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Local Rings of Singularities and $N=2$ Supersymmetric Quantum Mechanics*

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Abstract. We investigate the Kähler structure arising in n-component, $N=2$ supersymmetric quantum mechanics. We define L^2-cohomology groups of a modified $\bar{\partial}$-operator and relate them to the corresponding spaces of harmonic forms. We prove that the cohomology is concentrated in the middle dimension, and is isomorphic to the direct sum of the local rings of the singularities of the superpotential. In the physics language, this means that the number of ground states is equal to the absolute value of the index of the supercharge, and each ground state contains exactly n fermions.

I. Introduction

$N=2$ supersymmetric Wess–Zumino models in one and two dimensions have been extensively studied over the past few years. These quantum field theory models are particularly rich in structure, and serve as a nontrivial example in studying the phenomenon of supersymmetry breaking [CG 1, 2, GIM], constructive field theory [JLW, JL 1], as well as string theory [GSW]. Wess–Zumino models are far from being exactly solvable. Yet, $N=2$ supersymmetry allows for closed form computations of various numerical characteristics of the models. The simplest of these characteristics is the index of the supersymmetry generator, the supercharge. The supercharge plays a similar role and has a similar structure as the Dirac operator in differential geometry. Its index is a topological invariant which captures certain qualitative features of the model, and is independent of its details (see, e.g., Sect. II of [JL 2] for a precise formulation of this statement). It is known [JLL] that in one-component $N=2$ Wess–Zumino quantum mechanics with a polynomial superpotential, the number of ground states is equal to the index of the supercharge. It was also proven that this number is equal to the algebraic degree of the superpotential minus one.

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In a recent work [CGP], Cecotti, Girardello and Pasquinucci proposed a new approach to the $N=2$ Wess–Zumino quantum mechanics with many components. They suggested studying the L^2-cohomology of a certain complex which arises naturally in the theory. Motivated by their approach, and using some of their ideas, we establish a number of theorems on cohomology groups arising as a generalization of the cohomology of [CGP]. The complex in question is a perturbation of the Dolbeault complex with coboundary given by $\bar{\partial}+f$, where f is a holomorphic one-form on $\mathbb{C}^n$. We prove a Hodge type theorem for the square-integrable cohomologies of this operator, and relate them to the singularity structure of f. As a consequence of our results, we obtain the vanishing theorem in $N=2$ Wess–Zumino quantum mechanics: the number of ground states of the system is equal to the absolute value of the index of the supercharge.

It has been recognized recently [LVW] that $N=2$ Wess–Zumino models are closely related to the singularity theory of holomorphic maps. The relationship is roughly of the same type as the relationship between nonlinear supersymmetric σ-models and differential geometry. The ground states of the σ-model quantum mechanics are just harmonic forms on the target manifold and can be studied via de Rham cohomology. We show that the same is true in $N=2$ Wess–Zumino quantum mechanics with the Koszul complex of the singularity replacing the de Rham complex of the target manifold.

The paper is organized as follows. In Sect. II we study the smooth cohomologies of the perturbed $\bar{\partial}$-operator on an arbitrary Stein manifold X. In Sect. III we introduce the L^2-cohomology of this operator on $X=\mathbb{C}^n$ and formulate our main results: the Hodge-type theorem, the vanishing theorem, and the index theorem. These theorems are proven in Sects. IV and V. In Sect. VI, we connect our results with the theory of residues of meromorphic n-forms. Section VII has an informal, nonrigorous character: here we formulate some conjectures on various generalizations of our results.

II. Smooth Cohomologies

In this section we study the smooth cohomologies of a perturbed $\bar{\partial}$-operator on a Stein space X (see [GR] for the definition of a Stein space). Later in this paper will study the L^2-cohomologies of $X=\mathbb{C}^n$ (the reader who feels uncomfortable with Stein spaces may make this substitution already in this section), and relate them to the smooth cohomologies.

Let X be a Stein space of (complex) dimension n, and let $\wedge^{p,q}(X)$ denote the space of smooth (p,q)-forms on X. We set

$$\wedge^k(X)=\bigoplus_{p+q=k}\wedge^{p,q}(X),\quad \wedge(X)=\bigoplus_{k=0}^{2n}\wedge^k(X). \tag{II.1}$$

Let ∂ and $\bar{\partial}$ be the standard Dolbeault coboundary operators (we will be primarily concerned with the operator $\bar{\partial}$). Recall [GR] that the Dolbeault cohomologies $\mathscr{H}^{p,q}_{\bar{\partial}}(X)$ of a Stein space are very simple, namely

$$H^{p,q}_{\bar{\partial}}(X)=\begin{cases}0, & \text{if } q\geqq 1,\\ \Omega^p(X), & \text{if } q=0,\end{cases} \tag{II.2}$$

where $\Omega^p(X)$ is the space of holomorphic p-forms on X.

We are concerned with the perturbed Dolbeault operator $\bar{\partial}_f$ defined by

$$\bar{\partial}_f := \bar{\partial} + f \wedge . \tag{II.3}$$

Here $f \in \Omega^1(X)$ is a holomorphic one-form which acts on $\wedge(X)$ by exterior multiplication. Since

$$\bar{\partial}_f^2 = 0, \tag{II.4}$$

the following complex arises

$$\wedge^0 \xrightarrow{\bar{\partial}_f} \wedge^1 \xrightarrow{\bar{\partial}_f} \cdots \xrightarrow{\bar{\partial}_f} \wedge^{2n}. \tag{II.5}$$

Let $\mathscr{H}_f^k(X)$ denote the cohomology groups of this complex.

We will relate the cohomology of (II.5) to the cohomology of the following complex (the Koszul complex):

$$\Omega^0 \xrightarrow{f\wedge} \Omega^1 \xrightarrow{f\wedge} \cdots \xrightarrow{f\wedge} \Omega^n. \tag{II.6}$$

We denote the cohomology groups of this complex by $\mathscr{K}_f^k(X)$.

We should remark at this point that the cohomology of (II.5) arises as the total cohomology of a double complex whose vertical coboundary operator is $\bar{\partial}$ and whose horizontal coboundary operator is $f \wedge$. The results of this section, which we prove by rather elementary means, can be proven by studying the spectral sequences associated with this double complex (we thank Joe Harris for a discussion on this point). In this spectral sequence, the cohomology of (II.6) arises as $'E_1^{*,0}$.

