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Conformally Invariant Gauge Fixed Actions for 2-D Topological Gravity

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Abstract. We show that invariants of Mumford for moduli spaces of curves are obtainable from a gauge fixed action of a topological quantum field theory in two dimensions. The method is completely analogous to the relation of Donaldson invariants with the topological quantum field theory for gauge theories in four dimensions.

1. Introduction

Topological invariants possess an obvious gauge symmetry: Expressed as the integral of a field dependent local expression, they are independent of any given local deformation of these fields. One can generally associate a BRST symmetry to such an extended gauge symmetry. Thus, given a topological invariant, one can construct a partition function through the BRST formalism, and look for new invariants defined as BRST invariant observables. The difference between the BRST formalisms of ordinary gauge symmetries and of extended gauge symmetries of topological invariants is as follows. In the ordinary case one has a dynamics for the transverse modes already at the classical level. Gauge fixing is an operation for defining the dynamics of longitudinal modes at the quantum level. One has a consistent particle interpretation of the theory, which means a physical infinite dimensional Hilbert space with unitarity properties due to the BRST invariance. For the topological theories no dynamics exists at the classical level. There is no particle interpretation and the whole dynamics is only defined at the quantum level, by the choice of gauge functions. The BRST procedure amounts to defining a Lagrangian, which is d and BRST exact, and the quantum field theory is defined perturbatively from a functional integral concentrated around the solutions of the gauge functions. The observables only correspond to transitions between

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topological excitations: they must be invariants. Different gauge fixing are expected to lead to different quantum theories, and thus to different invariants. On the other hand, the invariants computed from these functional integrals are all expected to be topological in the sense that the BRST construction guarantees their independence of the choice of the metric which has to be done to give a meaning to the quantization.

In this paper we follow closely our approach to the TQFT for Donaldson invariants and show how to construct a TQFT for 2-D gravity which gives invariants of Mumford [6c]. See also E. Miller [6a] and S. Morita [6b]. The presentation is the same as in ref. [2]. In Sects. (2, 3, 4), choosing the appropriate description of a worldsheet, we construct the BRST symmetry, the cocycles and the action for a topological 2-D gravity. In Sect. (5), we give the geometrical meaning of our construction and identify the observables of our theory with the Mumford forms on moduli space.

Of the many papers on TQFT, in our view, the most pertinent to this paper are [2, 4, 5, 8, 9]. In particular, see Sect. 6 of [4] and Sect. 2.2 of [8].[1]

2. Topological BRST Symmetry for 2-D Gravity

2-a. The Geometrical Ghost Sector. For 2-D gravity, the basic symmetry is Weyl $\times$ Diffeomorphism. The natural variable, the source of the energy momentum tensor T_{zz}, is the Beltrami differential $\mu_{\bar z}^{z}$ [1]. By definition of $\mu_{\bar z}^{z}$ and $\mu_{z}^{\bar z}$, one has:

$$ds^2 = \exp\phi(dz + \mu_{\bar z}^{z} d\bar z)(d\bar z + \mu_{z}^{\bar z} dz), \tag{1}$$

where ds^2 measures the squared length of an infinitesimal line on the Riemann surface. ϕ is the conformal Weyl weight of the metric. $\mu_{\bar z}^{z}$ is inert under Weyl transformations, and transforms as follows under an infinitesimal diffeomorphism along the vector field $(\lambda^z, \lambda^{\bar z})$:

$$\delta\mu_{\bar z}^{z} = \partial_{\bar z}\Lambda^z + \Lambda^z\partial_z\mu_{\bar z}^{z} - \mu_{\bar z}^{z}\partial_z\Lambda^z. \tag{2}$$

(We have redefined $\Lambda^z = \lambda^z + \mu_{\bar z}^{z}\lambda^{\bar z}$.) One can express (2) in the form of a BRST operation s:

$$s\mu_{\bar z}^{z} = \partial_{\bar z}c^z + c^z\partial_z\mu_{\bar z}^{z} - \mu_{\bar z}^{z}\partial_z c^z, \quad sc^z = c^z\partial_z c^z. \tag{3}$$

(c is the anticommuting vector field ghost corresponding to Λ.) Equation (3) can be rewritten under the following Maurer Cartan like equation:

$$(d+s)\tilde A + \tfrac{1}{2}[\tilde A, \tilde A] = 0. \tag{4}$$

We have defined:

$$\begin{aligned} d &= dz\partial_z + d\bar z\partial_{\bar z}, \\ \tilde A &= (dz + d\bar z\mu_{\bar z}^{z} + c^z)\partial_z, \end{aligned} \tag{5}$$

d anticommutes with s and one has $(d+s)^2 = 0$. There is of course an equation conjugate to (4) in the antiholomorphic sector for $\mu_{z}^{\bar z}$ and $c^{\bar z}$.

The way the conformal factor ϕ transforms under a Weyl $\times$ Diffeomorphism is as follows:

[1] Added in proof: Further important developments can be found in [10] and [11].

$$s\phi = \Omega + \partial_z c^z + \partial_{\bar z} c^{\bar z} + \frac{1}{(1-\mu_{\bar z}^z \mu_z^{\bar z})}((c^z - \mu_{\bar z}^z c^{\bar z})(\partial_z \phi - \partial_{\bar z}\mu_z^{\bar z}) + (c^{\bar z} - \mu_z^{\bar z} c^z)(\partial_{\bar z}\phi - \partial_z \mu_{\bar z}^z)). \tag{6}$$

Ω is the ghost associated to the parameter of infinitesimal Weyl transformations.

Suppose we wish to define a partition function (and its correlation functions) of the following type:

$$\int [dg_{\alpha\beta}] \exp - I_{\text{top}} = \int [d\phi][d\mu_{\bar z}^z][d\mu_z^{\bar z}] \exp - I_{\text{top}} = ?, \tag{7}$$

where I_{top} is a topological invariant, and thus such that

$$\frac{\delta I_{\text{top}}}{\delta \mu_{\bar z}^z} = \frac{\delta I_{\text{top}}}{\delta \mu_z^{\bar z}} = \frac{\delta I_{\text{top}}}{\delta \phi} = 0. \tag{8}$$

