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Renormalization Group Flow of a Hierarchical Sine-Gordon Model by Partial Differential Equations*

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Abstract. We use a renormalization group differential equation to rigorously control the renormalization group flow in a hierarchical lattice Sine-Gordon field theory in the Kosterlitz-Thouless phase.

Introduction

We propose to use renormalization group differential equations to rigorously control the renormalization group flow in *lattice* field theories.

The differential equation approach to the renormalization group was first proposed by Wilson [1, 2]. Polchinski used it to give a short complete proof of perturbative renormalizability of ϕ_4^4-theory with a continuous momentum space cutoff [3]. Mitter and Ramadas extended his proof to the $O(N)_2$ nonlinear σ-model [4]. The first rigorous investigations were made by Brydges and Kennedy [5, 6], who proved short time estimates using the method of Cauchy and Kowalewski, and Felder [7], who constructed a family of fixed points for a hierarchical model in $d>2$ dimensions.

In lattice renormalization group theory (for, e.g., scalar fields) the block spin transformation is given by the fluctuation integral [8]

$$e^{-V_{\mathrm{eff}}[\phi]} = \int e^{-V[\mathscr{A}\phi+\zeta]} d\mu_\Gamma[\zeta]. \tag{0.1}$$

We propose to analyse (0.1) using an auxiliary potential, which interpolates between V and V_{eff} by

$$e^{-V[\psi,t]} = \int e^{-V[\psi+\zeta]} d\mu_{t\Gamma}[\zeta], \tag{0.2}$$

where $0 \leqq t \leqq 1$ is the interpolation parameter. The auxiliary potential satisfies the quasi-linear parabolic partial differential equation,

$$V_t = \frac{1}{2} \sum_{x,y\in\Lambda} \Gamma(x,y)\left(V_{\psi(x)\psi(y)} - V_{\psi(x)}V_{\psi(y)}\right), \tag{0.3}$$

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which is the renormalization group differential equation of Wilson and Polchinski, with the initial condition $V[\varphi, 0] = V[\varphi]$. The effective potential is recovered as the boundary value $V[\mathscr{A}\phi, 1] = V_{\text{eff}}[\phi]$.

We use techniques from the theory of parabolic partial differential equations to prove estimates on the solutions of the renormalization group equation (0.3), when Γ is diagonal and when the initial potential is local. This is the case in hierarchical models. We also require that the potential V be periodic in ψ. Our estimates can be used to control the renormalization group flow in this situation.

The estimates are valid for all (positive) times and prove the dissipative character of the renormalization group equation. As a virtue of our approach we do not need a small parameter, and we do not rely on the analyticity properties of the effective potential. We believe that our method, when extended, applies to more general lattice field theories.

As an application we choose the hierarchical $\cos(\varphi)_2$-model, which has been investigated recently by Dimock [9]. A continuous version of the hierarchical Coulomb gas model has been studied previously by Benfatto, Gallavotti, and Nicolo [10]. For general facts on the hierarchical approximation we refer to their work and to that of Gawedzki and Kupiainen [8, 11, 12].

Let us briefly recall that the full model is defined in terms of a massless Gaussian measure[1] $d\mu_{\beta(-\Delta)^{-1}}(\varphi)$ on $\mathbb{R}^{\Lambda}$, where Λ denotes a two dimensional toroidal lattice, and a potential $V(\phi) = 2z \sum_{x \in \Lambda} \cos(\varphi(x))$. This model has various isomorphic representations including that as a classical lattice gas of charged particles with Coulomb interaction $(-\Delta)^{-1}$ and an overall neutrality condition. For a general discussion on this aspect see the review of Fröhlich and Spencer [13]. As was discovered first by Kosterlitz and Thouless [14], and rigorously proved by Fröhlich and Spencer [15], the model has for large β a phase, in which the charged particles form dipoles, and in which correlations of fractional charges show a powerlaw decay. The Kosterlitz-Thouless phase has to be contrasted with the plasma phase for β and z small [16], in which (truncated) correlation functions show exponential decay due to Debye screening. This other phase has been studied by Brydges and Federbush [17, 18]. The ultraviolet behavior of the massive Sine-Gordon model for $\beta < 8\pi$ has been investigated by Fröhlich [19], and Benfatto, Gallavotti, Nicolo et al. [20–22].

