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Non-Orientable Strings

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Abstract. We describe the halfcomplex structure on non-orientable two-dimensional surfaces and calculate chiral determinants and Laplacians necessary for construction of the Polyakov measure.

Introduction

Most considerations in Polyakov's string theory [1] are confined to the case of two-dimensional compact orientable Riemannian surfaces, or (due to cancellation of conformal anomaly) to one-dimensional compact complex manifolds. The number of works devoted to other logical possibilities, namely to open and non-orientable strings is smaller [2–8]. Partly this is connected with the well-known reduction of open and non-orientable surfaces (so-called Klein surfaces) to orientable and closed ones. More precisely, the category of orientable Riemannian surfaces with antiholomorphic involution is equivalent to the category of Klein surfaces. Factorization of this surface (called double) under the involution gives an open surface if the involution has fixed points and a non-orientable one if it has not. (Open non-orientable strings correspond to orientable surfaces with two involutions.)

In this work, devoted to the non-orientable case only, we tried (when it was possible) to treat all objects (j-differentials, $\bar{\partial}$-operators and so on) without reference to double.

Thinking in this direction we have rediscovered a notion of semicomplex structure (known as dianalytical structure in mathematical works [9]), and have reached a rather unusual generalization of holomorphic bundles. The number of fermion bundles on a non-orientable surface K is two times more than this number for its double X (Sect. 1). An explanation is easy. There is a nontrivial bundle ε on K, called the orientation bundle, which becomes trivial when lifted on X. Fermions on K can be divided in two classes: obtained from the fermion bundle $\mathscr{L}$ on X or having the form $\pi_* \mathscr{L} \otimes \varepsilon$.

In Sect. 2 we discuss moduli space of non-orientable surfaces, then establish various relations between determinants on Klein surfaces and on their doubles (these relations were partly known earlier [5–7]) and get as a result an expression for Polyakov's measure.

1. Klein Surfaces. Semiholomorphic Structure. Halfholomorphic Bundles. $\bar{D}$-Operator

The topological facts we use below can be found in [10], halfcomplex (dianalytical) structures were introduced in [9].

We start from topology. It is well-known that every orientable two-dimensional surface X_g can be represented as a sphere S^2 with g handles. The projective plane $\mathbb{R}P^2$ being S^2 with opposite points identified presents the simplest example of non-orientable surface. Every non-orientable surface K has two canonical decompositions:

$$\underbrace{K_n = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{n} \tag{1}$$

$$K_n = X_g \# \mathbb{R}P^2 \quad \text{or} \quad X_g \# \mathbb{R}P^2 \# \mathbb{R}P^2, \; g = \left[\frac{n-1}{2}\right]. \tag{2}$$

Here $Y_1 \# Y_2$ denotes the connected sum of surfaces Y_1 and Y_2 obtained in the following manner. One should remove small disks D_1 and D_2 from Y_1 and Y_2 and then glue remainders over the boundaries. (Strictly speaking this procedure requires a homeomorphism h of the boundaries, but one can prove that the connected sums defined with different choices of h and disks D_1, D_2 are homeomorphic.) Taking the connected sum is a commutative operation. Here are some examples. Let T^2 be the two-dimensional torus. Then, for an orientable surface X_g one has

$$X_g = \underbrace{T^2 \# T^2 \# \ldots \# T^2}_{g} \qquad g \geqq 1 . \tag{3}$$

Applying the relation

$$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 = T^2 \# \mathbb{R}P^2 , \tag{4}$$

one can deduce (2) from (1).

Non-orientable surfaces can be classified by their Euler characteristic

$$\chi(K_n) = \operatorname{rank} H_0(K_n) - \operatorname{rank} H_1(K_n) + \operatorname{rank} H_2(K_n) = 2 - n . \tag{5}$$

We write down homology groups of K_n with integer coefficients for completeness:

$$H_0(K_n) = \mathbb{Z}, \qquad H_1(K_n) = \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2\mathbb{Z}, \qquad H_2(K_n) = 0 . \tag{6}$$

A non-orientable surface K_n can be represented as a factor-space of orientable surface X_g under an action of involution σ without fixed points, changing the orientation:

$$\pi : X_g \to X_g/\sigma = K_n , \qquad g = n - 1 . \tag{7}$$

Now we turn to conformal geometry. The conformal class of metrics on two-dimensional surfaces enables us to define angles in tangent space and hence to define rotation to $\pi/2$ equivalent to complex structure J, $J^2=-1$. But this rotation J is defined up to the sign: if J_α is chosen in every U_α ($\cup U_a=K$), then $J_\alpha=\pm J_\beta$ in $U_\alpha \cap U_\beta$. In the orientable case this ambiguity can be removed and a global complex structure can be defined.