Proposition II.1. *With the above notation,*

$$\mathscr{H}_f^k \cong \begin{cases} 0, & \text{if } k > n, \\ \mathscr{K}_f^k & \text{if } k \leqq n. \end{cases} \tag{II.7}$$

Proof. Consider first $1 \leqq k \leqq n$. As the cochains in (II.6) are holomorphic forms, $\mathscr{K}_f^k$ is naturally embedded in $\mathscr{H}_f^k$, and what remains to be proven is that every $\bar{\partial}_f$-cohomology class contains a holomorphic representative. Let ω_k be a $\bar{\partial}_f$-closed k-form. Writing $\omega_k = \sum_{p+q=k} \omega_{p,q}$, we obtain the following conditions:

$$f \wedge \omega_{k,0} = 0, \tag{II.8}$$

$$\bar{\partial}\omega_{p+1,k-p-1} + f \wedge \omega_{p,k-p} = 0, \quad 1 \leqq p \leqq k-1, \tag{II.9}$$

and

$$\bar{\partial}\omega_{0,k} = 0. \tag{II.10}$$

We claim that there exist $(k-1)$-forms $\eta_{p,q}$ such that

$$\omega_{0,k} = \bar{\partial}\eta_{0,k-1}, \tag{II.11}$$

and

$$\omega_{p,k-p} = \bar{\partial}\eta_{p,k-p-1} + f \wedge \eta_{p-1,k-p}, \quad 1 \leqq p \leqq k-1. \tag{II.12}$$

Indeed, $\eta_{0,k-1}$ exists by (II.2). Given $\eta_{p-1,k-p}$, $p \leqq k-2$, such that (II.12) holds, we observe that

$$\bar{\partial}(\omega_{p,k-p}-f\wedge\eta_{p-1,k-p})=\bar{\partial}\omega_{p,k-p}+f\wedge\bar{\partial}\eta_{p-1,k-p}$$
$$=\bar{\partial}\omega_{p,k-p}+f\wedge\omega_{p-1,k-p+1}=0,$$

as a consequence of (II.9). Thus by (II.2) there is $\eta_{p,k-p-1}$ satisfying (II.12) with p replaced by $p+1$. If $p=k-1$, then Eq. (II.9) reads $\bar{\partial}\omega_{k,0}+f\wedge\omega_{k-1,1}=0$, or by means of (II.12), $\bar{\partial}(\omega_{k,0}-f\wedge\eta_{k-1,0})=0$. As a consequence of (II.2), $\omega_{k,0}-f\wedge\eta_{k-1,0}$ is a holomorphic k-form. Adding up (II.11) and (II.12) we find that ω_k is $\bar{\partial}_f$-cohomologous to $\omega_{k,0}-f\wedge\eta_{k-1,0}$, and thus $\omega_{k,0}-f\wedge\eta_{k-1,0}$ is the desired holomorphic representative of the $\bar{\partial}_f$-cohomology class of ω_k.

The remaining cases $k=0$ and $k>n$ can be analyzed in the same fashion. □

Let us now assume that f has a finite number of zeros,

$$Z_f:=\{z\in X:f(z)=0\}=\{z_1,\dots,z_r\}. \tag{II.13}$$

Let $\mathcal{O}_z$ denote the ring of germs of functions holomorphic in the vicinity of z. By $[f]_z$ we denote the ideal in $\mathcal{O}_z$ generated by the components of f. The corresponding quotient space $R_z^f:=\mathcal{O}_z/[f]_z$ is called the local ring of f at z [AGV] and it is independent of the choice of coordinates near z.

Theorem II.2. *Let Z_f be finite. Then*

$$\mathscr{H}_f^k\cong\begin{cases}0, & \text{if } k<n,\\ \bigoplus\limits_{z_j\in Z_f} R_{z_j}^f, & \text{if } k=n,\end{cases} \tag{II.14}$$

as isomorphisms of vector spaces.

Proof. Clearly, $\mathscr{H}_f^0\cong 0$. Let $1\leqq k\leqq n-1$, $n>1$. Since the dimension of Z_f is zero, $\Omega^p(X)\cong\Omega^p(X\backslash Z_f)$. Also, by the extension of cohomology classes theorem of Scheja [Sch],

$$\mathscr{H}_{\bar{\partial}}^{p,q}(X\backslash Z_f)\cong\begin{cases}\mathscr{H}_{\bar{\partial}}^{p,q}(X)\cong 0, & \text{if } q\geqq 1,\quad p\leqq n-2,\\ \Omega^p(X)\cong\Omega^p(X\backslash Z_f), & \text{if } q=0,\quad 0\leqq p\leqq n.\end{cases} \tag{II.15}$$

We can thus restrict attention to the cohomologies of the submanifold $X\backslash Z_f$. Choosing a hermitian structure on X, we can find a smooth vector field v of type $(1,0)$ such that

$$fv+vf=I \quad \text{on} \quad \wedge(X\backslash Z_f), \tag{II.16}$$

where v acts on $\wedge(X\backslash Z_f)$ by interior multiplication (v has, in general, singularities on Z_f). We note that v is not a homotopy for (II.6) as it does not map holomorphic forms into holomorphic forms. Now, if $\omega_{k,0}$ is a holomorphic $f\wedge$-closed k-form, then

$$\bar{\partial}\omega_{k,0}=f\wedge\omega_{k,0}=0. \tag{II.17}$$

As a consequence of (II.16), $\omega_{k,0}=f\wedge(vw_{k,0})$. This, however, does not solve our problem yet, as $\omega_{k-1,0}:=vw_{k,0}$ does not have to be holomorphic. From (II.17) we conclude, however, that $f\wedge\bar{\partial}\omega_{k-1,0}=0$, and so $\bar{\partial}\omega_{k-1,0}+f\wedge\omega_{k-2,1}=0$, where $\omega_{k-2,1}:=-v\bar{\partial}\omega_{k-1,0}$. Continuing this process, we obtain a sequence of $(k-1)$-

forms $\omega_{p,k-p-1}$, $p=0,1,\dots,k-1$, such that

$$\bar{\partial}\omega_{p,k-p-1}+f\wedge\omega_{p-1,k-p}=0, \quad \text{if} \quad p>0, \tag{II.18}$$

and

$$\bar{\partial}\omega_{0,k-1}=0, \tag{II.19}$$

(explicitly, we set $\omega_{p-1,k-p}:=-v\bar{\partial}\omega_{p,k-p-1}$ and observe that $f\wedge\bar{\partial}\omega_{p-1,k-p}=0$).

We now use (II.15) to write

$$\omega_{0,k-1}=\bar{\partial}\eta_{0,k-2}. \tag{II.20}$$

Using this and (II.18), we construct recursively a sequence of forms $\eta_{p,k-p-2}$ such that

$$\omega_{p,k-p-1}=\bar{\partial}\eta_{p,k-p-2}+f\wedge\eta_{p-1,k-p-1}, \tag{II.21}$$

if $p=1,2,\dots,k-2$. Using (II.21) for $p=k-2$ we see that $\bar{\partial}(\omega_{k-1,0}-f\wedge\eta_{k-2,0})=0$, and thus, by (II.15), $\omega_{k-1,0}-f\wedge\eta_{k-2,0}$ is a holomorphic $(k-1)$-form. Since by construction

$$\omega_{k,0}=f\wedge(\omega_{k-1,0}-f\wedge\eta_{k-2,0}), \tag{II.22}$$

$\omega_{k,0}$ is cohomologous to zero. Therefore, $\mathscr{K}_f^k\cong 0$, for $k\leqq n-1$.