Dimock's renormalization group analysis gives another proof that the hierarchical $\cos(\varphi)_2$-model is in the Kosterlitz-Thouless phase for β sufficiently large and $z \leqq e^{-\beta}$. He proves that, in this regime, the sequence of effective measures converges exponentially fast towards a massless Gaussian measure in the infrared limit. We can extend his result to a larger parameter range. We can show that, as a consequence of an energy estimate from the theory of partial differential equations, the renormalization group flow converges for $\beta > 4\ln(L)$ and any z towards the trivial fixed point $V = 0$.[2]

Our estimates follow roughly the lines of Dimock. However, we use L_p-norms instead of estimates on Fourier coefficients, and we estimate the effective potential instead of the Boltzmann factor.

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[1] In two dimensions $\int e^{i(k,\phi)} d\mu_{\beta(-\Delta)^{-1}}(\phi) = \exp\left(-\frac{1}{2}\beta \sum_{x,y \in \Lambda} k(x)C(x-y)k(y)\right)$ if $\sum_{x \in \Lambda} k(x) = 0$ and zero else. Here $C(x) = \lim_{\varepsilon \to 0} (C_\varepsilon(x) - C_\varepsilon(0))$, where $C_\varepsilon(x-y)$ is the integral kernel of $(-\Delta + \varepsilon^2)^{-1}$.

[2] $L \geqq 2$ is a fixed integer in the hierarchical model, and $\beta > 4\ln(L)$ corresponds to $\beta = 8\pi$ in the full model

1. The Hierarchical Model

We consider a Sine-Gordon field theory on a two dimensional finite lattice $\Lambda^{(N)} = (\mathbb{Z}/L^N\mathbb{Z})^2$ with periodic boundary conditions. $L \geqq 2$ is a fixed integer. The hierarchical model is defined in terms of a Gaussian measure $d\mu_{\beta v^{(N)}}[\varphi^{(N)}]$ on $\mathbb{R}^{\Lambda^{(N)}}$ with covariance

$$\begin{aligned} v^{(N)}(x,y) &= \sum_{M=0}^{N} \Gamma^{(M)}(x,y), \\ \Gamma^{(M)}(x,y) &= \delta_{[L^{-N+M}x],[L^{-N+M}y]}, \end{aligned} \tag{1.1}$$

where $[\cdot]$ denotes the integral part. It follows that

$$\begin{aligned} v^{(N)}(x,y) &= N+1-K(x,y), \\ K(x,y) &= \inf\{M \in \mathbb{N} \mid [L^{-M}x] = [L^{-M}y]\}. \end{aligned} \tag{1.2}$$

$v^{(N)}$ mimicks the long distance behavior of $(-\Delta)^{-1}$ up to a factor $2\pi/\ln(L)$, although it is not translation invariant. For details on the hierarchical approximation see [8, 12, 23], and references therein. The model has a local potential

$$V^{(N)}[\varphi^{(N)}] = 2z \sum_{x \in \Lambda^{(N)}} \cos(\varphi^{(N)}(x)), \tag{1.3}$$

and the full measure is $e^{-V^{(N)}[\varphi^{(N)}]} d\mu_{\beta v^{(N)}}[\varphi^{(N)}]$ with two positive real parameters β and z. The partition function is denoted by $Z^{(N)}$.

By iterative renormalization group transformation the theory on $\Lambda^{(N)}$, defined by (1.1) and (1.3), can be mapped to a sequence of effective theories on $\Lambda^{(M)} = (\mathbb{Z}/L^M\mathbb{Z})^2$ with $0 \leqq M \leqq N-1$, such that the partition function is preserved. The effective theory has a Gaussian measure $d\mu_{\beta v(M)}[\varphi^{(M)}]$, with $v^{(M)}$ defined analogously to (1.1). The hierarchical covariance has the property that it splits as follows,

$$v^{(M)} = C^{*(M,M-1)} v^{(M-1)} C^{(M-1,M)} + \mathbb{1}^{(M)}, \tag{1.4}$$

where $C^{(M-1,M)} : l_2(\Lambda^{(M)}) \to l_2(\Lambda^{(M-1)})$ denotes the block average operator

$$\begin{aligned} (C^{(M-1,M)} f)(x') &= \frac{1}{\mathrm{vol}(\Delta(x'))} \sum_{x \in \Delta(x')} f(x), \\ \Delta(x') &= \{x \in \Lambda^{(M)} \mid [L^{-1}x] = x'\}, \end{aligned} \tag{1.5}$$

and $C^{*(M,M-1)}$ denotes the transpose of $C^{(M-1,M)}$. We then define the effective potential $V^{(M-1)}[\varphi^{(M-1)}]$ by

$$e^{-V^{(M-1)}[\varphi^{(M-1)}]} = \frac{\int e^{-V^{(M)}[C^{*(M,M-1)}\varphi^{(M-1)} + \zeta^{(M)}]} d\mu_{\beta \mathbb{1}^{(M)}}[\zeta^{(M)}]}{\int e^{-V^{(M)}[\zeta^{(M)}]} d\mu_{\beta \mathbb{1}^{(M)}}[\zeta^{(M)}]}. \tag{1.6}$$