The inhomogeneous transformation law of ϕ permits one to forget the integration over this variable. On the other hand, the symmetry of I_{top} is larger than the one shown in Eq. (3): I_{top} is invariant under any given transformation of $\mu_{\bar z}^z$ and $\mu_z^{\bar z}$. The BRST symmetry must therefore be replaced by the following one:

$$\begin{aligned}
s_{\text{top}} \mu_{\bar z}^z &= \Psi_{\bar z}^z + \partial_{\bar z} c^z + c^z \partial_z \mu_{\bar z}^z - \mu_{\bar z}^z \partial_z c^z, \\
s_{\text{top}} c^z &= \Phi^z + c^z \partial_z c^z, \\
s_{\text{top}} \Psi_{\bar z}^z &= \partial_{\bar z} \Phi^z + \Phi^z \partial_z \mu_{\bar z}^z - \mu_{\bar z}^z \partial_z \Phi^z + c^z \partial_z \Psi_{\bar z}^z + \Psi_{\bar z}^z \partial_z c^z, \\
s_{\text{top}} \Phi^z &= \Phi^z \partial_z c^z - c^z \partial_z \Phi^z.
\end{aligned} \tag{9}$$

Going from (3) to (9) is the exact analog of going from the ordinary Yang–Mills symmetry to the topological Yang–Mills symmetry which leaves invariant the second Chern class. The way Ψ transforms reflects the fact that it is defined up to a diffeomorphism, provided c^z is redefined, as seen in the transformation law of $\mu_{\bar z}^z$. s_{top} and d anticommute and one has of course $s_{\text{top}}^2 = 0$. $\mu_{\bar z}^z$ and Φ^z are commuting while c^z and $\Psi_{\bar z}^z$ are anticommuting. Ghost numbers are respectively $0, 1, 1$ and 2 for $\mu_{\bar z}^z$, $\Psi_{\bar z}^z$, c^z, and Φ^z.

The definition of s_{top} is equivalent to the following equation, which is the analog of the one existing in the topological Yang–Mills symmetry:

$$(d + s_{\text{top}})\tilde A + \tfrac{1}{2}[\tilde A, \tilde A] = \Psi + \Phi. \tag{10}$$

As before one has $\tilde A = (dz + d\bar z \mu_{\bar z}^z + c^z)\partial_z$ and we have defined $\Psi = d\bar z \Psi_{\bar z}^z \partial_z$ and $\Phi = \Phi^z \partial_z$.

The diffeomorphism symmetry is intimately related to the local Lorentz symmetry on the worldsheet. For example, the conformal anomaly can be seen either as a diffeomorphism $\times$ Weyl, i.e. gravitational $\times$ Weyl, anomaly, or as a Lorentz $\times$ Weyl anomaly. The shifts between the corresponding expressions of the consistent anomalies are obtained by adding local counterterms to the actions. For our present purposes, let us thus introduce a spin-connection, i.e. a gauge field for Lorentz rotations $\omega = \omega_z dz + \omega_{\bar z} d\bar z$. The two components of the zweibein are $e^z = \exp\frac{\phi}{2}(dz + \mu_{\bar z}^z d\bar z)$ and $e^z = \exp\frac{\phi}{2}(d\bar z + \mu_z^{\bar z} dz)$. The spin connection is such that the torsion vanishes, $T^z = de^z + \omega e^z = 0$ and $T^{\bar z} = de^{\bar z} - \omega e^{\bar z} = 0$. If we put equal to zero the conformal weight ϕ, and work at the lowest order in $\mu_{\bar z}^z$ and $\mu_z^{\bar z}$, we get from the condition of vanishing torsion:

$$\omega = d\bar{z}\partial_z \mu_{\bar{z}}^{z} - dz\partial_{\bar{z}}\mu_z^{\bar{z}}. \tag{11}$$

The Lorentz curvature $R = d\omega$ is thus:

$$R = d\omega = d\bar{z}dz(\partial_z^2 \mu_{\bar{z}}^{z} + \partial_{\bar{z}}^2 \mu_z^{\bar{z}}). \tag{12}$$

Since we know from (9) the transformation laws under s_{top} of all fields, we can write the expression of the BRST symmetry under the form of an equation similar to (10), for the Lorentz curvature:

$$\begin{aligned}\tilde{R} &\equiv (d + s_{\text{top}})(\omega + \partial_z c^z - \partial_{\bar{z}} c^{\bar{z}}) \\ &= R + d\bar{z}(\partial_z \Psi_{\bar{z}}^{z} + \partial_{\bar{z}}^2 c^{\bar{z}}) - dz(\partial_{\bar{z}} \Psi_z^{\bar{z}} + \partial_z^2 c^z) + \partial_z \Phi^z - \partial_{\bar{z}} \Phi^{\bar{z}}.\end{aligned} \tag{13}$$

From this equation we see that $\partial_z c^z - \partial_{\bar{z}} c^{\bar{z}}$ plays the role of a ghost for Lorentz rotations. Moreover, (13) is of the type $\tilde{R} = R_2^0 + R_1^1 + R_0^2$, where the upper index means ghost number and the lower one the usual form degree. It permits a geometrical interpretation of $\partial_z \Psi_{\bar{z}}^{z}$ and $\partial_{\bar{z}} \Psi_{\bar{z}}^{z}$, as well as of $\partial_z \Phi^z$ and $\partial_{\bar{z}} \Phi^{\bar{z}}$ (see Sect. (5)).

2-b. The Antighost Sector. To give a meaning to the functional integral, we must introduce antighosts and Lagrange multiplier fields. In this way it becomes possible to add to I_{top} an action which is BRST exact and has ghost number zero. There is a freedom in the choice of the antighost sector, which amounts to the freedom in the choice of gauge functions. The structure of the spectrum of ghosts and antighosts is however dictated from the requirement that (i) there is a grading equal to the sum of the form degree and the ghost number, and (ii) the sum of all degrees of freedom, counted positively for the fields with the right spin statistics and negatively for the fields with the wrong statistics, vanishes. There are in fact two possibilities. (We only present the antighosts and Lagrange multiplier fields in the holomorphic sector; the other sector is trivially obtainable by the conjugation, $z \leftrightarrow \bar{z}$.)

