v a_i(v) \chi_k dv \right)$$

$$= \operatorname{sgn}(u - k)(A_i(u) - A_i(k)). \quad (3.19)$$

Hence, in view of (3.17) and (3.19), the weak limit of (3.6) recovers the entropy inequality

$$\frac{\partial}{\partial t}|u-k| + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}[\operatorname{sgn}(u-k)(A_i(u)-A_i(k))] \leqq 0. \tag{3.20}$$

The above argument shows that the strong limit of *any* subsequence of u_ε satisfies the entropy inequality (3.20). Since the entropy solution of (1.1) which assumes the initial data $u_0(x)$ is unique, we conclude that $\lim_{\varepsilon \downarrow 0} u_\varepsilon(x,t) = u(x,t)$ as asserted. □

We now turn to show that the continuum "fluid" limit of the kinetic equation (1.2) exists and is governed by the conservation law (1.1). By Lemma 3.3 it remains to show that $u_\varepsilon(x,t)$ is precompact in $L^\infty_{\text{loc}}(R_t^+; L^1(R_x^d))$. In this context there is (by now) a standard procedure, e.g., [4], which is based on uniform Bounded Variation (BV) estimate for each fixed t, coupled with equicontinuity (typically, Lipschitz continuity), in time. This brings us to our next lemma which states

Lemma 3.5. *Assume that*

$$\| f_\varepsilon(x,v,0) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))} \equiv \sup_{|\Delta x| \neq 0} \left[\frac{1}{|\Delta x|} \int_x \int_v |f_\varepsilon(x+\Delta x, v, 0) - f_\varepsilon(x,v,0)| dv dx \right]$$

is bounded uniformly in ε. *Then the corresponding kinetic solution,* $f_\varepsilon(x,v,t)$, *satisfies*

$$\| f_\varepsilon(x,v,t) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))} \leqq \| f_\varepsilon(x,v,0) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))}. \tag{3.21}$$

Moreover, if $f_\varepsilon(x,v,0) \in L^1(R_v; \mathrm{BV}(R_x^d))$ *are compactly supported in* R_v, *then we also have for* $t_1, t_2 \geqq 0$,

$$\| u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1) \|_{L^1(R_x^d)} \leqq |t_2 - t_1| \cdot a_\infty \cdot \| f_\varepsilon(x,v,0) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))}. \tag{3.22}$$

Proof. Since the kinetic model (1.2) is translation invariant in spatial variables, we can apply the L^1-contraction (2.5) to $f_\varepsilon(x,v,t)$ with $g_\varepsilon \equiv f_\varepsilon(x+\Delta x, v, t)$ and obtain (3.21).

Integration of the kinetic equation (1.2) over the phase space yields

$$\frac{\partial}{\partial t} u_\varepsilon(x,t) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \int_v a_i(v) f_\varepsilon(x,v,t) dv = 0,$$

and since $f_\varepsilon(x,v,0) \in L^1(R_v; \mathrm{BV}(R_x^d)) \subset L^1(R_v; L^\infty(R_x^d))$ is compactly supported in R_v, we may use the finite speed of propagation bound in Sect. 2 to conclude

$$\begin{aligned} \| u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1) \|_{L^1(R_x^d)} &= \int_{\tau=t_1}^{t_2} \left\| \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \int_v a_i(v) f_\varepsilon(x,v,\tau) \right\|_{L^1(R_x^d)} d\tau \\ &\leqq \int_{\tau=t_1}^{t_2} a_\infty \| f_\varepsilon(x,v,\tau) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))} d\tau. \end{aligned}$$

Also, since $f_\varepsilon(x,v,0) \in L^1(R_v; \mathrm{BV}(R_x^d)) \subset \mathrm{BV}(R_x^d \times L^1(R_v))$, the last inequality together with (3.21) imply the Lipschitz continuity in time, (3.22), which completes the proof. □

Remark. In the course of proving Lemma 3.3, consult (3.18), we established only the weak* L^∞ convergence of the spatial fluxes. However, equipped with the BV setup of Lemma 3.5 we are able to derive strong convergence. Indeed, one may utilize the integral representation (2.2) to conclude that in this case we have

$$f_\varepsilon(x,v,t) \to \chi_{u(x,t)}(v) \quad \text{strongly in} \quad L^1([0,T] \times R_x^d \times R_v).$$

