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Dimensional Regularization and Renormalization of QED

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Abstract. We give an x-space definition of dimensional regularization suited to the tree expansion method of renormalization. We apply the dimensionally regularized tree expansion to QED, obtaining sharp bounds on the size of a renormalized graph. Subtractions are made with the Lagrangian counterterms of the tree expansion, not by minimal subtraction techniques, and so do not entail a knowledge of the meromorphic structure of a graph as a function of dimension. This renormalization procedure respects the Ward identities, and the counterterms required are gauge invariant.

1. Introduction

In [1] with J. Feldman and T. Hurd, we developed a general scheme for renormalizing a quantum field theory based on the tree expansion of G. Gallavotti and F. Nicolò [2], and we applied this scheme to quantum electrodynamics (QED) to give a complete proof of the renormalizability of QED in perturbation theory. The basic idea of the tree expansion approach is to slice up each field as a sum of fields of different scales, to integrate out the fields one scale at a time, and to renormalize scale by scale. The resulting renormalization procedure is remarkably simple: one never sees "overlapping divergences" or the usual combinatoric complexities of BPHZ renormalization, and the required bounds amount to little more than superficial power counting. We briefly review the tree expansion in Sect. 2 but shall rely on [1] or [3] for details. See also Hurd [4] for a simple version of the tree expansion that employs continuous rather than discrete slicing, as in Polchinski [5].

The main technical difficulty we faced in applying the tree expansion to QED in [1] is that the slicing breaks gauge invariance and so it was not clear whether the theory could be renormalized using only gauge invariant counterterms. We overcame this problem as follows: we introduced an auxiliary regularization on the fermions that preserved the Ward identities but allowed us to remove the tree expansion cutoffs on the fermi lines; upon doing so we recovered the Ward identities, and were thus able to rule out forbidden gauge variant counterterms. The auxiliary regularization we used in [1] was "loop regularization." (We call such a regularization "auxiliary" to the tree expansion regularizations since it cannot be used to give the slicing of individual lines that is needed to run the tree expansion.)

Now loop regularization has its shortcomings. For a non-abelian gauge theory like Yang–Mills, it will not on its own give finite graphs. But even for QED, where loop regularization is most conveniently implemented via fictitious spinor fields, there is an incompatibility between loop regularization and renormalization. Graphs with fictitious field external legs must be renormalized with "incorrect" counterterms in order to maintain the algebraic cancellations involved in loop regularization. It was this complication of loop regularization that gave us the most trouble when we removed the UV cutoffs in QED [1].

Are there better auxiliary regularizations that preserve the Ward identities? There are precious few. In this paper we show that dimensional regularization [6, 7] can be used as an auxiliary regularization in the tree expansion; and we illustrate its use in QED, as a simpler alternative to the methods of [1]. The basic idea of dimensional regularization (dr) is to regularize a graph in d dimensions by evaluating it as though it were coming from $\nu < d$ dimensions (ν not necessarily a positive integer). For sufficiently small ν the regularized graphs have no UV divergences and yet Ward identities are maintained since, intuitively, they hold in "ν dimensions."

In spirit, our treatment of dr follows that of Breitenlohner and Maison [7]. However, in contrast to these and other authors, we shall work in x- rather than in p-space. Aside from our beliefs that it is more natural to regularize the dimension of the underlying *coordinate* space and that the resulting algebraic structure is clearer in x- than in p-space, our main reason for this choice is that the tree expansion is best carried out in x-space; in particular, by regularizing in x-space, we can easily obtain the bounds on graphs needed to establish renormalizability. Also, in contrast to most other treatments of dr (see e.g. [8]), our analysis involves neither an explicit computation of the value of a graph nor an investigation of its meromorphic structure as a function of ν. To renormalize a graph we do not subtract off poles in ν; instead we renormalize directly with Lagrangian counterterms defined in x-space. It should be possible to prove that these two subtraction schemes are equivalent, i.e. differ by a finite renormalization. We offer no such proof here. Rather, the onus is on the minimalists to demonstrate that their scheme is equivalent to a Lagrangian counterterm scheme and hence respects unitarity. Such a demonstration can be quite intricate (see, for example, [9]).

We restrict our attention in this paper to the example of QED_4. Thus we are not concerned with the problem of defining objects like γ^5 or $\varepsilon_{\mu\nu\lambda\sigma}$ [7]. Another simplification in QED is that one can place UV and IR cutoffs on photon lines which do not break the Ward identities.

