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Generic Global Solutions of the Relativistic Vlasov–Maxwell System of Plasma Physics

Gerhard Rein

Mathematisches Institut der Universität München, Theresienstraße 39, W-8000 München 2, Federal Republic of Germany

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Abstract. The behaviour of classical solutions of the relativistic Vlasov–Maxwell system under small perturbations of the initial data is investigated. First it is shown that the solutions depend continuously on the initial data with respect to various norms. The main result is on global solutions: A global solution whose electromagnetic field decays in a certain way for large times is shown to remain global under small perturbations of the initial data and to retain the decay behaviour of the field. Therefore, such global solutions are generic. This result implies the existence of global solutions for nearly symmetric initial data.

1. Introduction

Consider a collisionless plasma with N different species of particles, where a particle of species α has rest mass m_α and charge e_α. Each species is described by a particle density $f_\alpha(t,x,v)$, where $t \geqq 0$ denotes time, $x \in \mathbb{R}^3$ position, and $v \in \mathbb{R}^3$ momentum. The particles may move at relativistic speeds and are assumed to interact only by the electromagnetic forces they create themselves so that the density functions $(f_\alpha)_{\alpha=1}^N = f$ together with the selfconsistent electromagnetic fields E_f and B_f evolve according to the relativistic Vlasov–Maxwell system (RVM):

$$\partial_t f_\alpha + \hat{v}_\alpha \cdot \partial_x f_\alpha + e_\alpha(E_f + \hat{v}_\alpha \times B_f) \cdot \partial_v f_\alpha = 0, \qquad 1 \leqq \alpha \leqq N,$$
$$\partial_t E_f - \operatorname{curl} B_f = -4\pi j_f, \quad \operatorname{div} E_f = 4\pi \rho_f,$$
$$\partial_t B_f + \operatorname{curl} E_f = 0, \qquad \operatorname{div} B_f = 0.$$

Here

$$\rho_f(t,x) := \sum_{\alpha=1}^N e_\alpha \int f_\alpha(t,x,v)\,dv$$

and

$$j_f(t,x) := \sum_{\alpha=1}^N e_\alpha \int \hat{v}_\alpha f_\alpha(t,x,v)\,dv$$

denote the total charge and current densities, and

$$\hat{v}_\alpha := \frac{v}{\sqrt{m_\alpha^2 + v^2}},$$

is the relativistic speed of a particle of species α with momentum v where the speed of light is assumed to be 1.

We are interested in the corresponding initial value problem; that is, we impose initial conditions

$$f(0) = \mathring{f}, \quad E_f(0) = \mathring{E}_f, \quad B_f(0) = \mathring{B}_f,$$

where the data have to satisfy the compatibility conditions

$$\operatorname{div} \mathring{E}_f = 4\pi \mathring{\rho}_f, \quad \operatorname{div} \mathring{B}_f = 0.$$

Throughout this paper solutions are always classical solutions; that is, f, E_f, and B_f are C^1 with respect to all variables and satisfy the equations in the classical sense.

The purpose of this paper is to investigate the behaviour of classical solutions of RVM under small perturbations of the initial data. First we consider local solutions, obtaining results on continuous dependence on the initial data in various norms. Then we consider a class of global solutions exhibiting a certain asymptotic behaviour of the fields for large times. It is shown that the asymptotic behaviour of these solutions is stable under small perturbations of the initial data; that is, the perturbed solution remains global and retains its asymptotic behaviour. This general result applies to spherically symmetric solutions implying global existence for nearly symmetric data and extending Schaeffer's results from the case of the Vlasov–Poisson system to the relativistic Vlasov–Maxwell system, cf. [14].

Before going into more detail a brief survey of the known results on RVM may be useful. Glassey and Strauss [6] showed that a local solution is actually global, if the momenta remain bounded on its interval of existence. By establishing the required bound on the momenta, Glassey, Schaeffer, and Strauss proved global existence for small data [8] and for nearly neutral data [5]. For further results on RVM and related problems see the references.

If $B_f = 0$ and Maxwell's equations are replaced by Poisson's equation for the potential of E_f, the resulting system is known as the relativistic Vlasov–Poisson system RVP; by further replacing $\hat{v}_\alpha$ by v one obtains the Vlasov–Poisson system VP. For results on VP and RVP see the references. Schaeffer [15] proved that as the speed of light tends to infinity, the corresponding solutions of RVM tend to solutions of VP.

With some minor modifications the results of the present paper also hold for VP and RVP, while the proofs are greatly simplified by the fact that Poisson's equation is elliptic and much easier to analyse than the hyperbolic Maxwell system.

The paper is organized as follows: In the next section we state and briefly discuss the main results. For easier reference we collect a few known results on RVM in the third section. In the fourth section local perturbation is investigated, while the fifth section is devoted to global perturbation.

2. Main Results

For easier reference, we now state and briefly discuss our main results, postponing the proofs to later sections. To this end we need some notation. For an interval $I \subset \mathbb{R}$ define

$$C^+(I) := \{C: I \to]0, \infty[\mid C \text{ continuous and increasing}\},$$
$$C^-(I) := \{C: I \to [0, \infty[\mid C \text{ continuous and decreasing}\}.$$

For a solution (f, E_f, B_f) define $K_f := (E_f, B_f)$. Initial data will be taken from the following class:

$$\begin{aligned}\mathscr{D} = \mathscr{D}(R_0, U_0, K_0) := \{(\mathring{f}, \mathring{K}_f) \in C_c^1(\mathbb{R}^6, \mathbb{R}^N) \times C^2(\mathbb{R}^3, \mathbb{R}^6) \mid \mathring{f} \geqq 0,\\ \operatorname{supp} \mathring{f} \subset B_{R_0}(0) \times B_{U_0}(0), \operatorname{div} \mathring{E}_f = 4\pi \mathring{\rho}_f, \operatorname{div} \mathring{B}_f = 0,\\ |\partial_x^i \mathring{K}_f(x)| \leqq K_0(1 + R_0 + |x|)^{-2-i}, i = 0, 1, 2, x \in \mathbb{R}^3\},\end{aligned}$$

where $R_0 > 0, U_0 > 0$, and $K_0 > 0$, and $B_r(z) := \{y \in \mathbb{R}^3 \mid |y - z| < r\}$.

It follows from [6] that for each $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ there exists a unique, classical solution (f, K_f) of the corresponding initial value problem on a maximal interval of existence which we denote by $[0, T(f, K_f)[$.

Consider $(\mathring{g}, \mathring{K}_g) \in \mathscr{D}$. In order to study the behaviour of the solution (g, K_g) corresponding to these initial data under perturbation of the data, define

$$d_1 := \|\mathring{f} - \mathring{g}\|_\infty + \|\mathring{K}_f - \mathring{K}_g\|_{1,\infty}$$

for $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$. Here and in the following $\|\cdot\|_{k,\infty}$ denotes the infinity norm of the derivatives of the argument up to the order k.

Theorem 1. *There exist a constant $\varepsilon_1 > 0$, a function $\sigma_1 \in C^-(]0, \varepsilon_1[)$ with $\lim_{\beta \to 0} \sigma_1(\beta) = T(g, K_g)$, and a function $\zeta_1 \in C^+([0, T(g, K_g)[)$ such that for all initial data $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_1 < \varepsilon_1$ the corresponding solution (f, K_f) satisfies*

$$T(f, K_f) > \sigma_1(d_1)$$

and

$$\|f(t) - g(t)\|_\infty + \|K_f(t) - K_g(t)\|_\infty \leqq \zeta_1(t) d_1, \quad t \in [0, \sigma_1(d_1)].$$

In order to prove the global perturbation result we need continuous dependence also with respect to the first derivatives in x and v. To this end define

$$d_2 := \|\mathring{f} - \mathring{g}\|_{1,\infty} + \|\mathring{K}_f - \mathring{K}_g\|_{2,\infty}$$

for $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ and consider the following regularity assumption on the solution (g, K_g):

$$\text{(R)} \begin{cases} \text{The mapping } [0, T(g, K_g)[\ni t \mapsto g(t) \in C_c^2(\mathbb{R}^6) \\ \text{is well defined and continuous with respect to } \|\cdot\|_{2,\infty}. \end{cases}$$

Theorem 2. *Assume that (g, K_g) satisfies condition* (R). *Then there exist a constant $\varepsilon_2 \in]0, \varepsilon_1]$, a function $\sigma_2 \in C^-(]0, \varepsilon_2[)$ with $\lim_{\beta \to 0} \sigma_2(\beta) = T(g, K_g)$, and a function $\zeta_2 \in C^+([0, T(g, K_g)[)$ such that for all initial data $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_2$ the* *corresponding solution* (f, K_f) *satisfies*

$$T(f, K_f) > \sigma_1(d_1) \geqq \sigma_2(d_2)$$

and

$$\| f(t) - g(t) \|_{1,\infty} + \| K_f(t) - K_g(t) \|_{1,\infty} \leqq \zeta_2(t) d_2, \quad t \in [0, \sigma_2(d_2)].$$

It follows from [16] that condition (R) is satisfied if the initial data $(\mathring{g}, \mathring{K}_g)$ are smooth enough. Note that, if the unperturbed solution (g, K_g) is global, the perturbed solution (f, K_f) exists for arbitrarily long times by the above theorems whenever d_1 or d_2 are small enough. However, these theorems do not imply that the perturbed solution is global, too. The perturbation of global solutions is investigated under the assumption that the field K_g satisfies the following decay condition:

$$\text{(D)} \quad \begin{cases} T(g, K_g) = \infty, \text{ and there are constants } K_1 \geqq 0 \text{ and } \alpha_1 > 1/2, \alpha_2 \geqq 0 \\ \text{with } \alpha_1 + \alpha_2 > 1 \text{ such that} \\ |K_g(t,x)| \leqq K_1 (1+t)^{-\alpha_1} (1 + R_0 + t - |x|)^{-\alpha_2}, \\ |\partial_x K_g(t,x)| \leqq K_1 (1+t)^{-\alpha_1} (1 + R_0 + t - |x|)^{-\alpha_2 - 1} \\ \text{for } t \geqq 0 \text{ and } |x| \leqq R_0 + t. \end{cases}$$

The following theorem is the main result of the present paper.

Theorem 3. *Assume that* (g, K_g) *satisfies conditions* (R) *and* (D). *Then there exist constants* $\varepsilon_3 > 0$ *and* $C > 0$ *such that for all initial data* $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ *with* $d_2 < \varepsilon_3$ *the corresponding solution* (f, K_f) *is global and satisfies*

$$|K_f(t,x)| \leqq C(1 + R_0 + t + |x|)^{-1} (1 + R_0 + t - |x|)^{-1},$$
$$|\partial_x K_f(t,x)| \leqq C(1 + R_0 + t + |x|)^{-1} (1 + R_0 + t - |x|)^{-7/4}$$

for $t \geqq 0$ *and* $|x| \leqq R_0 + t$.

Note that the perturbed, global solution (f, K_f) obtained in Theorem 3 satisfies condition (D) with $\alpha_1 = 1$ and $\alpha_2 = 3/4$, which may be a stronger estimate than the initial assumption on the unperturbed solution (g, K_g). Horst [10] suggested the estimate

$$\| K_g(t) \|_\infty \leqq p(t), \quad \| \partial_x K_g(t) \|_\infty \leqq q(t), \quad t \geqq 0$$

with

$$\int_0^\infty (t q(t) + p(t)) dt < \infty$$

instead of condition (D). While this might be sufficient for global existence of the perturbed solution, the perturbed solution will in general not exhibit the same sort of decay, as may be seen by perturbing off the trivial solution $g = K_g = 0$ with $\mathring{f} = 0$ and observing that a solution of the homogeneous Maxwell system in general decays only like t^{-1} with respect to the infinity norm on $\mathbb{R}^3$. It is the key idea of the proof of Theorem 3 that the term $(1 + R_0 + t - |x|)^{-\alpha_2}$ introduces an additional decay of the fields, but only well inside the light cone; that is, for $|x| \leqq R_0 + Ct$ with $0 < C < 1$. For these reasons condition (D) seems to be more natural than Horst's suggestion.

Having proved Theorem 3 we will also establish a global estimate for the deviation of (f, K_f) from (g, K_g) at the end of Sect. 5.

Finally, it remains to be seen that there exist nontrivial solutions (g, K_g) satisfying conditions (R) and (D). As an example consider spherically symmetric solutions. For $(x,v)\in\mathbb{R}^6$ with $x \neq 0$ define

$$r := |x|, \quad w := \frac{x\cdot v}{|x|}, \quad F := x^2v^2 - (x\cdot v)^2 = |x \times v|^2$$

and call $\mathring{g}$ spherically symmetric if

$$\mathring{g}(x,v) = \mathring{g}(r, w, F) \quad \text{for} \quad x \neq 0.$$

If all particles have charges of the same sign, the forces are repulsive, the plasma disperses and the fields decay. In the other case this may be achieved by assuming that initially all particles move outward fast enough to escape to infinity:

$$\text{(E)} \quad \begin{cases} \text{There is a constant } \delta > 0 \text{ such that } w > \delta + \dfrac{|e_\alpha| M}{\delta r}\sqrt{m_\alpha^2 + \dfrac{F}{r^2} + \delta^2} \\ \text{for } (x,v)\in \operatorname{supp} \mathring{g}_\alpha \text{ with } x \neq 0, 1 \leqq \alpha \leqq N, \end{cases}$$

where

$$M := \sum_{\alpha=1}^{N} |e_\alpha| \iint \mathring{g}_\alpha(x,v)\,dv\,dx.$$

Theorem 4. *Assume that* $0 \leqq \mathring{g} \in C_c^2(\mathbb{R}^6)$ *is spherically symmetric and satisfies condition* (E) *if there are particles with charges of different signs. Define*

$$\mathring{E}_g(x) := \int \frac{x-y}{|x-y|^3}\mathring{\rho}_g(y)\,dy, \quad \mathring{B}_g := 0$$

and choose $R_0 > 0$, $U_0 > 0$, *and* $K_0 > 0$ *such that* $(\mathring{g}, \mathring{K}_g) \in \mathscr{D}(R_0, U_0, K_0)$. *Then there exist constants* $\varepsilon > 0$ *and* $C > 0$ *such that for all initial data* $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ *with* $d_2 < \varepsilon$ *the corresponding solution* (f, K_f) *is global and satisfies*

$$|K_f(t,x)| \leqq C(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-1},$$
$$|\partial_x K_f(t,x)| \leqq C(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-7/4}$$

for $t \geqq 0$ *and* $|x| \leqq R_0 + t$.