As a consequence of the convolution formula for Gaussian measures [24] $Z^{(M-1)} = Z^{(M)}$ as promised. Ultimately we end up with a theory on the single point lattice $\Lambda^{(0)}$, which consists of the Gaussian measure

$$d\mu_\beta(\zeta^{(0)}) = (2\pi\beta)^{-\frac{1}{2}} e^{-\frac{1}{2\beta}\zeta^{(0)2}} d\zeta^{(0)} \tag{1.7}$$

on $\mathbb{R}$ with covariance β and the effective potential $V^{(0)}(\zeta^{(0)})$. Since we also want to integrate out this last degree of freedom we define an auxiliary quantity

$$e^{-V^{(-1)}(\varphi^{(-1)})} = \frac{\int e^{-V^{(0)}(\varphi^{(-1)} + \zeta^{(0)})} d\mu_\beta(\zeta^{(0)})}{\int e^{-V^{(0)}(\zeta^{(0)})} d\mu_\beta(\zeta^{(0)})}. \tag{1.8}$$

The generating function (1.8) depends of course on N. The infinite volume limit is then constructed by taking $N \to \infty$ in (1.8).

The hierarchical renormalization group is designed such that the effective potentials remain local, i.e., sums of densities on all scales. Namely

$$V^{(M)}[\varphi^{(M)}] = \sum_{x \in \Lambda^{(M)}} V^{(M)}(\varphi^{(M)}(x)). \tag{1.9}$$

The renormalization group transformation for the function $V^{(M)}(\phi)$ of a single variable then takes the form

$$e^{-V^{(M-1)}(\phi)} = \left(\frac{\int e^{-V^{(M)}(\phi+\zeta)} d\mu_\beta(\zeta)}{\int e^{-V^{(M)}(\zeta)} d\mu_\beta(\zeta)} \right)^{L^2}. \tag{1.10}$$

In the following we will study this nonlinear transformation using techniques from the theory of partial differential equations. The initial value is $V^{(N)}(\phi) = 2z\cos(\phi)$. We decompose the transformation (1.10) into three steps. We first perform the Gaussian integral. For this purpose we define an auxiliary quantity $U^{(M-1)}(\phi, t)$, which depends on an auxiliary parameter $0 \leqq t \leqq \beta$, by

$$e^{-U^{(M-1)}(\phi,t)} = \int e^{-V^{(M)}(\phi+\zeta)} d\mu_t(\zeta). \tag{1.11}$$

It follows that $U^{(M-1)}(\phi, t)$ satisfies the partial differential equation

$$U_t = \tfrac{1}{2}(U_{\phi\phi} - U_\phi^2) \tag{1.12}$$

with initial condition $U^{(M-1)}(\phi, 0) = V^{(M)}(\phi)$.

We will use (1.12) to bound $U^{(M-1)}(\phi, \beta)$. It is clear that (1.11) provides a solution to (1.12), which is $C^\infty(\mathbb{R} \times \mathbb{R}^+)$. A short argument, which we defer to Appendix 1, proves that this solution is also unique. In the following section we will prove *a priori* estimates on the solutions of (1.12), which will be valid for all (positive) times.

In step two we scale $U^{(M-1)}(\phi, \beta)$ by the block volume L^2, so obtaining the unnormalized effective potential

$$V^{*(M-1)}(\phi) = L^2 U^{(M-1)}(\phi, \beta). \tag{1.13}$$

Finally we subtract a physically irrelevant normalization constant

$$V^{(M-1)}(\phi) = V^{*(M-1)}(\phi) - V^{*(M-1)}(0). \tag{1.14}$$

This completes the setup for the hierarchical renormalization group. The main result of our paper which we will prove is the following theorem.

Theorem 1.1. *For all $\beta > 0$, $z > 0$, and $0 \leqq M \leqq N$ the effective potential satisfies the upper bounds*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |V_\phi^{(M)}(\phi)|^2 d\phi \right)^{1/2} \leqq \sqrt{2}z(L^2 e^{-\frac{1}{2}\beta})^{N-M} \tag{1.15}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |V^{(M)}(\phi)| d\phi \leqq 2\pi\sqrt{2}z \left(L^2 e^{-\frac{1}{2}\beta} \right)^{N-M}. \tag{1.16}$$

In particular, this implies that the effective potential vanishes in the infrared limit $N \to \infty$ for all $\beta > 4\ln(L)$ and $z > 0$.

