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The C^*-Algebra of Bosonic Strings*

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Abstract. We give a rigorous definition of Witten's C^*-string-algebra. To this end we present a new construction of C^*-algebras associated to special geometric situations (Kähler foliations) and generalize this later construction to the string case. Through this we get a natural geometrical interpretation of the occurrence of semi-infinite forms as well as the fermionic algebra structure. Using the (non-commutative) geometric concepts for investigating the string algebra we get a natural Fredholm module representation of dimension 26+.

1. Introduction

One of the nice features in the development of string theory turned out to be the simple ways in which strings can interact. For example the open bosonic strings can only either split or join. The joining of two strings to a new one may remind one of a product-like structure while the splitting may be compared to the factorization.

This idea was pushed forward by Witten in 1986, [Wi 1], where he introduced a groupoid structure for classical strings, see Sect. 1. In the Schrödinger picture of the first quantization of string theory one passes from the classical strings (continuous paths in $\mathbf{R}^{1,d-1}$) to wavefunctions on the space of classical strings. The groupoid structure yields in a canonical way a *-algebra-structure on the space of wavefunctions. The first part of this article gives a precision to Witten's definition and describes this (bosonic-) algebra explicitly.

Now the physics should not depend on the parametrization of the interacting strings. The reparametrization group is an infinite-dimensional Frechet-Lie-group, [Mi]. In the course of quantizing the theory we have to take care of the unphysical degrees of freedom caused by the symmetry. Usually this is done by introducing Faddeev-Popov-ghost-fields, which are fermionic fields, for dividng out the

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symmetry-part. Connected with this approach is the BRS-formalism. Witten also considered the Faddeev-Popov-ghost part of the theory. He defined for it a *-algebra structure using the bosonization of the fermionic fields and in this way traced it back to the bosonic algebra structure. Besides making the formal or heuristic definitions analytically precise there are a lot of questions concerning the interpretation.

In the BRS-formalism of gauge theory the chains of the BRS-complex are Maurer-Cartan-forms on the gauge group, [Bo-Co], but in string theory they are semi-infinite forms [F-G-Z]. In physics one argues that the vacuum of the fermionic Faddeev-Popov-ghost-fields is a filled Dirac-sea. This perspective is supposed to be the reason for using semi-infinite forms. This leads to the strange fact that the physical string fields are not ghost-free, they have nonvanishing ghost-number [Wi 1]! The common interpretation of the BRS-formalism as the restriction to symmetry-invariant objects doesn't work in a naive way. We have the famous conformal anomaly which only cancels by putting together bosonic- and fermionic-ghost-parts. Last but not least, why should one also introduce an algebra structure for the ghost part?

The second part of the paper deals with these questions. In the course of giving a precise definition for the fermionic or ghost part, we present a purely geometrical interpretation of the construction. This is motivated by Connes' construction of C^*-algebras associated to foliations, [Co 1]. We present a slightly different construction more appropriate to Kähler manifolds which can be generalized to the string case. This new construction is seen to yield the strange fermionic part of the algebra.

We investigate Connes' non-commutative geometry for the string algebra and get a natural Fredholm module representation of dimension 26+.

2. The Path Groupoid

As mentioned in the introduction the only possible interactions of open bosonic strings is their splitting and joining. This looks like factorization and multiplication in a group. Witten pushed this idea forward and defined a groupoid structure for paths.

Let

$$\omega_i[0, 1] \to \mathbf{R}^{1,d-1} \qquad i = 1, 2 \tag{1}$$

be continuous paths

$$\omega_i^s = \omega_i \mid [0, \tfrac{1}{2}] \quad \text{(source)} \quad \omega_i^r = \omega_i \mid [\tfrac{1}{2}, 1] \quad \text{(range)} \tag{2}$$

with overlapping halves

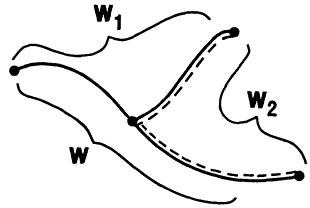
$$\omega_i^r(t) = \omega_2^s(\tfrac{1}{2} - t)\,. \tag{3}$$

Then their product is defined as the path

$$\omega_1 \bot \omega_2 := \omega \tag{4}$$

with

$$\omega(t) = \begin{cases} \omega_1(t) & t \in [0, \tfrac{1}{2}[\\ \omega_2(t) & t \in [\tfrac{1}{2}, 1] \end{cases} \tag{5}$$



Remark 1. This product for parametrized paths is associative

$$(\omega_1 \perp \omega_2) \perp \omega_3 = \omega_1 \perp (\omega_2 \perp \omega_3). \tag{6}$$

In quantizing the theory according to the Schrödinger picture one passes from classical objects to wavefunctions thereon, so that at a formal level we get

wavefunctions of 1. quantized string = functions on path space.

The groupoid structure of the path space yields formally an algebra structure for the wavefunctions as in the case of groups and group algebras:

Let Φ, Ψ be string wavefunctionals, then let

$$\Phi * \Psi(\omega) = \int_{\omega_1 \perp \omega_2 = \omega} \Phi(\omega_1) \cdot \Phi(\omega_2) d\mu(\omega_1) \otimes d\mu(\omega_2) \tag{7}$$

with μ a measure on the path space.

For doing this in a rigorously defined way we take the Euclidean space $\mathbf{R}^d$, instead of the Minkowski space, as the target space of the paths. Then the measure μ is defined as the direct sum of measures

$$d\mu = \bigotimes_{i=1}^{d} d\mu_{L_0} \oplus dx, \tag{8}$$

where $d\mu_{L_0}$ is the Gaussian measure with covariance L_0^{-1},

$$L_0 = \sqrt{-\Delta_N'} \tag{9}$$

and $-\Delta_N$ is the Laplacian on $L^2([0, 1])$ with Neumann boundary conditions in 0, 1, Δ_N' the restriction of Δ_N onto the complement of the kernel (without zero mode). The kernels sum up to $\mathbf{R}^d$. Let dx denote the Lebesgue measure on $\mathbf{R}^d$. We can split an arbitrary path into $\omega = \omega' + y$ with y=orthogonal projection of ω onto the kernel.

Then one rewrites at least formally the product as

$$\Phi * \Psi(\omega) = \int \Phi(\omega_1) \cdot \Psi(\omega_2) \delta(\omega_1^s - \omega^s) \delta(\omega_2^r - \omega^r) d\mu(\omega_1) d\mu(\omega_2). \tag{10}$$

Obviously the $\mathbf{R}^d$-integration (the kernel part) causes no trouble. It contributes to the algebra an injective-C^*-algebra tensor product with $C^0(\mathbf{R}^d)$. Therefore we will neglect this part in the following.

For notational simplification we temporarily take $d = 1$. The Gaussian measure μ_{L_0} is a cylinder measure on the space of Schwartzian distributions $\mathcal{S}'[0, 1]$, [Si]. The classical string space of continuous paths is to be substituted by a space of distributions, to be more precise, by the support of μ_{L_0}.

Lemma 2. *For μ_{L_0}-almost everywhere $T \in \mathcal{S}'[0,1]$ there exists no nonempty open set $U \subset [0,1]$ such that T restricted on U is a signed measure. In particular, T is not a continuous function on any such U.*

Proof. The covariance of μ_{L_0}

$$\begin{aligned} c(s,t) &= \sum_{n\geq 1} \frac{2}{n}(\cos n\pi t \cdot \cos n\pi s) \\ &= \ln\{4(1-\sin\pi(s+t))\,(1-\sin\pi(s-t))\} \end{aligned} \tag{11}$$

is obviously not continuous, which implies the lemma, see [Co-La]. □

On the support of μ_{L_0} we have to define the groupoid structure on "paths." These "paths" are distributions but are not in general continuous functions. It is not even clear what the splitting of strings into two parts should be. In general one would propose

$$\omega^s = \omega \cdot \chi_{[0,\frac{1}{2}]} \quad \text{for} \quad \omega \in \operatorname{supp}\mu_{L_0}, \tag{12}$$

where $\chi_{[0,\frac{1}{2}]}$ is the characteristic function of the interval $[0,\frac{1}{2}]$. This would be the product of two distributions and this is not always defined.

In the above situation we have a nice theorem which helps:

Theorem 3. *Let $L_{0,r}$ ($L_{0,s}$) be the square root of the Neumann Laplace operator on $L^2([\frac{1}{2},1])$ (respectively $L^2([0,\frac{1}{2})]$) without zero modes and denote H_0 the span of $\{\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}\}$. Then*

$$L_0^{-1} - (L_{0,s} + L_{0,r} + \alpha Id_{H_0})^{-1} \tag{13}$$

is of trace class for all $\alpha > 0$.

Proof. The $\{\psi_n\}_{n\in\mathbf{Z}}$ with

$$\begin{aligned} \psi_n(t) &= 2(\cos(2n\pi t)\chi_{[0,\frac{1}{2}]}(t) + \operatorname{sign}(n)\cos(2n\pi t)\chi_{[\frac{1}{2},1]}(t)) \quad n \neq 0, \\ \psi_0(t) &= \chi_{[0,\frac{1}{2}]}(t) - \chi_{[\frac{1}{2},1]}(t) \end{aligned} \tag{14}$$

build an orthonormal basis of eigenvectors of $L_{0,s} + L_{0,r} + \alpha Id_{H_0}$. The crucial part in the proof is to show the convergence of the sum

$$S = \sum_{m\in\mathbf{Z}\setminus\{0\}} \langle \psi_m, (L_0^{-1} - (L_{0,s} + L_{0,r} + \alpha Id_{H_0})^{-1})\psi_m\rangle < \infty. \tag{15}$$

Denote

$$\alpha_n^m := \int_0^{\frac{1}{2}} dt \cos n\pi t \cos m\pi t + \operatorname{sign}(m) \int_{\frac{1}{2}}^1 dt \cos m\pi t \cos n\pi t. \tag{16}$$

For the sum above one computes

$$
\begin{aligned}
S &= \sum_{m\in\mathbf{Z}\setminus\{0\}} \left(\sum_{n\in\mathbf{N}\setminus\{0\}} (\alpha_n^m)^2 \cdot \frac{1}{n} - \frac{1}{2|m|} \right) \\
&= \sum_{n\in\mathbf{N}\setminus\{0\}} \left(\sum_{m=1}^{\infty} \frac{1}{\pi^2} \frac{2n+1}{((2n+1)^2-(2m)^2)} - \frac{1}{2m} \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} \frac{1}{4m\pi^2} \left(\frac{1}{((2n+1)+2m)^2} - \frac{1}{((2n+1)-2m)^2} \right) - \frac{1}{2m} \right) \\
&= \sum_{m=1}^{\infty} \frac{1}{4m} \left(\sum_{n=m}^{\infty} \frac{1}{\pi^2} \frac{1}{(2n+1)^2} \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{c_1((2n+1)^{\frac{2}{3}}) \frac{8}{\pi^2} (2m)^{\frac{2}{3}}} \right) < \infty ,
\end{aligned} \tag{17}
$$

where one uses

$$
(2n+1+2m)^2 = (2m)^{\frac{1}{2}} \left(\frac{2n+1}{(2m)^{\frac{1}{4}}} + (2m)^{1-\frac{1}{4}} \right) \tag{18}
$$

and estimates

$$
\begin{aligned}
\frac{2n+1}{(2m)^{\frac{1}{4}}} + (2m)^{1-\frac{1}{4}} &\geq \left(\left(\frac{1}{3}\right)^{\frac{3}{4}} + \left(\frac{1}{3}\right)^{\frac{-1}{4}} \right) (2n+1)^{\frac{3}{4}} \\
&= c_1(2n+1)^{\frac{3}{4}} .
\end{aligned} \tag{19}
$$

The rest of the proof is straightforward □

For convenience let us denote $L_{0,s}+L_{0,r}+\alpha Id_{H_0}$ by L_α. Using Shale's Theorem this theorem has the following important

Corollary 4. *The Gaussian measures μ_{L_α} and μ_{L_0} are mutually absolutely continuous.*

Therefore the two measures have the same support. In the light of this result we cannot split a string into two halves but into three parts. We must incorporate the discontinuity at $t=\frac{1}{2}$,

$$
\begin{aligned}
\operatorname{supp} \mu_{L_0} &= \operatorname{supp} \mu_{L_{0,s}} \times \operatorname{supp} \mu_{L_{0,r}} \times \operatorname{supp} \mu_{\alpha Id_{H_0}} \\
&\subset \mathscr{S}'[0, \tfrac{1}{2}] \times \mathscr{S}'[\tfrac{1}{2}, 1] \times \mathbf{R} ,
\end{aligned} \tag{20}
$$

where we identified $\mathbf{R} \cong H_0$ by

$$
\begin{aligned}
&\mathbf{R} \cong H_0 \\
&x \mapsto x(\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]})
\end{aligned} \tag{21}
$$

so that the value x gives the jump at $t=\frac{1}{2}$.

