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Universal Teichmüller Space and Diff S^1/S^1 *

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Abstract. We point out that the coset space Diff S^1/S^1 is a dense complex submanifold of the Universal Teichmüller Space S of compact Riemann spaces of genus $g \geqq 1$. A holomorphic map of S into the infinite dimensional Segal disk D_1 is constructed. This is the Universal analogue of the map of Teichmüller spaces into the Siegel disk provided by the period matrix. The Kähler potential for the general homogenous metric on Diff S^1/S^1 is computed explicitly using the map into D_1. Some applications to string theory are discussed.

There are many reasons to believe that there is a string theory [1] of quantum gravity. Since classical gravity has a natural formulation in terms of Riemannian geometry, it is reasonable to expect that quantum gravity can be formulated in terms of its complex analogue, Kähler geometry. By combining these two surmises, it is natural to seek a formulation of string theory in terms of Kähler geometry. One approach to this was developed by one of us in collaboration with Bowick and Rajeev [2]. In that approach the basic object of study [1] is the coset space Diff S^1/S^1, which was proved to be a homogenous Kähler manifold. It was shown that this manifold has a finite Ricci tensor (a non-trivial fact in infinite dimensions) which gives a natural explanation of the critical dimension 26 of string theory.

Complex geometry also arises in the conventional perturbative string theory although in a completely different way. The g-loop scattering amplitude of string theory can be expressed as an integral of the square of a holomorphic function on a complex manifold, the Teichmüller space of Riemann surfaces of genus g. The

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[1] Throughout this paper Diff S^1 will denote the group of *orientation preserving* diffeomorphisms of the circle

measure of integration can be understood [3] in terms of the map of $\mathcal{T}_g$ to the Siegel disk $D(g)$. We will explain this in some more detail later in the paper.

It is interesting to ask how the two approaches are related. The approach based on $\mathrm{Diff}\, S^1/S^1$, while more abstract, holds the promise of a truly non-perturbative approach to string theory. This is important since there is reason to believe that the perturbative expansion does not converge [4]. It was conjectured initially [2] that $\mathrm{Diff}\, S^1/S^1$ is a "universal moduli space" for Riemann surfaces. The precise connection between the abstract approach based on $\mathrm{Diff}\, S^1/S^1$ and the more conventional approach based on Riemann surfaces remains, however, obscure. It was later conjectured by one of us (S.G.R.) that all the Teichmüller spaces could be embedded as Kähler submanifolds of $\mathrm{Diff}\, S^1/S^1$. But progress in this direction was obstructed by the presence of certain divergences. But since then there has been important progress in this direction from the work of Kirillov, Yuriev [5], Nag [6], and Verjovsky.

Their results imply that $\mathrm{Diff}\, S^1/S^1$ is a dense complex submanifold of S, the space of univalent functions on the unit disk. The reason why this is an exciting result is that S is the "Universal Teichmüller Space" in the theory of Bers. More precisely [7], the Teichmüller space $\mathcal{T}_g$ of compact Riemann surfaces of genus $g \geq 1$ can be holomorphically embedded into S.

These results open up the exciting possibility of a non-peturbative formalism for closed bosonic string theory. It is a natural conjecture [8] that the string amplitude can be written as an integral over $\mathrm{Diff}\, S^1/S^1$ of the modulus square of a holomorphic function (which can be expressed in terms of infinite dimensional analogues of Θ functions). The measure of integration would be determined by a homogenous Kähler metric on $\mathrm{Diff}\, S^1/S^1$. We will give a more precise statement of this idea at the end of this paper.

In constructing this approach to string theory, a holomorphic map of $\mathrm{Diff}\, S^1/PSL(2,R)$ into the infinite dimensional Segal disk [9] is important. It is the analogue of the map of the Teichmüller space to the Siegel upper half plane, (provided by the period matrix) in the perturbative approach. It is also useful to understand the geometry of $\mathrm{Diff}\, S^1/S^1$ and its various embeddings as explicitly as possible. The original approach to the Kähler geometry of $\mathrm{Diff}\, S^1/S^1$ was rather abstract and relied heavily on the homogeneity of the space. The work of Kirillov allows us to establish explicitly a holomorphic co-ordinate system on $\mathrm{Diff}\, S^1/S^1$. In this paper we will calculate the Kähler potential of the most general homogenous metric on $\mathrm{Diff}\, S^1/S^1$, in this co-ordinate system. Kirillov and Yuriev already obtained the Kähler potential for one parameter family of homogenous metrics. But for that special case the Ricci tensor does not exist. We will generalize their result by finding the potential for the general two-parameter family, including those for which the Ricci tensor does exist.