This together with the finite speed of propagation imply

$$\left\| \int_v a_i(v) f_\varepsilon(x,v,t) - A_i(u(x,t)) \right\|_{L^1([0,T]\times R_x^d)}$$
$$\leqq a_\infty \cdot \| f_\varepsilon(x,v,t) - \chi_{u(x,t)}(v) \|_{L^1([0,T]\times R_x^d \times R_v)} \xrightarrow[\varepsilon\downarrow 0]{} 0,$$

in contrast to the weak convergence stated in (3.18). We shall omit the details (consult Theorem 3.7 below), and we turn now to summarize our results by stating the following.

Theorem 3.6. *Suppose* $f_\varepsilon(x,v,0) \in L^1(R_v; L^1 \cap L^\infty(R_x^d))$ *such that*

$$u_\varepsilon(x,0) = \int_v f_\varepsilon(x,v,0)dv \to u_0(x) \quad \text{in } L^1(R_x^d). \tag{3.23}$$

Then the local density of the corresponding kinetic solution, $u_\varepsilon \equiv \int_v f_\varepsilon(x,v,t)dv$, *converges to the unique entropy solution of (1.1),* i.e., *we have*

$$\int_v f_\varepsilon(x,v,t)dv \to u(x,t) \quad \text{in } L^\infty([0,T]; L^1(R_x^d)), \tag{3.24}$$

and the entropy inequality (3.7) holds.

Proof. We begin by first assuming that $f_\varepsilon(x,v,0)$ is compactly supported in $L^1(R_v; \mathrm{BV}(R_x^d))$, uniformly with respect to ε. By Theorem 2.1 (consult (2.12)), $u_\varepsilon(x,t)$ are uniformly bounded, and by (3.21) they have uniformly bounded spatial variation, i.e.,

$$\| u_\varepsilon(x,t) \|_{\mathrm{BV}(R_x^d)} \leqq \| f_\varepsilon(x,v,t) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))} \leqq \mathrm{Const}.$$

Hence $\{u_\varepsilon(x,t), 0 \leqq t \leqq T\}$ is a bounded set in $L^1 \cap \mathrm{BV}(R_x^d)$ and by Helly's theorem it is therefore precompact in $L^1_{\mathrm{loc}}(R_x^d)$. By (3.22), $\| u_\varepsilon(x,t) \|_{L^1(R_x^d)}$ is Lipschitz continuous in time, and by Cantor diagonalization process of passing to further subsequence if necessary, (3.24) follows. By Lemma 3.3 this completes the convergence proof for compactly supported BV initial data. The general case is justified by standard cutoff and BV-regularization of arbitrary $L^1 \cap L^\infty(R_x^d)$ initial data, consult [4]. □

We continue with a couple of remarks.

1. *The Kinetic Initial Layer.* We observe that Lemma 3.5 supplies us with an ε-uniform bound on the spatial variation on the microscopic scale, (3.21). The temporal variation (Lipschitz continuity), however, is uniformly bounded only on the macroscopic scale, (3.22). In general, one cannot control the temporal variation in the microscopic scale (uniformly in ε), unless we can prevent the possibility of a kinetic initial layer in (1.2). To this end we proceed by

2. *Preparing the Kinetic Initial Data.* In order to avoid a kinetic initial layer, we have to bound $\frac{\partial}{\partial t} f_\varepsilon$ uniformly in ε and time, in particular at $t=0$. Taking into account the uniform bound (in ε and t) of the spatial variation, (3.21), it remains

to bound the nonlinear "interaction" kernel on the right of (1.2), $\frac{1}{\varepsilon}(\chi_{u_\varepsilon} - f_\varepsilon)$. In particular, we therefore need

$$\| f_\varepsilon(x, v, 0) - \chi_{u_\varepsilon(x,0)}(v) \|_{L^1(R_x^d \times R_v)} \xrightarrow[\varepsilon \downarrow 0]{} 0. \tag{3.25}$$