Consider now $k=n$, i.e.,

$$\mathscr{K}_f^n=\Omega^n/f\wedge\Omega^{n-1}. \tag{II.23}$$

Choosing local coordinates near each z_j we observe first that restriction of forms defines a mapping i of $\mathscr{K}_f^n$ into $\bigoplus_{z_j\in Z_f} R_{z_j}^f$. We claim that i is an isomorphism of vector spaces. First, we verify that i is surjective. This can be formulated as the following problem: given r n-forms $h_{n,0}^j$, $j=1,\dots,r$, where $h_{n,0}^j$ is holomorphic in a neighborhood U_j of z_j, find a global holomorphic n-form $h_{n,0}$ such that near each z_j,

$$h_{n,0}-h_{n,0}^j=f\wedge\zeta_{n-1,0}^j, \quad j=1,\dots,r, \tag{II.24}$$

with some $(n-1)$-forms $\zeta_{n-1,0}^j$, holomorphic near z_j. We proceed in two steps. In the first step we construct $h_{n,0}$, while in the second step we construct $\zeta_{n-1,0}^j$.

Step 1. For each j choose a small disk $D_j\subset U_j$ and a smooth function χ_j such that $\operatorname{supp}\chi_j\subset U_j$ and

$$\chi_j=1, \quad \text{on} \quad D_j. \tag{II.25}$$

Consider the following, globally defined smooth form:

$$\omega_{n,0}:=\sum_j \chi_j h_{n,0}^j. \tag{II.26}$$

In general, $\omega_{n,0}$ is not holomorphic. But, as a consequence of (II.25),

$$\bar{\partial}\omega_{n,0}=0, \quad \text{on} \quad \bigcup_j D_j. \tag{II.27}$$

Furthermore, from (II.16),

$$\bar{\partial}\omega_{n,0}+f\wedge\omega_{n-1,1}=0, \tag{II.28}$$

with $\omega_{n-1,1} := -v\bar{\partial}\omega_{n,0}$. Observe that, owing to (II.27), $\omega_{n-1,1}$ is defined on the whole space X (even though v is singular on Z_f) and

$$\omega_{n-1,1} = 0, \quad \text{on} \quad \bigcup_j D_j. \tag{II.29}$$

Now, since $f \wedge \bar{\partial}\omega_{n-1,1} = 0$, we infer that

$$\bar{\partial}\omega_{n-1,1} + f \wedge \omega_{n-2,2} = 0, \tag{II.30}$$

with some globally defined $\omega_{n-2,2}$ which vanishes on $\bigcup_j D_j$. Continuing this process we obtain a sequence of n-forms $\omega_{p,n-p}$ such that

$$\bar{\partial}\omega_{p,n-p} + f \wedge \omega_{p-1,n-p+1} = 0, \quad 1 \leqq p \leqq n, \tag{II.31}$$

$$\bar{\partial}\omega_{0,n} = 0, \tag{II.32}$$

and

$$\omega_{p,n-p} = 0, \quad \text{on} \quad \bigcup_j D_j, \quad 0 \leqq p \leqq n-1.$$

Using (II.2) we construct inductively a sequence of $(n-1)$-forms $\eta_{p,n-p-1}$ such that

$$\omega_{0,n} = \bar{\partial}\eta_{0,n-1}, \tag{II.33}$$

and

$$\omega_{p,n-p} = \bar{\partial}\eta_{p,n-p-1} + f \wedge \eta_{p-1,n-p}, \quad 1 \leqq p \leqq n-1. \tag{II.34}$$

Then from (II.28) we infer that $\bar{\partial}(\omega_{n,0} - f \wedge \eta_{n-1,0}) = 0$, and thus

$$\omega_{n,0} = h_{n,0} + f \wedge \eta_{n-1,0}, \tag{II.35}$$

with $h_{n,0}$ globally holomorphic. This concludes Step 1 of our construction. Observe that (II.35) is almost (II.24), up to the fact that $\eta_{n-1,0}$ is not holomorphic on D_j. In Step 2, we will show that it is possible to modify $\eta_{n-1,0}$ locally in such a way that the resulting form is holomorphic on D_j and that it still satisfies (II.35).

Step 2. By (II.27), we have $f \wedge \bar{\partial}\eta_{n-1,0} = 0$, on $\mathscr{D}_j$. Since D_j itself is a Stein space, we have by Scheja's theorem:

$$\mathscr{H}_{\bar{\partial}}^{p,q}(D_j \backslash \{z_j\}) \cong \begin{cases} 0, & \text{if} \quad p \leqq n-2, \quad q \geqq 1, \\ \Omega^p(D_j) \cong \Omega^p(D_j \backslash \{z_j\}), & \text{if} \quad q = 0. \end{cases} \tag{II.36}$$

Repeating the above procedure with X replaced by D_j, we construct a sequence of $(n-1)$-forms $\eta^j_{p,n-p-1}$ such that

$$\bar{\partial}\eta^j_{p,n-p-1} + f \wedge \eta^j_{p-1,n-p} = 0, \quad 1 \leqq p \leqq n-1, \tag{II.37}$$

$$\bar{\partial}\eta^j_{0,n-1} = 0, \tag{II.38}$$

and

$$\eta^j_{n-1,0} = \eta_{n-1,0}, \quad \text{on} \quad D_j, \tag{II.39}$$

and a sequence of $(n-2)$-forms $\tau^j_{p,n-p-2}$ defined on D_j such that

$$\eta^j_{0,n-1} = \bar{\partial}\tau^j_{0,n-2}, \tag{II.40}$$

and

$$\eta^j_{p,n-p-1} = \bar{\partial}\tau^j_{p,n-p-2} + f \wedge \tau^j_{p-1,n-p-1}, \quad 1 \leqq p \leqq n-2. \tag{II.41}$$

Since we also have

$$\bar{\partial}(\eta^j_{n-1,0} - f \wedge \tau^j_{n-2,0}) = 0, \tag{II.42}$$

we infer that

$$\eta^j_{n-1,0} = \zeta^j_{n-1,0} + f \wedge \tau^j_{n-2,0}, \quad \text{on} \quad D_j \backslash \{z_j\}, \tag{II.43}$$

with $\zeta^j_{n-1,0}$ holomorphic. Substituting (II.43) into (II.35), we find that

$$\omega_{n,0} = h_{n,0} + f \wedge \zeta^j_{n-2,0}, \quad \text{on} \quad D_j, \tag{II.44}$$

which is (II.24).

Now we show that i is injective. This amounts to proving the following: given $\omega \in \Omega^n$ and $(n-1)$-forms ζ^j_{n-1}, holomorphic in a neighborhood U_j of z_j, $j = 1, \ldots, r$, and such that

$$\omega = f \wedge \zeta^j_{n-1}, \quad j = 1, \ldots, r, \tag{II.45}$$

near z_j, find a global holomorphic $(n-1)$-form ζ_{n-1} such that

(α) $$\zeta_{n-1} - \zeta^j_{n-1} = f \wedge \zeta^j_{n-2,0}, \quad \text{near } z_j, \tag{II.46}$$

with some $(n-2)$-forms $\zeta^j_{n-2,0}$, holomorphic near z_j;

(β) $$\omega = f \wedge \zeta_{n-1}, \quad \text{globally on } X. \tag{II.47}$$

Observe that, in fact, (β) is a consequence of (II.45, 46) and the extension theorem for holomorphic forms. We prove (α). Proceeding as in Step 1 of the proof of surjectivity of i, we set

$$\zeta_{n-1,0} = \sum_j \chi_j \zeta^j_{n-1}, \tag{II.48}$$

and use it to construct a globally holomorphic form ζ_{n-1} and a smooth form $\eta_{n-2,0}$ such that

$$\zeta_{n-1} - \zeta_{n-1,0} = f \wedge \eta_{n-2,0}. \tag{II.49}$$

Then we proceed as in Step 2 of the proof of surjectivity to show that

$$\zeta_{n-2,0} = \zeta^j_{n-2,0} + f \wedge \tau^j_{n-3,0}, \quad \text{near } z_j, \tag{II.50}$$

with $\zeta^j_{n-2,0}$ holomorphic near z_j. This proves (II.46). □

Recall [AGV] that the dimension of R^f_z, known also as the Milnor number, coincides with the local degree, $\deg_z f$, of f at z,

$$\dim R^f_z = \deg_z f. \tag{II.51}$$

Consequently,

$$\dim \mathscr{H}^n_f = \dim \mathscr{K}^n_f = \sum_{z_j \in Z_f} \deg_{z_j} f = \deg f, \tag{II.52}$$

where $\deg f$ is the (global) degree of f.