Even apart from string theory, the geometry of $\mathrm{Diff}\, S^1/S^1$ is relevant to the study of irreducible representations of the Virasoro algebra, and to conformal field theory. For example Segal has constructed the $c=1$ projective representation of the Virasoro group $\mathrm{Diff}\, S^1$ by the embedding method. We believe that all irreducible (projective) representations of $\mathrm{Diff}\, S^1$ can be obtained in terms of modular functions on the homogenous spaces $\mathrm{Diff}\, S^1/S^1$ and $\mathrm{Diff}\, S^1/PSL(2,R)$. Witten [10] has proposed a similar idea.

Before we end this introduction, we have to warn the reader that any discussion of infinite dimensional manifolds is plagued by certain technical difficulties. For example, the tangent space of $\mathrm{Diff}\, S^1/S^1$ is not complete in the norm defined by its

Kähler metric. Similarly, in the following we will see that we will not be able to map all of S into D_1, only a dense subset. We are not able to completely settle these issues of infinite dimensional analysis. A rigorous theory can be developed only after a sufficiently rich set of examples of infinite dimensional manifolds have been studied. We believe that the examples we study will be useful in that direction. The experience in mathematical physics has been that many such technical issues get settled only *after* most of the physics is understood.

Let us begin by recalling that a holomorphic function f on the unit disk is *univalent* if it is injective, $f(z_1)=f(z_2)\Rightarrow z_1=z_2$. It is convenient to normalize these functions by imposing $f(0)=0, f'(0)=1$. Then, every element of S has an expansion

$$f(z)=z\left(1+\sum_{k=1}^{\infty} c_k z^k\right). \tag{1}$$

Thus the coefficients c_k, $k\geqq 1$ provide a complex co-ordinate system that covers all of S. The coefficients c_k must satisfy some inequalities in order that the Taylor series converge for $|z|<1$ to a univalent function. It is known [11] (the Bieberbach-De Branges theorem) that $|c_k|\leqq k+1$ is a necessary condition. Thus S is a sort of infinite dimensional bounded domain. We will produce a homogenous Kähler metric on S. Our strategy will be to embed S holomorphically into a more well-known infinite dimensional bounded domain, the Segal disk D_1.

In finite dimensions, the Siegel disk (so named after C.L. Siegel) $D(n)$ is the space of $n\times n$ complex matrices Z such that

$$Z^T=Z, \quad 1-Z^\dagger Z>0. \tag{2}$$

Clearly this is a bounded domain in complex space of dimension $n(n+1)/2$. (For $n=1$ this is just the unit disk on the complex plane.) This space can in fact be identified with the homogenous space $Sp(2n)/U(n)$. To see this, note that $Sp(2n)$ can be thought of as the group of matrices $g=\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix}$ with

$$a^T\bar{b}=b^\dagger a, \quad a^T\bar{a}-b^\dagger b=1. \tag{3}$$

Here $Sp(2n)$ acts on $D(n)$ by fractional linear transformations

$$Z\to(aZ+b)(\bar{b}Z+\bar{a})^{-1}. \tag{4}$$

Any $Z\in D(n)$ can be obtained from the origin $Z=0$ by this action. The stability group of the origin is the subgroup with $b=0$, $a^\dagger a=1$ which is just the $U(n)$ subgroup of $Sp(2n)$. Thus $D(n)=Sp(2n)/U(n)$. There is a Kähler meric on $D(n)$ which is homogenous under the action of $Sp(2n)$. The Kähler potential is just