Now we can generalize the groupoid structure of continuous paths to the support of μ_{L_0}. Let

$$
\omega_i = (\omega_i^s, \omega_i^r, x_i) \in \operatorname{supp} \mu_{L_\alpha} = \operatorname{supp} \mu_{L_0} , \qquad \omega_1^r = r(\omega_2^s) \tag{22}
$$

$$r : \mathscr{S}'[0,1] \to \mathscr{S}'[0,1] \qquad \omega \mapsto \omega(1-t) \tag{23}$$

then

$$\omega_1 \perp \omega_2 := (\omega_1^s, \omega_2^r, x_1 + x_2). \tag{24}$$

After these preparations we can give a rigorous definition of the bosonic part in Witten's construction. Let us start doing this by using the absolutely continuous measure μ_{L_α} instead of μ_{L_0}. Therefore let $\tilde{\Phi}$, $\tilde{\Psi} \in \mathscr{S}'[0,1]$ be two continuous functions on the support of μ_{L_α} then we can define their product by

$$\tilde{\Phi} \tilde{*}_\alpha \tilde{\Psi}(\omega^s, \omega^r, x) := \int d\mu_{L_{0,r}}(\tilde{\omega}) d\mu_{\alpha Id_{H_0}}(\tilde{x}) \{\tilde{\Phi}(\omega^s, \tilde{\omega}, \tfrac{1}{2}(x+\tilde{x})) \times \tilde{\Psi}(r(\tilde{\omega}), \omega^r, \tfrac{1}{2}(x-\tilde{x}))\}. \tag{25}$$

There is a natural involution for any such function $\tilde{\Phi}$ defined by the change of orientation

$$\tilde{\Phi}^*(\omega^s, \omega^r, x) := \tilde{\Phi}(r(\omega^r), r(\omega^s), -x). \tag{26}$$

Now consider first the x-part. A close look shows that for $\alpha < \infty$ the measure $d\mu_{\alpha Id_{H_0}}$ is not translation invariant, the resulting product structure is not associative! The half-string part of the algebra can be identified isomorphically to the C^*-algebra generated by the integral operators with smooth kernel on $L^2(\mathscr{S}'[\frac{1}{2}, 1], d\mu_{L_{0,r}})$, denoted by I_∞, the ideal of compact operators thereon.

Let us pass to the string case. We replace the measure $d\mu_{L_\alpha}$ by $d\mu_{L_0}$. There exists a Radon-Nikodym-derivative

$$d\mu_{L_0} = F_\alpha d\mu_{L_\alpha} \qquad F_\alpha > 0 \quad \text{a.e.} \tag{27}$$

which can be computed explicitly [Si]. For our purposes only the α-dependence is important. One get

$$F(\omega^s, \omega^r, x) = \sqrt{\alpha}\, e^{\frac{x^2}{2\alpha}} \tilde{F}(\omega^s, \omega^r, x) \tag{28}$$

with $\tilde{F}$ independent of α.

For defining the product of two string-functionals Φ, $\Psi \in \mathscr{S}[0,1]$ we replace the measures and obtain

$$\Phi *_\alpha \Psi = F_\alpha^{-1}((F_\alpha \cdot \Phi) *_\alpha (F_\alpha \cdot \Psi)). \tag{29}$$

This definition depends on α and putting the above dependence into the formulas and taking the limit $\alpha \to \infty$ yields

$$\Phi * \Psi(\omega^s, \omega^r; x) = \frac{1}{\sqrt{2\pi}} \tilde{F}^{-1}(\omega^s, \omega^r, x) \int d\mu_{L_{0,r}}(\tilde{\omega}) d\tilde{x} \left\{ \tilde{F}\left(\omega^s, \tilde{\omega}, \frac{1}{2}(x+\tilde{x})\right) \times \Phi\left(\omega^s, \tilde{\omega}, \frac{1}{2}(x+\tilde{x})\right) \tilde{F}\left(r(\tilde{\omega}), \omega^r, \frac{1}{2}(x-\tilde{x})\right) \times \Psi\left(r(\tilde{\omega}), \omega^r, \frac{1}{2}(x-\tilde{x})\right) \right\}. \tag{30}$$

It is not difficult to see that the x-part is just the convolution. Fourier transformation maps it isomorphically to the commutative algebra of functions. Going to completions this part is identified as the C^*-algebra of continuous $\mathbf{C}$-valued functions on $\mathbf{R}$ vanishing at infinity and equipped with the sup-norm. This C^*-algebra is denoted by $C^0(\mathbf{R})$.

Therefore this bosonic part of the string-algebra is isomorphic to the tensor product of the two algebras,

$$\text{(momentum-zero-)bosonic string-algebra} \cong C^0(\mathbf{R}) \otimes I_\infty . \tag{31}$$

This result agrees quite well with general topological considerations [Cr-Go]. Look at the foliation

$$\begin{array}{c} P(M) = \{f : [0, 1] \longrightarrow M \text{ continuous}\}, \\ \downarrow \pi \\ M \end{array} \tag{32}$$

where M is a C^∞-manifold. The foliation is defined by $\pi(f) = f(\frac{1}{2})$, the string groupoid can be viewed as the foliation groupoid. The K-theory of the bosonic string-algebra should describe the transversal cohomology of the foliation. This is obviously the case for $\mathbf{R}^d$.

To get the full bosonic part of Witten's string algebra we have to tensor this algebra with $C^0(\mathbf{R}^d)$, see (10)ff.

3. Symmetry Consideration

There is a gap in the construction of the previous section. The algebra should only depend on the geometry of the "paths", that is on the shape of the strings and not on their special parametrizations. For investigating this drawback we have to look at the reparametrization properties of the string. The space of strings is $\operatorname{supp}\mu_{L_0}$. For the half-strings L_0 is replaced by $L_{0,r}$ or $L_{0,s}$. These spaces are contained in the associated weighted spaces. For example the right halves in

$$H_r := \overline{\operatorname{span}\mathscr{S}[\tfrac{1}{2}, 1]} \quad \text{(right-halves-string Hilbert space)} \tag{33}$$

with

$$\|\phi\|_r^2 = \int dt \phi(t) \overline{(L_{0,r}^{-1}\phi)}(t) . \tag{34}$$

H_s is analogously defined. The general string turned out to be nowhere continuous and in addition to the two halves we need for the full description the jump in the middle. We take as the space of strings

$$\{\text{strings}\} = \{(\omega^s, \omega^r, x)/\omega^s \in H_s, \omega^r \in H_r, x \in \mathbf{R}\} \tag{35}$$

considered as a direct sum of Hilberts spaces.

What is a reparametrization of a string? For continuous strings one would suppose any homeomorphism of the parameter interval. But the strings are only distributionally defined and we have to take differentiable bijections, forming the group Diff[0, 1]. To be compatible with the path groupoid structure we further have to assume

$$\gamma \in \operatorname{Diff}[0, 1] \quad \text{with} \quad \gamma(\tfrac{1}{2}) = \tfrac{1}{2} . \tag{36}$$

So let

$$\gamma_s := \gamma \mid [0, \tfrac{1}{2}], \qquad \gamma_r := \gamma \mid [\tfrac{1}{2}, 1] \tag{37}$$

be the restrictions onto halves. Then a reparametrization of a string-half is given by

$$\gamma_r^*(\omega^r) = L_{0,r}^{-\frac{1}{2}}\left((L_{0,r}^{\frac{1}{2}}\omega^r)(\gamma_r^{-1}) \cdot |\gamma_r'|^{-\frac{1}{2}}\right) \tag{38}$$

for a $\omega^r \in \mathscr{S}[\frac{1}{2}, 1]$.

Remark 5. Naively one would expect from the path picture of strings an action without the $|\gamma_r'|^{-\frac{1}{2}}$-factor. But then the action would not be unitary. Furthermore a deeper geometrical inspection would show that the strings in the groupoid should be $\frac{1}{2}$-densities rather than paths which is reflected in the additional factor.

The reparametrized product reads then as follows:

$$\begin{aligned}
&\gamma^*(\Phi) * \gamma^*(\Psi)(\omega) \\
&= \frac{1}{\sqrt{2\pi}} \tilde{F}^{-1}(\omega) \int d\mu_{L_{0,r}}(\tilde{\omega}) d\tilde{x} \\
&\times \left\{ \tilde{F}\left(\omega^s, \tilde{\omega}, \frac{1}{2}(x+\tilde{x})\right) \Psi\left(\gamma_s^*(\omega^s), \gamma_r^*(\tilde{\omega}), \frac{1}{2}(x+\tilde{x})\right) \right. \\
&\times \left. \tilde{F}\left(r(\tilde{\omega}), \omega^r, \frac{1}{2}(x-\tilde{x})\right) \Psi\left(\gamma_s^* r(\tilde{\omega}), \gamma_r^*(\omega^r), \frac{1}{2}(x-\tilde{x})\right) \right\},
\end{aligned} \tag{39}$$

and we get a restriction on γ:

$$\gamma_r^* = \gamma_s^* \circ r, \tag{40}$$

that is

$$\gamma(1-t) = 1 - \gamma(t) \quad \forall t \in [0, 1]. \tag{41}$$

In particular we get periodic boundary conditions at 0 and 1 and rescaling the parameter interval we get

$$\gamma \in \mathrm{Diff}\, S^1/S^1, \tag{42}$$

where $\mathrm{Diff}\, S^1/S^1$ is the Frechet group of diffeomorphism of S^1 fixing the point $1 \in S^1$.

Denote $\mathrm{Diff}_W \subset \mathrm{Diff}\, S^1/S^1$ the subgroup of the above γ. This subgroup can be characterized infinitesimally:

Let $\{L_n\}_{n\in\mathbf{Z}}$ be the usual real basis of the Witt-algebra,

$$[L_n, L_m] = (m-n)L_{m+n} \tag{43}$$

which is the well known complexified Lie-algebra of $\mathrm{Diff}\, S^1$. The Lie-algebra of the restricted symmetry is then generated over the reals by

$$K_n := L_n - (-1)^n L_{-n} \tag{44}$$

which is just the symmetry Witten found for the product [Wi 2].

Under these restrictions we can perform a formal substituions $\tilde{\omega} \mapsto \gamma_r^*(\tilde{\omega})$ which yields at least formally,

$$\gamma^*(\Phi) * \gamma^*(\Psi) = \det(\gamma_r^*)\gamma^*(\Phi * \Psi). \tag{45}$$

(Notice that by the unitarity of γ^* we have $\gamma^*(\tilde{F}) = \tilde{F}$.) But in general $\gamma^* - Id$ is not of trace class in $\mathscr{H}_r$ and the determinant is not defined in the analytical sense. This result suggests the picture of an algebra bundle over the reparametrization group, each parametrization giving a bosonic string-algebra. The point is that each parametrization leads to incompatible measures. We will see below that the essence of such a bundle is a line bundle over the symmetry group.