Since by our assumption (3.23) we already have that

$$\| \chi_{u_\varepsilon(x,0)}(v) - \chi_{u_0(x)}(v) \|_{L^1(R_x^d \times R_v)} = \| u_\varepsilon(x) - u_0(x) \|_{L^1(R_x^d)} \xrightarrow[\varepsilon \downarrow 0]{} 0,$$

the requirement (3.25) boils down to

$$\| f_\varepsilon(x, v, 0) - \chi_{u_0(x)}(v) \|_{L^1(R_x^d \times R_0)} \xrightarrow[\varepsilon \downarrow 0]{} 0. \tag{3.26}$$

Thus, given the initial conditions $u(x, t = 0) = u_0(x)$, we have to prepare the kinetic initial data, $f_\varepsilon(x, v, 0)$, such that (3.26) holds. If we prepare the kinetic initial data in such a manner, then we can derive explicit bounds (*uniform in time*) on the error between the kinetic solution and the exact entropy solution, as told by

Theorem 3.7. (Error bound). *We consider kinetic initial data, $f_\varepsilon(x, v, 0) \in L^1(R_v; \mathrm{BV}(R_x^d))$ which are compactly supported in R_v. Suppose we prepare the kinetic initial data so that*

$$\| f_\varepsilon(x, v, 0) - \chi_{u_0(x)}(v) \|_{L^1(R_x^d \times R_v)} \xrightarrow[\varepsilon \downarrow 0]{} 0. \tag{3.27}$$

Then the following error bound holds

$$\begin{aligned} &\| f_\varepsilon(x, v, t) - \chi_{u_\varepsilon(x,t)}(v) \|_{L^1(R_x^d \times R_v)} \\ &\quad \leqq 2\varepsilon a_\infty \| f_\varepsilon(x, v, 0) \|_{L^1(R_v; \mathrm{BV}(R_x^d))} + 2 \| f_\varepsilon(x, v, 0) - \chi_{u_0(x)}(v) \|_{L^1(R_x^d \times R_v)} \xrightarrow[\varepsilon \downarrow 0]{} 0. \end{aligned} \tag{3.28}$$

Consequently, we have

$$f_\varepsilon(x, v, t) \to \chi_{u(x,t)}(v) \quad \text{strongly in } L^\infty(R_t^+; L^1(R_x^d \times R_v)). \tag{3.29}$$

Note. Preparing the kinetic initial data according to (3.27) is a strengthened version of our assumption (3.23). In this case, the kinetic distribution converges strongly and uniformly in time, to the equilibrium state χ_u, as expected. Also, all the weak limits indicated in the proofs of Lemma 3.3 and 3.4 are in fact strong ones; in particular we now have strong convergence of the corresponding fluxes

$$\int_v a_i(v) f_\varepsilon dv \to A_i(u) \quad \text{in } L^\infty(R_t^+; L^1(R_x^d \times R_v)),$$

compared with (3.18).

Proof. Since the kinetic model (1.2) is translation invariant in time, we can apply the L^1-construction (2.5) to $f_\varepsilon(x, v, t)$ with $g_\varepsilon \equiv f_\varepsilon(x, v, t + \Delta t)$ and obtain

$$\left\| \frac{\partial}{\partial t} f_\varepsilon(x, v, t) \right\|_{L^1(R_x^d \times R_v)} \leqq \left\| \frac{\partial}{\partial t} f_\varepsilon(x, v, t = 0) \right\|_{L^1(R_x^d \times R_v)}. \tag{3.30}$$

The kinetic equation (1.2a) enables us to upper bound the right-hand side of (3.30), namely,

$$\left\|\frac{\partial}{\partial t} f_\varepsilon(x,v,t=0)\right\|_{L^1(R_x^d\times R_v)} \leqq \|[a(v)\cdot\partial_x] f_\varepsilon(x,v,t=0)]\|_{L^1(R_x^d\times R_v)} + \frac{1}{\varepsilon}\|\chi_{u_\varepsilon(x,t=0)} - f_\varepsilon(x,v,t=0)\|_{L^1(R_x^d\times R_v)}. \quad (3.31)$$