$$K(Z)=-\operatorname{tr}\log(1-Z^\dagger Z). \tag{5}$$

There is a holomorphic map of the Teichmüller space $\mathscr{T}_g$ of compact Riemann surfaces of genus g into the Siegel disk $D(g)$. This map appears in perturbative string theory. Each point $p\in\mathscr{T}_g$ describes a Riemann surface Σ along with a choice of generators ("marked Riemann surface") for its fundamental group $\pi_1(\Sigma)$. There are some related spaces that are also of interest in string theory. Each point in the Torelli space $\mathscr{T}\mathscr{O}_g$ is Σ along with a choice of generators for the Homology group $H_1(\Sigma)$ of Σ. A choice of generators of $\pi_1(\Sigma)$ implies one for $H_1(\Sigma)$ so that there is a projection $\pi:\mathscr{T}_g\to\mathscr{T}\mathscr{O}_g$. One can identify $\mathscr{T}\mathscr{O}_g=\mathscr{T}_g/K_g$, where K_g is the group of diffeomorphisms not connected to the identity on Σ that leaves the choice of homology basis invariant. Each point of the Riemann moduli space M_g simply describes a Riemann surface. We have that $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$, where Γ_g is the "mapping class group," the group of diffeomorphisms of Σ that are not connected to the identity. Also $\mathcal{M}_g = \mathcal{T}\mathcal{O}_g/Sp(2g, Z)$. These relations are consistent because $Sp(2g, Z) = \Gamma_g/K_g$.

Given a point in $\mathcal{T}\mathcal{O}_g$ we have a Riemann surface Σ of genus g and a homology basis (a_i, b_i) satisfying

$$a_i \cdot a_j = b_i \cdot b_j = 0 \qquad a_i \cdot b_j = \delta_{ij} \tag{6}$$

on it. Here the dot represents intersection. There is then a dual basis of holomorphic 1-forms (abelian differentials) ω_i satisfying

$$\oint_{a_i} \omega_j = \delta_{ij}. \tag{7}$$

Then to each point in $\mathcal{T}\mathcal{O}_g$ we can associate a symmetric $g \times g$ matrix Π with positive imaginary part:

$$\Pi_{ij} = \oint_{b_j} \omega_i. \tag{8}$$

Now recall that the Siegel upper half plane (of symmetric matrices with positive imaginary part) can be mapped into the Siegel unit disk by the map

$$Z = (\Pi - i)(\Pi + i)^{-1}. \tag{9}$$

so that it is just a matter of convenience which description we choose. We have therefore a map $\tau: \mathcal{T}\mathcal{O}_g \to D(g)$. This is a holomorphic 2-to-1 immersion [12] of an open set (of non-hyperelliptic surfaces) of $\mathcal{T}\mathcal{O}_g$ into $D(g)$ for $g \geqq 3$. (If $g = 1, 2$ this is a holomorphic embedding of $\mathcal{T}\mathcal{O}_g$.) If we combine this with the projection $\pi: \mathcal{T}_g \to \mathcal{T}\mathcal{O}_g$ we get a holomorphic map $z: \mathcal{T}_g \to D(g)$. The pullback of the homogenous Kähler metric on $D(g)$ gives a Kähler metric on $\mathcal{T}_g$ with Γ_g as the isometry group.

The correlation functions of string theory can be written as an expansion in powers of a coupling constant λ. The g^{th}-order term in the expansion is the average of certain vertex operators with respect to a measure [3] in $\mathcal{M}_g$:

$$\lambda^g \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} dy_i d\bar{y}_i |F_g(y)|^2 \det(1 - z(y) z(y))^{-13}. \tag{10}$$

Here y^i is a complex co-ordinate system and $F(y)$ a holomorphic function on $\mathcal{M}_g$. (We will propose a non-perturbative analogue of this at the end of this paper.)

We will produce a universal analogue of these constructs by mapping S into the infinite dimensional Segal disk. We have not been able to produce an analogue for the Torelli space. Instead we will find a map of the Universal Teichmüller Space S directly into the Segal disk. This is the analogue of the map z discussed above. One important difference with the finite genus case is that the Kähler metric so obtained on S is also a homogenous metric. Of course there are no continuous isometries for the Kähler metric on $\mathcal{T}_g$ except for $g = 1$.

