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Heterotic Superstring Gauge Residue Trivialization Via Homogeneous CP^4 Topology Change

Randy A. Baadhio

Center for Radiophysics and Space Research, Space Sciences Building, Cornell University, Ithaca, New York 14853-6801 and Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853-5001, USA

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Abstract. A new mechanism for the cancellation of gauge residue symmetries in the framework of heterotic superstring compactification theories is revealed. The model preserves all the string features and fits naturally in the consistent topological structure of the homogeneous CP^4 Calabi–Yau manifold.

I. Introduction

The anomaly cancellation for the 10-dimensional heterotic superstring theory with $SO(32)$ or $E_8 \times E_8$ gauge group gives hope of allowing a consistent unified theory including gravity, especially if $N = 1$ supersymmetry is required to be unbroken at low energies.[1]

To make a realistic contact with the low energies phenomenology, it is assumed that the $D = 10$ theories compactify into $M^4 \times K^6$, where K is a compact complex 6-dimensional Calabi–Yau manifold for orbifold with $SU(N)$ holonomy. It is further assumed that all the known particles at low energies are singlets under the E_8 group and belong to the representation of E_6. Such realistic connection with low energies is then intrinsically related to lowering the rank of the E_6 gauge group [1].

A powerful method of implementing such symmetry breaking in superstring theory is to consider the string propagation on an orbifold. The most popular and effective method of breaking the gauge symmetry – and consequently reduce the number of generations – is known as the Wilson-lines mechanism [2] in the framework of orbifold compactification.

The Wilson-loop is a homomorphism of the translation defining the torus into

[1] One can consider the Atkin–Lehmer symmetry in a non-supersymmetric background as a good challenge, since its discrete symmetry of modular space makes the integral over τ vanish despite the precise absence of spacetime supersymmetry

$E_8 \times E_8$. (In the literature the $0^6 = T^6/\Lambda$ flat tori are usually considered.) Since the translation group is abelian this implies that the Wilson-lines satisfy this property: they commute between each other!

Such commutability in the embedding of the gauge group action into the internal degree of freedom lies in the explanation of why the rank of the gauge group is not reduced, and so on the survival of some extra $U(1)$'s under the symmetry breakdown by the Wilson-loops mechanism. The same remark applies when one, in an attempt to construct a chiral string model in 4-dimensions, obtains a large rank gauge group [3]. Thus, the theoretical perspectives of making a realistic connection between the Planck energies and the low energies phenomenology strike on the existence of these extra singlets.[2]

In this paper we present a new mechanism for removing the unwanted extra symmetries. In the language of topology this is called trivialization. We trivialize these symmetries. The paper is divided as follows: In Sect. II, we recall some basic properties of the algebraic topology and introduce the problem. We then treat the reduction of the E_6 gauge group associated with the $SU(3)$ gauge holonomy. A connection is provided between such reduction and the homogeneous CP^4 polynomial deformations. Particular emphasis is given to trivializing an arbitrary $U(1)$ residue. Here, the use of the universal coefficient theorem is required. Our treatment is generalized by considering the consequence of introducing the use of obstruction theory. This enables us to determine exactly for which specific class of homotopy the obstruction lives. In Sect. III, we discuss the string vacuum configuration with respect to the model, and, hence show how one can meet the geometrical requirement to not destabilize the string vacuum configuration.

The conclusion addresses some open questions, in particular, the geometrical interpretation of our results as well as the physical implications with respect to the Planck scale and low energies like the Salam–Weinberg scale.

II. The Model

Let us first start by recalling some basic facts. Within a complex projective space, a bundle $U(1)$ is defined as

$$BU(1) = CP^\infty \equiv k(Z, 2)$$

with cohomology

$$H^*[BU(1, Z)] = Z[c_1],$$

Let $\mathscr{M}$ be a complex manifold of dimension 3. Relating now $BU(1)$ to $\mathscr{M}$ leads us

[2] Singlets are considered extra by taking into account the standard $SU(3) \times SU(2) \times U(1)$ model. For an overview of this topic refer to S. Weinberg, A. Salam and L. Glashow, "Nobel Lectures in Physics", Review Mod. Phys., **52**, No. 3 (1980). Note that the inclusion of the Wilson-loops depends on the geometrical configurations chosen. In particular, it is related to the construction of the orbifold which turns out to be related to some specific inner automorphism inside the torus and to the number of singularities (usually the $A \cdot D \cdot E$ semi-simple laced Lie singularities) in the light cone

next to write

$$(\mathscr{M}, BU(1)) = H^2(\mathscr{M}, 2) \equiv 0 \quad \text{for} \quad \mathscr{M} = 3.$$