This result is analogous to the one obtained in [14] for VP and extends the class of globally solvable initial data for RVM to not necessarily small or nearly neutral data. In order to prove Theorem 4, conditions (R) and (D) have to be verified. Since in the spherically symmetric case RVM reduces to RVP, this belongs to the investigation of RVP rather than RVM. Thus, the proofs are not included here, and the reader is referred to [10] and [12].

Remark. Besides greater generality our reason for considering N different particle species lies in the fact that it may well make a difference whether there are particles with charges of different signs or not when checking condition (D), cf. Theorem 4. However, in the proofs of the other theorems it only makes the notation cumbersome to assume more than one particle species, but poses no additional difficulties. Thus, from now on we assume $N=1$, $m_1=e_1=1$ and drop the subscript in $\hat{v}_1, f_1$ etc.

3. Preliminary Results on RVM

The following lemma is a reformulation of results due to Glassey and Strauss and an immediate consequence of [6], Proposition 8.

Lemma 3.1. *Given* $(\mathring{f}, \mathring{K}_f)\in\mathscr{D}$ *there exists a unique, classical solution* (f, K_f) *of* RVM *with initial data* $(\mathring{f}, \mathring{K}_f)$ *on a maximal interval of existence* $[0, T(f, K_f)[$. *If*

$$\sup\{|v|\,|(x,v)\in\operatorname{supp} f(t), 0\leqq t< T(f,K_f)\}<\infty,$$

then the solution is global; that is, $T(f, K_f)=\infty$.

Next we recall some well known properties of the characteristics of RVM.

Lemma 3.2. *The characteristic system*

$$\dot{x}=\hat{v},$$
$$\dot{v}=E_f(t,x)+\hat{v}\times B_f(t,x)$$

has a unique solution $(X_f(\cdot,t,x,v), V_f(\cdot,t,x,v))$ *on the interval* $[0, T(f,K_f)[$ *with*

$$X_f(t,t,x,v)=x, \quad V_f(t,t,x,v)=v,$$

where $t\in[0,T(f,K_f)[$ *and* $(x,v)\in\mathbb{R}^6$. *For* $s,t\in[0,T(f,K_f)[$ *the mapping*

$$\mathbb{R}^6\ni(x,v)\mapsto(X_f(s,t,x,v), V_f(s,t,x,v))\in\mathbb{R}^6$$

is a measure preserving C^1*-diffeomorphism. Furthermore,*

$$\frac{d}{ds}f(s, X_f(s,t,x,v), V_f(s,t,x,v))=0, \quad s,t\in[0,T(f,K_f)[, \quad (x,v)\in\mathbb{R}^6,$$

and $\operatorname{supp} f(t)\subset B_{R_0+t}(0)\times\mathbb{R}^3$ *is compact for* $t\in[0,T(f,K_f)[$.

The following integral representation of the electromagnetic fields, due to Glassey and Strauss, is a key ingredient in our arguments.

Lemma 3.3. *There exist functions*

$$k_T, k_S\in C(\{\omega\in\mathbb{R}^3\,|\,|\omega|=1\}\times\mathbb{R}^3)$$

with

$$|k_T(\omega,v)|, |k_S(\omega,v)|\leqq C\sqrt{1+v^2}, \quad |\omega|=1, \quad v\in\mathbb{R}^3,$$

such that for each solution (f, K_f) *with initial data* $(\mathring{f}, \mathring{K}_f)\in\mathscr{D}$ *the following representation holds:*

$$E_f=E_{f,D}+E_{f,T}+E_{f,S},$$

where

$$E_{f,D}(t,x):=\frac{1}{4\pi t}\int_{|x-y|=t}\operatorname{curl}\mathring{B}_f(y)\,dS_y$$

$$+\frac{1}{4\pi t^2}\int_{|x-y|=t}(\mathring{E}_f(y)+\partial_x\mathring{E}_f(y)\cdot(y-x))dS_y$$

$$-\frac{1}{t}\int_{|x-y|=t}\int\frac{\omega+\hat{v}}{1+\omega\cdot\hat{v}}\mathring{f}(y,v)dvdS_y,$$

$$E_{f,T}(t,x):=\int_{|x-y|\leqq t}\int k_T(\omega,v)f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^2},$$

$$E_{f,S}(t,x):=\int_{|x-y|\leqq t}\int k_S(\omega,v)(fL_f)(t-|x-y|,y,v)dv\frac{dy}{|x-y|},$$

and

$$\omega:=\frac{y-x}{|y-x|},\quad L_f(t,x,v):=E_f(t,x)+\hat{v}\times B_f(t,x).$$

The representation for B_f is completely analogous with kernels having the same properties as the ones for E_f.

Proof. These formulas are given in [6], Theorem 3 with

$$k_T(\omega,v):=-\frac{1-\hat{v}^2}{(1+\omega\cdot\hat{v})^2}(\omega+\hat{v}),\quad k_S(\omega,v):=-\partial_v\left(\frac{\omega+\hat{v}}{1+\omega\cdot\hat{v}}\right).$$

For the estimates of the kernels cf. [7], p. 48 f., the explicit form of the data term which is not given in [6] may be verified by going through the proof. □

We will also need a representation for the derivatives of the fields, which is again due to Glassey and Strauss, cf. [6], Theorem 4.

Lemma 3.4. *There exist functions*

$$k_{TT},k_{TS},k_{SS}\in C(\{|\omega|=1\}\times\mathbb{R}^3)$$

with

$$\partial_v k_{SS}\in C(\{|\omega|=1\}\times\mathbb{R}^3),$$

$$|k_{TT}(\omega,v)|,|k_{TS}(\omega,v)|,|k_{SS}(\omega,v)|,|\partial_v k_{SS}(\omega,v)|\leqq C(1+\omega\cdot\hat{v})^{-4},\quad|\omega|=1,\quad v\in\mathbb{R}^3$$

and

$$\int_{|\omega|=1}k_{TT}(\omega,v)dS_\omega=0,\quad v\in\mathbb{R}^3,$$

such that for each solution (f,K_f) with initial data $(\mathring{f},\mathring{K}_f)\in\mathcal{D}$ and $k=1,2,3$, the following representation holds:

$$\partial_{x_k}E_f=E_{f,DD}+E_{f,TT}+E_{f,TS}+E_{f,SS}+E_{f,R},$$

where

$$E_{f,DD}(t,x):=\partial_{x_k}E_{f,D}(t,x)$$

$$+\frac{1}{t^2}\int_{|x-y|=t}\int\frac{\omega_k(\omega+\hat{v})}{(1+v^2)(1+\omega\cdot\hat{v})^3}\mathring{f}(y,v)dvdS_y$$

$$-\frac{1}{t}\int_{|x-y|=t}\int\frac{\omega_k}{1+\omega\cdot\hat{v}}\partial_v\left(\frac{\omega+\hat{v}}{1+\omega\cdot\hat{v}}\right)(\mathring{f}\mathring{L}_f)(y,v)dvdS_y,$$

$$E_{f,TT}(t,x):=\oint_{|x-y|\leq t}\int k_{TT}(\omega,v)f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^3},$$

$$E_{f,TS}(t,x):=\int_{|x-y|\leq t}\int k_{TS}(\omega,v)(fL_f)(t-|x-y|,y,v)dv\frac{dy}{|x-y|^2},$$

$$E_{f,SS}(t,x):=\int_{|x-y|\leq t}\int k_{SS}(\omega,v)S(fL_f)(t-|x-y|,y,v)dv\frac{dy}{|x-y|},$$

$$E_{f,R}(t,x):=\int\frac{1}{1+v^2}\int_{|\omega|=1}\frac{\omega+\hat{v}}{(1+\omega\cdot\hat{v})^3}\omega_k dS_\omega f(t,x,v)dv.$$

Here

$$\oint_{|x-y|\leq t}:=\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x-y|\leq t}$$

denotes the Cauchy principal value of the integral and

$$S:=\partial_t+\hat{v}\cdot\partial_x.$$

The derivatives of B_f can be represented in a completely analogous way with kernels having the same properties as the ones for E_f.

Corollary 3.5. *For a maximal solution (f, K_f) of* RVM *with initial data from $\mathscr{D}$ the mappings*

$$[0,T(f,K_f)[\ni t\mapsto f(t)\in C_c^1(\mathbb{R}^6,\mathbb{R}^N),$$

and

$$[0,T(f,K_f)[\ni t\mapsto K_f(t)\in C_b^1(\mathbb{R}^3,\mathbb{R}^6)$$

are well defined and continuous with respect to the norm $\|\cdot\|_{1,\infty}$.

4. Local Results

In order to prove Theorem 1, we have to introduce a few definitions and collect some auxiliary lemmas first. Throughout this section assume that $(\mathring{f},\mathring{K}_f)\in\mathscr{D}$ with $d_1<1$. Any constant or function $C\in C^+([0,T(g,K_g)[)$ may depend on the unperturbed solution (g,K_g), but not on the perturbed solution (f,K_f), and may change from line to line. Define

$$T_0(f,K_f):=\sup\{t\in[0,\min\{T(f,K_f),T(g,K_g)\}[\,|\,\|K_f(s)-K_g(s)\|_\infty\leq(1+s)^{-2}, s\in[0,t]\}$$

and

$$U(t):=U_0+1+\int_0^t\|K_g(s)\|_\infty ds,\quad t\in[0,T(g,K_g)[.$$

Note that Lemma 3.5 and the assumption $d_1<1$ imply $T_0(f,K_f)>0$ and that $U\in C^+([0,T(g,K_g)[)$. On the interval $[0,T_0(f,K_f)[$ we may now estimate the momenta as follows:

Lemma 4.1. *For all initial data $(\mathring{f},\mathring{K}_f)\in\mathscr{D}$ and $t\in[0,T_0(f,K_f)[$*

$$\sup\{|v|\,|\,(x,v)\in\operatorname{supp}f(t)\}\leq U(t).$$

Proof. The proof is immediate, since

$$\operatorname{supp} f(t) = \{(X_f(t,0,x,v), V_f(t,0,x,v)) \mid (x,v) \in \operatorname{supp} \mathring{f}\}$$

and

$$|\dot{V}_f(t,0,x,v)| \leqq \|K_f(t)\|_\infty \leqq \|K_g(t)\|_\infty + (1+t)^{-2}, \quad t \in [0, T_0(f, K_f)[. \quad \square$$

Using Lemma 3.3 and Lemma 4.1 we obtain an estimate for the difference of the fields on the interval $[0, T_0(f, K_f)[$:

Lemma 4.2. *There exists a function* $C \in C^+([0, T(g, K_g)[)$ *such that for all initial data* $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ *with* $d_1 < 1$ *and* $t \in [0, T_0(f, K_f)[$ *the following estimate holds*:

$$\|K_f(t) - K_g(t)\|_\infty \leqq C(t)d_1 + C(t)U(t)^4 \cdot \left(\sup_{0 \leqq \tau \leqq t} \|f(\tau) - g(\tau)\|_\infty + \int_0^t \|K_f(\tau) - K_g(\tau)\|_\infty d\tau \right).$$

Remark. The reason for making the dependence on U explicit in the above estimate will become apparent in the proof of Theorem 1.

Proof. Consider the integral representation for E_f and E_g given in Lemma 3.3. Obviously

$$\|E_{f,D}(t) - E_{g,D}(t)\|_\infty \leqq C(t)d_1.$$

By Lemma 4.1, it suffices to integrate over v with $|v| \leqq U(t - |x-y|) \leqq U(t)$ in the formulas for $E_{f,T}, E_{f,S}, E_{g,T}$, and $E_{g,S}$. Thus, we may estimate

$$|k_T(\omega, v)|, |k_S(\omega, v)| \leqq CU(t)$$

to obtain

$$\|E_{f,T}(t) - E_{g,T}(t)\|_\infty \leqq C(t)U(t)^4 \sup_{0 \leqq \tau \leqq t} \|f(\tau) - g(\tau)\|_\infty$$

and

$$|E_{f,S}(t,x) - E_{g,S}(t,x)| \leqq CU(t) \int_{|x-y| \leqq t} \int_{|v| \leqq U(t)} |fL_f - gL_g|(t - |x-y|, y, v) dv \frac{dy}{|x-y|}.$$

Putting

$$|fL_f - gL_g| \leqq |f||K_f - K_g| + |K_g||f - g|$$

and

$$|f| \leqq \|\mathring{f}\|_\infty \leqq \|\mathring{g}\|_\infty + 1$$

into the last estimate yields

$$\|E_{f,S}(t) - E_{g,S}(t)\|_\infty \leqq C(t)U(t)^4 \left(\sup_{0 \leqq \tau \leqq t} \|f(\tau) - g(\tau)\|_\infty + \int_0^t \|K_f(\tau) - K_g(\tau)\|_\infty d\tau \right).$$

Analogous estimates for the difference $B_f - B_g$ complete the proof. $\square$

Proof of Theorem 1. Let $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ satisfy $d_1 < 1$. For $0 \leqq s \leqq t < \min\{T(f, K_f),$

$T(g,K_g)\}$ and $(x,v)\in\mathbb{R}^6$ we have

$$\frac{d}{ds}((f-g)(s,X_f(s,t,x,v),V_f(s,t,x,v)))$$
$$= -(\partial_v g\cdot(L_f - L_g))(s,X_f(s,t,x,v),V_f(s,t,x,v)).$$

Integrating this equation from 0 to t and estimating, we obtain

$$(f-g)(t,x,v) = (\mathring{f}-\mathring{g})(X_f(0,t,x,v),V_f(0,t,x,v)) - \int_0^t (\partial_v g\cdot(L_f-L_g))(s,X_f(s,t,x,v),V_f(s,t,x,v))ds \tag{4.1}$$

and

$$\|f(t)-g(t)\|_\infty \leqq \|\mathring{f}-\mathring{g}\|_\infty + \int_0^t \|\partial_v g(s)\|_\infty \|K_f(s)-K_g(s)\|_\infty ds. \tag{4.2}$$

For $t\in[0,T_0(f,K_f)[$ Lemma 4.2 and the estimate (4.2) imply

$$\|K_f(t)-K_g(t)\|_\infty \leqq C(t)U(t)^4\left(d_1 + \int_0^t \|K_f(\tau)-K_g(\tau)\|_\infty d\tau\right).$$

Since the functions C and U are increasing, we have for $0\leqq t\leqq t' < T_0(f,K_f)$,

$$\|K_f(t)-K_g(t)\|_\infty \leqq C(t')U(t')^4\left(d_1 + \int_0^t \|K_f(\tau)-K_g(\tau)\|_\infty d\tau\right).$$