The first possibility is to introduce antighosts in such a way that there is a mirror symmetry between ghosts and antighosts.

$$\begin{array}{ccccc} & & \mu_{\bar{z}}^{z} & & \\ & \swarrow & & \searrow & \\ & \Psi_{\bar{z}}^{z}, c^z & & \bar{\Psi}_{zz}, \bar{c}_z & \\ \swarrow & & \searrow \;\; \swarrow & & \searrow \\ \Phi^z & & L & & \bar{\Phi}_z \end{array} \tag{14}$$

$\bar{c}$ is the ordinary diffeomorphism antighost. $\bar{\Psi}$ and $\bar{\Phi}$ are the mirror antighosts for Ψ and Φ respectively. L is the middle ghost of ghost which occurs when one has ghost of ghost phenomena. The ghost numbers are -1, -1, -2 and 0 for $\bar{c}$, $\bar{\Psi}$, $\bar{\Phi}$ and L. One can understand the ghost spectrum as follows. Introduce an antighost number, such that for instance c has form degree 0, ghost number 1 and antighost number 0 while $\bar{c}$ has form degree 0, ghost number 0 and antighost number 1. With this definition, the grading of a field is determined by the knowledge of the triplet of numbers (form degree, ghost number, antighost number) and equal to the sum modulo two of the usual form degree, ghost number and antighost number. For Φ, L and $\bar{\Phi}$, these triplets are respectively $(0,2,0)$, $(0,1,1)$ and $(0,0,2)$. It thus follows that Φ, L and $\bar{\Phi}$ are all possibilities of having a generalized two-form with ordinary form degree equal to zero. Ψ and $\bar{\Psi}$ are also 2-forms: their triplets are $(1,1,0)$ and $(1,0,1)$. If one reads Eq. (14) from the top to the bottom, one sees that the ordinary form degree decreases by one unit line after line, but the grading is unchanged. All the fields in (14) can thus be unified in generalized forms. The BRST transforms of the antighosts are as follows:

$$\begin{aligned} s\bar{c}_z &= b_z, & sb_z &= 0, \\ s\bar{\Psi}_{zz} &= \beta_{zz}, & s\beta_{zz} &= 0, \\ s\bar{\Phi}_z &= \bar{\eta}_z, & s\bar{\eta}_z &= 0, \\ sL &= \eta, & s\eta &= 0. \end{aligned} \tag{15}$$

The ghost numbers of the auxiliary fields b, β, η and $\bar{\eta}$ are obvious since the action of s increases ghost number by one unit.

The second possibility is a less symmetric combination of antighosts, obtained by suppressing $\bar{\Psi}$ and L. (These fields have opposite statistics, and compensate each other.) In this case, one has:

$$\begin{tikzcd} & & \mu^z_{\bar{z}} \arrow[dl] \arrow[dr] & & \\ & \Psi^z_{\bar{z}}, c^z \arrow[dl] & & \bar{c}_z \arrow[dr] & \\ \Phi^z & & & & \bar{\Phi}_z. \end{tikzcd} \tag{16}$$

The BRST transformations of the antighosts are now:

$$\begin{aligned} s\bar{c}_z &= b_z, & sb_z &= 0, \\ s\bar{\Phi}_z &= \bar{\eta}_z, & s\bar{\eta}_z &= 0. \end{aligned} \tag{17}$$

In the topological Yang–Mills theory for obtaining the Donaldson invariants, one has an antighost spectrum comparable to the one in (16). The anticommuting self-dual two-form antighost which occurs in this theory can be understood as the combination of an anticommuting one form and a commuting 0-form.

3. Cocycles

As in the topological Yang–Mills symmetry, the important equation is (10). If one applies the operator ∂_z to both sides of this equation, we find cocycles for the BRST differential operators s_{top} defined in (9). One has indeed:

$$s_{\mathrm{top}} \int_{1\ \mathrm{cycle}} \Delta^1_1 = 0, \quad s_{\mathrm{top}} \Delta^2_0 = 0, \tag{18}$$

with:

$$\begin{aligned} \Delta^1_1 &= d\bar{z}(\partial_z \Psi^z_{\bar{z}} + c^z \partial^2_z \mu - \mu \partial^2_z c^z), \\ \Delta^2_0 &= \partial_z \Phi^z + c^z \partial^2_z c^z. \end{aligned} \tag{19}$$

One has similar equations in the antiholomorphic sector, simply obtained by changing $z \leftrightarrow \bar{z}$.

4. Gauge Fixing

To gauge fix I_{top} we choose the ghost spectrum in (16). For simplicity we do not consider the one in (13). By the same method as the one that we shall follow, this would lead to other conformal field theories, with two other pairs $(\bar{\Psi}, L)$ of bosons and fermions. (Incorporating these fields in the formalism with the gauge function $D_z L$ is in fact an interesting exercise, but we don't know how to interpret the resulting theory. It would also be interesting to do different gauge fixing in the left and right sector.) In the holomorphic sector, we have Lagrange multipliers b_z and $\bar{\eta}_z$. The choice of gauge functions for $\mu_{\bar{z}}^{z}$ and $\Psi_{\bar{z}}^{z}$ defines the theory, and is thus the main subtlety which we must overcome. We are guided by the requirement that the expectation values of the second cocycle Φ be possibly non-zero. As in the Donaldson theory, this necessitates the presence of a cubic ghost interaction of the type $\bar{\Phi}\Psi\Psi$ in the BRST invariant Lagrangian that we shall build [2,9]. It is easy to convince onself that the conformal gauge, i.e., the choice of the gauge function $\mu_{\bar{z}}^{z} - \mu_{\bar{z}0}^{z}$ for $\mu_{\bar{z}}^{z}$, where $\mu_{\bar{z}0}^{z}$ is a given fixed background for $\mu_{\bar{z}}^{z}$, cannot lead to such interacting terms: choosing such a gauge in a BRST invariant way would lead us to an algebraic, and thus spurious, Ψ dependence in the action.

We thus choose covariant harmonic gauges for $\mu_{\bar{z}}^{z}$ and $\Psi_{\bar{z}}^{z}$. In the holomorphic sector, our gauge functions are:

$$D_z \mu_{\bar{z}}^{z}, \quad D_z \Psi_{\bar{z}}^{z}. \tag{20}$$

In the antiholomorphic sector, they are:

$$D_{\bar{z}} \mu_{z}^{\bar{z}}, \quad D_{\bar{z}} \Psi_{z}^{\bar{z}}. \tag{21}$$

Here, the differential operators D_z and $D_{\bar{z}}$ are defined by their action on conformal fields: $X^{l,R} = X^{\overbrace{\bar{z},\dots,\bar{z}}^{l},\overbrace{z,\dots,z}^{R}}$ of conformal weights R and l, with:

$$D_z X^{l,R} = (\partial_z - \mu_z^{\bar{z}}\partial_{\bar{z}} + (l-R)\partial_{\bar{z}}\mu_z^{\bar{z}})X^{l,R}, \tag{22}$$

$$D_{\bar{z}} X^{l,R} = (\partial_{\bar{z}} - \mu_{\bar{z}}^{z}\partial_z + (R-l)\partial_z\mu_{\bar{z}}^{z})X^{l,R}. \tag{23}$$

One has thus $D_z\mu_{\bar{z}}^{z} = (\partial_z - \mu_z^{\bar{z}}\partial_{\bar{z}})\mu_{\bar{z}}^{z} - \partial\mu_{\bar{z}}^{z}\partial_{\bar{z}}\mu_z^{\bar{z}}$ and $D_z\Psi_{\bar{z}}^{z} = (\partial_z - \mu_z^{\bar{z}}\partial_{\bar{z}})\Psi_{\bar{z}}^{z} - 2\Psi_{\bar{z}}^{z}\partial_{\bar{z}}\mu_z^{\bar{z}}$ in Eq. (20), and the conjugate equations in (21).