At this point, we introduce a useful notion, Hopf invariance. To roughly apply this, take 3-spheres of respective rank 1, 2 and 3. Under the Hopf invariance, the 3-spheres will give the following map:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2. \end{array}$$

Applying it to a complex projective space, one obtains:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^\infty \\ & & \downarrow \\ & & CP^\infty. \end{array}$$

Hence, a universe of the principle bundle will substantially be of the form:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n-1} \\ & & \downarrow \\ & & CP^{n-1}. \end{array}$$

Extending this to a vector bundle with E_8 as gauge group leads to:

$$\begin{array}{ccc} E_6 & \longrightarrow & E_8 \\ & & \downarrow \\ & & \mathscr{M}, \end{array} \tag{1}$$

where we view E_8 as the product of a sphere [4]

$$S^3 \cdot S^{15} \cdot S^{23} \cdot S^{27}$$

with respective dimensions 35, 39, 47, 59.

We are now interested in computing the exact cohomology of the E_8 vector bundle, and consequently its homotopic class. This leads us to consider an Eilenberg–MacLane space, a space with exactly non-zero homotopy group. It is, in fact, a path connected space, all of whose homotopy groups vanish except for a single dimension.[3]

According to Eq. 1, we note this space:

$$K(Z,3) \times K(Z,15) \times K(Z,23) \times K(Z,27) \cdots.$$

What one gains from this notation is that the vector bundle will have a cohomology of the form:

$$H^*(BE_8, R) \quad \text{for} \quad R = k_3, k_{15}, \ldots, k_{27}.$$

E_6 Structure Group Reduction. We turn now, at this point, our attention to a CP^4

[3] For a more detailed and historical overview see S. Eilenberg and MacLane, “On the Groups $H(\pi, n)$,” Ann. Math. **58**, 55–106 (1953)

Calabi–Yau manifold, which contains our vector bundle with E_8 as a maximal gauge group. In the heterotic superstring case, one way to break it down (as soon as one is interested) to obtain a Ricci flat scalar metric is to embed the spin connection with the gauge group[4]. Later we will come back to this specific topic. Although this will be the tool of the next part, let us take here $E_6 \times SU(3)$ as the maximal subgroup of E_8. So, one write:

$$E_6 \times SU(3) \hookrightarrow E_8, \tag{2}$$

where $SU(3)$ is the gauge group for the $SU(N)$ holonomy. Translated in the sequel language, the decomposition takes the form

$$E^1 \times H^*G: \quad \begin{array}{ccc} E_6 \times SU(3) & \longrightarrow & E^1 \\ & & \downarrow \\ & & M \end{array}$$

Lemma. *Every $E_6 \times SU(3)$ bundle gives an E_6 extension of E_8 bundles by a homomorphism.*

Taking Lemma 1 into account, we define then the sequence

$$\cdots \to H_j \subset H_1 \subset H_0 \to \cdots \tag{3}$$

with cohomologies

$$H^*(BH_0) \to H^*(BH_1) \to H^*(BH_j).$$

In order to reduce E_6 one must relate E_6 to a map of a cohomology of a certain classifying space.

Let us consider again the induction

$$E_6 \times SU(3) \hookrightarrow E_8$$

and let G_{standard} be:

$$G_{\text{std}} = SU(3) \times SU(2) \times U(1).$$

Thus we get the explicit induced sequence

$$G_{\text{std}} \hookrightarrow E_6 \to E_6 \times SU(3) \hookrightarrow E_8. \tag{4}$$

This process is a homomorphism of $H_j \subset H_1 \subset H_0$:

$$G_{\text{std}} \subset G_{\text{std}} \times U(1) \subset E_6$$

and thus induces an E_6 bundle which naturally reduces itself to a G_{std} bundle (noted (E_2)) and $G_{\text{std}} \times U(1)$ bundle (noted $E(1)$). Here, we took $U(1)$ (e.g. from $G_{\text{std}} \times U(1)$ as the gauge residue symmetry. Basically, we are not only interested – at this stage – in reducing E_6 to $G_{\text{std}} \times U(1)$ but also to G_{std}. So given Lemma 1 in that respect, a first point is to find out the E_1 extension of E_6 from G_{std} to $G_{\text{std}} \times U(1)$. To answer, consider a principal bundle $B(G_1 \times G_2)$ which satisfies the commutative