4. C^*-Algebras and Kähler Geometry

Look at a C^∞-manifold M and a compact Lie-groupd G acting on M. A proper and general description of the orbit space M/G is given by using the language of C^*-algebras. Take the reduced cross-product

$$C^0(M) \lhd G, \tag{46}$$

where $C^0(M)$ is the C^*-algebra of continuous functions on M. To recover the topological space M/G notice that

$$\operatorname{spec} C^0(M) \lhd G = M/G, \tag{47}$$

[Co 1, Mo-Scho].

For Kähler manifolds let us propose a different way. Assume M, G Kählerian, for example $M = \mathbf{C}^n$, $G = Sl(2, \mathbf{C})$, with

$$\begin{aligned} \varrho_m : G &\rightarrow N_m \\ g &\mapsto g^{-1} m , \end{aligned} \tag{48}$$

holomorphic for all $m \in M$, N_m the orbit of G through m. We first look at the simplest case of one orbit with trivial isotropy group. The additional Kähler structure suggests a different construction.

Let $\dim_{\mathbf{C}} G = n$, $T^*_{k,l}G$ the (k, l)-cotangent bundle and $G_{n,0}$ the space of holomorphic n-forms. Fixing a Kähler metric we define the Hodge-*-operator. Then

$$\langle \phi, \psi \rangle := \int_G \phi * \psi \qquad \phi, \psi \in G_{n,0} \tag{49}$$

defines a scalar product on $G_{n,0}$ and we get the Hilbert space $H(G)$ of holomorphic n-forms. One can generalize to $H \neq \mathrm{id}$ and to more than one orbit. For general isotropy group H, G/H is Kählerian by assumption. The naturally associated Hilbert space is $H(G/H)$, the holomorphic k-forms on G/H, if $\dim_{\mathbf{C}} G/H = k$. The ideal of compact operators on $H(G)$ acts naturally on the later Hilbert space, the image of this representation is obviously the ideal of compact operators on $H(G/H)$. Let us further assume that there is a global section Φ_0 in $G_{n,0}$ such that $\Phi_0 * \bar{\Phi}_0$ is a multiple of the Haar measure $d\mu_G$ on G. Using this section we can pass from sections ϕ in $G_{n,0}$ to meromorphic functions Φ on G

$$\Phi \in G_{n,0} \Rightarrow \phi = \Phi \Phi_0 . \tag{50}$$

The general construction for foliation algebras, [Co 1], leads then to the generating elements

$$F : M \times G \rightarrow \mathbf{C}, \tag{51}$$

$F(gm, g')$ continuous in m and holomorphic in g, antiholomorphic in g' for any m. The product of two such elements is given by

$$F * K(m, g) := \int_G F(m, g') K(g'm, g)/\mu_G(g') \tag{52}$$

and the involution by

$$F^*(m, g) := \bar{F}(gm, 1) . \tag{53}$$

Let us fix an $m \in M$. We identify $G/H \cong N_m$ and associate to the orbit N_m the Hilbert space $H(G/H)$. For fixed m the function F defines an integral operator on $H(G)$ in a natural way. This operator reduces to $H(G/H)$. Denote I_{F_m} this operator on $H(G/H)$. Then we define the norm of F by

$$\|F\| := \sup_{m \in M} \|I_{F_m}\|_{H(G/H)}. \tag{54}$$

This definition is quite analogous to the one for the foliation algebra [Co 1].

In the Kähler situation we take the completion of this algebra as the naturally associated C^*-algebra. The advantage of this approach is that one can generalize the later construction to the case of the symmetry group $\mathrm{Diff}\, S^1/S^1$. This application and generalization to the symmetry group $\mathrm{Diff}\, S^1/S^1$ is the content of the next paragraph.

5. The Diff S^1/S^1 Symmetry Group

In generalizing the constructions of the previous paragraph we have to answer three main questions concerning the structures of the group

a) What are the Kähler structures on $\mathrm{Diff}\, S^1/S^1$?

b) What are the analogues of holomorphic n-forms?

c) Can one define a "Haar-measure" on $\mathrm{Diff}\, S^1/S^1$?

In a) we will show the well known result of the existence of a whole family of Kähler structures. It will turn out that it is quite appropriate to use them all at once in the following. For b) we will propose natural candidates given by the holomorphic sections of canonical bundles, i.e. the DET*-bundle over $\mathrm{Diff}\, S^1/S^1$. The last problem is the most difficult one. Here it is important to look at the whole family of Kähler structures at once.

The main problems concern the infinite dimensionality of the group. We will restrict ourselves to the identity component $(\mathrm{Diff}\, S^1/S^1)_0$ of $\mathrm{Diff}\, S^1/S^1$.

5.1. Part a)

We can look at $(\mathrm{Diff}\, S^1/S^1)_0$ in two quite different ways. First it is the subgroup of $\mathrm{Diff}\, S^1$ of diffeomorphisms fixing the point $(0, 1) \in S^1$ (S^1 viewed as the unit ball in $\mathbf{C}$). Second it is a generalized flag manifold. S^1 is a maximal abelian subgroup of $\mathrm{Diff}\, S^1$, the subgroup of rigid motion with respect to a parametrization θ. S^1 is not a normal subgroup of $\mathrm{Diff}\, S^1$. As a flag manifold $(\mathrm{Diff}\, S^1/S^1)_0$ doesn't inherit the group structure induced on the equivalence classes, but it has a natural complex structure.

The tangent space of $(\mathrm{Diff}\, S^1/S^1)_0$ at the identity in $\mathrm{Diff}\, S^1$ is spanned over the reals by

$$L_n(\theta) := \cos n\theta \frac{d}{d\theta} + \mathrm{sign}(n) \sin n\theta \frac{d}{d\theta} \tag{55}$$

for $n \in \mathbf{Z}\backslash\{0\}$. The splitting of the tangent space with respect to the parametrization θ is

$$T_1(\mathrm{Diff}\, S^1/S^1)_0 = (T_1(\mathrm{Diff}\, S^1/S^1)_0)_+ \oplus (T_1(\mathrm{Diff}\, S^1/S^1)_0)_-, \tag{56}$$

where

$$(T_1(\mathrm{Diff}\, S^1/S^1)_0)_\pm := \mathrm{span}\{L_n\}_{\pm n > 0}. \tag{57}$$

It is much easier to work on $(\operatorname{Diff} S^1)_0 \supset (\operatorname{Diff} S^1/S^1)_0$. Let us therefore look at the Lie-algebra on $(\operatorname{Diff} S^1)_0$,

$$T_1(\operatorname{Diff} S^1)_0 = T_1(\operatorname{Diff} S^1/S^1)_0 \oplus \operatorname{span}\{L_0\}. \tag{58}$$

By using the scalar product

$$\langle L_n, L_m \rangle := \delta_{n,m} \tag{59}$$

we can complete the (complexified) space to Hilbert spaces

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-, \tag{60}$$

where $\mathscr{H}_+ = \operatorname{span}\{L_n\}_{n\geq 0}$, $\mathscr{H}_- = \operatorname{span}\{L_n\}_{n<0}$, [Pre-Se].

As a flag manifold $(\operatorname{Diff} S^1/S^1)_0$ is a quotient manifold of $\operatorname{Diff} S^1$. The splitting defines an almost complex structure J obeying $J = 1$ on $\mathscr{H}_+$ and $J = -1$ on $\mathscr{H}_-$. This yields the canonical complex structure on $(\operatorname{Diff} S^1/S^1)_0$ with respect to θ, [Mic]. The group structure doesn't respect this complex structure. We defined for any fixed parametrization θ – equivalent for any identification of S^1 as a subgroup of rigid motion – a complex structure which we will denote by $J(\theta)$.

Denote by Ad the Ad-action of $\operatorname{Diff} S^1$ on $\mathscr{H}$. Then we have

$$J(\theta') = \operatorname{Ad}(\theta'\theta^{-1})J(\theta)\operatorname{Ad}(\theta(\theta')^{-1}). \tag{61}$$

There is a well known

Lemma 6. *for all θ, $\theta' \in \operatorname{Diff} S^1[\operatorname{Ad}(\theta'), J(\theta)]$ is Hilbert Schmidt.*

Proof. See [Se]. □

Now fix a parametrization θ and the associated splitting $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$. Let $\mathscr{H}_+(\theta')$ be the positive part of the $J(\theta')$ splitting. Then we get a map

$$(\operatorname{Diff} S^1/S^1)_0 \overset{\mathscr{H}_+(\cdot)}{\longrightarrow} \operatorname{Gr}(\mathscr{H})_0, \tag{62}$$

where $\operatorname{Gr}(\mathscr{H})_0$ is the component of the universal Grassmannian $\operatorname{Gr}(\mathscr{H})$ of index 0, [Pr-Se]. This space is a Hilbert-manifold modelled over

$$\mathscr{L}^2(\mathscr{H}_+, \mathscr{H}_-) = \{A : \mathscr{H}_+ \to \mathscr{H}_-/A \text{ is Hilbert Schmidt}\} \tag{63}$$

equipped with the Hilbert-Schmidt norm. The sesquiliniear form

$$\langle X, Y \rangle := 2\operatorname{tr} X^* Y \quad X, Y \in \mathscr{L}^2(\mathscr{H}_+, \mathscr{H}_-) \tag{64}$$

gives a Kähler metric on $\operatorname{Gr}(\mathscr{H})$, [Pr-Se]. It can be shown that this map is holomorphic with respect to the complex structure on $(\operatorname{Diff} S^1/S^1)_0$ and injective [Pr-Se, Mic]. As a complex submanifold of a Kählerian $(\operatorname{Diff} S^1/S^1)_0$ is also Kählerian. We take the induced Kähler metric on $(\operatorname{Diff} S^1/S^1)_0$ for defining the Kähler structure on $(\operatorname{Diff} S^1/S^1)_0$.

5.2. Part b)

For a fixed parametrization $\theta \in (\operatorname{Diff} S^1/S^1)_0$ we get a Kähler structure on (Diff $S^1/S^1)_0$ defined above. A basis of the holomorphic cotangent vectors over θ at the identity are by the very definition the L^*_{-n}, $n \in \mathbf{N}$, the duals to L_{-n}. The formal analogue to the maximal antisymmetric tensor product would be

$$L^*_{-1} \wedge L^*_{-2} \wedge L^*_{-3} \wedge \dots . \tag{65}$$

Over a different parametrization $\theta' \in (\mathrm{Diff}\, S^1/S^1)_0$ this formal expression passes to the fibre

$$\begin{aligned}\mathrm{Ad}(\theta')\,(L_{-1}^* \wedge L_{-2}^* \wedge \ldots) &\approx (\det \mathrm{Ad}(\theta') \mid \mathscr{H}_0)^* \,(L_{-1}^* \wedge L_{-2}^* \wedge \ldots) + \ldots \\ &\approx \Lambda^{\max}(\tfrac{1}{2}(J(\theta') + \mathrm{Id})\,(\mathscr{H}))^* \\ &\approx \Lambda^{\max}(\mathscr{H}_+(\theta'))^* \,. \end{aligned} \tag{66}$$

The determinant is not defined analytically but rather categorially, as was proposed by Quillen [Q]. Let

$$\begin{array}{c}\mathrm{DET}^* \\ \downarrow \\ \mathrm{Gr}(\mathscr{H})\end{array} \tag{67}$$

be the dual bundle of the Quillen DET-bundle over $\mathrm{Gr}(\mathscr{H})$. This is a holomorphic line bundle, [Pr-Se]. Pulling back to $(\mathrm{Diff}\, S^1/S^1)_0$ gives a holomorphic line bundle over $(\mathrm{Diff}\, S^1/S^1)_0$. An inspection of (66) shows that the right candidate for the analogue of holomorphic n-forms are the holomorphic sections of the bundle. To define the analogue of all holomorphic forms on $(\mathrm{Diff}\, S^1/S^1)_0$ we use the following map: Let S be the shift operator on $\mathscr{H}$,

$$\begin{aligned} S : \mathscr{H} &\to \mathscr{H} \\ L_n &\mapsto L_{n+1} \,. \end{aligned} \tag{68}$$

Then $S \in U_{\mathrm{res}}(\mathscr{H})$ and

$$S : (\mathrm{Gr}(\mathscr{H}))_n \to (\mathrm{Gr}(\mathscr{H}))_{n+1} \tag{69}$$

is a biholomorphic map. Pulling back the DET^*-bundle of $(\mathrm{Gr}(\mathscr{H}))_n$ to $(\mathrm{Gr}(\mathscr{H}))_0$ via S^n defines a holomorphic line bundle

$$\begin{array}{c}(\mathrm{DET}^*)^n \\ \downarrow \\ (\mathrm{Gr}(\mathscr{H}))_0 \,.\end{array} \tag{70}$$

The formal fibre at the identity is then given by

$$L_n^* \wedge L_{n-1}^* \ldots \wedge L_1^* \wedge L_0^* \wedge L_{-1}^* \wedge \ldots \tag{71}$$

for $n > 0$, analogously for $n < 0$. We define the analogue of holomorphic n-forms on $(\mathrm{Diff}\, S^1/S^1)_0$ to be holomorphic sections of the pull back $(\mathrm{DET}^*)^n$-bundle over $(\mathrm{Diff}\, S^1/S^1)_0$.