2. A Priori Estimates for the Potential

The partial differential equation for $U(\phi,t)$ implies bounds on the L_p-norms of $U(\phi,t)$ and all its ϕ-derivatives. We present this analysis in detail to convince the reader of the power of our method.

All functions in this section will be $C^\infty(\mathbb{R}\times\mathbb{R}^+)$ and periodic in the first variable $\phi\to\phi+2\pi\mathbb{Z}$. U will also be real valued. We will use the notation

$$\begin{aligned}
(f,g)&=\frac{1}{2\pi}\int_0^{2\pi}\overline{f(\phi)}g(\phi)d\phi,\\
\|f\|_2^2&=(f,f),\\
\|f\|_\infty&=\sup_{\phi\in[0,2\pi]}|f(\phi)|,\\
f(\phi)&=\sum_{n\in\mathbb{Z}}\widehat{f_n}e^{in\phi}.
\end{aligned}\tag{2.1}$$

The L_2-norm of U refers to the first variable ϕ, and not to the second variable t. A superscript ^{0} will denote initial values at time $t=0$. C_1 and C_2 will denote Sobolev constants. Our first estimate is an energy bound.

Lemma 2.1. *Let $U(\phi,t)$ be a smooth solution of* (1.12). *Then the energy $\|U_\phi\|_2$ decreases strictly in $t\geqq 0$, and it is bounded from above by*

$$\|U_\phi\|_2\leqq e^{-\frac{1}{2}t}\|U_\phi^0\|_2.\tag{2.2}$$

Proof. Using (1.12) it follows that

$$\frac{d}{dt}\|U_\phi\|_2^2=(U_\phi,U_{\phi\phi\phi}-2U_\phi U_{\phi\phi})=-\|U_{\phi\phi}\|_2^2,\tag{2.3}$$

since the second term on the right-hand side is a total derivative. By Fourier expansion $\|U_\phi\|_2\leqq\|U_{\phi\phi}\|_2$, and thus

$$\frac{d}{dt}\|U_\phi\|_2^2\leqq-\|U_\phi\|_2^2,\tag{2.4}$$

which implies (2.2) by integration. □

As a consequence, all nonzero Fourier modes of U decay exponentially to zero for asymptotically large times.

Lemma 2.2. *The zero mode $\widehat{U_0}$ decreases strictly in $t\geqq 0$, and it is bounded from below by*

$$\widehat{U_0}\geqq\widehat{U_0^0}-\tfrac{1}{2}(1-e^{-t})\|U_\phi^0\|_2^2.\tag{2.5}$$

Proof. Integrate (1.12) to obtain

$$\frac{d\widehat{U_0}}{dt}=-\frac{1}{2}\|U_\phi\|_2^2\geqq-\frac{1}{2}e^{-t}\|U_\phi^0\|_2^2,\tag{2.6}$$

which yields (2.5). □

The complete analysis of (1.12) includes upper bounds on the L_2-norms of all higher ϕ-derivatives of U. As a tool we note the following estimate, which we will prove in the appendix. There exists a Sobolev constant C_2 such that

$$\|U_{\phi\phi}\|_3 \leqq C_2^{1/3} \|U_\phi\|_2^{1/3} \|U_{\phi\phi\phi}\|_2^{2/3}. \tag{2.7}$$

In the appendix we will prove that $C_2 \leqq \pi/\sqrt{3}$.

Lemma 2.3. *Let $\|U_\phi^0\|_2 < \sqrt{3}/\pi$. Then $\|U_{\phi\phi}\|_2$ decreases strictly in $t \geqq 0$, and it is bounded from above by*

$$\|U_{\phi\phi}\|_2 \leqq e^{-\frac{1}{2}t + \frac{1}{2}C_2 \int_0^t \|U_\phi\|_2 dt'} \|U_{\phi\phi}^0\|_2,$$

$$\frac{1}{2} C_2 \int_0^t \|U_\phi\|_2 dt' \leqq \frac{\pi}{\sqrt{3}} (1 - e^{-\frac{1}{2}t}) \|U_\phi^0\|_2. \tag{2.8}$$

Proof. Using (1.12) we have

$$\begin{aligned} \frac{d}{dt} \|U_{\phi\phi}\|_2^2 &= (U_{\phi\phi}, U_{\phi\phi\phi\phi} - 2(U_{\phi\phi})^2 - 2U_\phi U_{\phi\phi\phi}) \\ &= -\|U_{\phi\phi\phi}\|_2^2 - \frac{1}{2\pi} \int_0^{2\pi} (U_{\phi\phi})^3 d\phi. \end{aligned} \tag{2.9}$$