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[4] Restricting $\mathscr{M} < 14$ means that the first fourteen homotopy groups of E_8 are $\pi_k(E_8) = Z$ for $K = 3$. Such homotopy is exactly trivial if we take $1 \leqq K \leqq 14$ for $K \neq 3$

property

$$B(G_1 \times G_2) = BG_1 \times BG_2.$$

If so, then the only known classifying space of

$$BG_{\text{std}} \times U(1) = BG_{\text{std}} \times BU(1)$$

has a nontrivial cohomology of H^*BG expressed as a product [6]. More explicitly for $G = G_1 \times G_2$ the cohomologies of BG_1 and BG_2 appear under the commutative relation $B(G_1 \times G_2) = BG_1 BG_2$, which in turn extends to the cohomology

$$H^*(BG_1 \times BG_2) = H^*G.$$

Going back now to $H^*BU(1)$, let us notice that it can be viewed as a characteristic class, measuring whether the extension to $G_{\text{std}} \times U(1)$ from the restriction E_2 to G_{std} is isomorphic to E_1.

To see how this may happen, consider the following sequence:

$$\begin{array}{c} B(G_{\text{std}} \times U(1)) = BG_{\text{std}} \times BU(1) \\ \downarrow \\ BG_{\text{std}} \\ \downarrow \\ BG_{\text{std}} \times U(1). \end{array}$$

This gives us an extension (E_2 isomorphic to E_1, respectively) which furthermore satisfies

$$\tilde{c}_1(E_1) = 0.$$

When E_1 is actually restricted to G_{std}, the composition becomes an identity ($\tilde{c}_1$ is the characteristic form of the criterion for this restriction). Defining, now, the following sequence,

$$\rightarrow \cdots G_{\text{std}} \rightarrow G_{\text{std}} \times U(1) \rightarrow E_1 \times SU(3) \rightarrow E_6 \cdots \rightarrow,$$

one can easily point out that any G-bundle over homogeneous CP^4 can be of the form:

$$H^*B(E_6, R) \rightarrow H^*B(G_{\text{std}}, R), \tag{5}$$

where $H^*B(E_6, R)$ is taken to be the maximal cohomology of the maximal bundle structure of E_6. Before digressing to the characteristics class of our homogeneous CP^4 manifold [7], let us consider the map:

$$\begin{array}{ccc} & \nearrow & BE_6 \\ & & \uparrow i \\ n & \cdots\cdots & BG_{\text{std}} \end{array}, \tag{6}$$

and also a universe of E bundles such as

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG \end{array} \qquad \begin{array}{ccc} H & \longrightarrow & EG \\ & & \downarrow \\ & & BH. \end{array}$$

Hence, what we construct is nothing other than a homogeneous space[5] with fiber bundles [8, 9]:

$$\begin{array}{ccccc} C/H & \longrightarrow & EG/H & \longrightarrow & EG/G \\ & & \nwarrow & \nearrow & \\ & & & n & \end{array}$$

and

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ & & E \\ & \nearrow & \downarrow \\ n & \longrightarrow & B. \end{array} \tag{7}$$

A first question to ask is whether the section $E \to B$ exists and, if so, under which conditions. As it will become clear later this implies that one has to look at the global cohomology of F. Such global cohomology is obtained if one combines the π_{n-1} homotopy of F with EG/H:

$$H^n(n, \pi_{n-1} F).$$

In that case and only in that case, (4) ignores the torsion [10], since the DeRham cohomology of EG/H will be isomorphic to

$$H^i(EG/H, R) = R^K.$$

This is an interesting fact. We now wish to specify the triviality of the torsion, in other words, to work out these implications. A useful notion will be required: the universal coefficient theorem [11]. If we consider any principal ideal domain, S, with a non-trivial cohomology $H^i(M, S)$ defined by $H^i(M, Z)$ and use the fact that M is compact, it follows that the only case where $H^i(M, Z)$ is finitely generated is when it is isomorphic to[6]

[5] Space is homogeneous in the sense that G admits a transitive Lie group of a homomorphism and carries a complex analytic structure. The coset spaces C/H and EG/G are homogeneously complex (respectively homogeneous Kählerian) if they carry a complex analytic structure invariant under G. For a discussion see H. C. Wang, Am. J. Math. **76**, 1–32 (1954)