Before continuing to part c) let me make some

5.2.1. Remarks. 1. Using the above machinery we can again look at the bosonic part of the algebra. As we have shown the measures of integration depend crucially on the chosen parametrization. Formally one got

$$\Phi * \Psi \approx \det(\gamma_r^*)\gamma^*(\Phi * \Psi) \tag{72}$$

or equivalently

$$\gamma_r^*(d\mu_{L_{0,r}}) \approx (\det \gamma_r^*) d\mu_{L_{0,r}} \,. \tag{73}$$

A closer look shows that γ_r is Fredholm. Therefore we can redefine the determinant of γ_r by the Quillen-determinant. The equation

$$\gamma_r^*(d\mu_{L_{0,r}}) = \det \gamma_r^* \, d\mu_{L_{0,r}} \tag{74}$$

is then to be interpreted as a parallel transport of a line bundle over $\mathrm{Diff}_W \subset (\mathrm{Diff}\, S^1/S^1)_0$. The fibres are the measures of integration in the definition of the bosonic string algebra with respect to the chosen reparametrization.

2. The multiplicative property of the Quillen-determinant yields a central extension of the group $(\mathrm{Diff}\, S^1/S^1)_0$. As a central extension this bundle has a canonical $U(1)$-connection given by the projection onto the central extension of the Lie-algebra [Pr-Se]. The curvature of this connection is given by the central extension, [Pr-Se], equivalently by the central charge c of the Lie-algebra extension.

We considered two different line bundles. The first is induced by the $(\mathrm{Diff}\, S^1/S^1)_0$, the other by the above. The resulting central charges associated to these DET-bundles are computed as

$$\begin{aligned} c_{\mathrm{Ad}} &= -26\,, \\ c_{\mathrm{reparam}} &= 1\,. \end{aligned} \tag{75}$$

Furthermore the interpretation of

$$\gamma_r^*(d\mu_{L_{0,r}}) = \det \gamma_r^*\, d\mu_{L_{0,r}} \tag{76}$$

as defining a parallel transport is made precise by the above definition of the canonical connection. This connection on the $U(1)$-PFB gives a canonical connection on the associated DET-bundle which is exactly the (bosonic string-)measure-bundle equipped with the above formal parallel transport.

3.[1] To make contact with the physics literature let us point out the intimate relationship of the DET-bundle over $\mathrm{Gr}(\mathscr{H})$ to what is called the Bogoliubov transformation in fermion theory.

According to Dirac in physics one interprets the decomposition of the complexified Lie-algebra $\mathscr{H}$ into positive and negative parts as the splitting of the one fermionic particle space of the Faddeev-Popov-ghost-fields into ghost-anti-ghost states. The fermionic field algebra is the CAR algebra over $\mathscr{H}$, [Fre]. The vacuum of the field theory is given by specifying a state on the algebra. This is usually done by choosing the filled Dirac sea as a vacuum. Let $a(f)^*$ (respectively $a(g)$) denote the creation (respectively annihilation) operator in $\mathrm{CAR}(\mathscr{H})$ of a particle $f \in \mathscr{H}$ (respectively $g \in \mathscr{H}$). These operators generate the CAR algebra. Let E_+ be the orthogonal projection onto $\mathscr{H}_+$. Then the filled Dirac sea vacuum is given by the state evaluated on the generating elements

$$\omega_{E_+}(a(f_n)^* \ldots a(f_1)^* a(g_k) \ldots a(g_1)) := \det(\langle f_i, E_+ g_j \rangle)\,, \tag{76a}$$

$f_i,\ g_j \in \mathscr{H}$, $i = 1, \ldots, n$; $j = 1, \ldots, k$. This state is pure and using the GNS-construction we can rewrite this as

$$\omega_{E_+}(a(f_n)^* \ldots a(g_1)) = \langle \Omega_0, a(f_n)^* \ldots a(f_1)^* a(g_k) \ldots a(g_1)\Omega_0 \rangle\,. \tag{77}$$

Ω_0 is the unique vacuum vector with

$$\begin{aligned} a(f)^*\Omega_0 &= 0 \quad \forall f \in \mathscr{H}_-\,, \\ a(g)\Omega_0 &= 0 \quad \forall g \in \mathscr{H}_+\,, \end{aligned} \tag{78}$$

which reads formally as

$$\Omega_0 = \Lambda^{\max} \mathscr{H}_-\,. \tag{79}$$

[1] After discussion with K. Fredenhagen

Given a transformation $A \in U_{\text{res}}(\mathscr{H})$ of the one particle state this defines canonically an automorphism on $\text{CAR}(\mathscr{H})$, the second quantization,

$$\alpha_A(a(f)^*) = a(Af)^*, \qquad \alpha_A(a(g)) = a(Ag). \tag{80}$$

This automorphism induces a state

$$\omega_{A^{-1}E_+A}(a(f_n)^* \dots a(g_1)) := \omega_{E_+}(a(Af_n)^* \dots a(Ag_1)). \tag{81}$$

In the case of $A \in U_{\text{res}}(\mathscr{H})$ these states are equivalent. Again such a transformed state is quasi-free and pure,

$$\omega_{A^{-1}E_+A}(a(f_n)^* \dots a(g_1)) = \langle \Omega_A, a(f_n)^* \dots a(g_1)\Omega_A\rangle \tag{82}$$

with the transformed vacuum

$$\Omega_A = \mathscr{B}(A)\Omega_0, \tag{83}$$

where $\mathscr{B}(A)$ is the Bogoliubov transformation of the fermionic vacuum. Explicitly $\mathscr{B}(A)$ is given by the following formula [Fre]:

Let $\{f_i\}_{i=1,\dots,k}$, $\{\tilde{g}_j\}_{j=1,\dots,l}$ be an orthonormal basis in $c(\ker a)$ respectively $b(\operatorname{coker} a)$, where we use the block matrix representation of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{\text{res}}(\mathscr{H}). \tag{84}$$

Let

$$\begin{aligned} L : \mathscr{H} = (\ker a) \oplus (\ker a)^{\perp} \oplus \mathscr{H}_- &\to \mathscr{H}_- \\ (f_1, f_2, f_3) &\mapsto b(a)^{-1}(f_2), \end{aligned} \tag{85}$$

with $d\Gamma(L)$ the second quantization of L,

$$d\Gamma(L) = \sum_{n,m \in \mathbf{N}\setminus\{0\}} \langle L_{-m}, LL_n\rangle a^*(L_{-m})a(L_n). \tag{86}$$

Then

$$\mathscr{B}(A) := c_A e^{d\Gamma(L)} \prod_{i=1}^{k} a(\tilde{f}_i)^* \prod_{i=1}^{l} a(\tilde{g}) \tag{87}$$

with the normalization constant

$$c_A = \det(1 + LL^*)^{\frac{1}{2}}. \tag{88}$$

Obviously the Bogoliubov transformation doesn't really depend on $A \in U_{\text{res}}$ but on the equivalence class in

$$A \hookrightarrow \frac{U_{\text{res}}(\mathscr{H})}{U(\mathscr{H}_+) \times U(\mathscr{H}_-)} = \text{Gr}(\mathscr{H}). \tag{89}$$

Thereby we get a line bundle over $\text{Gr}(\mathscr{H})$ with the fibres being the Bogoliubov transformed vacua. This line bundle is isomorphic to DET.

5.3. *Part c)*

We are left with the most difficult part, the integration over $(\mathrm{Diff}\, S^1/S^1)_0$. Instead of integrating over $(\mathrm{Diff}\, S^1/S^1)_0$ we will integrate over $\mathrm{Gr}(\mathscr{H})$.

First let me give an easier description of holomorphic sections in the DET^*-bundle over $\mathrm{Gr}(\mathscr{H})_0$. These are equivalently given by equivariant functions

$$\Phi : (\mathrm{Gr}(\mathscr{H}))_0 \to \mathbf{C} \tag{90}$$

with

$$\Phi\left(\begin{pmatrix} 0 & 0 \\ 0 & q^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q^{-1} \end{pmatrix} \right) = \det q \Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \tag{91}$$

for $q \in \mathrm{Gl}(\mathscr{H}_-)$ of det-class [Pi].

Then we have a canonical section in DET^* over $(\mathrm{Gr}(\mathscr{H}))_0$ given by

$$\begin{gathered} \Phi_0 : (\mathrm{Gr}(\mathscr{H}))_0 \to \mathbf{C}, \\ \Phi_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = (\det(1 + LL^*))^{\frac{1}{2}}, \end{gathered} \tag{92}$$

where

$$\begin{aligned} L : \mathscr{H} = (\ker d) \oplus (\ker d)^{\perp} \oplus \mathscr{H}+ &- \to \mathscr{H}_+ \\ (f_1, f_2, f_3) &\mapsto c(d)^{-1}(f_2), \end{aligned} \tag{93}$$

see Remark 3. This section gives just the Bogoliubov transformation when $\ker d$ and $\mathrm{coker}\, d$ are trivial. At least formally the section over $\mathrm{Id} \in U_{\mathrm{res}}(\mathscr{H})$ has value

$$\Phi_0(\mathrm{Id}) = L_{-1}^* \wedge L_{-2}^* \wedge L_{-3}^* \wedge \dots . \tag{94}$$

The Hodge-*-operator defined with respect to the Kähler structure on $\mathrm{Gr}(\mathscr{H})$ can be applied to this semi-infinite form [F-G-Z] or (138),

$$*\Phi_0(\mathrm{Id}) = L_0^* \wedge L_1^* \wedge L_2^* \wedge \dots . \tag{95}$$

We get at a formal level the Haar-measure

$$\Phi_0 * \Phi_0(\mathrm{Id}) = \dots L_{-1}^* \wedge L_0^* \wedge L_1^* \wedge \dots . \tag{96}$$

Using the biholomorphic map S we get too canonical sections Φ_n of $(\mathrm{DET}^*)^n$. The idea is to pass from $\frac{1}{2}$-densities on $\mathrm{Gr}(\mathscr{H})$ – holomorphic sections in DET^* – to holomorphic functions by using the canonical sections Φ_n, see Sect. 3. The problem is then reduced to the construction of a measure on $(\mathrm{Gr}(\mathscr{H}))_0$, which is invariant under the $(\mathrm{Diff}\, S^1/S^1)_0$-action, at least quasi-invariant. This was done by Pickrell, [Pi],

Theorem 7. *There exists a measure μ on $(\mathrm{Gr}(\mathscr{H}))_0$, a cylinder measure on the coordinate patches, with the following invariance property:*