The partition function (7) is thus defined as follows:

$$\begin{aligned} &\int [d\mu_{\bar{z}}^{z}][dc^{z}][d\bar{c}_z][db_z][d\Psi_{\bar{z}}^{z}][d\Phi^{z}][d\bar{\Phi}_z][d\bar{\eta}_z][d\mu_z^{\bar{z}}][dc^{\bar{z}}][d\bar{c}_{\bar{z}}] \\ &\quad \cdot [db_{\bar{z}}][d\Psi_z^{\bar{z}}][d\Phi^{\bar{z}}][d\bar{\Phi}_{\bar{z}}][d\bar{\eta}_{\bar{z}}]\exp - I_{\text{GF}} = \int [dp]\exp - I_{\text{GF}} \end{aligned} \tag{24}$$

$$I_{\text{GF}} = I_{\text{top}} + \int d^2z\, s_{\text{top}}(\bar{c}_z D_z \mu_{\bar{z}}^{z} + \bar{\Phi}_z D_z \Psi_{\bar{z}}^{z} + (z \leftrightarrow \bar{z})). \tag{25}$$

Expanding this action by using our definition of s_{top}, we obtain

$$\begin{aligned} I_{\text{GF}} = I_{\text{top}} + \int d^2z(&b_z D_z\mu_{\bar{z}}^{z} - \bar{c}_z D_z \Psi'^{z}_{\bar{z}} + \bar{\Phi}_z D_z D_{\bar{z}}\Phi_z + \bar{\eta}_z D_z \Psi_{\bar{z}}^{z} \\ &+ \bar{\Phi}_z(\Psi'^{\bar{z}}_{z}\partial_{\bar{z}}\Psi_{\bar{z}}^{z} - 2\partial_{\bar{z}}(\psi'^{\bar{z}}_{z}\psi_{\bar{z}}^{z})) - \bar{c}_z(\Psi'^{\bar{z}}_{z}\partial_{\bar{z}}\mu_{\bar{z}}^{z} - 2\partial_{\bar{z}}(\Psi'^{\bar{z}}_{z}\mu_{\bar{z}}^{z}) + (z \leftrightarrow \bar{z})). \end{aligned} \tag{26}$$

We have defined $\Psi'^{z}_{\bar{z}} = \Psi_z \bar{z}^z + \partial_{\bar{z}} c^z + c^z \partial_z \mu^z_{\bar{z}} - \mu^z_{\bar{z}} \partial_z c^z = \Psi^z_{\bar{z}} + D_{\bar{z}} c^z$. I_{GF} can be rewritten as:

$$\begin{aligned} I_{\mathrm{GF}} = I_{\mathrm{top}} + \int d^2 z (b_z D_z \mu^z_{\bar{z}} - (\bar{c}_z - \bar{\eta}_z) D_z \Psi'^{z}_{\bar{z}} + \bar{\Phi}_z D_z D_{\bar{z}} \Phi^z + \bar{\eta}_z D_z D_{\bar{z}} c^z \\ + \bar{\Phi}_z (\Psi'^{\bar{z}}_{z} \partial_{\bar{z}} \Psi^z_{\bar{z}} - 2\partial_{\bar{z}} (\psi'^{\bar{z}}_{z} \psi^z_{\bar{z}})) - \bar{c}_z (\Psi'^{\bar{z}}_{z} \partial_{\bar{z}} \mu^z_{\bar{z}} - 2\partial_{\bar{z}} (\Psi'^{\bar{z}}_{z} \mu^z_{\bar{z}})) + (z \leftrightarrow \bar{z})). \end{aligned} \tag{27}$$

We see in (27) that our gauge fixed action is a conformally invariant action which involves in the holomorphic sector two couples of commuting fields $(\mu^z_{\bar{z}}, b_z)$ and $(\bar{\Phi}_z, \Phi^z)$, and two couples of anticommuting fields, $(\bar{\eta}'_z, \Psi'^{z}_{\bar{z}})$ and $(\bar{\eta}_z, c^z)$, with $\bar{\eta}'_z = \bar{c}_z - \bar{\eta}_z$. The quadratic approximation of I_{GF} is well defined, and we have the expected cubic interaction $\bar{\Phi}_z (\Psi'^{\bar{z}}_{z} \partial_{\bar{z}} \Psi^z_{\bar{z}} - 2\partial_{\bar{z}} (\Psi'^{\bar{z}}_{z} \Psi^z_{\bar{z}}))$ which is the analogous term of $\mathrm{Tr}\, \bar{\Phi}[\Psi, \Psi]$ of the topological Yang–Mills action. Thus, by the same mechanism as discussed in ref. [2], the action (27) permits one to compute invariants by inserting polynomials of the cocycles $\partial_z \Phi^z$ and $\partial_{\bar{z}} \Phi^{\bar{z}}$ in front of the functional integral measure in (24).

5. Geometric Setup

In order to interpret geometrically many of the equations in the previous sections, we need to describe the case of gravity in a way similar to the case of gauge theories [2, 9].

Let Met denote the space of smooth Riemannian metrics on the oriented compact manifold N of dimension n. Let B denote the bundle of oriented bases (vielbein) of N, and let $\tilde{Q}$ be the submanifold of $B \times \mathrm{Met}$ consisting of $(e_1, \dots, e_n) \times g$, where $\{e_j\}$ is an oriented orthonormal base relative to the metric g.

The group of diffeomorphisms of N, $\mathrm{Diff}(N)$ acts on $\tilde{Q}$. The quotient $\tilde{Q}/\mathrm{Diff}(N)$ is a principal $SO(N)$ bundle Q whose base space is itself a fibre bundle $\mathscr{L} = N \times \mathrm{Met}/\mathrm{Diff}(N)$ with fibre N and base $\mathrm{Met}/\mathrm{Diff}(N)$. For the moment we ignore the difficulties posed by the fixed points of $\mathrm{Diff}(N)$ on Met.

For $n = 2$, Q is an $SO(2) = U(1)$ bundle over $\mathscr{L}$. When the genus g of N is greater than one, the space of curvature -1 metrics Met_{-1} is contained in Met and the submanifold $N \times \mathrm{Met}_{-1}/\mathrm{Diff}(N) = \mathscr{N}_g$ of $\mathscr{L}$ can be identified with the universal curve over moduli space $M_g = \mathrm{Met}_{-1}/\mathrm{Diff}(N)$. We leave the minor modifications needed to cover $g = 1$ and $g = 0$ to the reader.