In the non-orientable case a nontrivial bundle ε (called an orientation bundle) can be defined by its transition functions:

$$\varepsilon_{\alpha\beta}=\begin{cases} 1 & \text{if} \quad J_\alpha=J_\beta \\ -1 & \text{if} \quad J_\alpha=-J_\beta \end{cases}. \tag{8}$$

Another equivalent definition of ε is

$$\varepsilon_{\alpha\beta}=\operatorname{sign}\det I_{\alpha\beta},$$

$I_{\alpha\beta}$ being the Jacobi transition matrix between coordinates in U_α and U_β. So we can define a holomorphic coordinate z_α, corresponding to a particular choice of $\{J_\alpha\}$, but the transition functions should be

$$\begin{aligned} z_\alpha&=f_{\alpha\beta}(z_\beta) \quad \text{if} \quad J_\alpha=J_\beta \\ z_\alpha&=f_{\alpha\beta}(\bar{z}_\beta) \quad \text{if} \quad J_\alpha=-J_\beta, \end{aligned} \tag{9}$$

for some holomorphic functions $f_{\alpha\beta}$.

We propose to call this structure on K halfholomorphic. (It was called dianalytical in [9].)

Halfholomorphic structure on K can be lifted to X (see (7)) giving rise to two opposite complex structures (the bundle $\pi^*\varepsilon$ becomes trivial), and the involution σ becomes antiholomorphic isometry.

The question of which bundles on K may be called halfholomorphic is not completely trivial. A simple idea to take σ-invariant holomorphic bundles on X (automatically being bundles on K) gives bundles with constant transition functions only. (Indeed, σ being antiholomorphic makes holomorphic transition functions for L into antiholomorphic ones for σ^*L, hence they should be constant). This is a rather small class of bundles. For example, the bundle Ω^1K of complex-valued 1-forms does not belong to it, but we wish to call it half-holomorphic. Obviously the local decomposition

$$\Omega^1=\Omega^{1,0}\oplus\Omega^{0,1}$$

is not invariant globally: an antiholomorphic change of coordinates interchanges $\Omega^{1,0}$ and $\Omega^{0,1}$, but leaves fixed their set-theoretical union. These properties can be formalized in the definition of cross.

We shall say that a bundle E on a halfcomplex manifold K has a cross if the following holds. Let $\{U_\alpha, z_\alpha\}$ be an atlas for K, z_α being the corresponding coordinates. For every U_α, z_α there should exist a decomposition

$$E|U_\alpha=V_{1,\alpha}\oplus V_{2,\alpha} \tag{10}$$

into the sum of subbundles, satisfying the condition on $U_\alpha \cap U_\beta$,

$$\begin{aligned} V_{1,\alpha}\leftrightarrow V_{1,\beta} \quad &\text{and} \quad V_{2,\alpha}\leftrightarrow V_{2,\beta} \quad \text{if} \quad z_\alpha=f(z_\beta), \\ V_{1,\alpha}\leftrightarrow V_{2,\beta} \quad &\text{and} \quad V_{2,\alpha}\leftrightarrow V_{1,\beta} \quad \text{if} \quad z_\alpha=f(\bar{z}_\beta). \end{aligned} \tag{11}$$

Note that the union $V_{1,\alpha} \cup V_{2,\alpha}$ in E does not depend on the choice of coordinates, is defined globally and resembles a cross. Only the order of $V_{i,\alpha}$ depends on the choice of coordinates.

When lifted to X a bundle E with a cross decomposes into a direct sum

$$\pi^* E = L_1 \oplus \sigma^* L_1$$

for some L_1 on X. The bundle L_1 depends on the cross and the orientation of X, and is defined by

$$L_1 = \begin{cases} V_1 & \text{for} \quad \pi^{-1}(U_\alpha)_h \\ V_2 & \text{for} \quad \pi^{-1}(U_\alpha)_a . \end{cases} \tag{12}$$

Let us explain the notations. We assume U_α to be small enough for $\pi^{-1}(U_\alpha)$ being two disjoint sets, and the function $z_\alpha \circ \pi$ is holomorphic on the first set (called $\pi^{-1}(U_\alpha)_h$) and antiholomorphic on the second ($\pi^{-1}(U_\alpha)_a$) one.