Let $g \in U_{\mathrm{res},0}$. The induced measure $g^*(\mu)$ is absolutely continuous to μ with Radon-Nikodym-derivative

$$g^*(\mu) = \pi_g \cdot \mu, \qquad \pi_g(u) = \frac{\Phi_0(g^{-1}ug)}{\Phi_0(u)}. \tag{97}$$

Proof. [Pi, Proposition 4.14]. □

Applied to holomorphic sections Φ, Ψ of DET* this gives as an Ansatz

$$\langle \Phi, \Psi \rangle := \int\limits_{(\mathrm{Gr}(\mathscr{H}))_0} \left(\frac{\Phi}{\Phi_0}\right) \overline{\left(\frac{\Psi}{\Phi_0}\right)} d\mu \tag{98}$$

for their scalar product, where $\frac{\Phi}{\Phi_0}$ and $\frac{\Psi}{\Phi_0}$ denote the holomorphic functions on $(\mathrm{Gr}(\mathscr{H}))_0$

$$\Phi = \frac{\Phi}{\Phi_0} \Phi_0 ; \qquad \Psi = \frac{\Psi}{\Phi_0} \Phi_0 . \tag{99}$$

That this is the right Ansatz is the content of the next

5.3.1. Remark. Let $V_+ \subset \mathscr{H}_+$, $V_- \subset \mathscr{H}_-$ be subspaces of dimension n. Define the new splitting

$$\mathscr{H}' = (\mathscr{H}_+ \ominus V_+ \oplus V_-) \oplus (\mathscr{H}_- \ominus V_- \oplus V_+) . \tag{100}$$

An element $A \in \mathrm{Gr}(\mathscr{H})_0$ gives obviously too an element $\imath(A) \in \mathrm{Gr}(\mathscr{H}')_0$ and vice versa. Let Φ_0' be the canonical equivariant function on $\mathrm{Gr}(\mathscr{H}')_0$ as defined in (91). Then

$$A \mapsto \Phi_0'(\imath(A)) \tag{101}$$

defines a new equivariant function on $\mathrm{Gr}(\mathscr{H})_0$ and thereby a holomorphic section in DET*, denoted by Φ_{V,V_-}. A closer look shows that this section depends only on $\Lambda^n V_+ \wedge (\Lambda^n V_-)^*$. Pickrell showed that this map defines an isomorphism of the zero charge fermionic Hilbert space $(\Lambda^* \mathscr{H}_+ \wedge (\Lambda^* \mathscr{H}_-)^*)_0$ to the L^2-space of holomorphic sections [Pi].

For the scalar product of general n-forms we use the canonical sections $\Phi_n = (S^{-n})^*(\Phi_0)$, S the shift operator, instead of Φ_0. Let $\tilde{\Phi}$ be a holomorphic section in DET^{*m} respectively $\tilde{\Psi}$ a holomorphic section in DET^{*n}. Then we define

$$\langle \tilde{\Phi}, \tilde{\Psi} \rangle := \delta_{m,n} \int\limits_{(\mathrm{Gr}(\mathscr{H}))_0} \left(\frac{\tilde{\Phi}}{\Phi_n}\right) \overline{\left(\frac{\tilde{\Psi}}{\Phi_n}\right)} d\mu . \tag{102}$$

The proof in [Pi] carries over to the L^2-space of holomorphic sections in (DET*)^n. We get that the direct sum of Hilberts paces of the n-forms is isomorphic to the fermionic Hilberts space $\Lambda^* \mathscr{H}_+ \wedge (\Lambda^* \mathscr{H}_-)^*$. Alternatively we can think of the direct sum as the Hilbert space of holomorphic sections of DET* over all GR$(\mathscr{H})$.

Now we are almost in the position of Sect. 3. A reparametrization $\gamma \in (\mathrm{Diff}\, S^1/S^1)_0$ gives an element $\gamma^* \in (\mathrm{Gl}_{\mathrm{res}}(\mathscr{H}))_0$. Pickrell's proof of the quasi-invariance property of the measure μ carries over to the $(\mathrm{Gl}_{\mathrm{res}}(\mathscr{H}))_0$-action on $(\mathrm{Gr}(\mathscr{H}))_0$ [Pi]. But different from the situation assumed in Sect. 3 the integration measure is only quasi-invariant. Again one can picture this as a line bundle over the symmetry group. The quasi-invariance can be interpreted as a parallel transport according to a canonical connection on that line bundle of measures, see Remark 2. As in the bosonic case we get a line bundle with fibres the measures of integration, equipped with a canonical connection.

6. The String-Algebra

First one remarks that $\mathrm{Diff}\,S^1/S^1$ is not the reparametrization symmetry group of open strings. This later one was recognized to be the Witten subgroup Diff_W of $\mathrm{Diff}\,S^1/S^1$ given by the restriction $\gamma(t) = (1-\gamma(t))$. Secondly we got an integration measure over $\mathrm{Gr}(\mathscr{H})_0$ (respectively using S^n over $\mathrm{Gr}(\mathscr{H})_n$) instead over Diff_W. But now we can argue that one associates the direct sum of a whole family of Hilbert spaces to one orbit. The Borel σ-algebra of the Frechet-Lie-group Diff_W is generated by the cylinder sets on the coordinate patches. Then the map

$$\pi : \mathrm{Gr}(\mathscr{H}) \to \mathrm{Gr}(\mathscr{H})/\mathrm{Diff}_W$$

induces a measure μ^π on $\mathrm{Gr}(\mathscr{H})/\mathrm{Diff}_W$. Furthermore we can find a measurable section

$$\pi_0^{-1} : \mathrm{Gr}(\mathscr{H})_0/\mathrm{Diff}_W \to \mathrm{Gr}(\mathscr{H})_0$$

and shifting π_0^{-1} by S^n,

$$\pi_n^{-1} := S^n \pi_0^{-1} S^{-n} : \mathrm{Gr}(\mathscr{H})_n/\mathrm{Diff}_W \to \mathrm{Gr}(\mathscr{H})_n$$

a measurable section

$$\pi^{-1} : \mathrm{Gr}(\mathscr{H})/\mathrm{Diff}_W \to \mathrm{Gr}(\mathscr{H}) . \tag{103}$$

We get a Borel isomorphism

$$\begin{aligned}(\mathrm{Diff}_W \times \mathrm{Gr}(\mathscr{H})/\mathrm{Diff}_W) &\xrightarrow{\cong} \mathrm{Gr}(\mathscr{H}) ,\\ (\gamma, [g]) &\mapsto \gamma\pi^{-1}([g]) .\end{aligned} \tag{104}$$

Vice versa we can define a measurabe map

$$\Gamma : \mathrm{Gr}(\mathscr{H})_0 \to \mathrm{Diff}_W \tag{105}$$

$$g \to \gamma \tag{106}$$

for $g = \gamma\pi_0^{-1}([g])$. The induced measure μ^Γ on Diff_W can be viewed as a generalization of the Haar measure on Diff_W! It is quasi-invariant. The idea is to use as the associated Hilbert space to one orbit a direct integral

$$\int_{\mathrm{Gr}(\mathscr{H})/\mathrm{Diff}_W} \mathscr{H}(\mathrm{Diff}_W) d\mu^\pi .$$

On $\mathrm{Gr}(\mathscr{H})_0$ we have a natural groupoid structure given by the above Borel isomorphism. For $g_1, g_2 \in \mathrm{Gr}(\mathscr{H})_0$ with $[g_1] = [g_2]$ we define their product $g_1 \cdot g_2 = \gamma_1\gamma_2\pi^{-1}([g_2])$, where $g_i = \gamma_i\pi^{-1}([g_i])$. The inverse is defined by $g^{-1} = \gamma^{-1}([g])$. For general $g_1 \in \mathrm{Gr}(\mathscr{H})_n$, $g_2 \in \mathrm{Gr}(\mathscr{H})_m$ with

$$S^{-n}\pi^{-1}([g_1]) = S^{-m}\pi^{-1}([g_2]) \in \mathrm{Gr}(\mathscr{H})_0 ,$$

we define

$$g_1 \cdot g_2 = \gamma_1\gamma_2 S^m \pi^{-1}([g_1]) \tag{107}$$

and the inverse

$$g^{-1} = \gamma^{-1} S^{-2n} \pi^{-1}([g_1]) . \tag{108}$$

This structure is compatible with an action of $\mathrm{Gr}(\mathscr{H})$ on the space of strings

$$
\begin{aligned}
g : \mathscr{H}^{r,s} &\to \mathscr{H}^{r,s}, \\
\omega &\mapsto \gamma^{-1}(\omega).
\end{aligned}
\tag{109}
$$

Then we can look at the total string groupoid. It is the semi-direct product of the two groupoids, the bosonic string groupoid and the reparametrization symmetry groupoid. The elements are given by

$$\omega = (\omega^s, \omega^r, x, g) \in (\mathscr{H}^s \times \mathscr{H}^r \times \mathbf{R} \times \mathrm{Gr}(\mathscr{H})), \tag{110}$$

where ω^s respectively ω^r are the two halves of the string, x the height of the jump in the middle and g reflects the choice of the reparametrization. For two elements ω_1, ω_2 with $g_2\omega_1^s = \omega_2^r$ and $S^{-n}\pi^{-1}([g_1]) = S^{-m}\pi^{-1}([g_2])$, $g_1 \in \mathrm{Gr}(\mathscr{H})_n$, $g_2 \in \mathrm{Gr}(\mathscr{H})_m$, we can define their product

$$(\omega_2^s, \omega_2^r, g_2) \perp (\omega_1^s, \omega_1^r, g_1) = (\omega_2^s, g_2\omega_1^r, g_1 g_2). \tag{111}$$

The inverse is then given by

$$(\omega^s, \omega^r, g)^{-1} := (g^{-1}r(\omega^r), g^{-1}r(\omega^s), g^{-1}). \tag{112}$$

After these preliminaries we are able to define the total string algebra. The generating elements are measurable functions

$$F : \mathscr{H}^s \times \mathscr{H}^r \times \mathbf{R}^{26} \times \mathbf{R}^{26} \times \mathrm{Gr}(\mathscr{H}) \to \mathbf{C} \tag{113}$$

with

$$F(g_1\omega^s, g_1\omega^r, x, y, g_2) \quad \text{continuous in } \omega^s, \omega^r, y, \tag{114}$$

holomorphic in g_1 and antiholomorphic in g_2 and compact support in$y$. The involution is then given by

$$F^*(\omega^s, \omega^r, x, y, g) := \bar{F}(r(\omega^r), r(\omega^s), -x, y, g^{-1}). \tag{115}$$

The product of two such functions F and K is then defined by

$$
\begin{aligned}
F * K&(\omega^s, \omega^r, x, y, g) \\
&= \sum_{n\in\mathbf{Z}} \int d\mu_{L_{0,r}}(\tilde{\omega}) d\tilde{x} d\mu^r(\tilde{g}) F\left(\tilde{g}^{-1}\omega^s, \tilde{\omega}, \frac{1}{2}(x-\tilde{x}), y, \tilde{g}S^{-n}\pi(g)\right) \\
&\quad \times K\left(\tilde{\omega}, g\tilde{g}\omega^r, \frac{1}{2}(x+\tilde{x}), y, \tilde{g}^{-1}S^n g\right).
\end{aligned}
\tag{116}
$$

But this product is a priori not associative.

For proving associativity one uses the invariance of the measure under reparametrization. Now the measures μ and $\mu_{L_{0,r}}$ are not invariant but define line bundles over Diff_W, see Sect. 4. These line bundles are equipped with canonical connections given by the projective representations of Diff_W (the quasi-invariance). The invariance of the tensor product measure $d\mu \otimes d\mu_{L_{0,r}}$ can be reformulated in geometric terms by the vanishing of the curvature on the tensor product bundle. In that case we can pick up a constant section of the bundle. Thereby we have canonical isomorphisms of translated measures. This is meant when we speak of invariant measures. In Sect. 3 we sketched the computation of the curvatures. The bosonic part gives for any dimension d of the target space the central charge

$$c_{\text{param}} = 1. \tag{117}$$

The fermionic part has central charge

$$c_{\mathrm{Ad}} = -26. \tag{118}$$

We get a cancellation in $d = 26$, that is, an invariant measure. Then the product of such functions F, K is associative. For a more detailed analysis, see [Wie].