[6] We do not have torsion when prime P does not divide the order of the Weyl group of G. For E_6, E_7, E_8 the order of the Weyl Group is

$$2^7 \cdot 3^4 \cdot 5; \quad 9! \cdot 8; \quad 10! \cdot 3 \cdot 26.$$

If P is greater than the coefficients of the highest roots then the simply connected group G has no torsion. We restrict G to be either G_{std} or $G_{\text{std}} \times U(1)$. Thus the simplest connected representatives of the structures E_6, E_7, E_8 would imply no P-torsion (no torsion P and no torsion, respectively) for $P \geqq 5$; $P \geqq 5$; $P \geqq 7$ respectively.

We are able to work out the vanishing of the torsion in CP^4 simply if we point out that there is a deep relation between the cohomology of CP^4 and the first Chern class. Since CP^4 is defined by 5 homogeneous polynomials, $\sum_{i=1}^{5} z_0^5 = 0$, such as the first Chern class $c_1 = 0$, the relation is immediately given by

$$H^*(CP^4, Z) = Z[c_1]$$

$$Z^K \oplus Z/P_i^{K_1}Z \oplus Z/P_i^{K_2}Z \oplus \cdots \oplus Z/P_n^{K_n}Z, \tag{5}$$

where $Z/P_1^{K_1}$ are integer modules of $P_1^{K_1}$ and P_1 is a prime,

$$Z \otimes R = R \quad \text{and} \quad Z/P_1^{K_1}Z \otimes R = 0,$$

the torsion being $Z/P_1^{K_1}$ in this tensor form.

Reducing the structure group from G to E corresponds to a specific section of some bundle. The first step is to find out which section we may consider.

Again it will be useful to construct an explicit map of the sequences which are of some relevance for our purpose. Let such a sequence be

$$\begin{array}{ccccc} & & & & G/H \text{ bundle} \\ & & & \nearrow & \\ \longrightarrow \cdots G \longrightarrow E_6 & \longrightarrow & E_6/H & & \\ \searrow & \swarrow & & & \\ & M = E_0/G. & & & \end{array} \tag{8}$$

We are interested in

$$G/H \longrightarrow E_0/H \longrightarrow E_0/G. \tag{9}$$

Substituting (7) into (4), such that (7) must be exactly solvable, then it follows by the use of obstruction theory [12] that a set of F lives in $H^i[B, \pi_{i-1}(F)]$, where π_{i-1} denotes the first homotopy of F. Given this statement, considering now any reduction of G-bundles to E over M implies asking whether

$$\begin{array}{ccccc} \longrightarrow \cdots H \longrightarrow & E_1 & \longrightarrow & E_1 X_H G \cdots \longrightarrow \\ & \downarrow & & \downarrow \\ & M & & M \end{array}$$

is isomorphic to the coset homogeneous space E/G.

If we define a global cohomology both from M and G/H,

$$H^i[(M, \pi_{i-1}, G/H)],$$

then it is not so difficult to work out the cohomology of M and G/H together as a tensor product:

$$H^i[(M, R) \otimes (\pi_{i-1} G/H) \otimes R]. \tag{10}$$

Consider now the following inductive sequence:

$$\begin{array}{ccc} \longrightarrow \cdots & G_{\rm std} \times U(1) & \cdots \longrightarrow \\ & \downarrow & \\ \longrightarrow \cdots & G_{\rm std} & \cdots \longrightarrow \cdot \end{array}$$

By the use of (10) one writes

$$H^i[(M, \pi_{i-1}(E_6/G_{\rm std})] \neq 0 \tag{11}$$

for any arbitrary $H \subset G$. Then by definition

$$G/H^i \longrightarrow EG/H \xrightarrow{p} EG/G.$$

If H is taken as a maximal rank subgroup of G, then

$$H^*(G/H) = H^*BH/P^*(H^*BG). \tag{12}$$

What is now the correct homotopy of E_6/G_{std}? First we may recall that the global cohomology of E_6 was given already by

$$H^i(M_5^3, Z).$$

So then, by recurrence, the cohomology of $SU(3, R)$ is[7]

$$H^*SU(3, R) \sim \wedge [e_5, e_3],$$

the cohomology of G_{std} is

$$H^*(G_{std}, R) = \wedge [e_5, e_3, e_3', e_1],$$

and, finally, the cohomology of E_6 is

$$H^*(E_6, R) = \wedge [F_3, F_9, F_{11}, \ldots].$$

So

$$\pi_{i-1}(E_6/G_{std}) \neq 0 \quad \text{for} \quad i = 2, 4, 6.$$

This results allows the obstruction to live only in

$$H^2(M_5^3, \pi_i) \neq 0, \quad H^4(M_5^3, \pi_3) \neq 0, \quad H^6(M_5^3, \pi_5). \tag{13}$$