First we need an infinite dimensional analogue for the Siegel disk. It turns out that all the above properties of $D(g)$ can be generalized to infinite dimensions if certain convergence conditions are imposed on Z [9]. Let V be an infinite dimensional *real* Hilbert space. Pick a complex structure J_0 on V that turns it into a complex Hilbert space W, so that $V = W \oplus \bar{W}$ and $J_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Choose also the symplectic form $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The linear transformations that leave ω and J invariant (i.e., $\omega = g\omega g^T$ and $J = gJg^{-1}$) form the group U of all unitary operators on a complex Hilbert space. Those that leave ω but not necessarily J, form the infinite dimensional symplectic group. In order that $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ leave ω invariant we have the familiar conditions

$$a^T \bar{b} = b^\dagger a, \quad a^T \bar{a} - b^\dagger b = 1. \tag{11}$$

But in fact we will only consider a subgroup of "restricted" symplectic transformations Sp_1, that also satisfy

$$\operatorname{tr} b^\dagger b < \infty. \tag{12}$$

Segal showed that much of the classical theory of $Sp(2n)$ generalizes to this restricted symplectic group [9]. The subgroup with $b=0$ is clearly U. Thus Sp_1 consists of those symplectic transformations that differ from unitary transformations by a finite amount in the Hilbert-Schmidt norm.

The infinite dimensional Segal disk (now named after Segal) D_1 is the space of all operators Z on an infinite dimensional complex Hilbert space W that satisfy

$$Z^T = Z, \quad 1 - Z^\dagger Z < 0, \quad \operatorname{tr} Z^\dagger Z < \infty. \tag{13}$$

The convergence condition (which is of course trivial in finite dimensions) on the trace has been added so that the Kähler geometry of D_1 is well-defined (see below).

There is a transitive action of Sp_1 on D_1 given as before by

$$Z \to (aZ + b)(\bar{b}Z + \bar{a})^{-1}. \tag{14}$$

Thus we can identify D_1 with the coset space Sp_1/U. Again we see that this infinite dimensional bounded domain admits a homogenous Kähler metric determined by the potential

$$K(Z) = -\operatorname{tr}\log(1 - Z^\dagger Z). \tag{15}$$

Clearly the trace is well defined because of the convergence condition imposed on Z. $\left(\text{Recall that } \sum_n \log(1 - x_n) \text{ converges absolutely iff} \sum_n |x_n| \text{ converges.}\right)$ It is convenient for a later purpose to write the Kähler potential in a slightly different form. The group element $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ that maps 0 to Z has $b\bar{a}^{-1} = Z$. ($b\bar{a}^{-1}$ is symmetric and is invariant under a right multiplication of $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ by an element of U.) Thus $K(Z)$ can be viewed as a function on Sp_1 invariant under the right action of U. In this language,

$$\begin{aligned} K(Z) &= -\log\det(1 - a^{T-1} b^\dagger b \bar{a}^{-1}) \\ &= -\log\det a^{T-1}(a^T\bar{a} - b^\dagger b)\bar{a}^{-1} \\ &= \log\det(a^T\bar{a}) \\ &= \operatorname{tr}\log(1 + b^\dagger b). \end{aligned}$$

Geometrically, we can regard each matrix $Z \in D_1$ as describing a complex structure on V. The origin $Z=0$ corresponds to the complex structure we started with, $J_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Elements of Sp_1 that leave J_0 invariant form U. So Sp_1/U can be viewed as the space of all complex structures $J = gJ_0g^{-1}$ that can be obtained from J_0 by the action of some $g \in Sp_1$.

Therefore, to get an embedding of S into D_1, it is enough to find a way to construct such a complex structure. We will find a real vector space on which each univalent function $f \in V$ defines a complex structure. This is the space V of *real* functions on the unit circle with zero average:

$$V = \left\{ \psi : S^1 \to R \,\middle|\, \oint_{S^1} \psi(v) \frac{dv}{2\pi i v} = 0 \right\}. \tag{16}$$

We take the completion of this space (also called V to save on notation) with the norm

$$\|\psi\|^2 = \oint_{S^1} \psi(v)^2 \frac{dv}{2\pi i v} \tag{17}$$

to be our real Hilbert space. Any ψ can be expanded in a Fourier series

$$\psi(e^{i\theta}) = \sum_{n=1}^{\infty} \psi_n e_n + \sum_{n=1}^{\infty} \bar{\psi}_n \bar{e}_n. \tag{18}$$

Here we use the basis

$$e_m = \frac{e^{im\theta}}{\sqrt{|m|}} \quad m \neq 0$$

for convenience.