Now apply Gronwall's lemma, set $t'=t$, and obtain

$$\|K_f(t)-K_g(t)\|_\infty \leqq (1+t)^{-2}\xi_1(t)d_1, \quad t\in[0,T_0(f,K_f)[, \tag{4.3}$$

where $\xi_1\in C^+([0,T(g,K_g)[)$ is defined by

$$\xi_1(t) := (1+t)^2 C(t)U(t)^4 \exp(tC(t)U(t)^4).$$

Obviously ξ_1 is strictly increasing, and $\lim_{t\to T(g,K_g)} \xi_1(t) = \infty$. If $T(g,K_g)=\infty$, this follows from $\lim_{t\to\infty}(1+t)^2=\infty$, if $T(g,K_g)<\infty$, then Lemma 3.1, applied to the solution (g,K_g), implies $\lim_{t\to T(g,K_g)} U(t)=\infty$. Define

$$\varepsilon_1 := \min\left\{1, \frac{1}{2\xi_1(0)}\right\}, \quad \sigma_1(\beta) := (\xi_1)^{-1}\left(\frac{1}{2\beta}\right) \quad \text{for} \quad 0<\beta<\varepsilon_1,$$

to obtain

$$\sigma_1\in C^-(]0,\varepsilon_1[), \quad \lim_{\beta\to 0}\sigma_1(\beta) = T(g,K_g).$$

If $d_1<\varepsilon_1$ then $\sigma_1(d_1)>0$, and on the interval $[0,\min\{\sigma_1(d_1),T_0(f,K_f)\}[$ the estimate (4.3) implies

$$\|K_f(t)-K_g(t)\|_\infty < \tfrac{1}{2}(1+t)^{-2}. \tag{4.4}$$

Assume that $T(f,K_f)\leqq\sigma_1(d_1)$. This entails

$$T_0(f,K_f) = \min\{T(g,K_g),T(f,K_f)\} = T(f,K_f),$$

so we may apply Lemma 4.1 on the whole interval $[0, T(f, K_f)[$, obtaining

$$\sup\{|v|\,|(x,v)\in \sup f(t)\} \leqq U(t) \leqq U(T(f,K_f)) < \infty$$

in contradiction to Lemma 3.1. Thus we have shown that $T(f, K_f) > \sigma_1(d_1)$, by (4.4) this yields $T_0(f, K_f) > \sigma_1(d_1)$, and by (4.3) we finally get

$$\| K_f(t) - K_g(t) \|_\infty \leqq (1+t)^{-2}\zeta_1(t) d_1, t\in[0, \sigma_1(d_1)].$$

Observing (4.2) completes the proof. □

In addition to Theorem 1 the above proof also established the following result:

Corollary 4.3. *Assume that* $(\mathring{f}, \mathring{K}_f)\in\mathscr{D}$ *with* $d_1 < \varepsilon_1$ *and* $t\in[0,\sigma_1(d_1)]$. *Then*

$$\| K_f(t) - K_g(t) \|_\infty \leqq (1+t)^{-2}$$

and

$$\sup\{|v|\,|(x,v)\in \sup f(t)\} \leqq U(t).$$

We now turn to the proof of Theorem 2. The following lemma gives an estimate for the differences of the spatial derivatives of the fields similar to Lemma 4.2.

Lemma 4.4. *There exists a function* $C\in C^+([0, T(g, K_g)[)$ *such that for all initial data* $(\mathring{f}, \mathring{K}_f)\in\mathscr{D}$ *with* $d_1 < \varepsilon_1$ *and* $t\in[0,\sigma_1(d_1)]$ *the following estimate holds*:

$$\| \partial_x K_f(t) - \partial_x K_g(t) \|_\infty \leqq C(t) U(t)^4 \Bigg(d_2 + \sup_{0\leqq\tau\leqq t} \| \partial_x f(\tau) - \partial_x g(\tau) \|_\infty + \int_0^t \| \partial_x K_f(\tau) - \partial_x K_g(\tau) \|_\infty d\tau \Bigg).$$

Proof. First note that it would be possible, but unnecessarily complicated to use Lemma 3.4 at this point. Instead, we use Lemma 3.3 and differentiate under the integral sign. Since the proof is then similar to the proof of Lemma 4.2 we only treat the difference $\partial_x E_{f,S} - \partial_x E_{g,S}$. With

$$\partial_x E_{f,S}(t,x) = \int_{|x-y|\leqq t} \int_{|v|\leqq U(t)} k_S(\omega, v) \partial_x (f L_f)(t - |x-y|, y, v) dv \frac{dy}{|x-y|},$$

the corresponding expression for $\partial_x E_{g,S}$ and Lemma 4.1 we obtain the estimate

$$\begin{aligned} &|\partial_x E_{f,S}(t,x) - \partial_x E_{g,S}(t,x)| \\ &\leqq C U(t) \int_{|x-y|\leqq t} \int_{|v|\leqq U(t)} |\partial_x (f L_f) - \partial_x (g L_g)|(t - |x-y|, y, v) dv \frac{dy}{|x-y|} \end{aligned}$$

for $t\in[0,\sigma_1(d_1)]$ and $x\in\mathbb{R}^3$. Putting

$$\begin{aligned} |\partial_x(fL_f) - \partial_x(gL_g)| &\leqq |\partial_x f - \partial_x g|\,|K_f| + |\partial_x g|\,|K_f - K_g| \\ &\quad + |f|\,|\partial_x K_f - \partial_x K_g| + |f-g|\,|\partial_x K_g|, \\ \|f(\tau)\|_\infty &= \|\mathring{f}\|_\infty \leqq \|\mathring{g}\|_\infty + 1, \\ \|K_f(\tau)\|_\infty &\leqq \|K_g(\tau)\|_\infty + \|K_f(\tau) - K_g(\tau)\|_\infty \\ &\leqq \|K_g(\tau)\|_\infty + \zeta_1(\tau) d_1 \leqq C(\tau) \end{aligned}$$

and

$$\| f(\tau)-g(\tau)\|_\infty+\| K_f(\tau)-K_g(\tau)\|_\infty \leqq \zeta_1(\tau)d_1, \quad \tau\in[0,\sigma_1(d_1)]$$

into the estimate for $\partial_x E_{f,S}-\partial_x E_{g,S}$, we end up with

$$\begin{aligned}&\|\partial_x E_{f,S}(t)-\partial_x E_{g,S}(t)\|_\infty\\ &\leqq C(t)U(t)^4\left(d_1+\sup_{0\leqq\tau\leqq t}\|\partial_x f(\tau)-\partial_x g(\tau)\|_\infty+\int_0^t\|\partial_x K_f(\tau)-\partial_x K_g(\tau)\|_\infty d\tau\right).\end{aligned}$$

The remaining estimates are similar and therefore omitted. □

In order to prove Theorem 2 we need to estimate the derivatives of the characteristics of f with respect to x and v.

Lemma 4.5. *For $(\mathring{f},\mathring{K}_f)\in\mathscr{D}$, $0\leqq s\leqq t<T(f,K_f)$, and $(x,v)\in\mathbb{R}^6$ the following estimates hold:*

$$|\partial_x X_f(s,t,x,v)|+|\partial_x V_f(s,t,x,v)|\leqq\exp\left(\int_s^t(3+3\|B_f(\tau)\|_\infty+\|\partial_x K_f(\tau)\|_\infty)d\tau\right),$$

$$|\partial_v X_f(s,t,x,v)|+|\partial_v V_f(s,t,x,v)|\leqq\exp\left(\int_s^t(3+3\|B_f(\tau)\|_\infty+\|\partial_x K_f(\tau)\|_\infty)d\tau\right).$$

Proof. Differentiating the characteristic system we obtain

$$\begin{aligned}\frac{d}{ds}\partial_{x_k}X_f(s)&=\frac{\partial_{x_k}V_f(s)}{\sqrt{1+V_f^2(s)}}-\partial_{x_k}V_f(s)\cdot V_f(s)\frac{V_f(s)}{(1+V_f^2(s))^{3/2}},\\ \frac{d}{ds}\partial_{x_k}V_f(s)&=\partial_x E_f(s,X_f(s))\cdot\partial_{x_k}X_f(s)+\frac{V_f(s)}{\sqrt{1+V_f^2(s)}}\times(\partial_x B_f(s,X_f(s))\cdot\partial_{x_k}X_f(s))\\ &\quad+\left(\frac{\partial_{x_k}V_f(s)}{\sqrt{1+V_f^2(s)}}-\frac{\partial_{x_k}V_f(s)\cdot V_f(s)V_f(s)}{(1+V_f^2(s))^{3/2}}\right)\times B_f(s,X_f(s)).\end{aligned}$$

Now some straightforward estimates and Gronwall's lemma yield the desired result. □

Proof of Theorem 2. Assume that the unperturbed solution (g,K_g) satisfies the regularity condition (R) and differentiate equation (4.1) to obtain

$$\begin{aligned}\partial_{x_k}(f-g)(t,x,v)&=\partial_x(\mathring{f}-\mathring{g})(X_f(0,t,x,v),V_f(0,t,x,v))\cdot\partial_{x_k}X_f(0,t,x,v)\\ &\quad+\partial_v(\mathring{f}-\mathring{g})(X_f(0,t,x,v),V_f(0,t,x,v))\cdot\partial_{x_k}V_f(0,t,x,v)\\ &\quad-\int_0^t(\partial_x(L_f-L_g)\cdot\partial_{x_k}X_f(s,t,x,v)\\ &\quad+\partial_{x_k}\hat{V}_f(s,t,x,v)\times(B_f-B_g))\cdot\partial_v g(s,X_f(s),V_f(s))ds\\ &\quad-\int_0^t(L_f-L_g)(\partial_x\partial_v g\cdot\partial_{x_k}X_f(s,t,x,v)\\ &\quad+\partial_v^2 g\cdot\partial_{x_k}V_f(s,t,x,v))ds.\end{aligned}\tag{4.5}$$

For $t \in [0, T(f, K_f)[$ define

$$P(t) := \sup\{|\partial_x X_f(s,t,x,v)| + |\partial_x V_f(s,t,x,v)|\,|\,0 \leqq s \leqq t,\ (x,v) \in \mathbb{R}^6\}.$$

Assuming $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_1 < \varepsilon_1$ and $t \in [0, \sigma_1(d_1)]$, Theorem 1, Eq. (4.5) and Lemma 4.4 imply

$$\|\partial_x K_f(t) - \partial_x K_g(t)\|_\infty \leqq C(t)U(t)^4(1 + P(t))\left(d_2 + \int_0^t \|\partial_x K_f(s) - \partial_x K_g(s)\|_\infty ds\right).$$

Gronwall's lemma now gives the following estimate for $t \in [0, \sigma_1(d_1)]$:

$$\|\partial_x K_f(t) - \partial_x K_g(t)\|_\infty \leqq C(t)U(t)^4(1 + P(t))\exp(tC(t)U(t)^4(1 + P(t)))d_2. \tag{4.6}$$

Thus, the function P, which depends on (f, K_f), has to be estimated independently of (f, K_f). By Lemma 4.5 and Theorem 1,

$$P(t) \leqq \exp\left(\int_0^t (3 + 3\|B_g(s)\|_s + 3\zeta_1(s)d_1 + \|\partial_x K_g(s)\|_\infty + \|\partial_x K_f(s) - \partial_x K_g(s)\|_\infty)ds\right)$$

$$\leqq \exp\left(\int_0^t (C(s) + \|\partial_x K_f(s) - \partial_x K_g(s)\|_\infty)ds\right), \quad t \in [0, \sigma_1(d_1)].$$

Since $d_2 < \varepsilon_1 \leqq 1$,

$$T_1(f, K_f) := \sup\{t \in [0, \sigma_1(d_1)]\,|\,\|\partial_x K_f(s) - \partial_x K_g(s)\|_\infty \leqq 1,\ 0 \leqq s \leqq t\} > 0,$$

and

$$P(t) \leqq \exp\left(\int_0^t (C(s) + 1)ds\right), \quad t \in [0, T_1(f, K_f)],$$

where $C \in C^+([0, T(g, K_g)[)$. Define

$$\bar{P}(t) := \exp\left(\int_0^t (C(s) + 1)ds\right), \quad t \in [0, T(g, K_g)[,$$

and

$$\xi_2(t) := C(t)U(t)^4(1 + \bar{P}(t))\exp(tC(t)U(t)^4(1 + \bar{P}(t))),$$

and observe that the function $\xi_2 \in C^+([0, T(g, K_g)[)$ is strictly increasing with $\lim_{t \to T(g, K_g)} \xi_2(t) = \infty$. For $t \in [0, T_1(f, K_f)]$ the estimate (4.6) implies

$$\|\partial_x K_f(t) - \partial_x K_g(t)\|_\infty \leqq \xi_2(t)d_2. \tag{4.7}$$

Now

$$\varepsilon_2 := \min\left\{\varepsilon_1, \frac{1}{\xi_2(0)}\right\}, \quad \sigma_2(\beta) := \min\{\sigma_1(\beta), (\xi_2)^{-1}(1/\beta)\}, \quad 0 < \beta < \varepsilon_2$$

yields

$$\sigma_2 \in C^-(]0, \varepsilon_2[), \quad \lim_{\beta \to 0} \sigma_2(\beta) = T(g, K_g).$$

Assume that $T_1(f, K_f) < \sigma_2(d_2)$ for $d_2 < \varepsilon_2$. Then the estimate (4.7) implies

$$\|\partial_x K_f(T_1(f, K_f)) - \partial_x K_g(T_1(f, K_f))\|_\infty < \xi_2(\sigma_2(d_2))d_2 = 1,$$

contradicting the definition on $T_1(f, K_f)$. Hence

$$\sigma_2(d_2) \leqq T_1(f, K_f) \leqq \sigma_1(d_1) < T(f, K_f).$$

For $t \in [0, \sigma_2(d_2)]$ we have established the estimate

$$\| \partial_x K_f(t) - \partial_x K_g(t) \|_\infty \leqq \xi_2(t) d_2,$$

putting this into (4.5) results in a corresponding estimate for $\| \partial_x f(t) - \partial_x g(t) \|_\infty$. Since $\sigma_2(d_2) \leqq \sigma_1(d_1)$, all the estimates from Theorem 1 remain valid on the interval $[0, \sigma_2(d_2)]$, so only the difference $\partial_v f(t) - \partial_v g(t)$ remains to be estimated. To this end note that on the interval $[0, \sigma_2(d_2)]$ Lemma 4.5 implies

$$Q(t) := \sup \{ | \partial_v X_f(s,t,x,v)| + |\partial_v V_f(s,t,x,v)| \,|\, 0 \leqq s \leqq t, \ (x,v) \in \mathbb{R}^6 \} \leqq \bar{P}(t).$$

This completes the proof, since the remaining estimate now follows from an equation for $\partial_{v_k}(f - g)(t,x,v)$ analogous to (4.5) and the already established estimates for $\| K_f(t) - K_g(t) \|_{1,\infty}$ and $\| f(t) - g(t) \|_\infty$. □

5. Global Results

5.1. Global Existence of the Perturbed Solution. Let us briefly describe the idea of the proof of Theorem 3. The decay condition (D) implies the existence of a time $T_1 > 0$ such that for larger times the field $K_g(t)$ and its derivative $\partial_x K_g(t)$ are so small that the solution (g, K_g) behaves almost like a free streaming. Theorem 2 implies that for d_2 small enough the perturbed solution (f, K_f) also behaves almost like a free streaming, at least on some interval I, which starts at T_1. On this interval the volume of the support of the function $f(t,x,\cdot)$ decays like t^{-3} and the momenta remain bounded. This implies estimates for the field and its derivative such that the interval I may be extended to infinity and the proof is complete.