The second term on the right-hand side can be estimated with the interpolation inequality (2.7). Thus using $\|U_{\phi\phi}\|_2 \leqq \|U_{\phi\phi\phi}\|_2$ it follows that

$$\begin{aligned} \frac{d}{dt} \|U_{\phi\phi}\|_2^2 &\leqq -(1 - C_2 \|U_\phi\|_2) \|U_{\phi\phi\phi}\|_2^2 \\ &\leqq -(1 - C_2 \|U_\phi\|_2) \|U_{\phi\phi}\|_2^2. \end{aligned} \tag{2.10}$$

The estimate (2.8) follows from Lemma 2.1 and the estimate (A2.7). □

Since there exists a Sobolev constant C_1, such that $\|U_\phi\|_\infty \leqq C_1 \|U_{\phi\phi}\|_2$, the estimate (2.8) implies exponential decay also for $\|U_\phi\|_\infty$. In the appendix we will prove that $C_1 \leqq \pi/\sqrt{3}$.

Lemma 2.4. *Let $\|U_\phi^0\|_2 < \sqrt{3}/\pi$, and let $\|U_{\phi\phi}^0\|_2 < \sqrt{3}/(5\pi)$. Then $\|U_{\phi\phi\phi}\|_2$ decreases strictly in $t \geqq 0$, and it is bounded from above by*

$$\|U_{\phi\phi\phi}\|_2 \leqq e^{-\frac{1}{2}t + \frac{5}{2}C_2 \int_0^t \|U_{\phi\phi}\|_2 dt'} \|U_{\phi\phi\phi}^0\|_2,$$

$$\frac{5}{2} C_2 \int_0^t \|U_{\phi\phi}\|_2 dt' \leqq \frac{5\pi}{\sqrt{3}} (1 - e^{-\frac{1}{2}t}) e^{\frac{\pi}{\sqrt{3}}\|U_\phi^0\|_2} \|U_{\phi\phi}^0\|_2. \tag{2.11}$$

Proof. As a consequence of (1.12),

$$\begin{aligned} \frac{d}{dt} \|U_{\phi\phi\phi}\|_2^2 &= (U_{\phi\phi\phi}, U_{\phi\phi\phi\phi\phi} - 6U_{\phi\phi}U_{\phi\phi\phi} - 2U_\phi U_{\phi\phi\phi\phi}) \\ &= -\|U_{\phi\phi\phi\phi}\|_2^2 - \frac{5}{2\pi} \int_0^{2\pi} U_{\phi\phi}(U_{\phi\phi\phi})^2 d\phi. \end{aligned} \tag{2.12}$$

The second term on the right-hand side can be estimated using Hölder's inequality with $p = 3$ and $q = \frac{3}{2}$. Thus

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |U_{\phi\phi}| (U_{\phi\phi\phi})^2 d\phi &\leqq \|U_{\phi\phi}\|_3 \|U_{\phi\phi\phi}\|_3^2 \\ &\leqq C_2 \|U_\phi\|_2^{1/3} \|U_{\phi\phi\phi}\|_2^{2/3} \|U_{\phi\phi}\|_2^{2/3} \|U_{\phi\phi\phi\phi}\|_2^{4/3} \\ &\leqq C_2 \|U_{\phi\phi}\|_2 \|U_{\phi\phi\phi\phi}\|_2^2, \end{aligned} \tag{2.13}$$

where we also used the interpolation inequality (2.7). Hence

$$\frac{d}{dt}\|U_{\phi\phi\phi}\|_2^2 \leqq -(1-5C_2\|U_{\phi\phi}\|_2)\|U_{\phi\phi\phi\phi}\|_2^2$$
$$\leqq -(1-5C_2\|U_{\phi\phi}\|_2)\|U_{\phi\phi\phi}\|_2^2, \tag{2.14}$$

and (2.11) follows from Lemma 2.3. □

Finally we can prove that the L_2-norms of all higher ϕ-derivatives are bounded uniformly in $t\in\mathbb{R}^+$. Let $U_{(n)}=\frac{\partial^n U}{\partial\phi^n}$.