Reversing this procedure we can construct bundles with crosses: consider a bundle L_1 on X and $\pi_* L_1$ on K. We recall that a fiber of $\pi_* L_1$ in $k \in K$ is equal to a sum of fibers of L_1 in $\pi^{-1}(k)$:

$$(\pi_* L_1)_k = \bigoplus_{\pi(x)=k} (L_1)_x .$$

The cross in $\pi_* L_1$ is defined by

$$\pi_* L_1|_{U_\alpha, z_\alpha} = L_1|_{\pi^{-1}(U_2)_h} \oplus L_1|_{\pi^{-1}(U_\alpha)_a} .$$

There are bundles with crosses different from $\pi_*(L_1)$, for example a bundle $\pi_*(L_1) \otimes \varepsilon$ cannot be represented as $\pi_*(M)$ for some M, but possesses a cross. (We recall that ε is the orientation bundle, see (8)).

The tensor product of bundles with crosses decomposes into a direct sum

$$E_1 \otimes E_2 = E_1 \otimes_h E_2 \oplus E_1 \otimes_a E_2 , \tag{13}$$

where $E_1 \otimes_h E_2$ and $E_1 \otimes_a E_2$ are defined as follows: Let $E_1|_{U_\alpha, z_\alpha} = V_1 \oplus V_2$ and $E_2|_{U_\alpha, z_\alpha} = W_1 \oplus W_2$. Then, defining

$$E_1 \otimes_h E_2|_{U_\alpha, z_\alpha} = V_1 \otimes W_1 \oplus V_2 \otimes W_2 , \tag{14}$$

$$E_1 \otimes_a E_2|_{U_\alpha, z_\alpha} = V_1 \otimes W_2 \oplus V_2 \otimes W_1 , \tag{15}$$

we get two bundles with crosses. (Subscripts h and a near $\otimes$ symbolize the words holomorphic and antiholomorphic; we shall demonstrate later that $\otimes_h$ preserves "halfholomorphic" bundles on K). It is easy to see that

$$\pi^*(E_1 \otimes_h E_2) = L_1 \otimes L_2 + \sigma^*(L_1 \otimes L_2) , \tag{16}$$

$$\pi^*(E_1 \otimes_a E_2) = L_1 \otimes \sigma^* L_2 + (\sigma^* L_1) \otimes L_2 \tag{17}$$

(see (12) for definition of L_i).

Now we are ready to define halfholomorphic bundles on K with nonconstant transition functions. Namely, a bundle E is called halfholomorphic if two conditions are satisfied:

(i) E possesses a cross,

(ii) $\pi^* E = L_1 \oplus \sigma^* L_1$ with holomorphic bundle L_1 on X.

We emphasize that the definition of halfholomorphic bundles depends on the orientation on X.

A bundle Ω^1 with the cross $\Omega^{1,0}+\Omega^{0,1}$ is obviously halfholomorphic. From (16) one sees that halfholomorphic bundles are preserved under the $\otimes_h$ product. Now it is clear how to define complex-valued j-differentials,

$$\underbrace{\Omega_j=\Omega^1\otimes_h\Omega^1\otimes_h\ldots\otimes_h\Omega^1}_{j}$$
$$=(\Omega^{1,0})^{\otimes j}\oplus(\Omega^{0,1})^{\otimes j}. \tag{18}$$

(Ω^1 coincides with Ω_1 in these notations.) True j-differentials are real subbundles of these complex bundles on K. We postpone this discussion until Sect. 2 (see (33) and below).

We define 1/2-differentials as solutions to the equation

$$\Omega_{1/2}\otimes_h\Omega_{1/2}=\Omega_1. \tag{19}$$

Two solutions may differ by a tensor product on a bundle δ satisfying the following condition: $\delta\otimes\delta$ is the trivial bundle. The group of δ's coincides with $H^1(K_n,\mathbb{Z}_2)\simeq\mathbb{Z}_2^n$.

The group $H^1(X_g,\mathbb{Z}_2)\simeq\mathbb{Z}_2^g$ classifying 1/2-differentials on X_g is 2 times smaller. The explanation is that the orientation bundle ε becomes trivial on X_g, but contributes to $H^1(K,\mathbb{Z}_2)$. Another approach to 1/2-differentials based on "$O(2)$-spinor representations" is discussed in [8, 11].