For the rest of this section let (g, K_g) be a solution satisfying the conditions (R) and (D). Constants such as C, c_1, C_1, c_2, C_2 may depend on (g, K_g), but not on the perturbed solution (f, K_f); dependence on (f, K_f) is always explicitly noted.

A first consequence of condition (D) is the following estimate for the momenta of the perturbed solution.

Lemma 5.1. *There is a constant $c_1 > 0$ such that*

$$\sup \{ |v| \,|\, (x,v) \in \operatorname{supp} f(t) \} \leqq c_1$$

for all initial data $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_2$ and $t \in [0, \sigma_2(d_2)]$.

Proof. Take $t \in [0, \sigma(d_2)]$. Because of $\sigma_2(d_2) \leqq \sigma_1(d_1)$ and Corollary 4.3 we have

$$\| K_f(t) - K_g(t) \|_\infty \leqq (1+t)^{-2}.$$

Together with condition (D) this implies

$$|K_f(t,x)| \leqq K_1 (1+t)^{-\alpha_1} (1 + R_0 + t - |x|)^{-\alpha_2} + (1+t)^{-2}, \quad |x| \leqq R_0 + t.$$

Without loss of generality we may assume $\alpha_2 \leqq 1/2$. Now observe that

$$\operatorname{supp} f(t) = \{ (X_f(t,0,x,v), \ V_f(t,0,x,v)) \,|\, (x,v) \in \operatorname{supp} \mathring{f} \}$$

and define

$$u(t) := \sup\{|V_f(s,0,x,v)| \mid 0 \leqq s \leqq t\}, \quad \hat{u}(t) := \frac{u(t)}{\sqrt{1+u(t)^2}}$$

for $(x,v) \in \operatorname{supp} \mathring{f}$. The characteristic system implies

$$|\dot{X}_f(s,0,x,v)| = |\hat{V}_f(s,0,x,v)| \leqq \hat{u}(s),$$

whence

$$|X_f(s)| \leqq R_0 + s\hat{u}(s),$$

and

$$\begin{aligned} |\dot{V}_f(s,0,x,v)| &\leqq K_1(1+s)^{-\alpha_1}(1+R_0+s-|X_f(s)|)^{-\alpha_2} + (1+s)^{-2} \\ &\leqq K_1(1+s)^{-\alpha_1}(1+s(1-\hat{u}(s)))^{-\alpha_2} + (1+s)^{-2}. \end{aligned}$$

Integrating this inequality we obtain

$$u(t) \leqq 1 + U_0 + K_1 \int_0^t (1+s)^{-\alpha_1}(1+s(1-\hat{u}(s)))^{-\alpha_2} ds.$$

A short computation shows that

$$1 \geqq 1 - \hat{u} \geqq \frac{1}{2}\frac{1}{1+u^2},$$

and hence

$$u(t) \leqq 1 + U_0 + 2^{\alpha_2} K_1 \int_0^t (1+s)^{-\alpha}(1+u(s)^2)^{\alpha_2} ds,$$

where $\alpha := \alpha_1 + \alpha_2$. Let $z : [0, t_{\max}[\to \mathbb{R}^+$ be the maximal solution of

$$\dot{z}(t) = 2^{\alpha_2} K_1 (1+t)^{-\alpha}(1+z(t)^2)^{\alpha_2}, \quad z(0) = 2 + U_0.$$

Obviously, $u(t) < z(t)$ for $0 \leqq t < \min\{t_{\max}, \sigma_2(d_2)\}$. Thus, the lemma is established if we can show that $\sup_{s \geqq 0} z(s) < \infty$. The assumption $\alpha = \alpha_1 + \alpha_2 > 1$ implies

$$\begin{aligned} \int_0^t \frac{\dot{z}(s)}{(1+z(s)^2)^{\alpha_2}} ds &= 2^{\alpha_2} K_1 \int_0^t (1+s)^{-\alpha} ds \\ &\leqq 2^{\alpha_2} K_1 \int_0^\infty (1+s)^{-\alpha} ds < \infty. \end{aligned}$$

On the other hand,

$$\int_0^t \frac{\dot{z}(s)}{(1+z(s)^2)^{\alpha_2}} ds = \int_{z(0)}^{z(t)} \frac{dz}{(1+z^2)^{\alpha_2}},$$

whence

$$\int_{z(0)}^{\sup_{t \geqq 0} z(t)} \frac{dz}{(1+z^2)^{\alpha_2}} < \infty.$$

Because of the assumption $\alpha_2 \leqq 1/2$ this entails $\sup_{t \geqq 0} z(t) < \infty$, and the proof is complete. □

For the rest of this section assume that $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_2$ so that Theorem 2 and Lemma 5.1 apply.

The Free Streaming Condition. Let us make precise what we mean by saying that a solution (f, K_f) behaves almost like a free streaming. To this end choose $\beta \in]1/2, \min\{\alpha_1, \alpha/2, 3/4\}[$, where $\alpha = \alpha_1 + \alpha_2$.

Definition 5.2. *Let (f, K_f) be a classical solution of* RVM, $0 \leqq a < b \leqq T(f, K_f)$ *and* $\eta > 0$. *The solution (f, K_f) satisfies condition* (FS) *with respect to the constant η on the interval $[a, b[$, if the estimates*

$$|K_f(t,x)| \leqq \eta(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta},$$

$$|\partial_x K_f(t,x)| \leqq \eta(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta-1}$$

hold for $t \in [a, b[$ and *$x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$.*

For η small enough we obtain the following estimates for the support of the function $f(t)$ on the interval $[a, b[$:

Lemma 5.3. *There is a constant $\eta_1 > 0$ such that if a solution (f, K_f) satisfies condition* (FS) *with respect to a constant $\eta \leqq \eta_1$ on an interval $[a, b[\subset [0, T(f, K_f)[$ with $a \leqq \sigma_2(d_2)$, then*

$$\operatorname{supp} f(t) \subset \{(x,v) \in \mathbb{R}^6 \,|\, |x| \leqq R_0 + \hat{C}_1 t, |v| \leqq C_1\}, \quad 0 \leqq t < b,$$

where $C_1 := 2c_1$ and $\hat{C}_1 := C_1(1 + C_1^2)^{-1/2}$.

Proof. Let $\eta > 0$ be arbitrary. For $t \in [0, \sigma_2(d_2)]$ the claim follows from Lemma 5.1 with c_1 instead of C_1. For $(x, v) \in \operatorname{supp} \mathring{f}$ define

$$\tilde{t} := \sup\{t \in [a, b[\,|\, |V_f(s, 0, x, v)| \leqq C_1,\ s \in [0, t]\}.$$

The assumption $a \leqq \sigma_2(d_2)$ implies $\tilde{t} > a$, and on the interval $[a, \tilde{t}[$ we obtain

$$|X_f(t, 0, x, v)| \leqq R_0 + \int_0^t |\hat{V}_f(s, 0, x, v)| ds \leqq R_0 + \hat{C}_1 t.$$

Now (FS) implies

$$\begin{aligned}|K_f(t, X_f(t))| &\leqq \eta(1+R_0+t+|X_f(t)|)^{-\beta}(1+R_0+t-|X_f(t)|)^{-\beta}\\ &\leqq \eta(1+t)^{-\beta}(1+R_0+t-R_0-\hat{C}_1 t)^{-\beta}\\ &\leqq \eta(1-\hat{C}_1)^{-\beta}(1+t)^{-2\beta},\end{aligned}$$

and hence

$$\begin{aligned}|V_f(t,0,x,v)| &\leqq |V_f(a,0,x,v)| + \eta(1-\hat{C}_1)^{-\beta}\int_a^t (1+s)^{-2\beta} ds\\ &\leqq c_1 + \eta(1-\hat{C}_1)^{-\beta}(2\beta-1)^{-1}, \quad a \leqq t < \tilde{t}.\end{aligned}$$

Choosing $\eta_1 := \frac{1}{2}c_1(1-\hat{C}_1)^\beta(2\beta-1)$ we get $|V_f(t,0,x,v)| \leqq \frac{3}{2}c_1$ on the interval $[a, \tilde{t}[$, which by definition of $\tilde{t}$ entails $\tilde{t} = b$ and the proof is complete. □

To further exploit condition (FS) we need the following Gronwall lemma for second order differential inequalities:

Lemma 5.4. *Assume that* $c_1, c_2 \in C^-([a,\infty[)$ *with*

$$I := \int_a^\infty (\sigma c_1(\sigma) + c_2(\sigma))d\sigma < \infty,$$

$c_3, c_4 \geqq 0$ *and* $x \in C^2([a,t])$ *with* $x(t) = \dot{x}(t) = 0$ *and*

$$|\ddot{x}(s)| \leqq c_1(s)(|x(s)| + (t-s)c_3 + c_4) + c_2(s)(|\dot{x}(s)| + c_3)$$

for $0 \leqq a \leqq s \leqq t < \infty$. *Then*

$$|x(s)| \leqq ((t-s)c_3 + c_4)Ie^I, \quad a \leqq s \leqq t.$$

Proof. Obviously

$$x(s) = -\int_s^t \dot{x}(\tau)d\tau = \int_s^t \int_\tau^t \ddot{x}(\sigma)d\sigma d\tau,$$

whence

$$|x(s)| \leqq \int_s^t |\dot{x}(\tau)|d\tau =: z(s).$$

Now some straightforward estimates yield

$$z(s) \leqq \int_s^t (\sigma c_1(\sigma) + c_2(\sigma))z(\sigma)d\sigma + ((t-s)c_3 + c_4)\int_s^t (\sigma c_1(\sigma) + c_2(\sigma))d\sigma.$$

If $a \leqq s' \leqq s \leqq t$, then

$$z(s) \leqq \int_s^t (\sigma c_1(\sigma) + c_2(\sigma))z(\sigma)d\sigma + ((t-s')c_3 + c_4)\int_{s'}^t (\sigma c_1(\sigma) + c_2(\sigma))d\sigma,$$

and Gronwall's lemma completes the proof. □

The next lemma will be used to estimate the diameter and thus the volume of the support of the function $f(t,x,\cdot)$.

Lemma 5.5. *There are constants* $\eta_2 \in]0, \eta_1]$ *and* $c_2 > 0$ *such that if a solution* (f, K_f) *satisfies condition* (FS) *with respect to the constant* η_2 *on an interval* $[a,b[\subset [0, T(f,K_f)[$ *with* $a \leqq \sigma_2(d_2)$, *then*

$$|X_f(s,t,x,v) - X_f(s,t,x,v')| \geqq c_2(t-s)|v-v'|$$

for $(x,v), (x,v') \in \operatorname{supp} f(t)$ *and* $a \leqq s \leqq t < b$.

Proof. For (x,v), $(x,v') \in \operatorname{supp} f(t)$ and $a \leqq s \leqq t < b$ define

$$x(s) := X_f(s,t,x,v), \quad \hat{v}(s) := \hat{V}_f(s,t,x,v)$$

and $x'(\cdot), \hat{v}'(\cdot)$ analogously. One immediately checks that

$$\ddot{x}(s) = J_f(s, x(s), \hat{v}(s)),$$

where

$$J_f(s,x,\hat{v}) := \sqrt{1-\hat{v}^2}(E_f(s,x) + \hat{v} \times B_f(s,x) - \hat{v}\cdot E_f(s,x)\hat{v}).$$

Define

$$y(s) := x(s) - x'(s) + (t-s)(\hat{v} - \hat{v}'), \quad a \leqq s \leqq t < b$$

to obtain $y(t)=\dot{y}(t)=0$ and

$$\begin{aligned}|\ddot{y}(s)| &= |\ddot{x}(s)-\ddot{x}'(s)|\\ &= |J_f(s,x(s),\hat{v}(s))-J_f(s,x'(s),\hat{v}'(s))|\\ &\leqq \sup_{0\leqq\tau\leqq 1}|\partial_x J_f(s,\tau x(s)+(1-\tau)x'(s),\hat{v}(s))|\,|x(s)-x'(s)|\\ &\quad+\sup_{0\leqq\tau\leqq 1}|\partial_{\hat{v}} J_f(s,x'(s),\tau\hat{v}(s)+(1-\tau)\hat{v}'(s))|\,|\hat{v}(s)-\hat{v}'(s)|.\end{aligned}$$

A short computation proves

$$|\partial_{x_k}J_f(s,x,\hat{v})|\leqq 2|\partial_x K_f(s,x)|,\quad |\partial_{\hat{v}_k}J_f(s,x,\hat{v})|\leqq\frac{4}{\sqrt{1-\hat{v}^2}}|K_f(s,x)|.$$

Now take $\eta\in]0,\eta_1]$. Since $(x,v)\in\operatorname{supp} f(t)$, and hence $(x(s),v(s))\in\operatorname{supp} f(s)$, Lemma 5.3 implies

$$|x(s)|\leqq R_0+\hat{C}_1 s,\quad |v(s)|\leqq C_1,$$

for $a\leqq s\leqq t<b$, and analogous estimates hold for $x'(s)$ and $v'(s)$. Hence

$$\begin{aligned}&|\tau x(s)+(1-\tau)x'(s)|\leqq R_0+\hat{C}_1 s,\\ &|\tau\hat{v}(s)+(1-\tau)\hat{v}'(s)|\leqq\hat{C}_1,\quad 0\leqq\tau\leqq 1.\end{aligned}$$