III. L^2-Cohomologies

From now on we assume that $X=\mathbb{C}^n$. As before, let $f(z)=\sum_{j=1}^{n} f_j(z)dz_j$ be a holomorphic one-from. We say that f is elliptic if all f_j's are polynomials and if the following condition is satisfied: Write

$$|f(z)|^2:=\sum_{j=1}^{n}|f_j(z)|^2. \tag{III.1}$$

Then:

(i)
$$|f(z)|\to\infty,\quad \text{as } |z|\to\infty. \tag{III.2}$$

(ii) For any $\varepsilon>0$, there is a constant C such that

$$|\partial_j f_k(z)|\leqq\varepsilon|f(z)|^2+C, \tag{III.3}$$

for all $z\in\mathbb{C}^n$ and $j,k=1,2,\dots,n$. As a consequence of (III.2), Z_f, the set of zeros of f, is compact and thus consists of finitely many points, $Z_f=\{z_1,\dots,z_r\}$.

Let $\bar\partial_f:=\bar\partial+f\wedge$ be the coboundary operator on $\wedge(\mathbb{C}^n)$ defined in Sect. II, and let $\partial_f:=\partial+\bar f\wedge$ be its complex conjugate. In coordinates, they can be written as

$$\bar\partial_f=\sum_{j=1}^{n}(\bar\partial_j d\bar z_j\wedge+f_j dz_j\wedge), \tag{III.4}$$

$$\partial_f=\sum_{j=1}^{n}(\partial_j dz_j\wedge+\bar f_j d\bar z_j\wedge). \tag{III.5}$$

Let $*:\wedge^{p,q}\to\wedge^{n-p,n-q}$ be the Hodge star operator, and let $(\omega,\eta):=\int *\omega\wedge\eta$ be the standard inner product defined on the space $\wedge^{(0)}$ of compactly supported smooth forms. By $\bar\partial_f^*$ and ∂_f^* we denote the formal adjoints with respect to $(\cdot,\cdot)$ of $\bar\partial_f$ and ∂_f, respectively. We introduce the following notation

$$b_j^*:=dz^j\wedge, \tag{III.6}$$

$$\bar b_j^*:=d\bar z^j\wedge, \tag{III.7}$$

as operators on $\wedge^{(0)}$. By b_j and $\bar b_j$ we denote the adjoints of b_j^* and $\bar b_j^*$, respectively. The operators b_j^*, $\bar b_j^*, b_j$ and $\bar b_j$ obey the following algebra:

$$\{b_j,b_k^*\}=\{\bar b_j,\bar b_k^*\}=\delta_{jk}, \tag{III.8}$$

$$\{b_j,b_k\}=\{\bar b_j,\bar b_k\}=\{b_j,\bar b_k^*\}=\{\bar b_j,b_k^*\}=\{b_j^*,b_k^*\}=\{\bar b_j^*,\bar b_k^*\}=0. \tag{III.9}$$

Using this notation we have

$$\bar\partial_f=\sum_j(\bar b_j^*\bar\partial_j+b_j^* f_j), \tag{III.10}$$

$$\partial_f=\sum_j(b_j^*\partial_j+\bar b_j^*\bar f_j), \tag{III.11}$$

$$\bar\partial_f^*=-\sum_j(\bar b_j\partial_j-b_j\bar f_j), \tag{III.12}$$

$$\partial_f^*=-\sum_j(b_j\bar\partial_j-\bar b_j f_j). \tag{III.13}$$

We define the corresponding Laplace operators

$$\bar{\square}_f := (\bar{\partial}_f + \bar{\partial}_f^*)^2 = \{\bar{\partial}_f, \bar{\partial}_f^*\}, \tag{III.14}$$

and

$$\square_f := (\partial_f + \partial_f^*)^2 = \{\partial_f, \partial_f^*\}, \tag{III.15}$$

and we find that, in coordinates,

$$\bar{\square}_f = -\Delta + |f|^2 + \sum_{j,k} (\bar{b}_j^* b_k \overline{\partial_j f_k} + b_j^* \bar{b}_k \partial_k f_j), \tag{III.16}$$

and

$$\square_f = -\Delta + |f|^2 + \sum_{j,k} (\bar{b}_j^* b_k \overline{\partial_k f_j} + b_j^* \bar{b}_k \partial_j f_k), \tag{III.17}$$

where $\Delta := -\sum_j \partial_j \bar{\partial}_j$.

If f is exact, i.e., $f = \partial V$, with a holomorphic polynomial $V: \mathbb{C}^n \to \mathbb{C}$, then $\bar{\square}_f$ coincides with the Hamiltonian of the $N = 2$ supersymmetric quantum mechanics. The function V is called a superpotential. An interesting feature of this situation is that $\bar{\square}_{\partial V} = \square_{\partial V}$. Furthermore, defining $d_f := \bar{\partial}_f + \partial_f$ and

$$\Delta_f := (d_f + d_f^*)^2 = \{d_f, d_f^*\}, \tag{III.18}$$

and observing that

$$\{\partial_f, \bar{\partial}_f^*\} = \{\bar{\partial}_f, \partial_f^*\} = 0, \tag{III.19}$$

(this holds for arbitrary f) we obtain

$$2\bar{\square}_{\partial V} = 2\square_{\partial V} = \Delta_{\partial V}. \tag{III.20}$$

Relations (III.20) are familiar from the theory of Kähler manifolds [W, GH] and thus the $N = 2$ supersymmetric quantum mechanics with a superpotential V may serve as a model of a nontrivial Kähler structure on $\mathbb{C}^n$.

Now let $\wedge_2(\mathbb{C}^n)$ be the Hilbert space of square-integrable forms on $\mathbb{C}^n$ defined as the completion of $\wedge^{(0)}(\mathbb{C}^n)$ in $(\cdot,\cdot)$. We also let $\wedge_2^{p,q}(\mathbb{C}^n)$ denote the Hilbert space of square-integrable (p,q)-forms on $\mathbb{C}^n$ and likewise we let $\wedge_2^k(\mathbb{C}^n)$ denote the Hilbert space of square-integrable k-forms on $\mathbb{C}^n$. Then, the operators $\bar{\partial}_f, \partial_f$ and their adjoints extend to densely defined, closed operators on $\wedge_2$.