Halfholomorphic bundles admit an action of the $\bar{D}$ operator:

$$\begin{aligned}&\bar{D}:C^\infty(E)\to C^\infty(E\otimes_h\Omega^1),\\&\bar{D}(v_1\oplus v_2)|_{U_\alpha,z_\alpha}=\bar{\partial}v_1\oplus\partial v_2,\end{aligned} \tag{20}$$

converting into the action of $\bar{\partial}$ on L_1 and ∂ on σ^*L_1.

The $\bar{D}$ operator enables us to define halfholomorphic sections $\Gamma_h(E)$ as solutions $\bar{D}e=0$.

2. Moduli Space of Non-Orientable Surfaces, and Determinants

We begin with moduli space of non-orientable surfaces. The antiholomorphic involution σ acting on double X (see (7)) maps its period matrix as follows:

$$\sigma:\tau_{ij}\to-\bar{\tau}_{ij}. \tag{21}$$

Some clarifications are needed here. Of course period matrix and σ-action depend on the basis in $H_1(X,\mathbb{Z})$. One can prove that the following choice of canonical basis with σ-action is possible:

$$\sigma:A_i\to A_i,\ B_j\to-B_j;\quad (A_i,B_j)=\delta_{ij}. \tag{22}$$

(Here (A,B) denotes the intersection index.) The formula (21) is written in a such basis. Moreover, it is easy to see that

$$\bar{\omega}_i(z^*)=\omega_i(z). \tag{23}$$

(Here $z^*=\sigma z$ and $\{\omega_i\}$ are canonical holomorphic 1-differentials.) The relation (23) implies that only pure imaginary period matrices are possible

$$\tau_{ij}=\sqrt{-1}\,t_{ij}. \tag{24}$$

The subgroup of the modular group commuting with σ consists of transformations

$$\tau \mapsto c\tau c^t, \quad c \in GL_g(\mathbb{Z}). \tag{25}$$

Now we are ready to start computation of determinants on K. More precisely, we will express them in terms of determinants on X, which are well-known [12] to be constructed from θ-functions and bases in j-differentials.

At first we describe the correspondence of zero modes when $j \neq 0$. Holomorphic j-differentials $\Gamma_{\text{hol}}(\Omega_j, X)$ can be transformed into solutions of $\bar{D}f_j = 0$, $f_j \in \Omega_j(K)$ or $f_j \in \Omega_j \otimes \varepsilon(K)$ via the following local rule. Let $w \in K$; $z, z^* \in X$, $\pi(z) = \pi(z^*) = w$. We denote by the same letters local coordinates near these points chosen so that $z = w$, $z^* = \bar{w}$. Let $\varphi_j(z) \in \Gamma_{\text{hol}}(\Omega_j, X)$. We define

$$\begin{aligned} f_j(w) &= \varphi_j(z) + \varphi_j(z^*) \in \Gamma(\Omega_j, K), \\ f_j^\varepsilon(w) &= \varphi_j(z) - \varphi_j(z^*) \in \Gamma(\Omega_j \otimes \varepsilon, K). \end{aligned} \tag{26}$$

It is easy to check that $\bar{D}f_j = \bar{D}f_j^\varepsilon = 0$. An inverse correspondence and a slight generalization for the case of the arbitrary holomorphic bundle E are straightforward. So we have the established isomorphisms,

$$\Gamma_{\text{hol}}(E, X) \simeq \Gamma_h(\pi_* E, K) \simeq \Gamma_h(\pi_* E \otimes \varepsilon, K). \tag{27}$$

In the same way one can obtain the following relations between determinants:

$$\det \Delta(E, X) = \det \Delta(\pi_* E, K) = \det \Delta(\pi_* E \otimes \varepsilon, K), \tag{28}$$

$$\det \bar{\partial}(E, X) = \det \bar{D}(\pi_* E, K) = \det \bar{D}(\pi_* E \otimes \varepsilon, K). \tag{29}$$

It remains to find $\Delta_0(K)$. Formulas (28), (29) are useless because

$$\pi_* \mathscr{F}_0 = \mathscr{F}_0 \oplus \varepsilon. \tag{30}$$

The point is that the involution σ maps $\mathscr{F}_0$ into itself (it differs from the case of j-differentials, when $\sigma^* \mathscr{F}_{j,0} = \mathscr{F}_{0,j}$) and $\mathscr{F}_0(X)$ decomposes into two subspaces via the eigenvalues of $\sigma = \pm 1$:

$$\mathscr{F}_0(X) = \mathscr{F}_0^+(X) \oplus \mathscr{F}_0^-(X).$$

And $\mathscr{F}_0^+(X)$ corresponds to $\mathscr{F}_0(K)$, $\mathscr{F}_0^-(X)$ corresponds to $\Gamma(\varepsilon, K)$. The constructions below literally repeat the constructions for open strings [5–7], and we present them in a shortened form. The idea is to compute the variation of $\det \Delta^\pm$ along moduli space using the corresponding Green functions,

$$G^\pm(z, w) = \tfrac{1}{4}(G(z, w) \pm G(z^*, w) \pm G(z, w^*) + G(z^*, w^*)). \tag{31}$$