There is a natural positive functional on that algebra given by

$$\varrho(F) := \int_{\mathcal{H}^r \times (\mathrm{Gr}(\mathcal{H})_0/\mathrm{Diff}_W) \times \mathbf{R}^{26}} \times F(r(\tilde{\omega}), \tilde{\omega}, 0, y, \pi^{-1}([g])) d\mu_{L_{0,r}}(\tilde{\omega}) d\mu^{\pi}([g]) dy. \tag{119}$$

We define the pre-Hilbert space spanned by all elementary functions F with $\varrho(FF^*) < \infty$ and scalar product

$$\langle F, K \rangle := \varrho(FK^*). \tag{120}$$

Let $\mathcal{H}^{\sim}_{\mathrm{str}}$ be the Hilbert space completion. Then the generating functions F with $\varrho(FF^*) < \infty$ define in a canonical way bounded operators on $\mathcal{H}^{\sim}_{\mathrm{str}}$. By the very definition we get a representation of the *-algebra generated by such elementary functions as bounded operators. We take the completion with respect to the operator norm of this representation algebra. This C^*-algebra defines the total string algebra denoted by $\mathcal{A}$.

7. The Representation Space of the String Algebra

For F an elementary function we get explicitly

$$\varrho(F * F^*) = \int_{\mathbf{R}^{26} \times \mathbf{R}^{26} \times \mathcal{H}^s \times \mathcal{H}^r \times \mathrm{Diff}_W \times (\mathrm{Gr}(\mathcal{H})/\mathrm{Diff}_W)} \times dy d\tilde{x} d\mu_{L_{0,s}}(\tilde{\omega}) d\mu_{L_{0,r}}(\hat{\omega}) d\mu^{\Gamma}(\tilde{g}) d\mu^{\pi}([g]) \tag{121}$$

$$F\left(\tilde{\omega}, \tilde{g}\hat{\omega}, \frac{1}{2}\tilde{x}, y, \tilde{g}\pi^{-1}([g])\right) \bar{F}\left(\tilde{\omega}, \tilde{g}\hat{\omega}, -\frac{1}{2}\tilde{x}, y, \tilde{g}\pi^{-1}([g])\right) \tag{122}$$

$$= \int_{\mathbf{R}^{26} \times \mathbf{R}^{26} \times \mathcal{H}^s \times \mathcal{H}^r \times \mathrm{Gr}(\mathcal{H})} dy d\tilde{x} d\mu_{L_{0,s}}(\tilde{\omega}) d\mu_{L_{0,r}}(\hat{\omega}) d\mu(g) \tag{123}$$

$$F\left(\tilde{\omega}, g\hat{\omega}, -\frac{1}{2}\tilde{x}, y, g\right) \bar{F}\left(\tilde{\omega}, g\hat{\omega}, -\frac{1}{2}\tilde{x}, y, g\right) \tag{124}$$

$$= \int_{\mathbf{R}^{26} \times \mathbf{R}^{26} \times \mathcal{H}^s \times \mathcal{H}^r \times \mathrm{Gr}(\mathcal{H})} dy d\tilde{x} d\mu_{L_{0,s}}(\tilde{\omega}) d\mu_{L_{0,r}}(\hat{\omega}) d\mu(g) \tag{125}$$

$$F\left(\tilde{\omega}, \hat{\omega}, -\frac{1}{2}\tilde{x}, y, g\right) \bar{F}\left(\tilde{\omega}, \hat{\omega}, -\frac{1}{2}\tilde{x}, y, g\right) \tag{126}$$

Denote $\tilde{\mathcal{F}}$ the Hilbert space of $d\tilde{x} d\mu_{L_{0,s}} d\mu_{L_{0,r}}$-square integrable functions, Λ^*_{∞} the Hilbert space of $d\mu$-square integrable holomorphic functions on $\mathrm{Gr}(\mathcal{H})$. Then the above equality shows

Lemma 8. $\mathscr{H}_{\text{str}}^{\sim} := \tilde{\mathscr{F}} \otimes \Lambda_{\infty}^{*} \otimes L^2(\mathbf{R}^{26}, dx)$.

Let F_{R-N} denote the Radon-Nikodym derivative

$$d\mu_{L_0} = F_{R-N} d\mu_{L_{0,s}} \otimes d\mu_{L_{0,r}} \otimes d\tilde{x}\,.$$

Multiplication with $(F_{R-N})^{\frac{1}{2}}$ gives a unitary isomorphism of $\tilde{\mathscr{F}}$ to the Hilbert space $\mathscr{F}$ of $d\mu_{L_0}$-square integrable functions. We get a C^*-algebra representation of the bosonic string algebra on $\mathscr{F} \otimes \Lambda_{\infty}^{*} \otimes L^2(\mathbf{R}^{26}, dx)$. The Hilbert space $\mathscr{F}$ is just the familiar bosonic string Fock space. Let

$$\varphi(\omega_n^i) \tag{127}$$

be the Gaussian variable to μ_{L_0} of the classical string

$$\omega_n^i := (t \mapsto e^i(\cos n\pi t)) \qquad n \in \mathbf{N}, \quad i \in 1, \ldots, 26, \tag{128}$$

where e^i is an orthonormal basis of $\mathbf{R}^{26}$. We can split the variable into an annihilation and creation part

$$\varphi(\omega_n^i) = a_{-n}^i + a_n^i\,. \tag{129}$$

One computes for them

$$[a_m^i, a_n^i] = \delta_{m,-n}\delta_{i,j} n \tag{130}$$

which are the familiar commutation rules in string theory. It is often much more comfortable to use the Fock space language, see below.

The fermionic Hilbert space in string theory is defined by

$$\Lambda^* \mathscr{H}_+ \wedge (\Lambda^* \mathscr{H}_-)^* \tag{131}$$

see [Pr-Se, F-G-Z].

A theorem of Pickrell, [Pi], shows

$$\Lambda_{\infty}^{*} \cong \Lambda^* \mathscr{H}_+ \wedge (\Lambda^* \mathscr{H}_-)^*\,, \tag{132}$$

where the isomorphism is given as in Remark (5.3.1).

Again it is often quite useful to work in the fermionic Fock space language.

As a final result we get a C^*-algebra representation of the bosonic string algebra on the Hilbert space

$$\mathscr{H}_{\text{str}}^{\wedge} := \mathscr{F} \otimes (\Lambda^* \mathscr{H}_+ \wedge (\Lambda^* \mathscr{H}_-)^*) \otimes L^2(\mathbf{R}^{26}, dx) : \tag{133}$$

This can also be read as a direct integral of Hilbert spaces

$$\mathscr{H}_{\text{str}}^{\wedge} = \int_{\mathbf{R}^{26}} \mathscr{F} \otimes \Lambda_{\infty}^{*}(p) dp\,. \tag{134}$$

The fermionic Hilbert space has a natural $\mathbf{Z}$-grading given by the ghost-number. But this can naturally be traced back to a $\mathbf{Z}$-grading of the underlying string algebra. Let F be an elementary function vanishing outside $\mathrm{Gr}(\mathscr{H})_n$. Then we define

$$\deg F := n \in \mathbf{Z}\,. \tag{135}$$

Going through the definitions we see that equipped with this grading the bosonic string algebra is a $\mathbf{Z}$-graded algebra. We can pass to the reminiscent $\mathbf{Z}_2$-grading.

Let

$$\sigma : \mathscr{H} \to \mathscr{H}$$

be given by

$$\sigma(L_n) := -L_{-n}. \tag{136}$$

Then define the map

$$\begin{aligned} * : \mathrm{Gr}(\mathscr{H}) &\to \mathrm{Gr}(\mathscr{H}) \\ W &\mapsto *W := [\sigma(W)]^{\perp}. \end{aligned} \tag{137}$$

This is a smooth involution as it is easily shown. It induces a map

$$* : \Gamma((\mathrm{DET}^*)^n) \to \Gamma((\mathrm{DET}^*)^{-n+1}) \tag{138}$$

of sections. The Quillen metric defines a canonical fibre pairing

$$(\mathrm{DET}^*)^n \to \mathbf{C}(\mathrm{DET}^*)^{-n+1} \tag{139}$$

To get the analogue of the Poincaré Duality one would like to define the pairing

$$\langle \psi, \varphi \rangle := \int_{\mathrm{Gr}(\mathscr{H})_0} (\psi, \varphi) d\,\mathrm{vol}, \tag{140}$$

where (,) denotes the Quillen pairing. Now at least formally we have [Pi]

$$d\,\mathrm{vol} \approx (\Phi_n \wedge \sigma\Phi_{-n+1}) \approx \left(\int_{\mathrm{Gr}(\mathscr{H})_0} \Phi_n \wedge \sigma\Phi_{-n+1} \right) d\mu. \tag{141}$$

This suggests the following

Definition 9. We have a natural pairing of square integrable n-forms ψ and $(-n+1)$-forms φ given by

$$\langle \psi, \varphi \rangle := \int_{\mathrm{Gr}(\mathscr{H})_0} \overline{\left(\frac{\psi}{\Phi_n}\right)} \left(\frac{\varphi}{\Phi_{-n+1}}\right) d\mu. \tag{142}$$

The Poincaré Duality also gives a natural definition for the generalized hodge-*-operator.

Definition 10. For an n-form $\psi \in \Lambda^*_\infty$ let $*\psi \in \Lambda^*_\infty$ be the Riesz-representant given by the above pairing. This map defines uniquely a *-operator on Λ^*_∞ which maps even forms onto odd and vice versa. It gives a natural involution on the fermionic Hilbert space.

Finally we get the

Definition 11. Let $\Gamma : \mathscr{H}_{\hat{\mathrm{str}}} \to \mathscr{H}_{\hat{\mathrm{str}}}$ be given by

$$\Gamma := \mathrm{Id}_{\mathscr{F} \otimes L^2(\mathbf{R}^{26}, d\lambda)} \otimes *. \tag{143}$$

This operator defines a natural $\mathbf{Z}_2$-grading operator on the string Hilbert space $\mathscr{H}_{\hat{\mathrm{str}}}$.

8. The Generalized String-Dirac-Operator

The symmetry group for bosonic strings is Diff_W. Fixing a parametrization of S^1 we can imbed

$$\mathrm{Diff}_W \subset \mathrm{Diff}\, S^1 . \tag{144}$$

It is much easier to work with the later group. This group also acts on the space of classical strings. (Notice that the classical string can be discontinuous). Therefore we have an action of $\mathrm{Diff}\, S^1$ on $\mathscr{H}^s \times \mathscr{H}^r \times \mathbf{R}^{26} \times \mathbf{R}^{26} \times \mathrm{Gr}(\mathscr{H})$. The fourth factor consists of the constant strings. We can restrict the action on the direct summands $\mathscr{H}^s \times \mathscr{H}^r \times \mathbf{R}^{26} \times \mathrm{Gr}(\mathscr{H})$. This action leads canonically to an infinitesimal action of $\mathrm{Diff}\, S^1$ on $\mathscr{F} \otimes \Lambda^*_\infty$, to be precise on $\mathscr{F} \otimes \Lambda^*_\infty(p)$ with $p \in \mathbf{R}^{26}$. The natural Ansatz for the transversal elliptic operator then leads to the BRST-operator in string theory, [F-G-Z]. In the following we fix p and simply write $\mathscr{F} \otimes \Lambda^*_\infty$ instead of $\mathscr{F} \otimes \Lambda^*_\infty(p)$.