The first term on the right of (3.31) does not exceed

$$\|[a(v)\cdot\partial_x] f_\varepsilon(x,v,t=0)\|_{L^1(R_x^d\times R_v)} \leqq a_\infty \|f_\varepsilon(x,v,0)\|_{L^1(R_v;\mathrm{BV}(R_x^d))};$$

the second term is less than

$$\frac{1}{\varepsilon}\|\chi_{u_\varepsilon(x,t=0)}(v) - f_\varepsilon(x,v,t=0)\|_{L^1(R_x^d\times R_v)} \leqq \frac{2}{\varepsilon}\|f_\varepsilon(x,v,0) - \chi_{u_0}(v)\|_{L^1(R_x^d\times R_v)}.$$

Substituting the last two estimates into (3.31) we end up with

$$\varepsilon\left\|\frac{\partial}{\partial t} f_\varepsilon(x,v,t)\right\|_{L^1(R_x^d\times R_v)} \leqq \varepsilon a_\infty \|f_\varepsilon(x,v,0)\|_{L^1(R_v;\mathrm{BV}(R_x^d))} + 2\|f_\varepsilon(x,v,0) - \chi_{u_0(x)}(v)\|_{L^1(R_x^d\times R_v)}. \quad (3.32)$$

Finally, we use the kinetic equation (1.2a) once more, obtaining

$$\begin{aligned}\|f_\varepsilon(x,v,t) - \chi_{u_\varepsilon(x,t)}(v)\|_{L^1(R_x^d\times R_v)} &\leqq \varepsilon\left\|\frac{\partial}{\partial t} f_\varepsilon(x,v,t)\right\|_{L^1(R_x^d\times R_v)} + \varepsilon\|[a(v)\cdot\partial_x] f_\varepsilon(x,v,t)\|_{L^1(R_x^d\times R_v)} \\ &\leqq 2\varepsilon a_\infty \|f_\varepsilon(x,v,0)\|_{L^1(R_v;\mathrm{BV}(R_x^d))} + 2\|f_\varepsilon(x,v,0) - \chi_{u_0(x)}(v)\|_{L^1(R_x^d\times R_v)}, \end{aligned} \quad (3.33)$$

and (3.28) follows.

By Theorem 3.6 we also have that $u_\varepsilon - u$ and consequently that $\chi_{u_\varepsilon} - \chi_u$ converges strongly and uniformly in time to zero, and by adding this to (3.28) we obtain (3.29) as asserted. □

We note in passing that the L^1-contraction and the related BV estimates stated in Sect. 2 and Lemma 3.5 are *not* identical with the usual L^1-contraction statements concerning viscosity regularizations of entropy solutions of (1.1). In fact, at any fixed time level, we have

$$\|f_\varepsilon - g_\varepsilon\|_{L^1(R_x^d\times R_v)} \geqq \int_x \left|\int_v (f_\varepsilon - g_\varepsilon)dv\right| dx = \|u_\varepsilon - w_\varepsilon\|_{L^1(R_x^d)}.$$

By (3.29), however, the two statements coincide in the limit as $\varepsilon\downarrow 0$,

$$\|f_\varepsilon - g_\varepsilon\|_{L^1_{\mathrm{loc}}(R_x^d\times R_v)} \to \int_{x,v} |\chi_u - \chi_w| dv dx = \|u - w\|_{L^1(R_x^d)},$$

and we recover the L^1-contraction (and the corresponding BV estimates) for entropy solutions of the conservation law (1.1).

We close this section by calling attention to a rather unusual result in the theory kinetic equations. Namely, if $u(x,t)$ is a *smooth* solution of the conservation law (1.1), then the equilibrium function $\chi_{u(x,t)}(v)$ is an L^1-solution of the corresponding kinetic equation (1.2). That is

Theorem 3.8 (exact solutions). *If $u(x,t) \in C \cap L^1([0,T] \times R_x^d)$ satisfies the conservation law (1.1), then $\chi_{u(x,t)}(v)$ is a kinetic solution (1.2) on $R_x^d \times [0,T]$.*

Note. Theorem 3.8 is no longer valid when $u(x,t)$ contains shock discontinuities. After the formation of shock waves, the corresponding kinetic solution has a "multivalued" form, e.g., $\chi_{u_1(x,t)}(v) + \chi_{[u_2(x,t),u_3(x,t)]}(v)$, as in the transport collapse method of Brenier [2].