We make the analogy to Yang–Mills explicit; $Q \leftrightarrow P \times \mathscr{A}/G$, $\mathscr{L} \leftrightarrow M \times \mathscr{A}/G$ and the finite dimensional space $\mathscr{N} \subset \mathscr{L}$ is analogous to the moduli space of antidual connections on P.

Rather than the space of metrics, it is instructive to also consider the space of complex structures Com on N. In fact $\mathrm{Com} = \mathrm{Met}/\mathscr{W}$, where $\mathscr{W}$ is the group of conformal factors. Moreover $\mathrm{Diff}(N)$ acts on Com [$\mathscr{W}$ is a normal subgroup in $\mathscr{W} \rhd \mathrm{Diff}(N)$] and $N \times \mathrm{Com}/\mathrm{Diff}(N) \cong \mathscr{N}_g$ for $g \neq 1$. From the complex point of view we have the bundle $\tilde{Q}_c = \{(n, e, J)$; $n \in N, e \neq 0$ a vector in $\wedge^{1,0}(n)$ relative to the complex structure $J.\}$ Since $\mathrm{Diff}(N)$ acts on $\tilde{Q}_c$, we get a C^*-bundle Q_c over $\mathscr{N}_g$. The line bundle associated to Q_c is the tangent spaces to the fibers in $\mathscr{N}_g$ over M_g. The choice of constant curvature metrics reduces the C^* bundle to a $U(1)$ bundle and reduces the tangent bundle to the unit circle bundle.

We can summarize our constructions so far by the following diagram:

$$\begin{tikzcd}
& \tilde{Q}_c \arrow[dl, "\mathrm{Diff}(N)"'] \arrow[d, "C^*"] & & \tilde{Q} \subset B \times \mathrm{Met} \arrow[ll, "\mathscr{W}"'] \arrow[d, "SO(2)"] \arrow[dr, "\mathrm{Diff}(N)"] & \\
Q_c \arrow[dr, "C^*"'] & N \times \mathrm{Com} \arrow[d, "\mathrm{Diff}(N)"] & & N \times \mathrm{Met} \arrow[ll, "\mathscr{W}"'] \arrow[d, "\mathrm{Diff}(N)"] & Q \arrow[dl, "SO(2)"] \\
& \mathcal{N}_g \arrow[d, "N"] \arrow[rr, hook] & & \mathscr{L} \arrow[d, "N"] & \\
& M_g \arrow[rr, hook] & & \mathrm{Met}/\mathrm{Diff}(N). &
\end{tikzcd}$$

The differential of the action of $\mathrm{Diff}(N)$ on the various spaces gives a map from smooth vector fields on N into vector field along the orbits. Let L denote this differential (into Met), i.e., $L: C^\infty(T(N)) \to T(\mathrm{Met}, g) = C^\infty(\mathrm{Sym}\, T(N) \otimes T(N))$; $L(V) = \mathrm{Sym}\, D_g V$ or $L\left(\sum v_j \frac{\partial}{\partial x_j}\right) = D_i v_j + D_j v_i$, where D_i is the Riemannian covariant differential.

In the complex category, we note that Com has a natural almost complex structure, since $T(\mathrm{Com}, J) \cong \mathrm{Hom}(\wedge^{1,0}(J), \wedge^{0,1}(J))$. Given J, we can identify $T(N)$ with $T^{1,0}(J)$ and symmetric traceless tensors with the complex line bundle $(\wedge^{0,1})^2$. Then the differential above is $\bar{\partial}: C^\infty(T^{1,0}) \to C^\infty(T^{1,0} \otimes \wedge^{0,1})$. In the presence of a metric $C^\infty(T^{1,0}) \cong C^\infty(T^{1,0})^{**} \cong C^\infty((T^{1,0})^*)^* \cong C^\infty(\wedge^{1,0})^* \supseteq C^\infty(\wedge^{0,1})$, so that we can identify L with $\bar{\partial}: C^\infty(\wedge^{0,1}) \to C^\infty((\wedge^{0,1})^2)$ after projecting the range of L into traceless symmetric tensors. In the presence of a metric ρ, $L^*: C^\infty(\mathrm{Sym}\, T \otimes T) \to C^\infty(T)$ sends $S \to \mathrm{div}_\rho S$. In the complex category, $L^* = \bar{\partial}: C^\infty((\wedge^{1,0})^2) \to C^\infty(\wedge^{1,0})$. We are restricting L^* to traceless symmetric tensors because the Weyl factor has no effect on Com. If we identify $\wedge^{1,0}$ with $\wedge^{0,1}$ by conjugation, or equivalently using the metric identity one space with its adjoint, then $L^* = \partial: C^\infty((\wedge^{0,1})^2) \to C^\infty(\wedge^{0,1})$ and $L^*L: C^\infty(\wedge^{0,1}) \to C^\infty(\wedge^{0,1})$ is $\partial\bar{\partial}$. In the notation of Sect. 3, $L^*: \psi_{\bar{z}}^{z} \to \partial_z \psi_{\bar{z}}^{z}$.

6. Mumford Invariants

We next pursue the analogy with the Donaldson invariants to obtain in the gravity case invariants of Mumford [6]. In the gauge theories case we took Chern classes (in particular the second Chern class) in the curvature $\mathscr{F}$ on $M \times \mathscr{A}/G$. We integrated $c_2 = \mathrm{tr}(\mathscr{F}^2)$ over a cycle σ in M, giving a differential form of $\mathscr{A}/G$ of degree $4-i$ if σ is an i-cycle. We restricted such forms ω_σ to the moduli space of antidual solutions $\mathscr{M}$ and obtained the Donaldson invariants $\int_{\mathscr{M}} \omega_{\sigma_1} \wedge \cdots \wedge \omega_{\sigma_l}$, where $\sum_{j=1}^{l} 4 - \dim \sigma_j = \dim \mathscr{M}$.

We proceed similarly. Since the group is $U(1)$, we have only one Chern class c_1. The natural forms are then c_1^{r+1}; let $\tilde{n}_r = \int_{\mathrm{fibre}} c_1^{r+1}$, a $2r$ form on $\mathrm{Met}/\mathrm{Diff}(N)$

and let n_r be the restriction of $\tilde{n}_r$ to $M_g \hookrightarrow \mathrm{Met}/\mathrm{Diff}(N)$. Suppose $\mathscr{Z}$ is a complex cycle in M_g of complex dimension s, then we have the Mumford invariant $N_z(r_1, \ldots, r_k) = \int_z n_{r_1} \wedge \cdot \wedge n_{r_k}$, where $s = \sum_{j=1}^{k} r_j$.