III. String Vacua Stability

In order to be consistent, the model as described in Sect. II must obey certain geometrical requirements which at last resort should ensure that world sheet instantons do not destabilize the string vacuum configuration. For the general case, conditions to preserve the string vacuum state have been extensively and explicitly discussed by Witten and collaborators [15]. We will follow these prescriptions.

Our starting point is with respect to Sect. II, to take once again a 3-D Calabi–Yau manifold which is described basically by an algebraic equation of the form:

$$Z_0^5 + Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 = 0.$$

As in π, for notational convenience, we write this manifold M_5^3, which is a submanifold of CP^4; that is, one sets:

$$M_5^3 \cong CP^4.$$

Let us now define $V(M_5^3)$ as a "stable holomorphic vector bundle" [16] (i.e. we refer to the vector bundle of Eq. 1, Sect. II). At this point, we can now meet one of the algebraic geometrical requirements for string vacua stability, namely, by posing:

$$V(M_5^3) \oplus T(M_5^3) = T(CP^4)/M_5^3. \tag{14}$$

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7 We have used the relations $SU(3) = S^5 \times S^3$ and $SU(2) = S^3$

Although V is holomorphically stable it is still a 1-dimensional complex normal bundle, while T denotes the 4-D complex tangent bundle of CP^4.

The question arises now how one can establish an equivalence relation between the second Chern class of V and that of $T(CP^4)$. To proceed, we first of all define a map i:

$$i: M_5^3 \to CP^4,$$

and its dual i^*:

$$i^*: H^*(CP^4) \to H^*(M_5^3).$$

Using the Whitney sum formula for Chern classes, one gains a better evaluation of i^*:

$$[1 + c_1(V(M_5^3)][1 + c_1(T(CP^4)) + c_2(T(CP^4)) + C_3 \cdots] = i^*, \tag{15}$$

where the first term under the first bracket expression denotes the total Chern class of $V(M_5^3)$ and the second one, the total Chern class of the tangent bundle of CP^4. So from Eq. (15), i^* is nothing other than the total Chern class of the tangent bundle of CP^4.

It is well known that one can write the total Chern class of a complex projective space of dimension n like:

$$C(CP^n) = (1 + \alpha)^{n+1} \tag{16}$$

where, by α we means the $H^2(M;Z)$ generator. Generally, $H^K(CP^n)$ will be an integer of K is even and $K < 2n$; or, otherwise will vanish.

Returning now to Eq. (16), it is straightforward to see:

$$C(T(CP^4)) = (1 + \alpha)^5 \tag{17}$$

for

$$\alpha \in H^2(CP^4; Z).$$

Equation (17) can actually take a more explicit form, that is:

$$C[T(CP^4)] = 1 + 5\alpha + 10\alpha^2 + 10\alpha^3 + 5\alpha^4. \tag{18}$$

Let us note that the reason why the coefficient α^5 is not in (18) has to do with the fact that it is an element of $H^1 0(CP^4; Z)$, which is known to have a zero value. So consequently, α^5 will vanish.

Next, our main concern is about the fundamental class of M_5^3 in $H_6(CP^4; Z)$. Introducing a certain Poincaré dual, denoted by β, in CP^4 one gets with respect to Eq. (18):

$$c_1[V(M_5^3)] = 5\beta;$$

β is an element of the characteristic form $H^2(CP^4; Z)$. Having introduced a Poincaré dual in CP^4, our next task is to apply it to M_5^3. While it was rather easy in the CP^4 case, one will need a different approach here. Let us then digress to this well established relation, roughly

$$H_K(M^n) = H^{n-k}(M^n).$$

Going back to the fact that M_5^3 is a submanifold of CP^4, we get a sufficient condition for it to have also a fundamental class, which, among other things, generates the following relation:

$$H_6(M_5^3;\ Z) \equiv Z.$$

To substantially apply these interesting facts, we introduce a "homological dual" for i, noted i_* which has the basic features of i: roughly, it is a linear map of the form:

$$i_*: H_6(M_5^3; Z) \to H_6(CP^4). \tag{19}$$

By the earlier well-established relation, one just writes

$$i_*[M] \in H_6(CP^4; Z). \tag{20}$$

Although (19–20) give a rich appreciation for having introduced i_*, there is another fact which deserves to be pointed out here. Namely, one can associate an element $\bar{\beta}$ dual to i_*. $\bar{\beta}$ lives in $H^2(CP^2; Z)$. Now, in terms of $\bar{\beta}$, β has the value:

$$\beta = \bar{\beta} \in H^2(M_5^3; Z). \tag{21}$$

So rewriting the Whitney sum formula, we obtain:

$$(1 + 5\beta)(1 + c_1 + c_2 + c_3) = 1 + 5\tilde{\alpha} + 10\tilde{\alpha}^2 + 10\tilde{\alpha}^3, \tag{22}$$

where, $\tilde{\alpha} = i^*\alpha$ and, furthermore, $\tilde{\alpha} \in H^2(M_5^3; Z)$. The first term on the left is taken to be a constant while, the second term on the left are characteristic forms for the tangent bundle of (CP^4).

What we definitely gain is a good way to find explicit forms for those characteristic forms; they are:

$$5\beta + c_1 = 5\tilde{\alpha} \to c_1 = 5(\tilde{\alpha} - \beta),$$

$$5\beta c_1 + c_2 = 10\tilde{\alpha}^2 \to c_2 = 10\tilde{\alpha}^2 - 5\beta c_1 = 10\tilde{\alpha}^2 - 25\beta(\tilde{\alpha} - \beta) = 10\tilde{\alpha}^2 - 25\tilde{\alpha}\beta + 25\beta^2,$$

$$c_3 = 10\tilde{\alpha}^3 - 5\beta c_2 = 10\tilde{\alpha}^3 - 5\beta(10\tilde{\alpha}^2\beta + 25\beta^2),$$

where

$$5\beta c_2 + c_3 = 10\tilde{\alpha}^3,$$

$$c_2[V(M_5^3)] = c_2[T(CP^4)] \equiv 10\tilde{\alpha}^2. \tag{23}$$

Equation (23) is precisely the statement that the model described in Sect. II does not destabilize the string vacuum configuration.

IV. Conclusion

The procedure for reducing extra $U(1)$'s can be generalized to any Calabi–Yau manifold under of course the assumptions that they are solutions for string theory. Essentially, the generalization itself will have to deal with obstructions. As given by Eq. (13) of Sect. II, they are natural characteristic forms generated by the reductional structure group of the E_8 vector bundle. It may not appear surprising that, definitively, as a result of generalization to other Calabi–Yau manifolds, the localisation of obstructions appears more or less the same as the one found in this paper. The reason is the universality of the string gauge group for any soluble string manifold and in relation to this point, the requirement about the dimension of the manifold.

One should point out at this point that the fact that one may get different obstructions for different CY manifolds may lead to different spectrum. Roughly, the spectrum of the model doesn't seem to be altered since, a relation like the one given by (10) enables us to choose the "good" $U(1)$ via hypercharge checking.

Another source of interest should be to look at the global geometrical consequence induced by the model. In [13], it has been pointed out that, actually, one can minimize those consequences through the introduction of a certain type of homomorphism, known as the "Chern–Weil homomorphism."

Appendix A: The $U(1)$ Generators

Let us define a universal principal G-bundle by

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & B \end{array}$$

and assume that EG is contractible. The contractibility of EG has an *a priori* meaning: the homotopy class of maps of X to BG is $\pi_{i-1}[X,BG] \approx 1{:}2$ of the principal G-bundle over X, where BG is the classifying space for the principal G-bundle. We take here EG and BG such that they must be essentially unique. Consider now the contractibility property of EG. We define EG:

$$EG = \gamma_G.$$

Given $F{:}X \to BG$, then $F^*\gamma_G$ is nothing other than the correspondence of G-bundles over Y. Let us take $G = U(1)$, and define EG:

$$EG = S^\infty \subset C^\infty.$$

$U(1)$ acts on C^∞ by

$$\lambda(Z_1, Z_2, \ldots,) = \sum |Z_i|^2 = 1 = S^\infty$$

for any complex number $C^n \subset C^{n+1}$ $(n = 1)$ so, one has:

$$(Z_1, Z_2, \ldots, Z_n) \longrightarrow (Z_1, Z_2, \ldots, Z_n, 0),$$

and furthermore

$$C^\infty = \overset{\infty}{U}\, C^n.$$

This implies that the relation between S^∞ and C^∞ is $S^\infty =$ infinite dim sphere $\to \sum^{\infty} |Z_i|^2 = 1$ and $U(1)$ acts definitely on C^∞ by

$$\lambda(Z_1, Z_2, \ldots,) = (\lambda Z_1, \lambda Z_2, \lambda Z_3, \ldots, \lambda Z_n).$$

Given the Hopf invariance, one can just write S^∞ invariance $S^\infty/S^1 = CP^\infty$. We are able to work out an interesting fact, namely the S^∞ contractibility by

$$\begin{array}{ccc} S^1 & \longrightarrow & S^\infty \\ & & \downarrow \\ & & CP^4 . \end{array}$$

This contractibility generates in turn the universal $U(1)$ bundle.

It follows that if

$$H^*(CP^\infty, Z) = Z[X] \quad \dim X = 2$$

then a $U(1)$ bundle over X had a 1 to 1 correspondence with $[X, CP^\infty]$:

$$X \underset{1:1}{\longleftrightarrow} [X, CP^\infty] = [X, K(Z,2)] = H^2(X, Z).$$

Notice that $K(Z,2)$ is a Eilenberg–MacLane space with dim 2. A brief look tells us that the cohomology class corresponding to a $U(1)$ bundle γ is simply $C_1(\gamma)$.

We wish now to point out the generators of $U(1)$ bundle. To do this we first recall a constraint in the cohomology of (M_5^3, Z). From Sect. II, we know that $H^2(M_5^3, Z) \geqq 1$, and a Kähler form Ω was associated with $H^2(M, Z)$.

Consider now the following transition functions:

$$X \xrightarrow{F} K(Z,2)$$

and

$$F^* : H^2[K(Z,2), Z] \longrightarrow H^2(Z, Z).$$

These transition functions generate a class which turns out to be represented by F and which gives furthermore the following map:

$$\begin{array}{ccc} CP^4 & \longrightarrow & CP^\infty \\ \cup & \nearrow f\circ i & \\ M_5^3 & & \end{array}$$

The F^* generator of $H^2(CP^\infty, Z)$ is precisely the generators of $H^2(CP^4)$, $Z)$. They turn out to be equivalent to a universal constant modulo the Kähler form of CP^4 (i^* is the congruent here of Ω of M_5^3). In conclusion, $f \circ i$ is the classifying map for the $U(1)$ bundle with a restriction corresponding to the Kähler form i^*.[8]

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[8] To be more precise, a principal $G_1 \times G_2$ bundle $E \to M$ will have a trivial G_2 piece only if the structure group can be reduced to G_1. In that case (1) for F_E a classifying map for E is

$$\begin{array}{ccc} M & \xrightarrow{E_E} & BG_1 \times BG_2 = B(G_1 \times G_2) \\ & & \uparrow i_1 \\ & & BG_1 . \end{array}$$

i_1 is induced by $g \to (g, \mathbb{1})$. Then if F_1 exists we have $i_1 \circ F_1 = F_E$ and $F_E^* = F_i^* \circ i_1$. (2) For $H^*(BG_2, Z) \subset \ker F_E^*$ all the characteristic classes in the G_2 piece would vanish for E by the use of the "Chern–Weil homomorphism." That is, ϕ: adj. G-invt. polynomials with connection on $M_5^3 \to H^*(M_5^3, R)$. This could be translated into differential forms by the use of the extension derivative of the DeRham cohomology (see ref. 13)