Proof. We have to show that $f_\varepsilon(x,v,t) = \chi_{u(x,t)}(v)$ satisfies the kinetic equation (1.2a), i.e., that for any C_0^∞ test function $\phi(x,v,t)$

$$\int_0^T \int_{R_x^d \times R_v} \chi_{u(x,t)}(v)[\partial_t + a(v)\cdot\partial_x]\phi(x,v,t)dx\,dv\,dt = 0. \tag{3.34}$$

Since the integration in R_v is compactly supported (on $[-u_\infty, u_\infty]$), it is enough to consider successively $\phi(x,v,t) = \phi(x,t)\cdot\{1, v, v^2, \ldots\}$, in which case (3.34) amounts to the equivalent conservation laws

$$\frac{\partial}{\partial t}\left[\frac{u^p(x,t)}{p}\right] + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[\int^{u(x,t)} v^{p-1} a_i(v)dv\right] = 0, \quad p = 1,2,\ldots \text{ in } \mathscr{D}'. \tag{3.35}$$

Indeed, (3.35) are the usual entropy *equalities* satisfied by *continuous* solutions of (1.1), but violating (for $p > 1$) the Rankine–Hugoniot conditions after the formation of shock discontinuities.

4. Microscopic Oscillations and Compensated Compactness

In this section we deal with the one-dimensional scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial A(u)}{\partial x} = 0. \tag{4.1}$$

The corresponding underlying kinetic equation reads

$$\left[\frac{\partial}{\partial t} + a(v)\frac{\partial}{\partial x}\right] f_\varepsilon(x,v,t) = \frac{1}{\varepsilon}[\chi_{u_\varepsilon(x,t)}(v) - f_\varepsilon(x,v,t)], \quad a(\cdot) \equiv A'(\cdot), \tag{4.2}$$

and we raise the question of convergence of the local "particles density," $u_\varepsilon(x,t) = \int_v f_\varepsilon(x,v,t)\,dv$, towards the entropy solution, $u(x,t)$, of (4.1). In this section we give an affirmative answer to this question, which is independent of compactness arguments, i.e., the BV estimates used in Lemma 3.4. Instead, we appeal to compensated compactness arguments, specifically, we employ Tartar's div-curl lemma [13]. In this context, it is instructive to see how oscillations which persist on the microscopic scale are "compensated" in a manner which enables us to pass to the limit on the macroscopic scale. We have

Theorem 4.1. *Consider the* nonlinear *conservation law (4.1), and let*

$$f_\varepsilon \in L^\infty(R_t^+; L^1(R_v; L^1 \cap L^\infty(R_x)))$$

be the solution of the corresponding kinetic equation (4.2), with convergent initial *averages*

$$u_\varepsilon(x,0) \equiv \int_v f_\varepsilon(x,v,0)dv \to u_0(x) \quad \text{in} \quad L^1(R_x^d).$$

Then $u_\varepsilon(x,t) \equiv \int_v f_\varepsilon(x,v,t)dv$ *converges strongly in* $L^p_{\text{loc}}(R_x \times R_t^+)$, $p < \infty$, *to the unique entropy solution of the nonlinear conservation law (4.1), corresponding to initial conditions* $u(x,t=0) = u_0(x)$.

Remarks.
1. The conservation law (4.1) is *nonlinear* in the sense that there exists no interval on which the flux $A(u)$ in linear, i.e., $A''(u) \neq 0$ a.e.
2. Theorem 4.1 indicates that there are two procedures which are responsible for the cancellation of oscillations: microscopic oscillations in $f_\varepsilon(x,v,t)$ are averaged out by integration over the phase space, as before; and in addition, macroscopic oscillations in $u_\varepsilon(x,t)$ are annihilated thanks to the nonlinearity of the conservation law (4.1).