We consider the L^2-cohomology of the operator $\bar{\partial}_f$:

$$\wedge_2^0 \xrightarrow{\bar{\partial}_f} \wedge_2^1 \xrightarrow{\bar{\partial}_f} \cdots \xrightarrow{\bar{\partial}_f} \wedge_2^{2n-1} \xrightarrow{\bar{\partial}_f} \wedge_2^{2n}. \tag{III.21}$$

Let $\mathscr{H}_{2,f}^k$, $k = 0, 1, \ldots, 2n$ denote the cohomology groups of the above complex and let H_f^k, $k = 0, 1, \ldots, 2n$ denote the spaces of harmonic k-forms,

$$H_f^k := \{\omega \in \wedge_2^k : \bar{\square}_f \omega = 0\}. \tag{III.22}$$

Our first result is an analog of Hodge's theorem.

Theorem III.1. (Hodge Theorem) *Let f be elliptic. Then:*

(i) $\dim H_f^k < \infty$.

(ii) *There is a self-adjoint compact operator G_f on $\wedge_2$ such that*

$$\wedge_2^k = H_f^k \oplus \bar{\partial}_f(\bar{\partial}_f^* G_f \wedge_2^k) \oplus \bar{\partial}_f^*(\bar{\partial}_f G_f \wedge_2^k). \tag{III.22}$$

(iii) *There is a canonical isomorphism,*

$$\mathscr{H}_{2,f}^k \cong \mathscr{H}_f^k. \tag{III.23}$$

(iv) (Poincaré Duality) *There is an isomorphism,*

$$\mathscr{H}_{2,f}^k \cong \mathscr{H}_{2,f}^{2n-k}, \quad k = 0, 1, \ldots, n-1. \tag{III.24}$$

As in the standard differential geometric context (see e.g., [W]), part (iii) of this theorem follows from part (ii). Part (iv) follows from part (iii), the fact that the Hodge star operator $*$ maps antiunitarily $\wedge_2^k$ onto $\wedge_2^{2n-k}$, and that

$$[*, \bar{\Box}_f] = 0.$$

Parts (i) and (ii) of the theorem are proven in Sect. IV. Observe that, even though $\mathbb{C}^n$ is not compact, the spaces of $\bar{\Box}_f$-harmonic k-forms on $\mathbb{C}^n$ are finite-dimensional. This is a consequence of the ellipticity of f which "compactifies" $\mathbb{C}^n$ at infinity.

Our second result gives a precise characterization of the cohomology groups $\mathscr{H}_{2,f}^k$.

Theorem III.2. (Vanishing Theorem) *Let f be elliptic. Then*

$$H_f^k \cong \mathscr{H}_{2,f}^k \cong \begin{cases} 0, & \text{if } k \neq n, \\ \Omega^n/f \wedge \Omega^{n-1}, & \text{if } k = n. \end{cases} \tag{III.26}$$

The above results can be interpreted in terms of index theory. Write $\wedge_2 = \wedge_2^+ \oplus \wedge_2^-$, where $\wedge_2^+$ and $\wedge_2^-$ are the spaces of square-integrable forms of even and odd degree, respectively. This defines a $\mathbb{Z}_2$-grading on $\wedge_2^-$. The operator

$$\bar{Q}_f := \bar{\partial}_f + \bar{\partial}_f^* \tag{III.27}$$

is odd under this grading and thus can be written as

$$Q_f = \begin{pmatrix} 0 & Q_f^- \\ Q_f^+ & 0 \end{pmatrix}, \tag{III.28}$$

with $Q_f^\pm : \wedge_2^\pm \to \wedge_2^\mp$ and $(Q_f^+)^* = Q_f^-$.

Theorem III.3. (Index Theorem) *The operator Q_f^+ is Fredholm. Furthermore,*

$$i(Q_f^+) = (-1)^n \dim H_f^n, \tag{III.29}$$

where $i(Q_f^+)$ denotes the index of Q_f^+.

We prove this theorem in Sect. IV.

In $N=2$ supersymmetric quantum mechanics (i.e., $f = \partial V$) the space of harmonic forms coincides with the space of ground states of the system. The degree of a form becomes the number of fermions of the corresponding state and the set of zeros of f becomes the critical set, $\mathrm{cr}(V)$, of V. Translating Theorem III.2 into the physics language and using (II.14) we obtain

Theorem III.4. *Let V be such that ∂V is elliptic. Then*:

(i) *Every ground state of the system contains n fermions.*
(ii) *The number of ground states is* $\sum_{z_j \in \mathrm{cr}(V)} \dim R_{z_j}^{\partial V}$.

Note that the first statement of this theorem is true only if we use the representation of the fermionic operators defined above. Performing a Bogolubov transformation on the operators $b_j^*, \bar{b}_j^*, b_j, \bar{b}_j$ changes the structure of the ground states (we thank Konrad Osterwalder for this remark). However, the number of ground states is invariant under such a transformation, as it leads to a unitarily equivalent Hamiltonian.

IV. Hodge Theory

In this section we prove Theorems III.1 and III.3. It will be convenient to use the following notation. For a smooth (p,q)-form

$$\omega_{p,q}(z) = \frac{1}{p!q!} \sum_{\alpha,\beta} \omega_{\alpha\beta}(z) dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_q} \tag{IV.1}$$

we set

$$|\omega_{p,q}(z)| := \frac{1}{p!q!} \left\{ \sum_{\alpha,\beta} |\omega_{\alpha\beta}(z)|^2 \right\}^{1/2}. \tag{IV.2}$$

For $\omega(z) = \sum_{p,q} \omega_{p,q}(z)$ we define $|\omega(z)| := \left\{ \sum_{p,q} |\omega_{p,q}(z)|^2 \right\}^{1/2}$.

Lemma IV.1. *Let $f \in \Omega^1(\mathbb{C}^n)$ be elliptic. Then:*

(i) *The operator $\bar{\Box}_f$ has a compact resolvent.*
(ii) *Every eigenvector ω of $\bar{\Box}_f$ is smooth. Moreover, for any $a > 0$ there is a constant C such that*

$$|\omega(z)| \leqq C \exp\{-a|z|\}. \tag{IV.3}$$

Proof. (i) As a consequence of (III.2) and Theorem XIII.67 in [RS], the operator $H := -\Delta + |f|^2$ has a compact resolvent. Since $\|b_j^*\| = \|\bar{b}_j^*\| = \|b_j\| = \|\bar{b}_j\| = 1$, it follows from (III.16) and (III.3) that for any $\varepsilon > 0$ there is a constant C such that for all $\omega \in D(H)$,

$$\left(\omega, \left[\sum_{j,k} \bar{b}_j^* b_k \overline{\partial_j f_k} + b_j^* \bar{b}_k \partial_k f_j \right] \omega \right) \leqq \varepsilon(\omega, H\omega) + C\|\omega\|^2. \tag{IV.4}$$