Using condition (FS) we obtain the estimates

$$\begin{aligned}&\sup_{0\leqq\tau\leqq 1}|\partial_x J_f(s,\tau x(s)+(1-\tau)x'(s),\hat{v}(s))|\\ &\quad\leqq C\eta(1+s)^{-\beta}(1+R_0+s-R_0-\hat{C}_1 s)^{-\beta-1}\leqq C\eta(1+s)^{-2\beta-1},\\ &\sup_{0\leqq\tau\leqq 1}|\partial_{\hat{v}} J_f(s,x'(s),\tau\hat{v}(s)+(1-\tau)\hat{v}'(s))|\\ &\quad\leqq\frac{C}{\sqrt{1-\hat{C}_1^2}}\eta(1+s)^{-\beta}(1+R_0+s-R_0-\hat{C}_1 s)^{-\beta}\leqq C\eta(1+s)^{-2\beta}.\end{aligned}$$

Thus, the function y satisfies

$$\begin{aligned}|\ddot{y}(s)| &\leqq C\eta(1+s)^{-2\beta-1}|x(s)-x'(s)|+C\eta(1+s)^{-2\beta}|\hat{v}(s)-\hat{v}'(s)|\\ &\leqq C\eta(1+s)^{-2\beta-1}(|y(s)|+(t-s)|\hat{v}-\hat{v}'|)\\ &\quad+C\eta(1+s)^{-2\beta}(|\dot{y}(s)|+|\hat{v}-\hat{v}'|),\end{aligned}$$

and Lemma 5.4 implies

$$|y(s)|\leqq|\hat{v}-\hat{v}'|\eta I e^{\eta I}(t-s),\quad a\leqq s\leqq t<b,$$

where

$$I:=\int_0^\infty (C\sigma(1+\sigma)^{-2\beta-1}+C(1+\sigma)^{-2\beta})d\sigma<\infty.$$

Choose $\eta_2\in]0,\eta_1]$ such that $\eta_2 I e^{\eta_2 I}<1/2$ to obtain

$$|X_f(s,t,x,v)-X_f(s,t,x,v')+(t-s)(\hat{v}-\hat{v}')|=|y(s)|<\tfrac{1}{2}(t-s)|\hat{v}-\hat{v}'|,$$

and hence

$$|X_f(s,t,x,v) - X_f(s,t,x,v')| \geqq \tfrac{1}{2}(t-s)|\hat{v} - \hat{v}'|.$$

Since

$$\partial_{\hat{v}_k} v_j(\hat{v}) = \frac{(1-\hat{v}^2)\delta_{kj} + \hat{v}_k \hat{v}_j}{\sqrt{1-\hat{v}^2}^3},$$

Lemma 5.3 implies

$$|v - v'| \leqq \sup_{0 \leqq \tau \leqq 1} |\partial_{\hat{v}} v(\tau \hat{v} + (1-\tau)\hat{v}')| \, |\hat{v} - \hat{v}'| \leqq C |\hat{v} - \hat{v}'|$$

for $(x,v), (x,v') \in \operatorname{supp} f(t)$, and

$$|X_f(s,t,x,v) - X_f(s,t,x,v')| \geqq c_2(t-s)|v - v'|.$$

This completes the proof. □

Finally, condition (FS) implies the following estimate for the derivative of f with respect to x, which we will need to estimate $\partial_x K_f$:

Lemma 5.6. *There is a constant $c_3 > 0$ such that if a solution (f, K_f) satisfies condition* (FS) *with respect to the constant η_2 on an intervall $[a,b[\subset [0, T(f,K_f)[$ with $a \leqq \sigma_2(d_2)$, then*

$$\|\partial_x f(t)\|_\infty \leqq c_3 \|f(a)\|_{1,\infty}, \quad t \in [a,b[.$$

Proof. The equation

$$f(t,x,v) = f(a, X_f(a,t,x,v), V_f(a,t,x,v)), \quad t \geqq a$$

implies

$$\begin{aligned}\partial_x f(t,x,v) = {} & \partial_x f(a, X_f(a,t,x,v), V_f(a,t,x,v)) \partial_x X_f(a,t,x,v) \\ & + \partial_v f(a, X_f(a,t,x,v), V_f(a,t,x,v)) \partial_x V_f(a,t,x,v),\end{aligned}$$

and hence

$$|\partial_x f(t,x,v)| \leqq \|f(a)\|_{1,\infty} (|\partial_x X_f(a,t,x,v)| + |\partial_x V_f(a,t,x,v)|)$$

for $t \in [a, T(f,K_f)[$ and $(x,v) \in \mathbb{R}^6$. Now proceed similarly to the proof of Lemma 5.5 to estimate the derivatives of the characteristics with respect to x. Since

$$\ddot{X}_f(s,t,x,v) = J_f(s, X_f(s), \hat{V}_f(s)),$$

we have

$$\partial_{x_k} \ddot{X}_f(s) = \partial_x J_f(s, X_f(s), \hat{V}_f(s)) \partial_{x_k} X_f(s) + \partial_{\hat{v}} J_f(s, X_f(s), \hat{V}_f(s)) \partial_{x_k} \dot{X}_f(s).$$

From the proof of Lemma 5.5 we recall the estimates

$$|\partial_x J_f(s,x,\hat{v})| \leqq C |\partial_x K_f(s,x)|, \quad |\partial_{\hat{v}} J_f(s,x,\hat{v})| \leqq \frac{C}{\sqrt{1-\hat{v}^2}} |K_f(s,x)|.$$

For $(x,v) \in \operatorname{supp} f(t)$ and $0 \leqq s \leqq t < b$ Lemma 5.3 implies

$$|X_f(s,t,x,v)| \leqq R_0 + \hat{C}_1 s, \quad |V_f(s,t,x,v)| \leqq C_1.$$

Define

$$x(s) := \partial_x X_f(s,t,x,v) - \mathrm{id}_{\mathbb{R}^3}$$

to obtain

$$|\ddot{x}(s)| \leqq C(1+s)^{-2\beta-1}(|x(s)|+1) + C(1+s)^{-2\beta}|\dot{x}(s)|$$

and $x(t) = \dot{x}(t) = 0$. Thus, for $a \leqq s \leqq t < b$ and $(x,v) \in \operatorname{supp} f(t)$ Lemma 5.4 implies $|x(s)| \leqq C$, and hence

$$|\partial_x X_f(s,t,x,v)| \leqq C + 1.$$

To estimate $\partial_x V_f$ observe that

$$\partial_{x_k}\dot{V}_f(s) = \partial_x E_f(s, X_f(s))\partial_{x_k}X_f(s) + \partial_{x_k}\hat{V}_f(s) \times B_f(s, X_f(s)) \\ + \hat{V}_f(s) \times (\partial_x B_f(s, X_f(s))\partial_{x_k}X_f(s)),$$

and thus,

$$|\partial_x \dot{V}_f(s)| \leqq C(1+s)^{-2\beta-1} + C(1+s)^{-2\beta}|\partial_x V_f(s)|.$$

Integrating this estimate and applying Gronwall's lemma gives the desired estimate for $\partial_x V_f(s,t,x,v)$ and completes the proof. □

Estimates of the Fields. In order to estimate the data terms in Lemma 3.3 and Lemma 3.4 we need the following two technical lemmas:

Lemma 5.7. *Assume that* $h \in C(\mathbb{R}^3)$ *and* $k \in \{2,3,4\}$ *with*

$$|h(x)| \leqq K_0(1 + R_0 + |x|)^{-k}, \quad x \in \mathbb{R}^3.$$

Then

$$\left|\int_{|x-y|=t} h(y)\,dS_y\right| \\ \leqq 4\pi K_0 t \begin{cases} (1+R_0+t-|x|)^{-1}, & \text{in case} \quad k=2 \\ (1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-k+2}, & \text{in case} \quad k=3,4 \end{cases}$$

for $t \geqq 0$ *and* $x \in \mathbb{R}^3$ *with* $|x| \leqq R_0 + t$.

Proof. Obviously

$$\left|\int_{|x-y|=t} h(y)\,dS_y\right| \leqq K_0 \int_{|x-y|=t} (1+R_0+|y|)^{-k}dS_y \\ = K_0 t^2 \int_{|\omega|=1} (1+R_0+|x+t\omega|)^{-k}dS_\omega.$$

It is easily seen that

$$u(x) := \int_{|\omega|=1} (1+R_0+|x+t\omega|)^{-k}dS_\omega$$

depends on $r := |x|$ only. With abuse of notation we have

$$u(x) = u(r) := \int_{|\omega|=1} (1+R_0+|re+t\omega|)^{-k}dS_\omega,$$

where $e := (0,0,1)$, and for $r \leqq R_0 + t$ this implies

$$u(r) = \int_0^{2\pi}\int_0^{\pi} (1 + R_0 + \sqrt{r^2 + t^2 + 2rt\cos\theta})^{-k} \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \frac{1}{rt} \int_{|t-r|}^{t+r} \frac{\lambda \, d\lambda}{(1 + R_0 + \lambda)^k} \leqq 2\pi \frac{1}{rt} \int_{t-r}^{t+r} \frac{d\lambda}{(1 + R_0 + \lambda)^{k-1}},$$

using the substitution $\lambda = \sqrt{r^2 + t^2 + 2rt\cos\theta}$. Straightforward calculations now complete the proof. □

Lemma 5.8. *Assume that* $h \in C_c(\mathbb{R}^3)$ *with* $\operatorname{supp} h \subset B_{R_0}(0)$. *Then*

$$\left| \int_{|x-y|=t} h(y) \, dS_y \right| \leqq 4\pi \| h \|_\infty \min\{R_0^2, t^2\}$$

for $x \in \mathbb{R}^3$ *and* $t \geqq 0$.

Proof. Let $1_{B_{R_0}(0)}$ be the characteristic function of $B_{R_0}(0)$ and $S_t(x) := \{y \in \mathbb{R}^3 \mid |x-y| = t\}$. Obviously

$$\left| \int_{|x-y|=t} h(y) \, dS_y \right| \leqq \| h \|_\infty \int_{|x-y|=t} 1_{B_{R_0}(0)}(y) \, dS_y$$
$$= \| h \|_\infty \operatorname{vol}_2 (S_t(x) \cap B_{R_0}(0))$$
$$\leqq \| h \|_\infty \min\{\operatorname{vol}_2 S_t(x), \operatorname{vol}_2 S_{R_0}(0)\}$$
$$= 4\pi \| h \|_\infty \min\{t^2, R_0^2\}. \quad \square$$

We are now going to estimate the fields of a solution under the condition that the momenta remain bounded and the volume of the support of $f(t,x,\cdot)$ decays. In the proof of Theorem 3 the perturbed solution will satisfy these conditions on some interval by (FS) and its consequences.

Lemma 5.9. *For all constants* $C_1, C_2 > 0$ *there exists a constant* $C^* > 0$ *having the following property: If a solution* (f, K_f) *on an interval* $[0, b[$ *with initial data* $(\mathring{f}, \mathring{K}_f) \in \mathcal{D}$ *satisfies*

(i) $\sup\{|v| \mid (x,v) \in \operatorname{supp} f(t)\} \leqq C_1, \quad t \in [0,b[,$

(ii) $\operatorname{vol}(\operatorname{supp} f(t,x,\cdot)) \leqq C_2(1+t)^{-3}, \quad x \in \mathbb{R}^3, \quad t \in [0,b[,$

then

$$|K_f(t,x)| \leqq C^*(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-1}$$

for $t \in [0,b[$ *and* $x \in \mathbb{R}^3$ *with* $|x| \leqq R_0 + t$.

Proof. We use the integral representation

$$E_f = E_{f,D} + E_{f,T} + E_{f,S}$$

from Lemma 3.3 and estimate these terms under the assumptions (i) and (ii). The estimates for B_f are completely analogous.

Estimate for $E_{f,D}$: Consider $t \geqq 0$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$. We have

$$\begin{aligned} E_{f,D}(t,x) &= \frac{1}{4\pi t} \int_{|x-y|=t} \operatorname{curl} \mathring{B}_f(y)\, dS_y + \frac{1}{4\pi t^2} \int_{|x-y|=t} \mathring{E}_f(y)\, dS_y \\ &\quad + \frac{1}{4\pi t^2} \int_{|x-y|=t} \partial_x \mathring{E}_f(y)\cdot(y-x)\, dS_y \\ &\quad - \frac{1}{t} \int_{|x-y|=t} \int_{|v| \leqq U_0} \frac{\omega + \hat{v}}{1+\omega\cdot\hat{v}} \mathring{f}(y,x)\, dv\, dS_y \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$, Lemma 5.7 applies to the terms I_1, I_2, I_3 and gives

$$\begin{aligned} |I_1| &\leqq K_0(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}, \\ |I_2| &\leqq K_0 t^{-1}(1+R_0+t-|x|)^{-1}, \\ |I_3| &\leqq K_0(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}. \end{aligned}$$

On the other hand, $|I_2| \leqq \|\mathring{E}_f\|_\infty \leqq C$, and hence

$$|I_2| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}.$$

By applying Lemma 5.8 to the term I_4 we obtain

$$|I_4| \leqq 4\pi(1-\hat{U}_0)^{-1}\frac{4\pi}{3} U_0^3 \|\mathring{f}\|_\infty t^{-1} \min\{R_0^2, t^2\} \leqq C(1+t)^{-1}.$$

Since $\mathring{f}(y,\cdot) = 0$ for $|x| < t - R_0$ and $|x-y| = t$, and

$$(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1} \geqq C(1+t)^{-1}$$

for $t - R_0 \leqq |x| \leqq t + R_0$, we get

$$|I_4| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}, \quad |x| \leqq R_0 + t.$$

Thus, for $t \geqq 0$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ we have

$$|E_{f,D}(t,x)| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}. \tag{5.1}$$

Estimate for $E_{f,T}$: For $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_2 < 1$ and $t \in [0, T(f, K_f)[$ we have

$$\|f(t)\|_\infty = \|\mathring{f}\|_\infty \leqq \|\mathring{g}\|_\infty + 1. \tag{5.2}$$

Define

$$\chi(\tau, \lambda) := \begin{cases} 1, & \lambda \leqq R_0 + \tau \\ 0, & \lambda > R_0 + \tau \end{cases}.$$

For $t \in [0, b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the assumptions (i) and (ii) together with (5.2) imply

$$\begin{aligned} |E_{f,T}(t,x)| &\leqq \int_{|x-y| \leqq t} \int_{|v| \leqq C_1} |k_T(\omega, v)| f(t-|x-y|, y, v)\, dv \frac{dy}{|x-y|^2} \\ &\leqq C \int_{|x-y| \leqq t} (1+t-|x-y|)^{-3} \chi(t-|x-y|, |y|) \frac{dy}{|x-y|^2}, \end{aligned}$$

since $f(\tau, y, \cdot)=0$ for $|y|>R_0+\tau$. Now $|y| \leqq R_0+t-|x-y|$ entails

$$(1+t-|x-y|)^{-3} \leqq C(1+R_0+t-|x-y|+|y|)^{-3},$$

and hence

$$|E_{f,T}(t,x)| \leqq C \int_{|x-y| \leqq t} (1+R_0+t-|x-y|+|y|)^{-3} \frac{dy}{|x-y|^2}.$$