Proof. Integration of (4.2) over the phase space yields

$$\partial_t u_\varepsilon + \partial_x \int_v a(v) f_\varepsilon dv = 0. \tag{4.3a}$$

The corresponding entropy inequality reads

$$\partial_t \int_v |f_\varepsilon - \chi_k| dv + \partial_x \int_v a(v)|f_\varepsilon - \chi_k| dv \leqq 0. \tag{4.3b}$$

Since by (2.11) the left-hand side of (4.3b) lies in $W^{-1,\infty}$, Murat's lemma [10], [13] implies that the negative measure on the right of (4.3b) lies in a compact set of $H^{-1}_{\text{loc}}(R_x \times R_t^+)$. Hence we can apply the div-curl lemma [13] to the left-hand sides of (4.3a) and (4.3b), which gives

$$\begin{aligned}&\overline{u_\varepsilon \int_v a(v)|f_\varepsilon - \chi_k| dv} - \overline{\int_v a(v) f_\varepsilon dv \cdot \int_v |f_\varepsilon - \chi_k| dv} \\ &\quad = \bar{u}_\varepsilon \cdot \overline{\int_v a(v)|f_\varepsilon - \chi_k| dv} - \overline{\int_v a(v) f_\varepsilon dv} \cdot \overline{\int_v |f_\varepsilon - \chi_k| dv}.\end{aligned} \tag{4.4}$$

We recall that the overbar denotes the weak* L^∞-limit of the indicated quantities after extraction of appropriate subsequences, if necessary. Following [12], we can rewrite (4.4) in the equivalent form

$$\overline{(u_\varepsilon - \bar{u}_\varepsilon) \cdot \int_v a(v)|f_\varepsilon - \chi_k| dv} = \overline{\int_v |f_\varepsilon - \chi_k| dv \cdot \left(\int_v a(v) f_\varepsilon dv - \overline{\int_v a(v) f_\varepsilon dv} \right)}. \tag{4.5}$$

Using (3.10) and (3.11), the last equality is further simplified into

$$\overline{(u_\varepsilon - \bar{u}_\varepsilon) \cdot \operatorname{sgn}(u_\varepsilon - k) \int_v a(v)(f_\varepsilon - \chi_k) dv} = \overline{|u_\varepsilon - k| \cdot \left(\int_v a(v) f_\varepsilon dv - \overline{\int_v a(v) f_\varepsilon dv} \right)}. \tag{4.6}$$

We now examine (4.6) at an arbitrary *fixed* location (x,t); with $k = \bar{u}_\varepsilon(x,t)$ we find after little rearrangement

$$\overline{|u_\varepsilon - \bar{u}_\varepsilon| \cdot \left(\int_v a(v) \chi_k dv - \int_v a(v) f_\varepsilon dv \right)} = 0. \tag{4.7}$$

Of course, by (4.2)

$$f_\varepsilon - \chi_{u_\varepsilon} = -\varepsilon[\partial_t + a(v)\partial_x]f_\varepsilon \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \quad \text{in } \mathscr{D}',$$

hence

$$\overline{\int_v a(v)f_\varepsilon dv} = \overline{\int_v a(v)\chi_{u_\varepsilon} dv} = \overline{A(u_\varepsilon)}. \tag{4.8}$$

Also, we recall that with $k = \bar{u}_\varepsilon(x,t)$ we have

$$\int_v a(v)\chi_k dv = A(\bar{u}_\varepsilon). \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.7) we find

$$\overline{|u_\varepsilon - \bar{u}_\varepsilon|\cdot(A(\bar{u}_\varepsilon) - A(u_\varepsilon))} = 0.$$

This implies that

$$\overline{A(u_\varepsilon)} = A(\bar{u}_\varepsilon), \tag{4.10}$$

for otherwise, $\overline{|u_\varepsilon - \bar{u}_\varepsilon|}(x,t) = 0$, which in turn leads again to (4.10). Taking the weak limit of (4.2), we obtain with the help of (4.8) and (4.10),

$$\frac{\partial}{\partial t}\bar{u}_\varepsilon + \frac{\partial}{\partial x}A(\bar{u}_\varepsilon) = 0.$$