As in the gauge theories case, where the Donaldson invariants were interpreted as expectation values in a four dimensional topological quantum field theory, our aim is to interpret the Mumford invariants as expectation values for fields in the topological two dimensional quantum gravity described in Sect. 4.

To do so we have to compute the Chern class c_1 explicitly as a curvature two form $\mathscr{F}$ which will involve a Green's function as in the gauge theory case.

Before doing this we make two remarks:

1. The natural category here is a 2-surface with s punctures, its diffeomorphism group, its space of complete finite area metrics, and its moduli space. We have not explored the extension of our methods to this case [7].
2. As in the Donaldson case, the "infra-red" problem of integration over M_g needs careful consideration. The line bundle associated to Q_c extends to $\mathscr{N}_g$ over $\bar{M}_g$, the Deligne–Mumford compactification; the dualizing sheaf and the Grothendieck Riemann Roch Theorem describe the index of the family $\bar{\partial}$ along the fibers in terms of Mumford forms [3]. However in the quantum field theory computation explicit differential terms are to be integrated. We have not explored their behavior at infinity. We note that if one uses the (-1) curvature metric for the holomorphic line bundle, then its first Chern class will diverge at ∞ [3].

Moreover, if we use the Quillen metric on the determinant line bundle for $\bar{\partial}$, one finds that its first Chern class C has a logarithm singularity at ∞. From Grothendieck Riemann–Roch, C is the cohomology class of the two forms on M_g we have denoted by $n_1 = \int_{\text{fibre}} c_1^2$.

We have yet to explore the relationship between our n_1 and the 2-form obtained via the Quillen metric.

7. Curvature on Q

As promised, we now describe a natural $U(1)$ connection μ on Q over $\mathscr{L}$, whose curvature is $\mathscr{F} = c_1$, needed for the Mumford invariants. Note that Met has a natural metric invariant under the action of $\mathrm{Diff}(N)$. The tangent space to Met at $g \in \mathrm{Met}$, $T(\mathrm{Met}, g)$ are two-tensor fields ψ symmetric under ρ. Then $\langle \psi_1, \psi_2 \rangle = \int_N \psi_1 \cdot \psi_2 \, \mathrm{vol}_g$.

The orthogonal compliments to the orbits of $\mathrm{Diff}(N)$ gives a connection μ_2 of Met over $\mathrm{Met}/\mathrm{Diff}(N)$ with group $\mathrm{Diff}(N)$. The group $SO(2) \times \mathrm{Diff}(N)$ acts on $\tilde{Q}$. Now the submanifold of $\tilde{Q}$ with the metric g fixed is an $SO(2)$ bundle over N with connection, the Riemannian connection determined by g. As g varies we get an $SO(2)$ connection on $\tilde{Q}$. Combining this $SO(2)$ connection with the μ_2 connection gives a connection μ_1 of $\tilde{Q}$ over $\mathscr{L}$ with group $SO(2) \times \mathrm{Diff}(N)$ invariant under $\mathrm{Diff}(N)$. Dividing μ_1 by $\mathrm{Diff}(N)$ gives the described $SO(2)$ connection μ.

$N \times \mathrm{Met}$ also has a natural metric (at (n, g) the metric on $T(N, n)$ is g), invariant under Diff(N). So we have a natural connection μ_3 on $N \times$ Met over $\mathscr{L}$ with group Diff(N), the orthogonal compliment to the orbit. Finally $\mathscr{L} = N \times \text{Met}/\text{Diff}(N)$ has a natural metric. As a result, the curvature $\mathscr{F}$ on μ is $\mathscr{F}_{2,0} + \mathscr{F}_{1,1} + \mathscr{F}_{0,2}$, where $\mathscr{F}_{2,0}$ is $\mathscr{F}$ along the fibres, $\mathscr{F}_{0,2}$ is the curvature 2-form in directions orthogonal to the fibre, and $\mathscr{F}_{1,1}$ is the mixed curvature.

In the complex category, although Com has a natural almost complex structure described earlier, it does not have a natural inner product. A choice of metric g on N consistent with the complex structure gives one: If $\mu_j \in T(\text{Com}, J)$ then $\langle \mu_1, \mu_2 \rangle = \int_N \mu_1 \bar{\mu}_2 \,\text{vol}_g$. Thus Diff$(N)$ connections on Com depend on underlying metrics. Frequently one chooses the constant curvature metric equal to -1 consistent with each conformal structure. One could also fix the volume once and for all.

There are many ways to compute the curvature of the connections we have just described. One method can be found in Sect. 9 below. Here we give the curvature formulas and relate them to the formulas in Sect. 1–3, thus interpreting these formulas geometrically.

In formula (3), s is the exterior differential along the orbit of Diff(N) on $N \times$ Com, having identified diff(N) with $C^\infty(T^{1,0})$; c^z represents the Cartan tautological 1-form, and d is the exterior differential along N. The connection A in formula (4) has curvature zero because the left invariant connection on any smooth group is flat. However in formulas (9) we have extended s to s_{top} the differential on $N \times$ Com. Given a Diff(N) connection $\tilde{A}$ on $N \times$ Com over $\mathscr{N}_g$, its curvature has components of type $(0,2)$ and $(1,1)$ relative to the fibering of $\mathscr{N}_g$ over M_g; these are the terms ϕ and ψ in formula 10. The $(2,0)$ component of the curvature is 0 and a compliment to the fibre N in $\mathscr{N}_g$ has to be chosen to allow the decomposition. "Ghost number," in this geometric interpretation refers to the component in the $M_g = \text{Com}/\text{Diff}(N)$ direction (or Met/Diff(N)) direction in the metric case). Again we emphasize that in formulas (9) and (10) and the definition of the connection 1-form $\tilde{A}$ we have identified diff$(N) = C^\infty(T(N))$ with $C^\infty(T^{1,0})$.