For the furhter investigation we use the Fock-space formulation of $\mathscr{F} \otimes \Lambda^*_\infty$. We define the fermionic creation and annihilation operators

$$\begin{aligned} b^* &:= L_n \wedge \qquad n \geq 0 \text{ exterior multiplication with } L_n , \\ b(-n) &:= L^*_{-n} \wedge \qquad n < 0 \text{ exterior multiplication with } L_{-n} , \end{aligned} \tag{145}$$

and by the adjoint on $\Lambda^*_\infty \cong \Lambda^* \mathscr{H} \wedge \Lambda^* \mathscr{H}^*_-$

$$\begin{aligned} b^*(-n) &:= (b(-n))^* \qquad n > 0 , \\ b(n) &:= (b(n))^* \qquad n \leq 0 . \end{aligned} \tag{146}$$

These are bounded operators on Λ^*_∞. The b^*-operators are called ghost-creation-operators, the b-operators ghost-annihilation-operators.

The subspace $\mathscr{F}_0$ of finitely many excitations is then given by the algebraic span of vectors

$$\psi = \prod_{i=1}^{k} a^{v_i}_{-n_i} \otimes \bigwedge_{i=1}^{l} L_{s_i} \wedge \bigwedge_{i=1}^{m} L^*_{-r_i} \wedge \Omega \tag{147}$$

with

$$\begin{aligned} k, l, m \in \mathbf{N} , \qquad v_i \in \{1, \ldots, 26\} , \\ n_i, r_i \in \mathbf{N} \backslash \{0\} , \qquad s_i \in \mathbf{N} . \end{aligned}$$

It is a dense subspace of $\mathscr{F} \otimes \Lambda^*_\infty$. Finally we define some useful operators on $\mathscr{F}_0$.

Definition 12.

$$a^\mu_0 := p^\mu , \tag{148}$$

where $p \in \mathbf{R}^{26}$ was fixed above,

$$N_b := \left(\sum_{n \in \mathbf{N}} \sum_{i=1} a^i_{-n} a^i_n \right) \otimes \mathrm{Id}_{\Lambda^*_\infty} , \tag{149}$$

$$N_f := \mathrm{Id}_{\mathscr{F}} \otimes \left(\sum_{n \in \mathbf{N}} n(b^*(n)b(n) + b(-n)b^*(-n)) \right) , \tag{150}$$

$$\mathscr{N} := N_b + N_f \tag{151}$$

are operators defined on $\mathscr{F}_0$.

Lemma 13. *N_b, N_f, $\mathcal{N}$ are essentially s.a. on $\mathcal{F}_0$.*

Proof. $\mathcal{F}_0$ is a dense set of analytical vectors for the operators. By Nelson's analytical vector theorem [R-S, II X.39] this proves the lemma. □.

$\mathcal{N}$ has discrete spectrum and little combinatorics shows that $(1+\mathcal{N})^{-q}$ for $q > 1$ is of trace class.

The symmetry action on $\mathcal{F}$ leads to an infinitesimal action given by

$$\pi(L_n) := \frac{1}{2}\left(\sum_{m \in \mathbf{Z}} \sum_{i=1}^{26} : a^i_m a^i_{n-m} :\right) \otimes \mathrm{Id}_{\Lambda^*_\infty} \tag{152}$$

with

$$: a^i_m a^i_n := \begin{cases} a^i_m a^i_n & n \geq m \\ a^i_n a^i_m & n < m, \end{cases} \tag{153}$$

The symmetry action on Λ^*_∞ leads to an action

$$\varrho(L_n) := \mathrm{Id}_{\mathcal{F}} \otimes \left(\sum_{m \infty \mathbf{Z}} (m-n) : b(m) b(m+n) :\right) \tag{154}$$

with

$$: b^*(m) b(n) := \begin{cases} b^*(m) b(n) & m \geq n \\ -b(n) b^*(m) & m < n. \end{cases} \tag{155}$$

The operators $\pi(L_n)$ and $\varrho(L_n)$ are well defined on $\mathcal{F}_0$ and $\theta(L_n) := \pi(L_n) + \varrho(L_n)$ defines a Lie-algebra representation on $\mathcal{F}_0$. The BRST operator is then given by the Lie-algebra coboundary with values in the Lie-algebra module $\mathcal{F}$.

Definition 14.

$$\tilde{d} := \sum_{n \in \mathbf{Z}} \pi(L_n) b^*(n) + \sum_{m<n} (m-n) : b(m+n) b^*(m) b^*(n) : \tag{156}$$

is well defined on $\mathcal{F}_0$. Let $\tilde{d}^*$ be the adjoint on $\mathcal{F}_0$,

$$\tilde{d}^* := \sum_{n \in \mathbf{Z}} \pi(L_{-n}) b(n) + \sum_{m<n} (m-n) : b(n) b(m) b^*(m+n) : . \tag{157}$$

Lemma 15. *$\tilde{d}$ maps $\mathcal{F}_0$ to $\mathcal{F}_0$. We get $\tilde{d}^2 = 0$ on $\mathcal{F}_0$.*

Proof. See [F-G-Z, Proposition 1.2] □

The most important result of this section is stated in

Theorem 16. *The odd operator $\tilde{Q} := \tilde{d} + \tilde{d}^*$ is essentially* s.a. *on $\mathcal{F}_0$. Let $\tilde{H} := \tilde{Q}^2$. Then $(1+\tilde{H})^{-q}$ for $q > \frac{1}{2}$ is of trace class.*

The proof of this theorem takes nearly the rest of this section. Central for the arguments is the simple observation

$$[\mathcal{N}, \tilde{Q}] = 0 \quad \text{on} \quad \mathcal{F}_0 . \tag{158}$$

Let $\psi \in \mathcal{F}_0$ with $\mathcal{N}\psi = n\psi$. Then, by the very definition, for $m > n$ we have

$$a_m \psi = b(m)\psi = b^*(-m)\varphi = 0 . \tag{159}$$

We get

$$\pi(L_m)\psi = \frac{1}{2} \sum_{m-n\le m'\le m} \sum_{i=1}^{26} : a^i_{m'} a^i_{m-m'} : \otimes \mathrm{Id}_{\Lambda^*_\infty} \psi \tag{160}$$

$$\begin{aligned} \tilde{Q}\psi = & \sum_{|m|\le n} (\pi(L_n) b^*(m) + \pi(L_{-m}) b(m))\psi \\ & + \sum_{-n\le m'+m\le 2n,\, m'<m} (m'-m)\,(: b(n+m') b^*(m') b^*(m) \\ & + b(m') b(m) b^*(m+m') :)\psi\,. \end{aligned} \tag{161}$$

We estimate

$$\|\pi(L_m)\psi\| \le n\dot{m}\|\psi\|\,, \tag{162}$$

$$\|\tilde{Q}\psi\| \le (1+4n^3)\,\|\psi\|\,. \tag{163}$$

Lemma 17. $\tilde{Q}$ is essentially s.a. on $\mathscr{F}_0$.

Proof. Using $[\tilde{Q}, \mathscr{N}] = 0$ this is shown as in Lemma (13). □

Analogously one shows that $\tilde{H}$ is essentially s.a. on $\mathscr{F}_0$.

It remains to prove the trace class property of $(1+\tilde{H})^{-q}$. We do this by proving the following estimate.

Theorem 18. *In the sense of quadratic forms one estimates*

$$\tilde{H} \ge c_1 \mathscr{N}^2 + c_2 \quad \text{on} \quad \mathscr{F}_0 \tag{164}$$

with $c_1 > 0$.

Using this fundamental estimate we can finish the proof of the Theorem 16. We get

$$0 \le \mathrm{tr}(1+\tilde{H})^{-q} < \infty \tag{165}$$

for $q > \frac{1}{2}$ by the estimation of $\mathrm{tr}(1+\mathscr{N})^{-q} < \infty$.

We are left with the proof of the fundamental estimate. Using the commutation rules

$$[\pi(L_n), \pi(L_m)] = (n-m)\pi(L_{n+m}) + \frac{26}{12}(n^3-n)\delta_{m,-n} \tag{166}$$

on $\mathscr{F}_0$, one calculates $\tilde{H} = H_1 + H_2 + H_3$ with

$$\begin{aligned} H_1 := & \sum_{m>0} 2\pi(L_{-m})\pi(L_m) + (\pi(L_0))^2 \\ & + \sum_{m\ne -n} (m-n)\pi(L_{m+n}) : b(-m) b^*(n) : \\ & + \sum_{m>0} 2m\pi(L_0)\,(b(-m)b^*(-m) + b^*(m)b(m)) \\ & + \sum_{m>0} \frac{26}{12}(m^3-m)\,(b(-m)b^*(-m) + b^*(m)b(m))\,, \end{aligned} \tag{167}$$

$$H_2 := \sum_{n,k} k\pi(L_{-n-m}\,(b^*(k)b^*(n) + b(k)b(n))\,, \tag{168}$$

$$H_3 := \left(\sum_{m<n} (m-n) : b(m+n)b^*(m)b^*(n) + b(n)b(m)b^*(m+n) : \right)^2 . \tag{169}$$

For the next estimates let us first reduce to the case of the target space $\mathbf{R}$ of the strings instead of $\mathbf{R}^{26}$. We use the same terminology where the sum over the indices $i = 1, \ldots, 26$ is replaced by only the first summand. We simply write, a_n instead of a_n^1.

Let n, $n' \in \mathbf{N}$, $\mathbf{k} \in (\mathbf{N})^n$, $\mathbf{s} \in (\mathbf{Z})^{n'}$ be ordered tuples, i.e. $k_i \le k_{i+1}$, $s_i < s_{i+1 \ldots}$. Define

$$\psi_{\mathbf{k},\mathbf{s}} := \prod_{i=1}^{n} a_{-k_i} \otimes (b(s_1) + b^*(s_1)) \wedge \cdots \wedge (b(s_{n'}) + b^*(s_{n'}))\Omega , \tag{170}$$

$$|\mathbf{k}| := \sum_{i=1}^{n} k_i , \qquad |\mathbf{s}| := \sum_{i=1}^{n'} |s_i| , \tag{171}$$

$$\{\mathbf{k}\} := \{k_1, \ldots, k_n\} , \qquad \{\mathbf{s}\} := \{s_1, \ldots, s_{n'}\} . \tag{172}$$

The $\psi_{\mathbf{k},\mathbf{s}}$ build an orthogonal basis of eigenvectors of $\mathscr{N}$ to the eigenvalues $|\mathbf{k}| + |\mathbf{s}|$.

Theorem 19 (Main Technical Lemma). *On $\mathscr{F}_0$ we can estimate for $\frac{1}{2} < \alpha < \frac{3}{4}$,*

$$\sum_{m \neq -n} (m-n)\pi(L_{m+n}) : b(-m)b^*(n) : + H_2 \le N_b(N_b^{1-\alpha} + 2N_f^{\alpha})^2 \tag{173}$$

in the sense of quadratic forms.