Thus, (a subsequence of) $u_\varepsilon(x,t)$ converges to a weak solution of the conservation law (4.1). Moreover, in view of the nonlinearity of $A(u)$, equality (4.10) implies that $u_\varepsilon(x,t)$ converges strongly in $L^p_{\mathrm{loc}}(R_x \times R_t^+)$, $1 \leqq p < \infty$, consult Tartar [13, Theorem 26]. Using this fact together with Lemma 3.3 we conclude that u_ε converges strongly in $L^p_{\mathrm{loc}}(R_x \times R_t^+)$, $p < \infty$, to the unique entropy solution of (4.1), as asserted.

5. Conservation Laws with a Source Term

In this section we extend the above results to inhomogeneous scalar conservation laws

$$\frac{\partial}{\partial t}[u(x,t)] + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}[A_i(u(x,t))] = S(x,t,u), \quad (x,t) \in R_x^d \times R_t^+, \tag{5.1}$$

where $S(x,t,\cdot)$ is an $L^\infty(R_x^d \times R_t^+; C^1)$ source term satisfying $S(x,t,0) \equiv 0$.

The corresponding kinetic model equation reads

$$[\partial_t + a(v)\cdot\partial_x]f_\varepsilon(x,v,t) = \frac{1}{\varepsilon}[\chi_{u_\varepsilon(x,t)}(v) - f_\varepsilon(x,v,t)] + S'(x,t,v)f_\varepsilon(x,v,t), \tag{5.2}$$

and is augmented with the constitutive relations (1.2b), (1.2c).

A unique kinetic solution for (5.2) can be constructed, as before, by Banach fixed point iterations which yield

Theorem 5.1. *The kinetic model (5.2), (1.2b–c) is well-posed in $L^\infty(R_t^+; L^1(R_x^d \times R_v))$. Moreover, if f_ε and g_ε are two different inhomogeneous kinetic solutions of (5.1), and* *if we let $S'_\infty(t)$ denote*

$$S'_\infty(t) = \left\{ \max_{x,v} S'(x,t,v) \,\middle|\, v \in \operatorname*{supp}_v f_\varepsilon(x,v,t) \cup \operatorname*{supp}_v g_\varepsilon(x,v,t) \right\}, \tag{5.3}$$

then we have

$$\max_{0 \leqq t \leqq T} \| f_\varepsilon(x,v,t) - g_\varepsilon(x,v,t) \|_{L^1(R_x^d \times R_v)}$$
$$\leqq \exp\left\{ \int_{t=0}^{T} S'_\infty(t)dt \right\} \cdot \| f_\varepsilon(x,v,0) - g_\varepsilon(x,v,0) \|_{L^1(R_x^d \times R_v)} \tag{5.4}$$

We shall only indicate the proof of the $L^\infty(R_t^+; L^1(R_x^d \times R_v)$ stability stated in (5.4). The difference between the kinetic solutions $f_\varepsilon - g_\varepsilon$ (with corresponding local densities, $u_\varepsilon(x,t) = \int_v f_\varepsilon(x,v,t)dv$ and $w_\varepsilon(x,t) = \int_v g_\varepsilon(x,v,t)dv$) satisfies

$$[\partial_t + a(v)\cdot\partial_x](f_\varepsilon - g_\varepsilon) = \frac{1}{\varepsilon}[(\chi_{u_\varepsilon(x,t)}(v) - \chi_{w_\varepsilon(x,t)}(v)) - (f_\varepsilon - g_\varepsilon)] + S'(x,t,v)(f_\varepsilon - g_\varepsilon).$$

Multiplying this by $\operatorname{sgn}(f_\varepsilon - g_\varepsilon)$ and integrating over R_v and R_x^d (in this order), we obtain

$$\frac{d}{dt} \| f_\varepsilon(x,v,t) - g_\varepsilon(x,v,t) \|_{L^1(R_x^d \times R_v)} \leqq S'_\infty(t) \cdot \| f_\varepsilon(x,v,t) - g_\varepsilon(x,v,t) \|_{L^1(R_x^d \times R_v)},$$

and (5.4) follows. □

We conclude with several remarks concerning the entropy inequality.