Lemma 2.5. *For all $n\in\mathbb{N}\backslash\{0\}$ there exist constants $A_n<\infty$, such that*

$$\|U_{(n)}\|_2^2+\int_0^t \|U_{(n+1)}\|_2^2 dt' \leqq A_n \tag{2.15}$$

uniformly in $t\in\mathbb{R}^+$. In particular

$$\sup_{t\in\mathbb{R}^+}\|U_{(n)}\|_2 \leqq A_n,$$
$$\sup_{t\in\mathbb{R}^+}\|U_{(n)}\|_\infty \leqq \frac{\pi}{\sqrt{3}}A_{n+1}. \tag{2.16}$$

Proof. We have already proved (2.15) for $n\in\{1,2,3\}$. In fact (2.15) can be proved to hold without further assumptions on the initial data. The higher derivatives are estimated by induction on $n-1\to n$. From (1.12) it follows that

$$\frac{d}{dt}\|U_{(n)}\|_2^2=\left(U_{(n)},U_{(n+2)}-\sum_{l=0}^{n}\binom{n}{l}U_{(l+1)}U_{(n-l+1)}\right)$$
$$=-\|U_{(n+1)}\|_2^2-\sum_{l=0}^{n}\binom{n}{l}(U_{(n)},U_{(l+1)}U_{(n-l+1)}). \tag{2.17}$$

The terms under the sum are estimated respectively by

$$|U_{(n)},U_{(1)}U_{(n+1)})|\leqq \tfrac{1}{2}\|U_{(2)}\|_\infty\|U_{(n)}\|_2^2,$$
$$|(U_{(n)},U_{(2)}U_{(n)})|\leqq \|U_{(2)}\|_\infty\|U_{(n)}\|_2^2,$$
$$|(U_{(n)},U_{(l+1)}U_{(n-l+1)})|\leqq \|U_{(l+1)}\|_\infty\|U_{(n-l+1)}\|_2\|U_{(n)}\|_2. \tag{2.18}$$

We use the last estimate for $2\leqq l\leqq n-2$. Thus

$$\|U_{(n)}\|_2^2+\int_0^t\|U_{(n+1)}\|_2^2dt'$$
$$\leqq \|U_{(n)}^0\|_2^2+(2n+1)\int_0^t\|U_{(2)}\|_\infty\|U_{(n)}\|_2^2dt'$$
$$+\sum_{l=2}^{n-2}\binom{n}{l}\int_0^t\|U_{(l+1)}\|_\infty\|U_{(n-l+1)}\|_2\|U_{(n)}\|_2dt'$$
$$\leqq \|U_{(n)}^0\|_2^2+(2n+1)\sup_{t\in\mathbb{R}^+}\|U_{(2)}\|_\infty\int_0^t\|U_{(n)}\|_2^2dt'$$
$$+\sum_{l=2}^{n-2}\binom{n}{l}\sup_{t\in\mathbb{R}^+}\|U_{(l+1)}\|_\infty\left(\int_0^t\|U_{(n-l+1)}\|_2^2dt'\right)^{1/2}\left(\int_0^t\|U_{(n)}\|_2^2dt'\right)^{1/2}. \tag{2.19}$$

Suppose that there exist constants A_l with $1 \leqq l \leqq n-1$, such that (2.15) and (2.16) hold. Equation (2.19) then implies the existence of A_n. □

The estimates in this section immediately imply estimates on the single Fourier modes.

3. The Kosterlitz-Thouless Phase

When the estimates of Sect. 2 are applied to the hierarchical renormalization group transformation, the dissipative behavior of the partial differential Eq. (1.12) for large times (i.e., large β) competes with the rescaling by the block volume. It turns out that the transformation is contractive for $\beta > 4\ln(L)$. Thus the effective potential flows into the trivial fixed point $V=0$ for $\beta > 4\ln(L)$ in the infrared *for all values of z*, and the model is asymptotically free. This is the Kosterlitz-Thouless phase in the hierarchical model.

In Fig. 1, we display the infrared limit of the L_2-norm of the first ϕ-derivative of the effective potential on the lattice consisting of a single site. The data are obtained by numerical integration of (1.12) [25]. We choose $L=2$. For $\beta < 4\ln(L)$ (plasma phase) the flow converges to nontrivial fixed points which depend on β only. Recall that β remains unchanged under renormalization group transformations.

Lemma 3.1. *For all* $1 \leqq M \leqq N$,

$$\| V_\phi^{(M-1)} \|_2 \leqq L^2 e^{-\frac{1}{2}\beta} \| V_\phi^{(M)} \|_2 , \tag{3.1}$$

and thus

$$\| V_\phi^{(M)} \|_2 \leqq (L^2 e^{-\frac{1}{2}\beta})^{N-M} \| V_\phi^{(N)} \|_2 = \sqrt{2}z(L^2 e^{-\frac{1}{2}\beta})^{N-M} . \tag{3.2}$$