Applying [6], Lemma 7 and defining $r := |x|$ yields

$$|E_{f,T}(t,x)| \leqq \frac{C}{r} \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{t-\tau}.$$

Case 1: $r \geqq \frac{1}{2}(1+R_0+t)$

We have

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{t-\tau} \leqq \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{d\lambda}{(1+R_0+\tau+\lambda)^2} \frac{d\tau}{t-\tau}.$$

Since

$$r+t-\tau-|r-t+\tau| \leqq 2(t-\tau),$$

we may estimate

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{d\lambda}{(1+R_0+\tau+\lambda)^2} \frac{d\tau}{t-\tau} \leqq 2 \int_0^t \frac{d\tau}{(1+R_0+\tau+|r-t+\tau|)^2}$$

$$= 2 \int_0^{(t-r)_+} (1+R_0+\tau+|r-t+\tau|)^{-2} dt + 2 \int_{(t-r)_+}^t (1+R_0+\tau+|r-t+\tau|)^{-2} d\tau$$

$$\leqq 2(t-r)_+ (1+R_0+t-r)^{-2} - (1+R_0+r-t+2\tau)^{-1} \Big|_{(t-r)_+}^t$$

$$\leqq 3(1+R_0+t-r)^{-1} \leqq 9r(1+R_0+t+r)^{-1}(1+R_0+t-r)^{-1}.$$

Case 2: $r < \frac{1}{2}(1+R_0+t)$

Since

$$\lambda(t-\tau)^{-1} \leqq (t+r-\tau)(t-\tau)^{-1} \leqq 2(t-\tau)(t-\tau)^{-1} = 2$$

for $0 \leqq \tau \leqq t-r$ and $\lambda \leqq t+r-\tau$, we get

$$\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{t-\tau}$$

$$= \int_0^{(t-r)_+} \int_{t-r-\tau}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{t-\tau} + \int_{(t-r)_+}^t \int_{r-t+\tau}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{t-\tau}$$

$$\leqq 2 \int_0^{(t-r)_+} \int_{t-r-\tau}^{r+t-\tau} \frac{d\lambda d\tau}{(1+R_0+\tau+\lambda)^3} + \int_{(t-r)_+}^t \int_{r-t+\tau}^{r+t-\tau} \frac{d\lambda}{(1+R_0+\tau+\lambda)^2} \frac{d\tau}{t-\tau}$$

$$\leqq 2 \int_0^{(t-r)_+} 2r(1+R_0+\tau+t-r-\tau)^{-3} d\tau + 2 \int_{(t-r)_+}^t (1+R_0+\tau+r-t+\tau)^{-2} d\tau$$

$$\leqq 6r(1+R_0+t-r)^{-2} \leqq 18r(1+R_0+t+r)^{-1}(1+R_0+t-r)^{-1}.$$

Thus

$$|E_{f,T}(t,x)| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1} \tag{5.3}$$

for $t \in [0,b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$.

Estimate for $E_{f,S}$: For $t \in [0,b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the assumptions (i) and (ii) together with (5.2) imply the following estimate:

$$\begin{aligned}
|E_{f,S}(t,x)| &\leqq \int_{|x-y|\leqq t} \int_{|v|\leqq C_1} |k_S(\omega,v)(fL_f)(t-|x-y|,y,v)|\,dv \frac{dy}{|x-y|} \\
&\leqq C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)}{(1+t-|x-y|)^3} |K_f(t-|x-y|,y)| \frac{dy}{|x-y|} \\
&\leqq C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)}{(1+R_0+t-|x-y|+|y|)^3} |K_f(t-|x-y|,y)| \frac{dy}{|x-y|}. \qquad (5.4)
\end{aligned}$$

Conclusion. Combining the estimates (5.1), (5.3), and (5.4) and the corresponding estimates for B_f yields

$$\begin{aligned}
|K_f(t,x)| &\leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1} \\
&\quad + C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)}{(1+R_0+t-|x-y|+|y|)^3} |K_f(t-|x-y|,y)| \frac{dy}{|x-y|}
\end{aligned}$$

for $t \in [0,b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$, and applying [5], Lemma 11 to this estimate completes the proof. □

Estimates for the Derivatives of the Fields. We are now going to estimate the derivatives of the fields of a solution under the condition that the momenta remain bounded, the volume of the support of $f(t,x,\cdot)$ decays, and the derivative of $f(t)$ with respect to x remains bounded. Again, in the proof of Theorem 3 the perturbed solution will satisfy these conditions on some interval by (FS) and its consequences.

Lemma 5.10. *For all constants $C_1, C_2, C_3 > 0$ there is a constant $C^{**} > 0$ having the following property: If (f, K_f) is a solution with initial data $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ on an interval $[0,b[$ satisfying*

(i) $$\sup\{|v| \,|\, (x,v) \in \operatorname{supp} f(t)\} \leqq C_1, \quad t \in [0,b[,$$

(ii) $$\operatorname{vol}(\operatorname{supp} f(t,x,\cdot)) \leqq C_2(1+t)^{-3}, \quad x \in \mathbb{R}^3,\ t \in [0,b[,$$

(iii) $$\|\partial_x f(t)\|_\infty \leqq C_3, \quad t \in [0,b[,$$

then

$$|\partial_x K_f(t,x)| \leqq C^{**}(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}$$

for $t \in [0,b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$.

Proof. We use the integral representation from Lemma 3.4 and estimate the different terms under the assumptions (i), (ii), and (iii). We restrict ourselves to the term $\partial_x E_f$ the estimates for $\partial_x B_f$ being similar.

Estimate for $E_{f,DD}$: By Lemma 3.4 we have

$$E_{f,DD}(t,x) = \frac{1}{4\pi t} \int_{|x-y|=t} \partial_{x_k} \operatorname{curl} \mathring{B}_f(y) dS_y + \frac{1}{4\pi t^2} \int_{|x-y|=t} \partial_{x_k} \mathring{E}_f(y) dS_y$$
$$+ \frac{1}{4\pi t^2} \int_{|x-y|=t} (\partial_{x_k} \partial_x \mathring{E}_f(y)) \cdot (y-x) dS_y$$
$$- \frac{1}{t} \int_{|x-y|=t} \int_{|v| \leqq U_0} \frac{\omega + \hat{v}}{1 + \omega \cdot \hat{v}} \partial_{x_k} \mathring{f}(y,v) dv dS_y$$
$$+ \frac{1}{t^2} \int_{|x-y|=t} \int_{|v| \leqq U_0} \frac{\omega_k(\omega + \hat{v})}{(1+v^2)(1+\omega\cdot\hat{v})^3} \mathring{f}(y,v) dv dS_y$$
$$- \frac{1}{t} \int_{|x-y|=t} \int_{|v| \leqq U_0} \frac{\omega_k}{1+\omega\cdot\hat{v}} \partial_v \left(\frac{\omega + \hat{v}}{1+\omega\cdot\hat{v}} \right) (\mathring{f} \mathring{L}_f)(y,v) dv dS_y$$
$$=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

Consider $t \in [0, b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$. Since $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$, Lemma 5.7 applies to the terms I_1, I_2, I_3 and gives

$$|I_1| \leqq K_0(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-2}$$
$$|I_2| \leqq K_0 t^{-1}(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-1},$$
$$|I_3| \leqq K_0(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-2};$$

together with $|I_2| \leqq \|\partial_x \mathring{E}_f\|_\infty \leqq C$ this implies

$$|I_2| \leqq C(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-2}.$$

Applying Lemma 5.8 to the terms I_4, I_5, I_6 we obtain

$$|I_4| \leqq C t^{-1} \min\{1, t^2\} \leqq C(1+t)^{-1},$$
$$|I_5| \leqq C t^{-2} \min\{1, t^2\} \leqq C(1+t)^{-2},$$
$$|I_6| \leqq C t^{-1} \min\{1, t^2\} \leqq C(1+t)^{-1}.$$

Since $\mathring{f}(y,\cdot) = 0$ for $|x| < t - R_0$ and $|x - y| = t$, and

$$(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-2} \geqq C(1+t)^{-1}$$

for $t - R_0 \leqq |x| \leqq t + R_0$, we have

$$|E_{f,DD}(t,x)| \leqq C(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-2} \tag{5.5}$$

for $t \in [0, b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$.

Extimate for $E_{f,R}$: For $t \in [0, b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the assumptions (i) and (ii) together with (5.2) imply

$$|E_{f,R}(t,x)| \leqq \int_{|v| \leqq C_1} \frac{1}{1+v^2} \int_{|\omega|=1} \left| \frac{\omega + \hat{v}}{(1+\omega\cdot\hat{v})^3} \right| \omega_k dS_\omega f(t,x,v) dv$$
$$\leqq C(1+t)^{-3} \|f(t)\|_\infty \leqq C(1+t)^{-3},$$

and thus,

$$|E_{f,R}(t,x)| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2}. \tag{5.6}$$

Estimate for $E_{f,TS}$: Lemma 5.9 together with the assumptions (i) and (ii) implies

$$\begin{aligned}|E_{f,TS}(t,x)| &\leqq \int_{|x-y|\leqq t}\int_{|v|\leqq C_1} |k_{TS}(\omega,v)(fL_f)(t-|x-y|,y,v)|dv\frac{dy}{|x-y|^2}\\ &\leqq C\int_{|x-y|\leqq t}(1+t-|x-y|)^{-3}\chi(t-|x-y|,|y|)\\ &\quad\cdot(1+R_0+t-|x-y|+|y|)^{-1}\frac{dy}{|x-y|^2}.\end{aligned}$$

By the definition of χ we only have to integrate with respect to $|y|\leqq R_0+t-|x-y|$ and since for these y the estimate

$$(1+t-|x-y|)^{-3}\leqq C(1+R_0+t-|x-y|+|y|)^{-3}$$

holds, we obtain

$$|E_{f,TS}(t,x)|\leqq C\int_{|x-y|\leqq t}(1+R_0+t-|x-y|+|y|)^{-4}\frac{dy}{|x-y|^2}.$$

Now [6], Lemma 7 implies

$$\begin{aligned}|E_{f,TS}(t,x)| &\leqq \frac{C}{r}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^4}\frac{d\tau}{t-\tau}\\ &\leqq \frac{C}{r}(1+R_0+t-r)^{-1}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3}\frac{d\tau}{t-\tau}.\end{aligned}$$

The remaining integrals has been estimated by

$$Cr(1+R_0+t+r)^{-1}(1+R_0+t-r)^{-1}$$

when we treated the term $E_{f,T}$. Altogether this yields

$$|E_{f,TS}(t,x)|\leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2} \tag{5.7}$$

for $t\in[0,b[$ and $x\in\mathbb{R}^3$ with $|x|\leqq R_0+t$.

Estimate for $E_{f,TT}$: First we consider $t\in[2,b[$ and split $E_{f,TT}$ into two parts,

$$E_{f,TT}(t,x)=I_1+I_2,$$

where

$$I_1 := \oint_{|x-y|\leqq 1}\int_{|v|\leqq C_1} k_{TT}(\omega,v)f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^3},$$

$$I_2 := \int_{1\leqq|x-y|\leqq t}\int_{|v|\leqq C_1} k_{TT}(\omega,v)f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^3}.$$

The term I_2 is estimated as above, using (i), (ii), and [6], Lemma 7; that is,

$$|I_2|\leqq C\int_{1\leqq|x-y|\leqq t}(1+t-|x-y|)^{-3}\chi(t-|x-y|,|y|)\frac{dy}{|x-y|^3}$$

$$\leqq C \int_{1 \leqq |x-y| \leqq t} (1+R_0+t-|x-y|+|y|)^{-3} \frac{dy}{|x-y|^3}$$

$$\leqq \frac{C}{r} \int_0^{t-1} \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{(t-\tau)^2}$$

$$= \frac{C}{r} \int_0^{t/2} \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{(t-\tau)^2}$$

$$+ \frac{C}{r} \int_{t/2}^{t-1} \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda d\lambda}{(1+R_0+\tau+\lambda)^3} \frac{d\tau}{(t-\tau)^2}$$

$$=: I_{21} + I_{22}.$$

Now

$$I_{21} \leqq \frac{C}{r} \frac{4}{t^2} \int_0^{t/2} \int_{|r-t+\tau|}^{r+t-\tau} (1+R_0+\tau+\lambda)^{-2} d\lambda d\tau$$

$$\leqq Ct^{-1}(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}$$

$$\leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2},$$

since $t \geqq 2$. Furthermore,

$$I_{22} \leqq \frac{C}{r} \int_{t/2}^{t-1} (1+R_0+\tau)^{-3} \int_{|r-t+\tau|}^{r+t-\tau} \lambda d\lambda \frac{d\tau}{(t-\tau)^2}$$

$$= \frac{C}{r} \int_{t/2}^{t-1} (1+R_0+\tau)^{-3} 2r(t-\tau) \frac{d\tau}{(t-\tau)^2}$$

$$\leqq C(1+R_0+t/2)^{-3} \int_{t/2}^{t-1} \frac{d\tau}{t-\tau}$$

$$= C(1+R_0+t/2)^{-3} \ln t/2$$

$$\leqq C(1+R_0+t/2)^{-11/4}$$

$$\leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}.$$

Combining the estimates for I_{21} and I_{22} implies

$$|I_2| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}$$

for $t \in [2, b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$. Since

$$\int_{|\omega|=1} k_{TT}(\omega, v) dS_\omega = 0,$$

the term I_1 may be estimated as follows:

$$|I_1| \leqq C \oint_{|x-y| \leqq 1} \int_{|v| \leqq C_1} \frac{|f(t-|x-y|, y, v) - f(t-|x-y|, x, v)|}{|x-y|} dv \frac{dy}{|x-y|^2}.$$

Now the assumptions (ii) and (iii) yield

$$\mathrm{vol}\left(\mathrm{supp} \frac{f(t-|x-y|, y, \cdot) - f(t-|x-y|, x, \cdot)}{|x-y|} \right) \leqq 2C_2(1+t-|x-y|)^{-3}$$

and

$$\frac{|f(t-|x-y|,y,v)-f(t-|x-y|,x,v)|}{|x-y|} \leqq \|\partial_x f(t-|x-y|)\|_\infty \leqq C_3,$$

which for $t \geqq 2$ implies

$$|I_1| \leqq C \int_{|x-y|\leqq 1} (1+t-|x-y|)^{-3} \frac{dy}{|x-y|^2} \leqq Ct^{-3} \int_{|x-y|\leqq 1} \frac{dy}{|x-y|^2}$$
$$\leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2}.$$