Theorem. *The curvature* $\mathscr{F} = \mathscr{F}_{2,0} + \mathscr{F}_{1,1} + \mathscr{F}_{0,2}$ *of the* $U(1)$*-connection* μ *on* $\mathscr{N}_g \subset \mathscr{L}$ *is*
(a) $\mathscr{F}_{2,0}$ *at* $(n, g) =$ *the curvature 2-form of the fibre* N *of the metric class* g,
(b) $\mathscr{F}_{1,1}(v, S)$ *at* $(n, g) = -\langle \text{div}_g S, v \rangle$; *from the holomorphic point of view,* $\mathscr{F}_{1,1}(dz, \psi) = -\partial_z \psi_{\bar{z}}^{z} * (d\bar{z} \wedge dz)$,
(c) $\mathscr{F}_{0,2}(\tilde{S}, S)$ *at* $(n, g) = \text{div}_g G\{(\nabla_\rho \tilde{S}) \circ S - (\nabla_\rho S) \circ \tilde{S}\}$; *in complex terms,* $\mathscr{F}_{0,2}(\tilde{\psi}, \psi) = \text{div}_g G\{\tilde{\psi}_z^{\bar{z}} \partial_{\bar{z}} \psi_{\bar{z}}^{z} - \psi_z^{\bar{z}} \partial_{\bar{z}} \tilde{\psi}_{\bar{z}}^{z}\} = * \partial_z G\{\tilde{\psi}_z^{\bar{z}} \partial_{\bar{z}} \psi_{\bar{z}}^{z} - \psi_z^{\bar{z}} \partial_{\bar{z}} \tilde{\psi}_{\bar{z}}^{z}\}$.
(i) *In terms of a frame,* $\nabla_g \tilde{S} \circ S = S(\text{div}\, \tilde{S}) = \nabla_i \tilde{S}_{ij} S_{jk}$.
(ii) S *and* $\tilde{S}$ *in* $T(\text{Met}_{-1}, g)$ *are traceless symmetric tensor fields.* $G = (L^* L)^{-1}$ *on* $C^\infty(T(N))$ *or its dual.*

The proof can be found in Sect. 9. Note the similarity with the Yang–Mills case in [2]. Turning back to Sect. 2, formula (13) express the curvature $\mathscr{F}$ above in local coordinates $z, \bar{z}$. In $\mathscr{F}$ there are no c terms because we are working modulo Diff(N). Also, the last term in (15), $\partial_z \Phi^z - \partial_{\bar{z}} \Phi^{\bar{z}}$, express how $\mathscr{F}_{0,2}$ can be computed from the curvature K of μ_2.

8. Mumford Invariants as Expectation Values

In the path integral, we interpret the gauge functions (20) and (21) as projections from $N \times \mathrm{Com}$ to the quotient $\mathcal{N}_g$ with μ the Beltrami variable and $T(N \times \mathrm{Com})$ to $T(\mathcal{N}_g)$ with ψ in the tangent space. Using the complex conjugate as well means that the gauge functions project $N \times \mathrm{Met}$ to $\mathcal{L}$, and the restriction of the Weyl factor to be 0 means restriction to $\mathcal{N}_g \hookrightarrow \mathcal{L}$.

Now consider the expectation $\langle (\operatorname{div} \Phi)(n) \rangle = \int [d\rho] e^{I_{\mathrm{GF}}} (\operatorname{div} \Phi)(n)$ (see Eq. (20)). We are only concerned with the weak coupling limit. The relevant cubic term in I_{GF} is $\int_N \bar{\Phi}_z(\psi^{\bar{z}}_z \partial_{\bar{z}} \psi^z_{\bar{z}} - 2\partial_{\bar{z}}(\Psi^{\bar{z}}_z \Psi^z_{\bar{z}})$ whose integrand for brevity we will denote by $\bar{\Phi} \cdot \psi \circ \psi$. When $e^{-I_{\mathrm{GF}}}$ is expanded as a power series, we find in $e^{-I_{\mathrm{GF}}} (\operatorname{div} \Phi)(n)$, the term $\bar{\Phi} \cdot \psi \circ \psi \operatorname{div} \Phi(n)$. In view of the quadratic term $\bar{\Phi}_z D_z D_{\bar{z}} \Phi^z = \bar{\Phi} \cdot L^* L \Phi$ in the action, integration with respect to $[d\Phi^z][d\bar{\Phi}_z]$ gives the term $\operatorname{div} G\psi \circ \psi$ at n.

Now $\psi \circ \psi = \psi^{\bar{z}}_z \partial_{\bar{z}} \psi^z_{\bar{z}} - 2\partial_{\bar{z}}(\psi^{\bar{z}}_z \psi^z_{\bar{z}})$. Hence $\operatorname{div} G\psi \circ \psi = *\partial_z G(\psi^{\bar{z}}_z \partial_{\bar{z}}(\psi^z_{\bar{z}}) - 2*\partial_z G \partial_{\bar{z}}(\psi^{\bar{z}}_z \psi^z_{\bar{z}}))$. But $*\partial_z G \partial_{\bar{z}}$ is projection on the orthogonal complement of the kernel of $\partial_{\bar{z}}$ (on functions) and hence is the identity. Hence $\operatorname{div} G\psi \circ \psi = \mathcal{F}_{0,2}(n) - 2(\psi^{\bar{z}}_z \psi^z_{\bar{z}})$. Now $\psi^{\bar{z}}_z \psi^z_{\bar{z}}$ is purely imaginary since $\psi^{\bar{z}}_z \psi^z_{\bar{z}} = -\psi^z_{\bar{z}} \psi^{\bar{z}}_z = -\overline{\psi^{\bar{z}}_z \psi^z_{\bar{z}}}$ and so drops out under $z \leftrightarrow \bar{z}$.

Hence $\langle\langle \operatorname{div} \Phi(n) \rangle\rangle = \mathcal{F}_{0,2}(n)$, where $\langle\langle \quad \rangle\rangle$ means partial integration, only with respect to $[d\Phi^z][d\bar{\Phi}_z]$. The present situation is completely analogous to the Yang–Mills case and the comments in that case apply here as well.

There one integrates over all i-cycles on M $i = 0, \ldots, 4$. In the gravity case, the natural cycles are the fibres N in $\mathcal{N}_g$. (The cocycle Δ^2_0 with $s_{\mathrm{top}} \Delta^2_0 = 0$ in (20) and (21).) The cocycle $\int_{1\text{-cycle}} \Delta'_1$ is not invariant under $\mathrm{Diff}(N)$ and hence does not give a topological invariant. We note however that when our category is extended to Riemann surfaces with punctures, the punctures will be fixed under the corresponding diffeomorphism group so that evaluation at these 0 cycles and averaging will give topological invariants.

For the moment, we have only the forms $\tilde{n}_r$ which are obtained by integrating $\mathcal{F}^2_{1,1} \wedge (\mathcal{F}_{0,2})^{r-1} + \mathcal{F}_{2,0}(\mathcal{F}_{0,2})^r$ over the fibres. Because of the integration with respect to $[d\Psi^{\bar{z}}_z][d\Psi^z_{\bar{z}}]$, these are all expectation values.