The corresponding inhomogeneous kinetic entropy inequality now reads

$$\int_v [\partial_t + a(v)\cdot\partial_x] | f_\varepsilon - \chi_k | dv \leqq \int_v S'(x,t,v) | f_\varepsilon - \chi_k | dv. \tag{5.5}$$

Moreover, by arguing along the lines of the stability estimate (5.4) we find that for $\mathrm{BV}(R_x^d)$ source terms we have

$$\| f_\varepsilon(x,v,T) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))}$$
$$\leqq \exp\left\{ \int_{t=0}^{T} S'_\infty(t)dt \right\} \| f_\varepsilon(x,v,0) \|_{\mathrm{BV}(R_x^d \times L^1(R_v))}$$
$$+ \int_{t=0}^{T} \exp\left\{ \int_{\tau=t}^{T} S'_\infty(\tau)d\tau \right\} \cdot \max_v \| S'(x,t,v) \|_{\mathrm{BV}(R_x^d)} dt \cdot \| f_\varepsilon(x,v,0) \|_{L^1(R_x^d \times R_v)}. \tag{5.6}$$

This allows us to keep the convergence statement of Theorem 3.6,

$$\int_v f_\varepsilon(x,v,t)dv \underset{\varepsilon \downarrow 0}{\longrightarrow} u(x,t), \quad \text{in} \quad L^\infty([0,T], L^1(R_x^d)),$$

in the inhomogeneous case (5.2). In view of (5.5), we are also able to recover the macroscopic "continuum limit" entropy inequality for the above limit $u = u(x,t)$, which in this case amounts to

$$\frac{\partial}{\partial t}|u-k| + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}[\operatorname{sgn}(u-k)(A_i(u) - A_i(k))]$$
$$\leqq \operatorname{sgn}(u-k)[S(x,t,u) - S(x,t,k)], \quad \text{for any real } k.$$

References

1. Boltzmann, L.: Vorlesungen uber Gas theorie, Liepzig, 1886
2. Y. Brenier, Y.: Averaged multivaried solutions for scalar conservation laws. SIAM J. Numer. Anal. **21**, 1013–1037 (1986)
3. Caflisch, R.: The fluid dynamic limit of the nonlinear Boltzmann equation. Commun. Pure Appl. Math. **33**, 651–666 (1980)
4. Crandall, M., Majda, A.: Monotone difference approximations for scalar conservation laws. Math. Comp. **34**, 1–21 (1980)
5. Crandall, M., Tartar, L.: Some relations between non-expansive and order preserving mappings. Proc. Am. Math. Soc. **78** (3) 385–390 (1980)
6. DiPerna, R., Lions, P. L.: On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Ann. Math. (1989)
7. Giga, Y., Miyakawa, T.: A kinetic construction of global solutions of first order quasilinear equations. Duke Math. J. **50**, 505–515 (1983)
8. Krushkov, S. N.: First order quasilinear equations in several independent variables. Math. USSR Sb. **10**, 217–243 (1970)
9. Lax, P. D.: Hyperbolic systems of conservation laws and the mathematical theory of shock waves. SIAM Regional Conference Series in Applied Mathematics, vol. 11
10. Murat, F.: Compacité per compensation. Ann. Scuola Norm. Sup. Disa Sci. Math. **5**, 489–507 (1978) and **8**, 69–102 (1981)
11. Perthame, B.: Global existence of solutions to the BGK model of Boltzmann equations. J. Diff. Eq. **81**, 191–205 (1989)
12. Tadmor, E.: Semi-discrete approximations to nonlinear systems of conservation laws; consistency and L^{∞}-stability imply convergence. ICASE Report No. 88-41
13. Tartar, L.: Compensated compactness and applications to partial differential equations. In: Research Notes in Mathematics, vol. 39, Nonlinear Analysis and Mechanics, Heriot-Watt Sympos., vol. 4. Knopps, R. J. (ed.) pp. 136–211. Boston, London: Pittman Press 1975

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