As a consequence of Theorem XIII.68 in [RS], $\bar{\Box}_f$ has a compact resolvent. (ii) Now let ω be an eigenvector of $\bar{\Box}_f$ (as a consequence of (i), the spectrum of $\bar{\Box}_f$ is purely discrete). Since $\bar{\Box}_f$ is an elliptic differential operator, it follows from the elliptic regularity theorem (see e.g., [H1]) that ω is smooth. To prove (IV.3), we follow closely the method of the proof of Theorem XIII.70 in [RS]. Set

$$W(z) := \sum_{j,k} (\bar{b}_j^* b_k \overline{\partial_j f_k}(z) + b_j^* \bar{b}_k \partial_k f_j(z)) + |f(z)|^2 + C, \tag{IV.5}$$

where the constant C has been chosen in such a way that for all $z \in \mathbb{C}^n$,

$$W(z) \geqq 0, \tag{IV.6}$$

as operators on $\mathbb{C}^n$ (this is possible since f is elliptic). We now introduce real coordinates in $\mathbb{C}^n$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, 2, \ldots, 2n$ and set

$$H_0 := -\Delta = -\sum_{j=1}^{2n} \partial^2/\partial x_j^2. \tag{IV.7}$$

Consider the family of operators $H_f^k(a) := H_0^k(a) + W(z)$, $a \in \mathbb{R}$, where

$$H_0^k(a) := (i\partial/\partial x_k - ia)^2 - \sum_{j \neq k} \partial^2/\partial x_j^2, \tag{IV.8}$$

for $k = 1, 2, \ldots, 2n$. Then the semigroup $\exp\{-tH_0^k(a)\}$, $t > 0$, has a pointwise positive kernel

$$\exp\{-tH_0^k(a)\}(x, y) = (4\pi t)^{-n} \exp\{ax_k - |x - y|^2/4t - ay_k\}, \tag{IV.9}$$

$x, y \in \mathbb{R}^{2n}$. Consequently, for $\eta \in \wedge_2$ and almost all $z \in \mathbb{C}^n$,

$$|(\exp\{-tH_0^k(a)\}\eta)(z)| \leqq C_{a,t} \|\eta\|, \tag{IV.10}$$

with $C_{a,t}$ independent of z. We now let $W_m(z) := \exp\{-|z|^2/m\} W(z)$, $m = 1, 2, \ldots$, where $W(z)$ is defined by (IV.5) Then $|W_m(z)| \leqq C_m$, uniformly in z, and as operators in $\mathbb{C}^n$,

$$0 \leqq W_1(z) \leqq W_2(z) \leqq \cdots \leqq W_m(z) \leqq \cdots, \tag{IV.11}$$

and

$$W_m(z) \to W(z), \quad m \to \infty. \tag{IV.12}$$

Each $W_m(z)$ defines a bounded positive operator on $\wedge_2$ and $H_0^k(a) + W_m$ tends to $H_0^k(a) + W$ in the strong resolvent sense. Therefore, by Trotter's theorem,

$$\exp\{-tH_f^k(a)\} = \text{s-}\lim_{m \to \infty} \text{s-}\lim_{l \to \infty} [\exp\{-tH_0^k(a)/l\} \exp\{-tW_m/l\}]^l,$$

and since $0 \leqq \exp\{-tW_m/l\} \leqq I$ (as operators on $\mathbb{C}^n$) this and (IV.10) imply that

$$|(\exp\{-tH_f^k(a)\}\eta)(z)| \leqq C_{a,t} \|\eta\|. \tag{IV.13}$$

Therefore, if ω is an eigenvector of $\bar{\Box}_f$, then $\eta = \exp(ax_k)\omega$ is an eigenvector of $H_f^k(a)$ and (IV.13) implies that $|\eta(z)| \leqq C$, uniformly in z. This means that for $a > 0$,

$$|\omega(z)| \leqq C \exp\{-a|x_k|\}.$$

Repeating this argument for all $k = 1, 2, \ldots, 2n$ we obtain (IV.3). $\square$

The proof of parts (i) and (ii) of Theorem III.1 is a consequence of Lemma III.1. Since $(\kappa^2 + \bar{\Box}_f)^{-1}$ is compact for $\kappa^2 > 0$, this implies that $\dim \ker \bar{\Box}_f < \infty$, which proves (i). To prove (ii), we set

$$G_f := \begin{cases} 0, & \text{on } H_f^k, \\ (\bar{\Box}_f)^{-1}, & \text{on } (H_f^k)^\perp, \end{cases} \tag{IV.14}$$

and use the spectral theorem for compact operators.

Finally, let us prove Theorem III.3. The first statement is a standard consequence of Lemma IV.1 (i) (see e.g., [JL2]). The second statement follows from the formula

$$i(Q_f^+) = \sum_{j=0}^{2n} (-1)^j \dim H_f^j, \tag{IV.15}$$

and Theorem III.2.

V. Vanishing Theorem

In this section we prove Theorem III.2. The proof uses a technique analogous to the one employed in Sect. II. As compared to Sect. II, an additional difficulty arises, namely the square integrability of the cohomology classes.

As a consequence of Poincaré duality, we can restrict attention to $n \leqq k \leqq 2n$. Let $k > n$ and let $\omega_k \in H_f^k$. We write

$$\omega_k = \sum_{p+q=k} \omega_{p,q}, \quad \omega_{p,q} \in \wedge^{p,q}. \tag{V.1}$$

Condition $\bar{\partial}_f \omega_k = 0$ yields the following equations:

$$\bar{\partial}\omega_{k-n,n} = 0, \tag{V.2}$$

$$\bar{\partial}\omega_{p,k-p} + f \wedge \omega_{p-1,k-p+1} = 0, \quad k-n < p \leqq n, \tag{V.3}$$

and

$$f \wedge \omega_{n,k-n} = 0. \tag{V.4}$$

Lemma V.1. *Let $\omega \in \wedge^{p,q}$ be $\bar{\partial}$-closed, and let*

$$|\omega(z)| \leqq C(1+|z|)^N, \tag{V.5}$$

with a constant C and a positive integer N. Then:

(i) *If $q \geqq 1$, then there is $\eta \in \wedge^{p,q}$ such that $\omega = \bar{\partial}\eta$ and*

$$|\eta(z)| \leqq C'(1+|z|)^{N+4}. \tag{V.6}$$

(ii) *If $q = 0$, then $\omega \in \Omega^p$ and all the coefficients of ω are polynomials of algebraic degree not exceeding N.*

This is a refined version of Dolbeault's lemma. It follows from Theorem 4.2.2 in [H2].