For $t \in [0, \min\{2, b\}[$ we get

$$|E_{f,TT}(t,x)| \leqq C \int_{|x-y|\leqq t} (1+t-|x-y|)^{-3} \frac{dy}{|x-y|^2}$$
$$\leqq C \int_{|x-y|\leqq 2} \frac{dy}{|x-y|^2} \leqq C,$$

and conclude that for $t \in [0, b[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the following estimate holds:

$$|E_{f,TT}(t,x)| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}. \tag{5.8}$$

Estimate of $E_{f,SS}$: The assumption (i) yields

$$E_{f,SS}(t,x) = \int_{|x-y|\leqq t} \int_{|v|\leqq C_1} k_{SS}(\omega,v) S(fL_f)(t-|x-y|,y,v)\,dv \frac{dy}{|x-y|}.$$

Now $Sf = -L_f \cdot \partial_v f$ entails

$$S(fL_f) = (Sf)L_f + fSL_f = -\mathrm{div}_v(fL_f)L_f + fSL_f,$$

and we therefore split $E_{f,SS}$ into two terms obtaining

$$E_{f,SS}(t,x) = I_1 + I_2,$$

where

$$I_1 := -\int_{|x-y|\leqq t} \int_{|v|\leqq C_1} k_{SS}(\omega,v)(\mathrm{div}_v(fL_f)L_f)(t-|x-y|,y,v)\,dv \frac{dy}{|x-y|},$$

$$I_2 := \int_{|x-y|\leqq t} \int_{|v|\leqq C_1} k_{SS}(\omega,v)(fSL_f)(t-|x-y|,y,v)\,dv \frac{dy}{|x-y|}.$$

To treat the term I_1 integrate by parts with respect to v and by (i), (ii), Lemma 5.9, and [6], Lemma 7 obtain

$$|I_1| \leqq \int_{|x-y|\leqq t} \int_{|v|\leqq C_1} |\partial_v(k_{SS}(\omega,v)L_f)(fL_f)(t-|x-y|,y,v)|\,dv \frac{dy}{|x-y|}$$
$$\leqq C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)}{(1+t-|x-y|)^3}(1+R_0+t-|x-y|+|y|)^{-2}$$
$$\cdot(1+R_0+t-|x-y|-|y|)^{-2} \frac{dy}{|x-y|}$$

$$\leqq C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)}{(1+R_0+t-|x-y|+|y|)^5}(1+R_0+t-|x-y|-|y|)^{-2}\frac{dy}{|x-y|}$$

$$=\frac{C}{r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{\chi(\tau,\lambda)\lambda d\lambda d\tau}{(1+R_0+\tau+\lambda)^5(1+R_0+\tau-\lambda)^2}$$

$$\leqq\frac{C}{r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{\chi(\tau,\lambda)d\lambda d\tau}{(1+R_0+\tau+\lambda)^4(1+R_0+\tau-\lambda)^2}.$$

Taking τ, λ from the domain of integration satisfying $\chi(\tau,\lambda)\neq 0$ and defining $\xi:=\tau+\lambda$, $\sigma:=\tau-\lambda$ we obtain $\xi\in[t-r,t+r]$ and $\sigma\in[-R_0,t-r]$. Thus,

$$|I_1|\leqq\frac{C}{r}\int_{t-r}^{t+r}\int_{-R_0}^{t-r}(1+R_0+\xi)^{-4}(1+R_0+\sigma)^{-2}d\sigma d\xi$$

$$\leqq\frac{C}{r}\int_{t-r}^{t+r}(1+R_0+\pi)^{-3}d\xi$$

$$=\frac{C}{r}\frac{2r(1+R_0+t)}{(1+R_0+t+r)^2(1+R_0+t-r)^2}$$

$$\leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2}.$$

To estimate I_2 observe that

$$SL_f=\partial_t E_f+\partial_x E_f\hat{v}+\hat{v}\times\partial_t B_f+\partial_x(\hat{v}\times B_f)\hat{v}$$

$$=\operatorname{curl} B_f-4\pi j_f+\partial_x E_f\hat{v}-\hat{v}\times\operatorname{curl} E_f+\partial_x(\hat{v}\times B_f)\hat{v}$$

to obtain

$$|I_2|\leqq C\int_{|x-y|\leqq t}\int_{|v|\leqq C_1}|f||j_f|(t-|x-y|,y,v)dv\frac{dy}{|x-y|}$$

$$+C\int_{|x-y|\leqq t}\int_{|v|\leqq C_1}|f||\partial_x K_f|(t-|x-y|,y,v)dv\frac{dy}{|x-y|}$$

$$=:I_{21}+I_{22}.$$

By (ii) and (5.2) we may estimate

$$\|j_f(t)\|_\infty\leqq C(1+t)^{-3},\quad t\in[0,b[,$$

and continuing as above we obtain

$$|I_{21}|\leqq C\int_{|x-y|\leqq t}(1+R_0+t-|x-y|+|y|)^{-6}\frac{dy}{|x-y|}$$

$$=\frac{C}{r}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}(1+R_0+\tau+\lambda)^{-6}\lambda d\lambda d\tau$$

$$\leqq\frac{C}{r}\int_0^t(1+R_0+\tau)^{-2}\int_{|r-t+\tau|}^{r+t-\tau}(1+R_0+\tau+\lambda)^{-3}d\lambda d\tau.$$

Now

$$\int_{|r-t+\tau|}^{r+t-\tau} (1+R_0+\tau+\lambda)^{-3}d\lambda \leqq \frac{1}{2}\frac{(1+R_0+t+r)^2-(1+R_0+t-r)^2}{(1+R_0+t+r)^2(1+R_0+t-r)^2}$$

$$= \frac{2(1+R_0+t)r}{(1+R_0+t+r)^2(1+R_0+t-r)^2}$$

$$\leqq 2r(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2},$$

yields

$$I_{21} \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2}.$$

Estimating I_{22} we get

$$I_{22} \leqq C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)}{(1+R_0+t-|x-y|+|y|)^3} |\partial_x K_f(t-|x-y|,y)| \frac{dy}{|x-y|},$$

and combining all our estimates for $E_{f,SS}$ we have

$$|E_{f,SS}(t,x)| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-2} + C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)|\partial_x K_f(t-|x-y|,y)|}{(1+R_0+t-|x-y|+|y|)^3}\frac{dy}{|x-y|} \tag{5.9}$$

for $t\in[0,b[$ and $x\in\mathbb{R}^3$ with $|x|\leqq R_0+t$.

Conclusion. Combining the estimates (5.5), (5.6), (5.7), (5.8), and (5.9) and the corresponding estimates for $\partial_x B_f$ yields

$$|\partial_x K_f(t,x)| \leqq C(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4} + C \int_{|x-y|\leqq t} \frac{\chi(t-|x-y|,|y|)|\partial_x K_f(t-|x-y|,y)|}{(1+R_0+t-|x-y|+|y|)^3}\frac{dy}{|x-y|}$$

for $t\in[0,b[$ and $x\in\mathbb{R}^3$ with $|x|\leqq R_0+t$, and applying [5], Lemma 11 to this estimate completes the proof. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Since $\beta<\min\{\alpha_1,\frac{1}{2}(\alpha_1+\alpha_2)\}$ there exists $T_1>0$ such that

$$K_1(1+t)^{-\alpha_1}(1+R_0+t-|x|)^{-\alpha_2} \leqq \frac{\eta_2}{3}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta},$$

$$K_1(1+t)^{-\alpha_1}(1+R_0+t-|x|)^{-\alpha_2-1} \leqq \frac{\eta_2}{3}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta-1}, \tag{5.10}$$

for $t\geqq T_1$ and $x\in\mathbb{R}^3$ with $|x|\leqq R_0+t$.

Choose $\varepsilon_3\in]0,\varepsilon_2]$ so small that $d_2<\varepsilon_3$ implies $\sigma_2(d_2)>T_1$ and

$$\zeta_2(T_1)d_2 \leqq \frac{\eta_2}{3}(1+R_0+T_1+|x|)^{-\beta}(1+R_0+T_1-|x|)^{-\beta-1}$$

$$\leqq \frac{\eta_2}{3}(1+R_0+T_1+|x|)^{-\beta}(1+R_0+T_1-|x|)^{-\beta} \tag{5.11}$$

for $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + T_1$. Combining the estimates (5.10) and (5.11) and using Theorem 2 and the decay condition (D) gives

$$|K_f(T_1, x)| \leqq \frac{2}{3}\eta_2(1 + R_0 + T_1 + |x|)^{-\beta}(1 + R_0 + T_1 - |x|)^{-\beta},$$

$$|\partial_x K_f(T_1, x)| \leqq \frac{2}{3}\eta_2(1 + R_0 + T_1 + |x|)^{-\beta}(1 + R_0 + T_1 - |x|)^{-\beta-1}$$

for $|x| \leqq R_0 + T_1$. Hence

$$T_2(f, K_f) := \sup\{t \in [T_1, T(f, K_f)[\,|\,(f, K_f) \text{ satisfies (FS) with respect to } \eta_2 \text{ on } [T_1, t]\} > T_1.$$

We now estimate the volume of the support of $f(t, x, \cdot)$. Observe that for (x, v), $(x, v') \in \operatorname{supp} f(t)$ and $t \in [T_1, T_2(f, K_f)[$, Lemma 5.5 and Lemma 5.3 yield

$$\begin{aligned}|v - v'| &\leqq c_2(t - T_1)^{-1}|X_f(T_1, t, x, v) - X_f(T_1, t, x, v')| \\ &\leqq 2c_2(R_0 + \hat{C}_1 T_1)(t - T_1)^{-1},\end{aligned}$$

and for $t \in [0, T_2(f, K_f)[$ and $(x, v), (x, v') \in \operatorname{supp} f(t)$ Lemma 5.3 implies

$$|v - v'| \leqq 2C_1.$$

Hence

$$\operatorname{diam}(\operatorname{supp} f(t, x, \cdot)) \leqq C(1 + t)^{-1},$$

and there exists a constant $C_2 > 0$ such that for all $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_3$ the estimate

$$\operatorname{vol}(\operatorname{supp} f(t, x, \cdot)) \leqq C_2(1 + t)^{-3}, \quad t \in [0, T_2(f, K_f)[,\ x \in \mathbb{R}^3$$

holds. Furthermore, Lemma 5.3 yields

$$\sup\{|v|\,|\,(x, v) \in \operatorname{supp} f(t)\} \leqq C_1, \quad t \in [0, T_2(f, K_f)[.$$

By Lemma 5.6

$$\|\partial_x f(t)\|_\infty \leqq C\|f(T_1)\|_{1,\infty}, \quad t \in [T_1, T_2(f, K_f)[,$$

while Theorem 2 implies

$$\|f(t)\|_{1,\infty} \leqq \|g(t)\|_{1,\infty} + \zeta_2(T_1)d_2 \leqq C, \quad t \in [0, T_1].$$

Thus, there exist constants $C_1, C_2, C_3 > 0$ such that for $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_3$ and $t \in [0, T_2(f, K_f)[$ the following estimates hold:

$$\sup\{|v|\,|\,(x, v) \in \operatorname{supp} f(t)\} \leqq C_1 \tag{5.12}$$

$$\operatorname{vol}(\operatorname{supp} f(t, x, \cdot)) \leqq C_2(1 + t)^{-3}, \quad x \in \mathbb{R}^3, \tag{5.13}$$

$$\|\partial_x f(t)\|_\infty \leqq C_3. \tag{5.14}$$

We may therefore apply Lemma 5.9 and Lemma 5.10 to obtain constants $C^* > 0$ and $C^{**} > 0$ such that for all $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_3$, $t \in [0, T_2(f, K_f)[$, and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ we have

$$|K_f(t, x)| \leqq C^*(1 + R_0 + t + |x|)^{-1}(1 + R_0 + t - |x|)^{-1}. \tag{5.15}$$

$$|\partial_x K_f(t,x)| \leqq C^{**}(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}. \tag{5.16}$$

Since $\beta < 3/4 < 1$ there exists $T_3 > T_1$ such that for $t \geqq T_3$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the following estimates hold:

$$C^*(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}$$
$$\leqq \frac{\eta_2}{2}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta},$$
$$C^{**}(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}$$
$$\leqq \frac{\eta_2}{2}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta-1}.$$

For ε_3 small enough Theorem 2 ensures that $\sigma_2(d_2) > T_3$ for all $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ with $d_2 < \varepsilon_3$ and

$$\zeta_2(T_3)d_2 \leqq \frac{\eta_2}{3}(1+R_0+T_3+|x|)^{-\beta}(1+R_0+T_3-|x|)^{-\beta-1}$$
$$\leqq \frac{\eta_2}{3}(1+R_0+T_3+|x|)^{-\beta}(1+R_0+T_3-|x|)^{-\beta}$$

for $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + T_3$. For $t \leqq T_3$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the monotonicity of ζ_2 entails

$$\zeta_2(t)d_2 \leqq \frac{\eta_2}{3}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta-1}$$
$$\leqq \frac{\eta_2}{3}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta},$$

so that for $T_1 \leqq t \leqq T_3$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the estimate (5.10) and Theorem 2 imply

$$|K_f(t,x)| \leqq \tfrac{2}{3}\eta_2(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta}$$
$$|\partial_x K_f(t,x)| \leqq \tfrac{2}{3}\eta_2(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta-1},$$

whence by definition $T_2(f, K_f) > T_3$. For $t \in [T_3, T_2(f, K_f)[$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ the following estimates hold:

$$|K_f(t,x)| \leqq C^*(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-1}$$
$$\leqq \frac{\eta_2}{2}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta},$$
$$|\partial_x K_f(t,x)| \leqq C^{**}(1+R_0+t+|x|)^{-1}(1+R_0+t-|x|)^{-7/4}$$
$$\leqq \frac{\eta_2}{2}(1+R_0+t+|x|)^{-\beta}(1+R_0+t-|x|)^{-\beta-1}.$$

By definition of $T_2(f, K_f)$ this implies $T_2(f, K_f) = T(f, K_f)$, and by (5.12) we infer

$$\sup\{|v| \,|\, (x,v) \in \operatorname{supp} f(t)\} \leqq C_1, \quad 0 \leqq t < T(f, K_f).$$

Now Lemma 3.1 yields the desired global existence, the desired estimates for the fields and their derivatives hold by (5.15) and (5.16), and the proof of Theorem 3 is complete. □

As a corollary we note some additional asymptotic results that we obtained in the above proof.