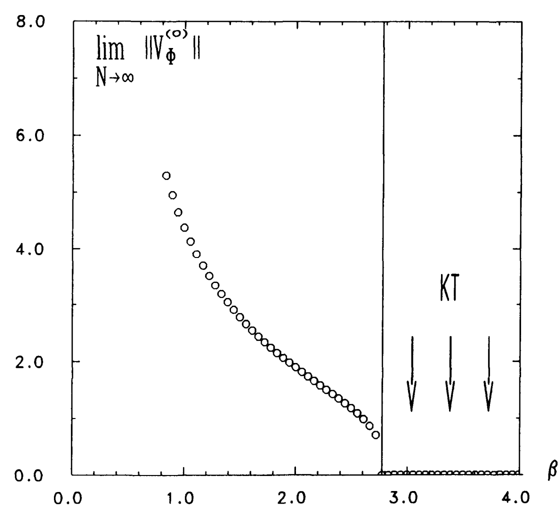

Fig. 1

Proof. From (1.14), (1.13), (1.11), and the energy estimate (2.2) it follows that

$$\|V_{\phi}^{(M-1)}\|_2 = \|V_{\phi}^{*(M-1)}\|_2 = L^2 \|U_{\phi}^{(M-1)}(\cdot,\beta)\|_2 \leqq L^2 e^{-\frac{1}{2}\beta} \|U_{\phi}^{(M-1)}(\cdot,0)\|_2 = L^2 e^{-\frac{1}{2}\beta} \|V_{\phi}^{(M)}\|_2, \tag{3.3}$$

which proves (3.1). □

Lemma 3.2. *For all* $0 \leqq M \leqq N-1$ *the modulus of the zero mode* $|\widehat{V_0^{(M)}}|$ *is bounded from above by*

$$|\widehat{V_0^{(M)}}| \leqq 2\pi L^2 e^{-\frac{1}{2}\beta} \|V_{\phi}^{(M+1)}\|_2. \tag{3.4}$$

For all $0 \leqq M \leqq N-1$ *the* L_∞*-norm of* $V^{(M)}$ *is bounded from above by*

$$\|V^{(M)}\|_\infty \leqq \left(2 + \frac{1}{\sqrt{3}}\right) \pi L^2 e^{-\frac{1}{2}\beta} \|V_{\phi}^{(M+1)}\|_2. \tag{3.5}$$

Proof. As a consequence of the subtraction (1.14)

$$|\widehat{V_0^{(M)}}| = \left|\frac{L^2}{2\pi} \int_0^{2\pi} \left(\int_0^{\phi} U_{\phi'}^{(M)}(\phi',\beta)d\phi'\right) d\phi\right| \leqq 2\pi L^2 \|U_{\phi}^{(M)}(\cdot,\beta)\|_1 \leqq 2\pi L^2 \|U_{\phi}^{(M)}(\cdot,\beta)\|_2 \leqq 2\pi L^2 e^{-\frac{1}{2}\beta} \|V_{\phi}^{(M+1)}\|_2. \tag{3.6}$$

Note that we do not use that $\widehat{V_0^{(N)}} = 0$. With the Sobolev estimate (A 2.1) we conclude that

$$\|V^{(M)}\|_\infty \leqq |\widehat{V_0^{(M)}}| + C_1 \|V_{\phi}^{(M)}\|_2 \leqq (2\pi + C_1) L^2 e^{-\frac{1}{2}\beta} \|V_{\phi}^{(M+1)}\|_2, \tag{3.7}$$

where we have used the estimates (3.1) and (3.4). □

In the analysis of correlation functions [25, 26] one also needs an iterated estimate for $\|V_{\phi\phi}^{(M)}\|_2$.

Lemma 3.5. *Let* $\beta > 4\ln(L)$, *and let* $z < \sqrt{3/(2\pi^2)}$. *For all* $0 \leqq M \leqq N$ $\|V_{\phi\phi}^{(M)}\|_2$ *is bounded from above by*

$$\|V_{\phi\phi}^{(M)}\|_2 \leqq \sqrt{2} z e^{\sqrt{\frac{2\pi^2}{3}} z (1 - L^2 e^{-\frac{1}{2}\beta})^{-1}} (L^2 e^{-\frac{1}{2}\beta})^{N-M}. \tag{3.8}$$

Proof. From Lemma 2.3 we infer the bound

$$\|V_{\phi\phi}^{(M-1)}\|_2 \leqq L^2 e^{-\frac{1}{2}\beta + \frac{\pi}{\sqrt{3}}\|V_{\phi}^{(M)}\|_2} \|V_{\phi}^{(M)}\|_2. \tag{3.9}$$

Using the estimate

$$\frac{\pi}{\sqrt{3}} \sum_{P=M+1}^{N} \|V_{\phi}^{(P)}\|_2 \leqq \frac{\pi}{\sqrt{3}} (1 - L^2 e^{-\frac{1}{2}\beta})^{-1} \|V_{\phi}^{(N)}\|_2 \tag{3.10}$$

together with the initial condition

$$\|V_{\phi}^{(N)}\|_2 = \|V_{\phi\phi}^{(N)}\|_2 = \sqrt{2} z, \tag{3.11}$$

the estimate (3.8) follows by iteration. □

Lemma 3.3 implies an immediate estimate on $\|V_{\phi}^{(M)}\|_\infty$. This completes the discussion of the effective potential.