Using this lemma we infer from (V.2) that

$$\omega_{k-n,n} = \bar{\partial}\eta_{k-n,n-1}, \quad \eta_{k-n,n-1} \in \wedge^{k-n,n-1}, \tag{V.7}$$

with $|\eta_{k-n,n-1}(z)| \leqq C_1(1+|z|)^4$. Substituting (V.7) to (V.3) with $p = k-n+1$ we obtain

$$\bar{\partial}(\omega_{k-n+1,n-1} - f \wedge \eta_{k-n,n-1}) = 0 \tag{V.8}$$

As a consequence of the lemma,

$$\omega_{k-n+1,n-1} = \bar{\partial}\eta_{k-n+1,n-2} + f \wedge \eta_{k-n,n-1}, \tag{V.9}$$

with $\eta_{k-n+1,n-2} \in \wedge^{k-n+1,n-2}$, $|\eta_{k-n+1,n-2}(z)| \leqq C_2(1+|z|)^{m+4}$, where $2m$ is the algebraic degree of the polynomial $|f(z)|^2$. Continuing this process we find that for $j = 1, 2, \ldots, 2n-k$,

$$\omega_{k-n+j,n-j} = \bar{\partial}\eta_{k-n+j,n-j-1} + f \wedge \eta_{k-n+j-1,n-j}, \tag{V.10}$$

with

$$|\eta_{k-n+j,n-j}(z)| \leqq C_j(1+|z|)^{N_j},$$

where $N_{2n-k} > N_{2n-k-1} > \cdots > N_1 = 4$. Equation (V.4) does not lead to any restrictions. As a consequence of the above computations,

$$\omega_k = \bar{\partial}\eta_{k-n,n-1} + \sum_{j=1}^{2n-k} (\bar{\partial}\eta_{k-n+j,n-j-1} + f \wedge \eta_{k-n+j-1,n-j}) = \bar{\partial}_f \eta_{k-1}, \tag{V.12}$$

where

$$\eta_{k-1} := \sum_{j=0}^{2n-k} \eta_{k-n+j,n-j-1}, \tag{V.13}$$

and

$$|\eta_{k-1}(z)| \leqq C(1+|z|)^N, \quad N > 0. \tag{V.14}$$

Therefore, ω_k is $\bar{\partial}_f$-exact. We claim that, in fact, $\omega_k = 0$. Since ω_k is harmonic, Lemma IV.1 implies that $|\omega_k(z)|$ tends to zero faster than $e^{-a|z|}$, for any $a > 0$, as $|z| \to \infty$. Also, by harmonicity, $\bar{\partial}_f^* \omega_k = 0$. Therefore,

$$\begin{aligned}
\|\omega_k\|^2 &= (\omega_k, \omega_k) = (\omega_k, \bar{\partial}_f \eta_{k-1}) = \int \overline{*\omega_k} \wedge \bar{\partial}_f \eta_{k-1} \\
&= \lim_{R\to\infty} \int_{|z|\leqq R} \overline{*\omega_k} \wedge \bar{\partial}_f \eta_{k-1} \\
&= \lim_{R\to\infty} \left\{ \int_{|z|\leqq R} \overline{*\bar{\partial}_f^* \omega_k} \wedge \eta_{k-1} + O(R^{N+n-1} e^{-aR}) \right\} = 0,
\end{aligned}$$

and thus $\omega_k = 0$, as claimed.

Now, let $k = n$. We claim that there is an isomorphism $\mathscr{H}^n_{2,f} \cong \mathscr{K}^n_f$, where $\mathscr{K}^n_f$ is the Koszul cohomology group of Sect. II. Observe first that, by an argument analogous to the one above, a $\bar{\partial}_f$-closed form ω_n can be written as

$$\omega_n = \bar{\partial}_f \eta_{n-1} + \rho_n, \tag{V.15}$$

where $\rho_n \in \Omega^n(\mathbb{C}^n)$. Furthermore, if $\omega_n = \bar{\partial}_f \eta'_{n-1} + \rho'_n$ is another representation of this form, then $\rho'_n - \rho_n$ is an f-coboundary. Thus the correspondence $\omega_n \to \rho_n$ defines a monomorphism $\mathscr{H}^n_{2,f} \to \mathscr{K}^n_f$. To show that this mapping is surjective, we have to prove that for every holomorphic n-form ρ_n, there exists a smooth η_{n-1} such that $\bar{\partial}_f \eta_{n-1} + \rho_n \in \Lambda_2$.

As in Sect. II, let v be a smooth vector field of type $(1,0)$ such that $fv + vf = I$ on $\mathbb{C}^n \backslash Z_f$. We have then

$$\rho_n = f \wedge \tau_{n-1,0}, \quad \text{on } \mathbb{C}^n \backslash Z_f, \tag{V.16}$$

where $\tau_{n-1,0} := v\rho_n$. The form $\rho_{n-1,0}$ is not holomorphic. However, $f \wedge \bar{\partial}\tau_{n-1,0} = 0$, and thus there exists $\tau_{n-2,1}$ such that

$$\bar{\partial}\tau_{n-1,0} + f \wedge \tau_{n-2,1} = 0. \tag{V.17}$$

We continue this process and define

$$\tau_{n-1} = \sum_{p+q=n-1} \tau_{p,q}, \tag{V.18}$$

where for $p = 1, 2, \ldots, n-1$,

$$\bar{\partial}\tau_{p,n-1-p} + f \wedge \tau_{p-1,n-p} = 0, \tag{V.19}$$

and

$$\bar{\partial}_f \tau_{0,n-1} = 0. \tag{V.20}$$

Then,

$$\rho_n = \bar{\partial}_f \tau_{n-1}, \quad \text{on } \mathbb{C}^n \setminus Z_f. \tag{V.21}$$

Now let χ be a smooth function vanishing in a neighborhood of Z_f and equal 1 outside a compact set. Let $\eta_{n-1} = -\chi\tau_{n-1}$. Then $\bar{\partial}_f \eta_{n-1} + \rho_n$ has a compact support and is thus square-integrable. This proves that $\mathscr{H}^n_{2,f} \cong \mathscr{K}^n_{2,f}$.

VI. Connection with the Theory of Residues

In this section we explain the relation between the $\bar{\partial}_f$-complex and the theory of residues. A clear presentation of this theory can be found in Chap. 5 of [GH], and we will follow here the notation of this reference.

We consider the following smooth $2n$-form on $\mathbb{C}^n$:

$$\psi(z) := (i/2\pi)^n \exp\{-|z|^2\} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \tag{VI.1}$$

Also, let θ be the following smooth $(2n-1)$-form on $\mathbb{C}^n \setminus \{0\}$:

$$\theta(z) := \exp\{-|z|^2\}\left(\exp|z|^2 - \sum_{j=0}^{n-1} \frac{1}{j!}|z|^{2j}\right)\omega_{\mathrm{MB}}(z), \tag{VI.2}$$

where ω_{MB} is the Martinelli–Bochner form,

$$\omega_{\mathrm{MB}}(z) := \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|z|^{2n}} \sum_{j=1}^{n} (-1)^{j-1} \bar{z}_j dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \tag{VI.3}$$

(here $\widehat{d\bar{z}_j}$ means omission of $d\bar{z}_j$).

Lemma VI.1. *The above forms have the following properties*:

(i) $$\int_{\mathbb{C}^n} \psi = 1, \tag{VI.4}$$

(ii) $$\psi = d\theta, \quad \text{on} \quad \mathbb{C}^n \setminus \{0\}, \tag{VI.5}$$

(iii) $$\theta(z) = O(z), \quad \text{as} \quad |z| \to 0. \tag{VI.6}$$

Proof. Statements (i) and (ii) follow by straightforward though tedious computations. Statement (iii) is clear, as $\exp t - \sum_{j=0}^{n-1} \frac{1}{j!} t^j = O(|t|^n)$, as $|t| \to 0$. □

We now consider the following meromorphic n-form

$$\omega_f := \frac{df_1(z) \wedge \cdots \wedge df_n(z)}{f_1(z) \cdots f_n(z)}, \tag{VI.7}$$

where $f_1, \ldots, f_n$ has the same meaning as in Sect. III. Let $\mathrm{Res}_{\{z_j\}} \omega_f$ denote the residue of ω_f at $z_j \in Z_f$ (see [GH]). In the theorem below we identify f with the holomorphic mapping $z \to (f_1(z), \ldots, f_n(z))$ of $\mathbb{C}^n$ into itself and we let $f^\# \psi$ denote the pull-back of the form ψ under this mapping.