Proof. Let us look at the first term.

$$\sum_{m \neq -n} (m-n) : \pi(L_{m+n}) b(-m) b^*(n) : \tag{174}$$

$$= \frac{1}{2} \sum_{m,n,z \neq 0} (z - 2n) a_m a_{z-m} : b(-z+n) b^*(n) : . \tag{175}$$

By orthogonality we get

$$\langle \psi_{\mathbf{k},\mathbf{s}}, a_m a_{z-m} : b(-z+n)b^*(n) : \psi_{\mathbf{k}',\mathbf{s}'} \rangle \neq 0$$

only if

a) $$\{\mathbf{k}\} \cup \mathrm{pos}\{m, z-m\} = \{\mathbf{k}'\} \cup \mathrm{pos}\{-m, m-z\} , \tag{176}$$

b) $$\{\mathbf{s}\} \cup \mathrm{pos}\{n-z, -n\} = \{\mathbf{s}'\} \cup \mathrm{pos}\{n, z-n\} , \tag{177}$$

and further

c) $$\mathrm{pos}\{m, z-m\} \subset \{\mathbf{k}'\} , \qquad \mathrm{pos}\{-m, m-z\} \subset \{\mathbf{k}\} , \tag{178}$$

d) $$\mathrm{pos}\{-n, n-z\} \subset \{\mathbf{s}'\} , \qquad \mathrm{pos}\{n, z-n\} \subset \{\mathbf{s}\} , \tag{179}$$

where pos$\{.\}$ denotes the subset of positive integers of $\{.\}$. Let us take a closer look at these restriction. Using $z \neq 0$ we can conclude from a) and c),

$$\{\mathbf{k}\} \backslash \{\{\mathbf{k}\} \cap \{\mathbf{k}'\}\} = \mathrm{pos}\{-m, m-z\} , \tag{180}$$

$$\{\mathbf{k}'\} \backslash \{\{\mathbf{k}\} \cap \{\mathbf{k}'\}\} = \mathrm{pos}\{m, z-m\} , \tag{181}$$

and analog from b) and d)

$$\{\mathbf{s}\}\backslash\{\{\mathbf{s}\}\cap\{\mathbf{s}'\}\} = \mathrm{pos}\{n-z,\, n\}\,, \tag{182}$$

$$\{\mathbf{s}'\}\backslash\{\{\mathbf{s}\}\cap\{\mathbf{s}'\}\} = \mathrm{pos}\{z-n,\, n\}\,. \tag{183}$$

Especially for fixed $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{s}$, $\mathbf{s}'$, we get the estimates

$$|m|\cdot|m-z| \le |\mathbf{k}|\,|\mathbf{k}'|\,, \tag{184}$$

$$|z-2n| \le |\mathbf{s}| + |\mathbf{s}'|\,. \tag{185}$$

Analogously let us look at

$$H_2 = \frac{1}{2}\sum_{n,m,z}(-z-n)\,:a_m a_{z-m}:\,b^*(-z-n)b^*(n) + b(-z-n)b(n)\,. \tag{186}$$

Let us define the restrictions on n, m, z for

$$\langle\psi_{\mathbf{k},\mathbf{s}},:a_m a_{z-m}:(b^*(-z-n)b^*(n) + b(-z-n)b(n))\psi_{\mathbf{k}',\mathbf{s}'}\rangle \tag{187}$$

to be nonzero.

If $z=0$ we get

$$m\in\{\mathbf{k}'\}=\{\mathbf{k}\}\,,\qquad n\in\{\mathbf{s}\}=\{\mathbf{s}'\} \tag{188}$$

and

$$\sum_{n,m} -n\,:a_m a_{-m}:\,(b^*(-n)b^*(n)+b(-n)b(n)) = -2N_f N_b\,. \tag{189}$$

If $z\neq 0$ we can use the above analysis.

$$\{\mathbf{k}\}\backslash\{\{\mathbf{k}\}\cap\{\mathbf{k}'\}\} = \mathrm{pos}\{-m,\, m-z\}\,, \tag{190}$$

$$\{\mathbf{k}'\}\backslash\{\{\mathbf{k}\}\cap\{\mathbf{k}'\}\} = \mathrm{pos}\{m,\, z-m\}\,, \tag{191}$$

$$\{\mathbf{s}\}\backslash\{\{\mathbf{s}\}\cap\{\mathbf{s}'\}\} = \mathrm{pos}\{-n-z,\, n\}\,, \tag{192}$$

$$\{\mathbf{s}'\}\backslash\{\{\mathbf{s}\}\cap\{\mathbf{s}'\}\} = \mathrm{pos}\{z+n,\, -n\}\,, \tag{193}$$

for the first summand, analogously for the second summand the two last restrictions are replaced by

$$\{\mathbf{s}\}\backslash\{\{\mathbf{s}\}\cap\{\mathbf{s}'\}\} = \mathrm{pos}\{n+z,\, -n\}\,, \tag{194}$$

$$\{\mathbf{s}'\}\backslash\{\{\mathbf{s}\}\cap\{\mathbf{s}'\}\} = \mathrm{pos}\{-z-n,\, n\}\,, \tag{195}$$

Again one estimates

$$|m|\cdot|m-z| \le |\mathbf{k}|\,|\mathbf{k}'|\,, \tag{196}$$

$$|z-2n| \le |\mathbf{s}| + |\mathbf{s}'|\,, \tag{197}$$

for fixed $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{s}$, $\mathbf{s}'$.

A short look at the restrictions shows that the sum

$$\langle\psi_{\mathbf{k},\mathbf{s}},\, a_m a_{z-m}(b^*(-z-m)b^*(n) + b(-z+n)b^*(n) + b(-z-n)b(n))\psi_{\mathbf{k}',\mathbf{s}'}\rangle \tag{198}$$

has at most one nonvanishing term. For that term we have at most 4 possible choices for n, m, z. Collecting the results we get the estimate

$$\begin{aligned}&\left\langle\psi_{\mathbf{k},\mathbf{s}},\left(\sum_{m\neq -n}(m-n)\pi(L_{m+n})b(-m)b^*(n) + H_2\right)\psi_{\mathbf{k}',\mathbf{s}'}\right\rangle\\ &\quad\le 2\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{s},\mathbf{s}'}|\mathbf{k}|\,|\mathbf{s}|\cdot\|\psi_{\mathbf{k},\mathbf{s}}\|^2\\ &\qquad + 2\sqrt{|\mathbf{k}|\,|\mathbf{k}'|}\,(|\mathbf{s}|+|\mathbf{s}'|)\,\|\psi_{\mathbf{k},\mathbf{s}}\|\,\|\psi_{\mathbf{k}',\mathbf{s}'}\|\,.\end{aligned} \tag{199}$$

Now we use that the operators commute with N, i.e.

$$|\mathbf{k}| + |\mathbf{s}| = |\mathbf{k}'| + |\mathbf{s}'|. \tag{200}$$

One estimates

$$\begin{aligned} |\mathbf{s}| + |\mathbf{s}'| &\le \big||\mathbf{s}| - |\mathbf{s}'|\big| + |\mathbf{s}|^\alpha \cdot |\mathbf{s}'|^\alpha \\ &\le (|\mathbf{s}|^\alpha + |\mathbf{k}|^{1-\alpha})\,(|\mathbf{s}'|^\alpha + |\mathbf{k}'|^{1-\alpha}) + |\mathbf{s}|^\alpha \cdot |\mathbf{s}'|^\alpha . \end{aligned} \tag{201}$$

Plugging this into the above we get

$$2\sqrt{|\mathbf{k}|\,|\mathbf{k}'|}\,(|\mathbf{s}| + |\mathbf{s}'|)\,\|\psi_{\mathbf{k},\mathbf{s}}\|\,\|\psi_{\mathbf{k}',\mathbf{s}'}\| \tag{202}$$

$$\begin{aligned} &\le 2\sqrt{|\mathbf{k}|}\,(|\mathbf{s}|^\alpha + |\mathbf{k}|^{1-\alpha})\,\|\psi_{\mathbf{k},\mathbf{s}}\|\,\sqrt{|\mathbf{k}'|}\,(|\mathbf{s}'|^\alpha + |\mathbf{k}'|^{1-\alpha})\|\psi_{\mathbf{k}',\mathbf{s}'}\| \\ &\quad + 2\sqrt{|\mathbf{k}|}\,|\mathbf{s}^\alpha\,\|\psi_{\mathbf{k},\mathbf{s}}\|\,\sqrt{|\mathbf{k}'|}\,|\mathbf{s}'|^\alpha\,\|\psi_{\mathbf{k}',\mathbf{s}'}\| . \end{aligned} \tag{203}$$

From this estimation one gets the Main Technical Lemma □

These estimates also work in the case of the target space $\mathbf{R}^{26}$. It was only for simplifying the notations that we reduced the dimension.

Lemma 20. *For* $\frac{1}{2} < \alpha < \frac{3}{4}$,

$$\frac{14}{12}N_f^3 + \frac{1}{2}N_b^2 - \frac{26}{12}N_f - 4N_b(N_b^{1-\alpha} + 2N_f^\alpha)^2 \tag{204}$$

is bounded from below.

Proof. The spectra of the operators are contained in $\mathbf{N}$. One compares the highest powers. □

From the lemmata we get the

Corollary 21. $\tilde{H} \ge \frac{1}{2}N_b^2 + N_f^3 + c$, *where c is a constant.*

Proof. Obviously we have

$$\sum_{m>0} 2\pi(L_{-m})\pi(L_m) \ge 0\,, \tag{205}$$

$$\sum_{m>0} 2m\pi(L_0)\ \ (b(-m)b^*(m)b(m)) \ge 0\,, \tag{206}$$

$$H_3 \ge 0\,. \tag{207}$$

The rest follows from the lemmata. □

Using that the spectra are contained in $\mathbf{N}$ we get the finally proof of Theorem 18. □

Thus we get a good candidate for the Dirac-operator on the $\mathscr{F} \otimes \Lambda_\infty^*(p)$-part. The total string algebra acts on the direct integral

$$\mathscr{H}_{\hat{\text{str}}} = \int_{\mathbf{R}^{26}} \mathscr{F} \otimes \Lambda_\infty^*(p)dp\,. \tag{208}$$

For the $C^0(\mathbf{R}^{26})$ factor of the algebra we have a canonical candidate of a quantum algebra Dirac operator. Let $\hat{d}$ be the exterior derivative on the Hilbert space $L^2(\mathbf{R}^{26}, \mathbf{C}^{2^{26}})$ of differential forms on $\mathbf{R}^{26}$, ($\hat{d}$ is a twisted Dirac operator).

$(L^2(\mathbf{R}^{26}, \mathbf{C}^{2^{26}}), \hat{d} + \hat{d}^*)$ defines an odd Fredholm module for $C^0(\mathbf{R}^{26})$, see [Co 2]. Therefore we define the operator

$$Q := \int_{\mathbf{R}^{26}} \tilde{Q}(p)dp + * \otimes (\hat{d} + \hat{d}^*) \tag{209}$$

on

$$\mathscr{H}_{\text{str}} := \mathscr{F} \otimes \Lambda_\infty^* \otimes L^2(\mathbf{R}^{26}, \mathbf{C}^{2^{26}}) . \tag{210}$$

This operator is defined on the dense subspace

$$\mathscr{F}_0 \otimes C^\infty_{cpct}(\mathbf{R}^{26}, \mathbf{C}^{2^{26}}) , \tag{211}$$

where C^∞_{cpct} denotes the subspace of smooth forms with compact support. The estimates show that the vectors in this subspace are analytical vectors for Q. This proves the essential s.a. of Q. The bosonic string algebra also acts on the later Hilbert space. The odd operator Q is a natural candidate for the Dirac operator of the bosonic string algebra.

9. The Finite Summable Fredholm Module

Let $\mathscr{A}$ be equipped with the $\mathbf{Z}_2$-grading as in Sect. 7. Looking at the generating elements of $\mathscr{A}$ we can pick up a dense local subalgebra $\mathscr{A}_\infty$. This algebra consists of the functions with compact support in y, where we used the notation of (113). By [Co2, App. 3] the K-theory of $\mathscr{A}_\infty$ coincides with that of $\mathscr{A}$. Using the foregoing paragraph and the techniques developed in [Co2, 6], we take as the $\mathbf{Z}_2$-graded Hilberts space *-module of $\mathscr{A}_\infty$,

$$\mathscr{H}_{\tilde{\text{str}}} := \mathscr{H}_{\text{str}} \otimes \mathbf{C}^2 \tag{212}$$

and the Dirac operator

$$Q_m := Q \otimes \text{Id} + m \, \text{Id} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{213}$$

Then define $F = \text{sign}(Q_m)$. We get a 26+-summable Fredholm module of $\mathscr{A}_\infty$ whose class is independent of m.

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