The above estimates can be used to control the long distance decay of correlation functions. This has been done in [9, 26]. The auxiliary effective potential can also be continuously rescaled while integrating the renormalization group differential equation. This yields an extra linear term in the equation. Stationary solutions and solutions periodic in time with period β are then fixed points of the transformation. The stationary solutions can be completely classified [25].

Work on the full Sine-Gordon model in the Kosterlitz-Thouless phase using the renormalization group differential equation is in progress. As in the hierarchical case the main problem is to control the L_2-norms of the effective potential and its first three ϕ-derivatives.

It remains an open question if similar estimates can be used to control models with unbounded potentials.

Appendix

1. Uniqueness of the Solution. Let $U^{(1)}$ and $U^{(2)}$ denote two solutions of $U_t=\frac{1}{2}(U_{\phi\phi}-U_\phi^2)$ with $U^{(1)}(\phi,0)=U^{(2)}(\phi,0)$. Then $W=U^{(1)}-U^{(2)}$ satisfies the differential equation

$$W_t=\tfrac{1}{2}(W_{\phi\phi}-FW_\phi) \tag{A1.1}$$

with $F=U_\phi^{(1)}+U_\phi^{(2)}$ and the initial condition $W(\phi,0)=0$. It follows that

$$\frac{d}{dt}\|W\|_2^2=-\|W_\phi\|_2^2+\frac{1}{2}(W,F_\phi W)\leqq\frac{1}{2}\|F_\phi\|_\infty\|W\|_2^2, \tag{A1.1}$$

and thus $W=0$.

2. Estimates on Sobolev Constants. The Sobolev constant C_1 in the estimate

$$\|U_\phi\|_\infty\leqq C_1\|U_{\phi\phi}\|_2 \tag{A2.1}$$

can be estimated as follows,

$$\begin{aligned}\|U_\phi\|_\infty&\leqq\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{|n|}|n\widehat{U_{\phi n}}|\\&\leqq\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n^2}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}}n^2|\widehat{U_{\phi n}}|^2\Big)^{1/2}\\&=\frac{\pi}{\sqrt{3}}\|U_{\phi\phi}\|_2\,.\end{aligned} \tag{A2.2}$$

Thus $C_1\leqq\frac{\pi}{\sqrt{3}}$. Next we bound the Sobolev constant C_2 in the interpolation estimate

$$\|U_\phi\|_3\leqq C_2^{1/3}(\|U\|_2^2-|\widehat{U_0}|^2)^{1/6}\|U_{\phi\phi}\|_2^{2/3}\,. \tag{A2.3}$$

By Young's theorem

$$\|U_\phi\|_3\leqq\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}}|n\widehat{U_n}|^{3/2}\Big)^{2/3}. \tag{A2.4}$$

Using Hölder's inequality with $p=4/3$ and $q=4$ it follows that

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} (|n|^{3/2}|\widehat{U_n}|)(|\widehat{U_n}|)^{1/2} \leqq \Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} n^2|\widehat{U_n}|^{4/3}\Big)^{3/4}\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{U_n}|^2\Big)^{1/4}. \quad (A2.5)$$

The first factor is estimated using Hölder's inequality with $p=3$ and $q=3/2$,

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{-2/3}(|n|^{2+2/3}|\widehat{U_n}|^{4/3}) \leqq \Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}\Big)^{1/3}\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} n^4|\widehat{U_n}|^2\Big)^{2/3}. \quad (A2.6)$$

Thus putting (A2.4), (A2.5), and (A2.6) together

$$\begin{aligned}\|U_\phi\|_3 &\leqq \Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}\Big)^{1/6}\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{U_n}|^2\Big)^{1/6}\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} n^4|\widehat{U_n}|^2\Big)^{1/3}\\ &= \Big(\frac{\pi^2}{3}\Big)^{1/6}(\|U\|_2^2-|\widehat{U_0}|^2)^{1/6}\,\|U_{\phi\phi}\|_2^{2/3}. \quad (A2.7)\end{aligned}$$

Hence $C_2^{1/3} \leqq \Big(\frac{\pi^2}{3}\Big)^{1/6}$.

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