Theorem VI.2. (Residue Theorem) *Let f be elliptic. Then*

$$\sum_{z_j \in Z_f} \mathrm{Res}_{\{z_j\}} \omega_f = \int_{\mathbb{C}^n} f^\# \psi. \tag{VI.8}$$

Proof. As a consequence of Lemma VI.1,

$$\int_{\mathbb{C}^n} f^\# \psi = \int_{\mathbb{C}^n \setminus \{0\}} d(f^\# \theta). \tag{VI.9}$$

Using Stokes' theorem, we obtain

$$\begin{aligned} \int_{\mathbb{C}^n \setminus \{0\}} d(f^\# \theta) &= \lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |f(z)| \leq R} d(f^\# \theta) \\ &= \lim_{R \to \infty} \int_{|f(z)| = R} f^\# \theta - \lim_{\varepsilon \to 0} \int_{|f(z)| = \varepsilon} f^\# \theta. \end{aligned} \tag{VI.10}$$

Since $f^\# \theta(z) = O(|f(z)|)$, as $|f(z)| \to 0$, it follows that $\lim_{\varepsilon \to \infty} \int_{|f(z)| = \varepsilon} f^\# \theta = 0$. Choose R sufficiently large so that Z_f is contained in the ball of radius R around the origin. Then (see [GH], p. 655):

$$\int_{|f(z)| = R} f^\# \omega_{\mathrm{MB}} = \sum_{z_j \in Z_f} \mathrm{Res}_{\{z_j\}} \omega_f.$$

On the other hand, since f is elliptic,

$$\int_{|f(z)| = R} \exp\{-|f(z)|^2\} \sum_{j=0}^{n-1} \frac{1}{j!} |f(z)|^{2j} f^\# \omega_{\mathrm{MB}}(z) = O(R^{4(n-1)} e^{-R^2}) \to 0, \quad \text{as} \quad R \to \infty.$$

The theorem follows. □

Corollary VI.3. *The following identities hold*:

$$\begin{aligned} (-1)^n i(Q_f^+) &= \dim \mathscr{H}^n_{2,f} = \dim \Omega^n / f \wedge \Omega^{n-1} \\ &= \sum_{z_j \in Z_f} \deg_{z_j} f = \deg f = \int_{\mathbb{C}^n} f^\# \psi. \end{aligned} \tag{VI.11}$$

VII. Concluding Remarks

In this section we would like to comment on two ways in which the results of the previous sections could be generalized. Our discussion is purely conjectural, and we attempt no mathematical rigor in presenting the arguments.

Our first remark is that the ellipticity assumption of Sect. III, though necessary for our proofs, does not seem to be optimal. It is easy to construct counterexamples showing that some kind of analytic conditions on f are necessary in order that the L^2-cohomology of Sect. III is well defined. On the other hand, it is easy to construct an f whose zero set is noncompact (and thus (III.2) is violated), yet the integral $\int f^{\#}\psi$ has an integer value. In fact, take

$$V(z) = z_1^{k_1} \cdots z_n^{k_n}, \tag{VII.1}$$

with $k_j \geqq 1$, for $j = 1, 2, \ldots, n$, and set $f = \partial V$. Then a straightforward computation shows that

$$\int_{\mathbb{C}^n} f^{\#}\psi = \left(\sum_{j=1}^{n} k_j \right) - 1. \tag{VII.2}$$

This indicates strongly that $\bar{\Box}_f$ has a compact resolvent, and $\int_{\mathbb{C}^n} f^{\#}\psi$ is the index of Q_f^+. The question of finding less restrictive conditions of f under which $\bar{\Box}_f$ has a compact resolvent can presumably be settled by means of the powerful methods of [F]. Related questions in nonsupersymmetric quantum mechanics were also discussed in [S] (we thank Arthur Jaffe for bringing this reference to our attention). Also. the geometric content of $\mathscr{H}^k_{2,f}$ in case of noncompact Z_f should be clarified.

A second interesting problem is to extend the results of this paper to two-dimensional supersymmetric quantum field theory. It is possible to write field theoretical analogs of $\bar{\partial}_f$ and ∂_f, namely

$$\bar{\partial}_{\partial V} = \int_0^{2\pi} \bar{b}^*(\sigma) \cdot (i\bar{\pi}(\sigma) - \partial_\sigma \bar{\phi}(\sigma)) d\sigma + \int_0^{2\pi} b^*(\sigma) \cdot \partial V(\phi(\sigma)), d\sigma, \tag{VII.3}$$

and

$$\partial_{\partial V} = \int_0^{2\pi} b^*(\sigma) \cdot (i\pi(\sigma) + \partial_\alpha \phi(\sigma)) d\sigma + \int_0^{2\pi} \bar{b}^*(\sigma) \cdot \overline{\partial V(\phi(\sigma))} d\sigma. \tag{VII.4}$$

Here $\phi(\sigma)$ is an n-component complex scalar field, $\pi(\sigma)$ is the canonical momentum and $\bar{b}^*(\sigma), b^*(\sigma), \bar{b}(\sigma), b(\sigma)$ are fermionic creation and annihilation operators. Informally, $\bar{\partial}_{\partial V}$ and $\partial_{\partial V}$ are coboundary operators for certain infinite dimensional complexes. Defining these operators on an appropriate Hilbert space is a nontrivial task, and can presumably be settled in the framework of constructive field theory. We conjecture that for polynomial V satisfying a suitable growth condition, the vanishing theorem of Sect. III holds. We also notice that there is an intriguing infinite dimensional Kähler structure associated with (VII.3, 4). Namely, we let

$$h = -\int_0^{2\pi} (b^*(\sigma) b(\sigma) - \bar{b}(\sigma) \bar{b}^*(\sigma)) d\sigma, \tag{VII.5}$$

$$L = \int_0^{2\pi} b^*(\sigma) \bar{b}^*(\sigma) d\sigma, \tag{VII.6}$$

and

$$\Lambda = \int \bar{b}(\sigma) b(\sigma) d\sigma. \tag{VII.7}$$

These operators obey the usual sl_2 algebra,

$$[h, L] = -2L, \quad [h, \Lambda] = 2\Lambda, \quad [\Lambda, L] = h. \tag{VII.8}$$

Furthermore, we have

$$\begin{aligned}
&[L, \bar{\partial}_{\partial V}] = 0, \quad [\Lambda, \bar{\partial}_{\partial V}] = \partial^*_{\partial V}, \\
&[L, \partial_{\partial V}] = 0, \quad [\Lambda, \partial_{\partial V}] = -\bar{\partial}^*_{\partial V},
\end{aligned} \tag{VII.9}$$

relations familiar from Kähler geometry [W, GH] (however $\bar{\square}_{\partial V} \neq \square_{\partial V}$). We hope that these relations, once put on sound mathematical foundations, will lead to a nontrivial infinite dimensional Kähler structure.

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