The complex structure J_0 can now be defined as

$$(J_0\psi)(e^{i\theta}) = \sum_{n=1}^{\infty} i\psi_n e_n + \sum_{n=1}^{\infty} (-i)\bar{\psi}_n \bar{e}_n. \tag{19}$$

We can simply think of ψ_n, $n = 1, 2, \ldots, \infty$ as the components of ψ in the corresponding complex vector space W. In fact W can be thought of as the space of functions holomorphic inside the unit disk and vanishing at the origin. The complex structure J_0 simply gives a rule (analytic continuation to the interior) that maps V to W. This way of thinking about the complex structure allows for a generalization.

The symplectic form in this space is defined to be

$$\omega(\psi, \chi) = \int \psi(e^{i\theta}) \frac{\partial \chi(e^{i\theta})}{\partial \theta} \frac{d\theta}{2\pi}. \tag{20}$$

In the decomposition $V = W \oplus \bar{W}$ described above

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21}$$

Now we can construct Sp_1, U and $D_1 = Sp_1/U$ for this space as before. What is special about this particular choice of ω is that it is invariant under an action of $\mathrm{Diff}\, S^1$ on V. Thus we have an embedding of $\mathrm{Diff}\, S^1$ to Sp_1 and $\mathrm{Diff}\, S^1/S^1$ to D_1 [9]. Viewed another way this will give us an embedding of S into D_1.

The action of an element $\phi^{-1} \in \mathrm{Diff}\, S^1$ on a $\psi \in V$ is given by

$$T(\phi^{-1})\psi(e^{i\theta}) = \psi(e^{i\phi(\theta)}) - \int \psi(e^{i\phi(\theta)}) \frac{d\theta}{2\pi}. \tag{22}$$

This is just the action of Diff S^1 on a scalar except that we subtract a constant to maintain the condition that the average value of the function be zero. It is straightforward to verify that this leaves the symplectic form ω invariant. In ref. [9] it is shown that the operator $T(\phi)$ satisfies the convergence conditions required for it to be a element of Sp_1. Thus we have an embedding $T: \mathrm{Diff}\, S^1 \to Sp_1$. Furthermore, it is obvious that rotations leave the complex structures invariant so that they go into the U subgroup under the embedding. But in fact none of the Möbius transformations mix positive and negative frequency components. So in fact T maps $PSL(2, R) \to U$. Thus we have a map of Diff $S^1/PSL(2, R) \to D_1$. This is a holomorphic isometric embedding. We can therefore obtain the Kähler potential on Diff $S^1/PSL(2, R)$ by pulling back the Kähler potential on D_1. To obtain the Kähler potential on Diff S^1/S^1 we must regard it as a bundle over Diff $S^1/PSL(2, R)$ with the unit disk $D = PSL(2, R)/U(1)$ as fibre. Thus Diff S^1/S^1 has one extra complex co-ordinate than Diff $S^1/PSL(2, R)$.

This way of thinking about Diff $S^1/PSL(2, R)$ however does not make it obvious that it is a complex manifold. However if we consider the complex manifold S of which Diff $S^1/PSL(2, R)$ is a subspace, we have a complex co-ordinate system, provided by the coefficients c_k. How can we construct a complex structure on V given a univalent function $f \in S$?

The complex structure J_0 on V was constructed based on a splitting of $\psi \in V$ into a positive frequency part that is holomorphic inside the unit circle and a negative frequency part holomorphic outside the unit circle.

Consider a univalent function $f \in S$ that can be extended as a continuous injective function on the unit circle. Then f maps the unit circle to some non- self-intersecting contour $K = f(S^1)$. It is known [13] (the Koebe one-quarter theorem) that this contour surrounds a disk of radius at least $\frac{1}{4}$. Hence functions holomorphic inside K can be expanded in a power series that will converge at least for $|z| < \frac{1}{4}$. Given any function $\psi \in V$ we can decompose it into a piece ψ_f^+ holomorphic inside K and a piece $-\psi_f^-$ holomorphic outside K. Explicitly,

$$\psi_f^{\pm}(z_\pm) = \oint_{S^1} \frac{dv}{2\pi i} f'(v) \frac{z_\pm}{f(v)} \frac{\psi(v)}{f(v) - z_\pm}. \tag{23}$$

It can be shown then that

$$\psi(v) = \psi_f^+(v) - \psi_f^-(v) \tag{24}$$

for $v \in S^1$. When $f(z) = z$ we can verify that $\psi_f^{\pm}$ is precisely the positive frequency part of ψ defined earlier.