Here $G(z, w)$ is the Green function for $\Delta_0(X)$. An explicit formula for $G(z, w)$ enables one to get the relation

$$\frac{\det \Delta^+}{\det \Delta^-} = \frac{1}{\det(t_{ij})}. \tag{32}$$

Before writing down the formulas for the measure we return to a postponed (Sect. 1) discussion of reality conditions for j-differentials. Recall that the bundles $\Omega_j(K)$ (see (18)) are complexifications of true j-differentials,

$$\Omega_j(K) = \Sigma_j(K) \otimes \mathbb{C}. \tag{33}$$

Real bundles Σ_j can be obtained as follows. There is an antilinear (over $\mathbb{C}$) involution on $\Gamma(\Omega_j, X)$,

$$\tau : f_j(z) \to \bar{f}_j(z^*) , \tag{34}$$

and multiplying on $i = \sqrt{-1}$ we get an isomorphism of eigenspaces for τ:

$$i : \Gamma^+(\Omega_j, X_g) \leftrightarrows \Gamma^-(\Omega_j, X_g) , \tag{35}$$

$\Gamma^\pm$ are sections of real bundle, and

$$\Gamma(\Sigma_j, K) \simeq \Gamma^+(\Omega_j, X) . \tag{36}$$

So (using the reality of Laplacians) we have

$$\det \Delta_j(X) = \det \Delta(\Omega_j, K) = \det^2 \Delta(\Sigma_j, K) . \tag{37}$$

It is worth mentioning that the square root of $\det \Delta_j(X)$ is equal to $\det \bar{\partial}_j(X)$. Indeed, we can choose a basis for j-differentials satisfying the condition

$$f_j(z) = \bar{f}_j(z^*) , \tag{38}$$

(it is possible via (35)) and represent $\det \bar{\partial}_j$ as a correlator (or as a ratio of two sections of the determinant bundle)

$$\det \bar{\partial}_j = \frac{\langle b(z_1) \dots b(z_{n_j}) \rangle}{\det(f_K(z_j))} . \tag{39}$$

The right-hand side of (39) does not depend on z_ℓ. In particular, one can transform z_ℓ into z_ℓ^*. Then

$$\frac{\langle b(z_1^*) \dots b(z_{n_j}^*) \rangle}{\det(f_K(z_\ell^*))} = \frac{\langle \bar{b}(z_1) \dots \bar{b}(z_{n_j}) \rangle}{\det(\bar{f}_K(z_\ell))} = \det \bar{\partial}_j . \tag{40}$$

Hence we get that chiral determinants are real:

$$\det \bar{\partial}_j(X) = \det \partial_j(X) = \det \bar{\partial}_j(X) . \tag{41}$$

And

$$\det \Delta(\Sigma_j, \mathrm{K}_n) = \det \bar{\partial}(\Omega_j, X) \times (N_j N_{1-j})^{1/2} \times \exp(-c_j S_L/24\pi) . \tag{42}$$

Using (32) we get

$$\det \Delta_0(K) = \det \bar{\partial}_1(X) \times \exp(-c_j S_L/24\pi) \tag{43}$$

($\det \bar{\partial}_1(X)$ is computed in the canonical basis of 1-differentials).

And finally, as for open strings [5],

$$d\mu_{\mathrm{bos}}(K) = [\det \bar{\partial}_1(x)]^{-13} \det \bar{\partial}_2(x) \prod_1^{d(g)} dy_\alpha , \tag{44}$$

where $d(g) = 3g - 3$ if $g \geqq 2$, $d(1) = 1$, $d(0) = 0$. We do not discuss the problem of cosmological constant cancellation for non-oriented strings. We only point out that this question can be reduced to the same statement for oriented strings [8] from one side, and from the other side using the relations (28), (29) one can achieve a "silly" cancellation, simply taking contributions from $\Sigma_{1/2}$ and $\Sigma_{1/2} \otimes \varepsilon$ with opposite signs.

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