9. Proof of the Theorem

To compute the curvature of the connection μ, we first compute the curvature of $\mu_2 = GL^*$, since LGL^* is the projection of $T(\mathrm{Met}, g)$ onto the tangent space of the orbit at g, with $G = (L^*L)^{-1}$. If S^1 and S^2 are two symmetric tensor fields, they are constant vector fields on Met. Assume they are orthogonal to the orbit at g, i.e., $L^*_g(S^j) = 0, j = 1, 2$. Then $S^j - L_\rho G_\rho L^*_\rho S^j$ are horizontal vector fields on Met over Met/Diff. So the curvature K_{μ_2} of μ_2 is given by $K_{\mu_2}(S^1, S^2) = \mu([S^1 - L_\rho G_\rho L^*_\rho S^1, S^2 - L_\rho G_\rho L^1_\rho S^2]$ at $\rho = g$. Since $[S^1, S^2] = 0$ on Met and since $L^*_g S^j = 0$, we have

$$K_{\mu_2}(S^1, S^2) = GL^* \left(LG \frac{\delta L^*_\rho(S^1)}{\delta S_2} - LG \frac{\delta L_\rho(S^2)}{\delta S_1} \right) \bigg|_{\rho = g} = G \left(\frac{\delta L^*_\rho(S_1)}{\delta S_2} - \frac{\delta L^*_\rho(S_2)}{\delta S_1} \right) \bigg|_{\rho = g}.$$

Since $L_\rho^* S = 2\ \mathrm{div}_\rho S - \text{dual}\ dt dr S$, $\left.\frac{d}{dt}\right|_{t=0} L^*_{g+tS^2}(S^1) = 2\nabla g S^2 \circ S^1 - S^2 d\,\mathrm{tr}\,(S^1)$, where in local coordinates $\nabla g S^2 \circ S^1 = (D_i S^2_{ik}) S^1_{kl}$. Hence

$$K_{\mu_2}(S^1, S^2) = G\{\nabla g S^2 \circ S^1 - \nabla g S^1 \circ S^2 + S^1 d\,\mathrm{tr}\,(S^2) - S^2 d\,\mathrm{tr}\,S^1\}.$$

When S^1 and S^2 are traceless, we have $K_{\mu_2}(S^1, S^2) = G\{\nabla_1 S^2 \circ S^1 - \nabla g S^1 \circ S^2\}$. In the complex category since $L^*\psi = \partial\psi = \partial_z \psi^z_{\bar z}$, and since for small variations of complex structures, $\bar\partial \to \bar\partial - \psi^z_{\bar z}\partial/1 - \psi^z_{\bar z}\psi^{\bar z}_z$, we get $\frac{\delta L^*}{\delta\psi^2}(\psi^1) = -(\psi^2)^{\bar z}_z \partial_{\bar z}(\psi^1)^z_{\bar z})$. Because the curvature is of type (1,1), we obtain $K_{\mu_2}(\psi, \tilde\psi) = G\{\psi^{\bar z}_z \partial_{\bar z}(\tilde\psi^z_{\bar z}) - \tilde\psi^{\bar z}_z \partial_{\bar z}(\psi^z_{\bar z})\}$.

We can extend the connection μ_1 in $\tilde Q$ (see Sect. 7) to a connection μ_3 on $B \times \mathrm{Met}$ by uniquely extending the Riemannian connection ω_g of g (the spin connection) to a connection on B the bundle of all bases. At $g \in \mathrm{Met}$, $\mu_3 = \omega_g + \mu_2$. To compute the $(0,2)$ component, $\mathscr{F}_{0,2}$ of μ, we bracket two horizontal vector fields along Met, project on the orbit of $SO(2) \times \mathrm{Diff}(N)$, and divided by $\mathrm{Diff}(N)$.

If V is a smooth vector field, its image in $\tilde Q$ at g is $(V, \mathrm{div}\,V, LV)$, where we have decomposed $T(\tilde Q, g) = T(B_g) \oplus T(\mathrm{Met})$ and $T(B_g) \cong T(N) \oplus so(2)$ using the connection ω_g. Dividing out $\mathrm{Diff}(N)$ implies that $\mathrm{div}\,V + L(V)$ is in the kernel of the map from $\tilde Q$ to Q. We conclude $\mathscr{F}_{0,2}(S^1, S^2) = -\mathrm{div}\,K_{\mu_2}(S^1, S^2) = \mathrm{div}\,G(\nabla_p S^1 \circ S^2 - \nabla_p S^2 \circ S^1)$ for traceless symmetric tensors. In the complex category,

$$\begin{aligned}\mathscr{F}_{0,2}(\psi, \tilde\psi) &= -\mathrm{div}\,K_{\mu_2}(\psi, \tilde\psi) = \mathrm{div}_g\, G\{\tilde\psi^{\bar z}_z \partial_{\bar z}\psi^z_{\bar z} - \psi^{\bar z}_z \partial_{\bar z}\tilde\psi^z_{\bar z}\} \\ &= *\partial_z G\{\tilde\psi^{\bar z}_z \partial_{\bar z}\psi^z_{\bar z} - \psi^{\bar z}_z \partial_{\bar z}\tilde\psi^z_{\bar z}\}.\end{aligned}$$

To compute the other two components of $\mathscr{F}$, we note that $\mathscr{F}_{2,0}$ is the curvature restricted to the fibre n and hence is the curvature 2-form of the metric class g.

Finally, for $\mathscr{F}_{1,1}$ let $\tilde V(\rho)$ be the horizontal lift at ρ of a vector field V on N to $B \times \mathrm{Met}$ relative to the connection μ_3; and let S be a symmetric tensor field, i.e., a constant vector field on Met. Then $\mathscr{F}_{1,1}(V(n)S)$ at (n, e_1, e_2, g) is the $so(2)$ component of $d\mu_3(\tilde V, S) - d\omega_g(\tilde V, S) = \omega_g[\tilde V, S] + S\omega_g(\tilde V) - \tilde V\omega_g(S) = \omega_g([\tilde V, S])$. Since $\tilde S$ is a constant field of $B \times \mathrm{Met}$, $[\tilde V, S] = \left.\frac{d}{dt}\right|_{t=0} (\tilde V_{g+tS})$. But $\omega_{g+tS}(\tilde V_{g+ts}) = 0$ by definition; so $[\tilde V, S] = -\left.\frac{d}{dt}\right|_{t=0} \omega_{g+ts}(\tilde V_g) = -\langle(\mathrm{div}_g S)V\rangle$. Hence $\mathscr{F}_{1,1}(V, S) = -\langle \mathrm{div}_g S, v\rangle$. Again from the holomorphic point of view $\mathscr{F}_{1,1}(dz, \psi) = -\partial_z \psi^z_{\bar z} *(dz \wedge d\bar z)$.

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