Recall that $e_m(v) = \dfrac{v^m}{\sqrt{|m|}}$ for $m \neq 0$ forms a basis for V. Because of the one-quarter theorem e_m for $m = 1, 2, \ldots, \infty$ also forms a basis for the space of functions holomorphic inside K. So we can associate to each real function $\psi \in V$ a set of complex numbers ψ_{nf} for $n = 1, 2, \ldots, \infty$:

$$\psi_f^+(z) = \sum_{n=1}^{\infty} \psi_{nf} \frac{z^n}{\sqrt{n}}. \tag{25}$$

This defines a complex structure J_f on V that differs from J_0 obtained earlier. The sum on the right converges inside the contour K. It is now possible to take the old basis e_m, e_{-m} for $m=1,2,\ldots,\infty$, decompose each basis vector into pieces holomorphic inside and outside K and re-expand in powers of z. This will produce an element of Sp_1, $g=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ where

$$e_{-mf}=\sum_{n=1}^{\infty} a_{mn}(f)\,e_{-n}+\sum_{n=1}^{\infty} b_{mn}(f)\,e_n. \tag{26}$$

We can get a more explicit form by multiplying both sides by a power of z and integrating over the circle of radius $\frac{1}{4}$ (where we know the right-hand side converges):

$$a_{mn}(f)=\sqrt{\frac{m}{n}}\oint_{S^1_{\frac{1}{4}}}\frac{du}{2\pi i}u^m\oint_{S^1}\frac{dv}{2\pi i}\frac{f'(v)}{f(v)}\frac{v^{-n}}{f(v)-u} \tag{27}$$

and

$$b_{mn}(f)=\sqrt{\frac{m}{n}}\oint_{S^1_{\frac{1}{4}}}\frac{du}{2\pi i}u^{-m}\oint_{S^1}\frac{dv}{2\pi i}\frac{f'(v)}{f(v)}\frac{v^{-n}}{f(v)-u}. \tag{28}$$

Now we have an explicit formula for the embedding $S\to D_1$:

$$Z(f)=b(f)\bar{a}^{-1}(f). \tag{29}$$

It is worthwhile to calculate explicitly the special case when f is only infinitesimally different from the identity:

$$f(z)=z\left(1+\sum_{p=1}^{\infty} c_p z^p\right).$$

A fairly straightforward calculation using our formula above gives

$$Z_{mn}=b_{mn}=\sqrt{(mn)}\,c_{m+n}+O(c^2). \tag{30}$$

Let us digress to make a technical remark. In fact, not all functions $f\in S$ lead to a Z that satisfies the convergence conditions, only those in a dense subset does. This is the sort of technical difficulty that arises all the time in infinite dimensions. From the above expression we see that in order for $\operatorname{tr} Z^\dagger Z$ to be finite $\sum_p |c_p|^2 p^3$ must converge. This is definitely true for the subset $\operatorname{Diff} S^1/PSL(2,R)$. ($\operatorname{Diff} S^1/PSL(2,R)$ consists of functions for which c_p goes to zero faster than any power of p.) In any case it is clear that the convergence condition is satisfied by a dense subset of functions in S. For example the space of univalent polynomials (for which Z is a finite rank matrix) is already dense [13] in S.

We can now pullback the Kähler form of D_1 to get a function K_1 on S. It is best to use the expression for K as a function of b that we derived earlier. Thus the Kähler potential for the homogenous metric on S is

$$K_1(f)=\operatorname{tr}\log(1+b(f)^\dagger b(f)), \tag{31}$$

where the matrix b_{mn} can be calculated by evaluating the contour integral given above. Since K is describing a homogenous Kähler metric it is completely determined by its value at the origin. We can now calculate, near the origin, to leading order

$$\begin{aligned} K_1(f) &= \operatorname{tr} b(f)^\dagger b(f) \\ &= \sum_{m,n=1}^{\infty} mn|c_{m+n}|^2 \\ &= \frac{1}{6}\sum_{p=1}^{\infty} (p^3-p)|c_p|^2 . \end{aligned}$$