Corollary 5.11. *There exist constants* $C_1 > 0$, $C_2 > 0$, *and* $C_3 > 0$ *such that for all initial data* $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ *with* $d_2 < \varepsilon_3$ *and* $t \geqq 0$ *the following estimates hold*:

$$\begin{gathered} \operatorname{supp} f(t) \subset \{(x,v) \in \mathbb{R}^6 \mid |x| \leqq R_0 + \hat{C}_1 t, |v| \leqq C_1\}, \\ \operatorname{vol}(\operatorname{supp} f(t,x,\cdot)) \leqq C_2(1+t)^{-3}, \quad x \in \mathbb{R}^3, \\ \| \partial_x f(t) \|_\infty \leqq C_3. \end{gathered}$$

5.2. Global Estimates for the Deviation of the Perturbed from the Unperturbed Solution. Global existence of the perturbed solution being established, the question arises, whether the deviation of (f, K_f) from (g, K_g) can be controlled globally in time. To treat this problem define

$$\| h \|_{\infty,t} := \sup_{|x| \leqq R_0 + t} |h(x)|, \quad h \in C(\mathbb{R}^3), \quad t \geqq 0,$$

and

$$\begin{aligned} d := \| \mathring{f} - \mathring{g} \|_\infty &+ \sup_{x \in \mathbb{R}^3} (1 + R_0 + |x|)^2 |\mathring{K}_f(x) - \mathring{K}_g(x)| \\ &+ \sup_{x \in \mathbb{R}^3} (1 + R_0 + |x|)^3 |\partial_x \mathring{K}_f(x) - \partial_x \mathring{K}_g(x)|. \end{aligned}$$

for $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$.

Proposition 5.12. *Assume that* (g, K_g) *satisfies* (R) *and* (D). *Then there exist constants* $C > 0$ *and* $\kappa > 0$ *such that for all initial data* $(\mathring{f}, \mathring{K}_f) \in \mathscr{D}$ *with* $d_2 < \varepsilon_3$ *and* $t \geqq 0$ *the corresponding solution* (f, K_f) *satisfies*

$$\begin{aligned} \| K_f(t) - K_g(t) \|_{\infty,t} &\leqq C(1+t)^\kappa d, \\ \| f(t) - g(t) \|_\infty &\leqq C(1+t)^{\kappa+2} d. \end{aligned}$$

Equation (4.1) suggests that we will need an estimate for the derivative of g with respect to v in order to prove Proposition 5.12.

Lemma 5.13. *Let* (g, K_g) *be a solution satisfying the condition* (R) *and* (D). *Then there is a constant* $C > 0$ *such that*

$$\| \partial_v g(t) \|_\infty \leqq C(1+t), \quad t \geqq 0.$$

Proof. Since

$$g(t,x,v) = \mathring{g}(X_g(0,t,x,v), V_g(0,t,x,v))$$

we have

$$\begin{aligned} \partial_v g(t,x,v) = {} & \partial_x \mathring{g}(X_g(0,t,x,v), V_g(0,t,x,v)) \partial_v X_g(0,t,x,v) \\ & + \partial_v \mathring{g}(X_g(0,t,x,v), V_g(0,t,x,v)) \partial_v V_g(0,t,x,v) \end{aligned}$$

and obtain

$$|\partial_v g(t,x,v)| \leqq \| \mathring{g} \|_{1,\infty} (|\partial_v X_g(0,t,x,v)| + |\partial_v V_g(0,t,x,v)|)$$

for $t \geqq 0$ and $(x,v) \in \mathbb{R}^6$.

With arguments similar to the ones used in the proof of Lemma 5.6 we may now estimate the derivatives of the characteristics with respect to v. Differentiating

$$\ddot{X}_g(s,t,x,v) = J_g(s, X_g(s), \hat{V}_g(s))$$

with respect to v, where

$$J_g(s,x,\hat{v}) := \sqrt{1-\hat{v}^2}\,(E_g(s,x) + \hat{v} \times B_g(s,x) - \hat{v}\cdot E_g(s,x)\hat{v}),$$

estimating as in the proof of 5.5, observing that for $(x,v) \in \operatorname{supp} g(t)$ and $0 \leqq s \leqq t < \infty$ Corollary 5.11 implies

$$|X_g(s,t,x,v)| \leqq R_0 + \hat{C}_1 s, \quad |V_g(s,t,x,v)| \leqq C_1,$$

and finally using the assumption (D) we get

$$|\ddot{x}(s)| \leqq C(1+s)^{-\alpha-1}(|x(s)| + (t-s)) + (1+s)^{-\alpha}(|\dot{x}(s)| + 1),$$

where

$$x(s) := \partial_v X_g(s,t,x,v) + (t-s)\partial_v(\hat{v}(v)).$$

Since $x(t) = \dot{x}(t) = 0$ and $\alpha = \alpha_1 + \alpha_2 > 1$, Lemma 5.4 implies

$$|x(s)| \leqq C(t-s),$$

and hence

$$|\partial_v X_g(s,t,x,v)| \leqq C(t-s).$$

On the other hand, we have

$$\begin{aligned} |\partial_v \dot{V}_g(s)| &\leqq |\partial_x K_g(s, X_g(s))|\,|\partial_v X_g(s)| + |B_g(s, X_g(s))|\,|\partial_v \hat{V}_g(s)| \\ &\leqq C(1+s)^{-\alpha-1}(t-s) + C(1+s)^{-\alpha}|\partial_v V_g(s)|. \end{aligned}$$

Integrating this inequality, observing $|\partial_v V_f(t,t,x,v)| = 1$, and applying Gronwall's lemma yields

$$|\partial_v V_g(s,t,x,v)| \leqq C(1+t).$$

Putting the estimates for the derivatives of the characteristics into the estimate for $\partial_v g$ completes the proof. □

Proof of Proposition 5.12. We use the integral representation from Lemma 3.3 to estimate the difference of the fields. For $t \geqq 0$ and $x \in \mathbb{R}^3$ with $|x| \leqq R_0 + t$ and by the definition of d we obtain the following estimate for the difference of the data terms:

$$\begin{aligned} |E_{f,D}(t,x) - E_{g,D}(t,x)| \leqq{}& \frac{d}{4\pi t} \int_{|x-y|=t} (1 + R_0 + |y|)^{-3} dS_y \\ &+ \frac{d}{4\pi t^2} \int_{|x-y|=t} (1 + R_0 + |y|)^{-2} dS_y \\ &+ C\frac{d}{t} \int_{|x-y|=t} 1_{B_{R_0}(0)} dS_y. \end{aligned}$$

Lemma 5.7 and Lemma 5.8 yield

$$|E_{f,D}(t,x) - E_{g,D}(t,x)| \leqq Cd(1 + R_0 + t + |x|)^{-1} + Cdt^{-1} + Cd(1+t)^{-1},$$

and since

$$\frac{d}{4\pi t^2}\int_{|x-y|=t}(1+R_0+|y|)^{-2}dS_y \leqq Cd,$$

we get

$$\| E_{f,D}(t)-E_{g,D}(t)\|_{\infty,t} \leqq C(1+t)^{-1}d, \quad t\geqq 0. \tag{5.17}$$

Corollary 5.11 implies that for $t\geqq 0$ and $x\in\mathbb{R}^3$ the following estimates hold:

$$\begin{aligned}|E_{f,T}(t,x)-E_{g,T}(t,x)| &\leqq C\int_{|x-y|\leqq t}(1+t-|x-y|)^{-3} \\ &\quad\cdot\| f(t-|x-y|)-g(t-|x-y|)\|_\infty \frac{dy}{|x-y|^2} \\ &= C\int_0^t (1+\tau)^{-3}\| f(\tau)-g(\tau)\|_\infty d\tau \end{aligned}\tag{5.18}$$

and

$$\begin{aligned}&|E_{f,S}(t,x)-E_{g,S}(t,x)| \\ &\leqq C\int_{|x-y|\leqq t}\int_{|v|\leqq C_1}(|f|\,|K_f-K_g|+|K_g|\,|f-g|)(t-|x-y|,y,v)dv\frac{dy}{|x-y|} \\ &=: I_1+I_2.\end{aligned}$$

First we estimate the term I_1:

$$\begin{aligned} I_1 &\leqq C\int_{|x-y|\leqq t}\frac{\chi(t-|x-y|,|y|)}{(1+t-|x-y|)^3}|K_f-K_g|(t-|x-y|,y)\frac{dy}{|x-y|} \\ &\leqq C\int_0^t (1+\tau)^{-3}\| K_f(\tau)-K_g(\tau)\|_{\infty,\tau}\tau d\tau \\ &\leqq C\int_0^t (1+\tau)^{-2}\| K_f(\tau)-K_g(\tau)\|_{\infty,\tau}d\tau.\end{aligned}$$

To estimate I_2 define

$$\tilde\chi(\tau,\lambda):=\begin{cases}1, & \lambda\leqq R_0+\hat C_1\tau \\ 0, & \lambda> R_0+\hat C_1\tau\end{cases}$$

and by Corollary 5.11 and condition (D) obtain that

$$\begin{aligned} I_2 \leqq C\int_{|x-y|\leqq t} &\frac{\tilde\chi(t-|x-y|,|y|)}{(1+t-|x-y|)^{3+\alpha_1}(1+R_0+t-|x-y|-|y|)^{\alpha_2}} \\ &\cdot\| f(t-|x-y|)-g(t-|x-y|)\|_\infty \frac{dy}{|x-y|}.\end{aligned}$$

For $|y|\leqq R_0+\hat C_1(t-|x-y|)$ we have

$$\begin{aligned}(1+R_0+t-|x-y|-|y|)^{-\alpha_2} &\leqq (1+(1-\hat C_1)(t-|x-y|))^{-\alpha_2} \\ &\leqq C(1+t-|x-y|)^{-\alpha_2},\end{aligned}$$

and thus,

$$I_2 \leqq C \int_{|x-y|\leqq t} (1+t-|x-y|)^{-3-\alpha} \| f(t-|x-y|) - g(t-|x-y|) \|_\infty \frac{dy}{|x-y|}$$
$$= C \int_0^t (1+\tau)^{-3-\alpha} \| f(\tau) - g(\tau) \|_\infty \tau \, d\tau$$
$$\leqq C \int_0^t (1+\tau)^{-3} \| f(\tau) - g(\tau) \|_\infty d\tau.$$

Hence for $t \geqq 0$ and $|x| \leqq R_0 + t$

$$|E_{f,S}(t,x) - E_{g,S}(t,x)| \leqq C \int_0^t (1+\tau)^{-2} \| K_f(\tau) - K_g(\tau) \|_{\infty,\tau} d\tau$$
$$+ C \int_0^t (1+\tau)^{-3} \| f(\tau) - g(\tau) \|_\infty d\tau. \tag{5.19}$$

Combining the estimates (5.17), (5.18), and (5.19) with the corresponding estimates for $B_f - B_g$ yields

$$\| K_f(t) - K_g(t) \|_{\infty,t} \leqq C(1+t)^{-1} d + C \int_0^t (1+\tau)^{-3} \| f(\tau) - g(\tau) \|_\infty d\tau$$
$$+ C \int_0^t (1+\tau)^{-2} \| K_f(\tau) - K_g(\tau) \|_{\infty,\tau} d\tau. \tag{5.20}$$

Now recall Eq. (4.1):

$$(f-g)(t,x,v) = (\mathring{f} - \mathring{g})(X_f(0,t,x,v), V_f(0,t,x,v))$$
$$- \int_0^t (\partial_v g \cdot (L_f - L_g))(s, X_f(s,t,x,v), V_f(s,t,x,v)) ds.$$

If $|X_f(s,t,x,v)| > R_0 + s$ then $\partial_v g(s, X_f(s), V_f(s)) = 0$, and hence

$$|(\partial_v g \cdot (L_f - L_g))(s, X_f(s,t,x,v), V_f(s,t,x,v))| \leqq \| \partial_v g(s) \|_\infty \| K_f(s) - K_g(s) \|_{\infty,s}$$

for $(x,v) \in \mathbb{R}^6$. For $t \geqq 0$ this implies

$$\| f(t) - g(t) \|_\infty \leqq \| \mathring{f} - \mathring{g} \|_\infty + \int_0^t \| \partial_v g(s) \|_\infty \| K_f(s) - K_g(s) \|_{\infty,s} ds, \tag{5.21}$$

and applying Lemma 5.13 we obtain

$$\| K_f(t) - K_g(t) \|_{\infty,t} \leqq C(1+t)^{-1} d + C \int_0^t (1+\tau)^{-3} d\tau \| \mathring{f} - \mathring{g} \|_\infty$$
$$+ C \int_0^t \int_0^\tau (1+\tau)^{-3} (1+\sigma) \| K_f(\sigma) - K_g(\sigma) \|_{\infty,\sigma} d\sigma \, d\tau$$
$$+ C \int_0^t (1+\tau)^{-2} \| K_f(\tau) - K_g(\tau) \|_{\infty,\tau} d\tau$$
$$\leqq Cd + C \int_0^t (1+\tau)^{-2} \| K_f(\tau) - K_g(\tau) \|_{\infty,\tau} d\tau$$
$$+ C \int_0^t \int_\sigma^t (1+\tau)^{-2} \| K_f(\sigma) - K_g(\sigma) \|_{\infty,\sigma} d\tau \, d\sigma$$
$$\leqq Cd + \kappa \int_0^t (1+\tau)^{-1} \| K_f(\tau) - K_g(\tau) \|_{\infty,\tau} d\tau.$$

Gronwall's lemma now yields

$$\|K_f(t)-K_g(t)\|_{\infty,t} \leqq C\exp\left(\kappa\int_0^t (1+\tau)^{-1}d\tau\right)d = C(1+t)^\kappa d, \quad t \geqq 0.$$

Putting this estimate into (5.21) we obtain

$$\begin{aligned}\|f(t)-g(t)\|_\infty &\leqq \|\mathring{f}-\mathring{g}\|_\infty + C\int_0^t (1+s)(1+s)^\kappa ds d\\ &\leqq C(1+t)^{\kappa+2}d, \quad t \geqq 0,\end{aligned}$$

and the proof is complete. □

Remark. The above investigation was of course motivated by the desire to obtain some sort of stability result for solutions satisfying the conditions (R) and (D). Note that if one could establish an estimate like

$$\|\partial_v g(t)\|_\infty \leqq C(1+t)^\gamma, \quad t \geqq 0$$

with $\gamma < 1$, the same proof as above would yield

$$\|K_f(t)-K_g(t)\|_{\infty,t} \leqq Cd, \quad t \geqq 0,$$

which might be interpreted as a stability result for the solution (g, K_g).

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