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References

1. Gross, J., Harvey, J. A., Martinec, E., Rohm, R.: Phys. Rev. Lett. **54**, 502 (1985); Nucl. Phys. **B256**, 253 (1985); **B267,** 75 (1986); Narain, K. S.: Phys. Lett. **B169** (1986); Narain, K. S. Sarmadi, ?. ?., Witten, E.: Nucl. Phys. **B279**, 369 (1987); Dixon, L., Harvey, J. A., Vafa, C., Witten, E.: Nucl. Phys. **B274**, 285 (1986); **B261**, 651 (1985); Strominger, A., Witten, E.: Commun. Math. Phys. **101**, 341 (1985)
2. Ibanez, L. E., Nilles, H. P., Quevedo, F.: Phys. Lett. **B187**, 25 (1989); Phys. Lett. **B192**, 332 (1987)
3. Kawai, H., Lewellen, C., Tye, H.: Nucl. Phys. **B288**, 1 (1987); Lerche, W., Lust, D., Schellekens, N.: Phys. Lett. **B181**, 71 (1986); Nucl. Phys. **B287**, 667 (1987)
4. Borel, A.: Topology of Lie Group and Characteristic Classes. Am. Math. Bull. 397–432 (1955). For a general overview of differential geometry see S. Kobayashi: Differential Geometry of Complex Vector Bundles. Princeton, NJ: Princeton Univ. Press, 1987. Kobayashi and Nomizu: Foundations of Differential Geometry, Vol. 2. New York: Intersciences Publishers 1963–1969
5. An extensive discussion about the heterotic superstring compactification is given in Unified String Theories, Gross and Green (ed.). Singapore: World Scientific, 1985, pp. 294, 357, 400–438, 635. For a deep mathematical approach to this topic see Candelas, E.: Lecture on Complex Manifolds, pp. 1–88. Superstring 88, Singapore: World Scientific, Trieste 87 Spring School on Superstrings, Gaune, A., Green, Grisaru (ed.). See also Symposium on Anomalies, Geometry, and Topology, eds. Bardeen and White, Argonne, 1985, also Calabi, L.: Rend. Math. e Appl. Vol. 11 (1952) pp. 1–5
6. Cartan, E.: Notions d'Algebre Differentielles: Application Aux Groupes de Lie. (Collogue de Topologie sur les Espaces Fibrés) Bruxelles (1950), Paris–Liege (1951) pp. 15–27. Cartan, E.: La Transpression dans un Groupe de Lie et dans an Espace Fibré Principal; ibid. pp. 57–71
7. For a specific discussion see R. Bott, W. Tu: Differential Forms in Algebraic Topology, Berlin, Heidelberg, New York: Springer, 1982, pp. 8, 266. A general connection between homogeneous spaces and characteristic class may be found in Cartan, E.: Sur les invariants Integraux de certains Espaces Homogènes Clos et les Proprietés topologiques de ces espaces 7, Annal. Societe Polonaise Math., Vol. **8** (1929) pp. 181–225; Cartan, E.: Oeuvres Completes, Part I, Vol. 3, Paris: Gallimard–Gauthier Villars, 1952, pp. 1081–1125. See also the good paper of Borel, A. and Hirzebruch, F. on characteristic classes of homogeneous space (1956)
8. Cartan, E.: La Transgression dans un Groupe de Lie et dans un Space Fibré Principal. Colloque de Topologie sur les espace fibrés. Paris (1951)
9. Witten, E.: Commun. Math. Phys. **100**, 197 (1985); S. Panier: Algebraic Topology, New York: McGraw Hill, 1966. See also Chap. 3 of Bott & Tu: Differential Forms in Algebraic Topology
10. Borel, A.: Sur la Torsion des Groupes de Lie, J. Math. Pures Appl., (1955), and Torsion in Lie Groups, Proc. Natl. Acad. Sciences USA **40**, 586–588 (1954)
11. For an introduction to the theory of Universal Coefficients see Spanier, E.: Algebraic Topology. New York: McGraw Hill, 1966, pp. 222–338
12. Kodiaira, K.: Complex Manifolds and Deformation of Complex Structures, in particular, Chap. 5, Berlin, Heidelberg, New York: Springer 1986
13. Baadhio, R. A.: On the Chern–Weil Homomorphism in Homogeneous M_5^3 with a Trivial $U(1)$ Bundle and 2-form Curvature. CRSR/Newman Lab Report, December 1989

14. Baadhio R. A.: On the E_8, E_7 and E_6 Topological obstructions structure. CRSR/Newman Lab, March 90
15. Witten, E.: Physics and Geometry, in Proceedings of the 1986 Congress of Mathematicians. Berkeley, and New issues in Manifold of $SU(N)$ Holonomy, Nucl. Phys. **B268**, 79–112 (1986) and references therein
16. Okonek, C.: Vectors Bundles on Complex Projective Spaces. Birkhauser, 1980; Schneider, M.: Sem. Bourbaki 530, Nov. 1978, no. 18-19-20, page 80

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