Note that this does not define a non-degenerate metric on S. At the origin, the direction c_1 is degenerate. It provides a metric on the submanifold Diff $S^1/PSL(2,R)$ of S. On Diff $S^1/PSL(2,R)$ the most general homogenous Kähler form is [14]

$$\omega(L_m, L_n) = a(m^3-m)\delta(m+n), \tag{32}$$

where a is some constant. Also L_m are the generators of the left action of the Lie algebra of Diff S^1. If we regard this manifold as a subspace of S, we can get an expression for the Kähler potential of this metric:

$$K(f) = 6a \operatorname{tr}\log(1+b(f)^\dagger b(f)). \tag{33}$$

Without the use of the co-ordinates c_m provided by the relation of Diff $S^1/PSL(2,R)$ with S, it would have been impossible to get such an explicit expression for K. The calculations of [2, 5, 14] show that the Ricci tensor is finite for this metric. We also note that the metric is in fact Kähler-Einstein,

$$\mathrm{Ric} = -\frac{26}{12a}\omega . \tag{34}$$

The most general homogenous metric on Diff S^1/S^1 depends on two parameters:

$$\omega(L_m, L_n) = (am^3+bm)\delta(m+n). \tag{35}$$

Kirillov and Yuriev have already found the Kähler potential in the special case $a=0$. However this is a singular case since the Ricci tensor does not exist [2] even though the Kähler metric itself does. By combining their result with ours, we can get the potential for the general homogenous Kähler metric.

Let us first recall the potential of Kirillov ad Yuriev. Consider a non-self-intersecting contour K on the complex plane. It can be viewed as the image of the unit circle by a univalent function $f\in S$. But it can also be thought of as the image of the unit circle under by a function g_f that is univalent *outside* the unit disk. This g_f is unique up to a constant phase, if we impose that $g_f(\infty)=\infty$. A point $v\in S^1$ will in general be mapped to two different points by g_f and f, but they will both lie on K. The quantity $\ln g_f'(\infty)$ is called the "analytic capacity" of the countour $K=f(S^1)$. It has been shown by Kirillov and Yuriev that $-b\ln g_f'(\infty)$ is the Kähler potential of the homogenous metric on S with $a=0$.

It follows that the Kähler potential of the most general homogenous metric on S is

$$K(f) = 6a\operatorname{tr}\log(1+b(f)^\dagger b(f)) - (b+1)\ln g_f'(\infty). \tag{36}$$

We would like to point out that our embedding of S into D_1 produces as a byproduct an infinite number of inequalities on the coefficients c_p of a univalent function. It is clear from the derivation above that the matrix

$$Z_{mp}(f) = \sum_{n=1}^{\infty} b_{mn}(f)\bar{a}_{np}^{-1}(f) \tag{37}$$

satisfies

$$Z^{\dagger}(f)Z(f) < 1. \tag{38}$$

It is quite possible that these inequalities imply the inequalities $c_p \leqq p+1$. Then this would give an alternative proof to the Bieberbach-De Branges theorem. Since the Bieberbach conjecture was an open problem for several decades before it was proved by a very long argument by De Branges, it might be of interest to follow this approach to its proof.

We now conclude with an idea on the non-perturbative approach to string theory [8] that is part of the motivation for this paper. It is proposed that the scattering amplitudes of string theory can be expressed as an integral of vertex operators, with measure

$$\int_{\mathcal{M}} \prod_{n=1}^{\infty} dc_n d\bar{c}_n |\tilde{F}(c)|^{2-13} \det(1 - Z^{\dagger}(f)Z(f)). \tag{39}$$

Here $\tilde{F}$ is a holomorphic function on S. Two points on S are to be thought of as equivalent if their images in D_1 differ only by an action of $Sp_1(Z)$, the subgroup of Sp_1 with integer entries. A fundamental region in S under such an equivalence relation is $\tilde{\mathcal{M}}$, a candidate for the "Universal Moduli Space".

We know that it is possible to embed the Teichmüller space of Riemann surfaces of genus $g \geqq 1$ into S. It should be possible to show that the restriction of $Z(f)$ to this subspace of S is related to the period matrix of the corresponding Riemann surface.

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