Given a (Euclidean) QFT with fields Φ defined on $\mathbf{R}^d$, free (quadratic) Lagrangian $\mathscr{L}_0$, and interaction Lagrangian $\mathscr{L}_I$, the tree expansion analyzes the generator of connected, amputated Green's functions, the "effective potential,"

$$V(\Phi^e) = [\log \mathscr{E}(e^{V_I(\Phi + \Phi^e)})]_0. \tag{1.1}$$

Here $\mathscr{E}$ is the Gaussian expectation with respect to Φ with density

$\exp[-\int \mathscr{L}_0(\Phi(x))dx]$; $V_I(\Phi) = -\int \mathscr{L}_I(\Phi(x))dx$ is the interaction potential; Φ^e is the set of external fields; and the notation $[\cdots]_0$ means "drop terms independent of Φ^e." For the purposes of perturbative renormalization theory, we interpret $V(\Phi^e)$ as a formal power series (fps) in the coupling constant(s) and fields Φ^e. As such, $V(\Phi^e)$ can be evaluated in terms of connected Feynman graphs whose external legs correspond to fields Φ^e.

We now give our definition of x-space dr. Consider a (connected) graph G contributing to (1.1) for QED$_d$. Its lines $\mathscr{L}(G)$ are either bosonic or fermionic, $\mathscr{L} = \mathscr{L}_b \cup \mathscr{L}_f$; its legs $\Lambda(G)$ are half-lines corresponding to external or uncontracted fields; each of its vertices $v \in \mathscr{V}(G)$ has two attached fermion lines or legs and one attached boson line or leg and carries a coordinate x and an index μ (corresponding to its photon field $A^\mu(x)$). Let $V = |\mathscr{V}(G)|$ and $L = |\mathscr{L}(G)|$. Each line $l \in \mathscr{L}_b$, arising from the contraction of the photon fields $A^{\mu_l}(x_l)$ and $A^{\nu_l}(y_l)$, contributes the propagator (in Feynman gauge) $\delta^{\mu_l \nu_l} C(x_l, y_l)$, where

$$C(x_l, y_l) = (-\Delta)^{-1}(x_l, y_l) = \int_0^\infty d\alpha_l e^{\alpha_l \Delta}(x_l, y_l)$$

$$= \int_0^\infty d\alpha_l (4\pi\alpha_l)^{-d/2} e^{-z_l^2/4\alpha_l}, \tag{1.2a}$$

where $z_l = x_l - y_l$. A line $l \in \mathscr{L}_f$, arising from the contraction of the fermion fields $\psi(x_l)$ and $\bar{\psi}(y_l)$, contributes the propagator

$$S(x_l, y_l) = (-i\partial\!\!\!/ + m)^{-1}(x_l, y_l) = \int_0^\infty d\alpha_l (i\partial\!\!\!/_x + m) e^{\alpha_l(\Delta - m^2)}(x_l, y_l)$$

$$= \int_0^\infty d\alpha_l (4\pi\alpha_l)^{-d/2} f_l e^{-z_l^2/\alpha_l - m^2\alpha_l}, \tag{1.2b}$$

where

$$f_l = -i z\!\!\!/_l / 2\alpha_l + m, \quad z\!\!\!/ = z_\mu \gamma^\mu,$$

the γ^μ's being Euclidean Dirac matrices with

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\delta^{\mu\nu}.$$

The value of the graph G is then

$$G = c \int d\alpha \prod_{l \in \mathscr{L}_f} e^{-m^2 \alpha_l} \beta_{\mathscr{L}}^{d/2} \int dx\, p e^{-b/4} \Pi^e(x^e), \tag{1.3}$$

where $c = (4\pi)^{-dL/2}$, $\alpha = (\alpha_l)_{l \in \mathscr{L}}$ with each α_l integrated from 0 to ∞, $\beta_l = \alpha_l^{-1}$, $\beta_{\mathscr{L}} = \prod_{l \in \mathscr{L}} \beta_l$, $x = (x_1, \ldots, x_V)$, $\int dx = \int dx_1 \cdots dx_V$,

$$p = \prod_{l \in \mathscr{L}_f} f_l \prod_{l \in \mathscr{L}_b} \delta^{\mu_l \nu_l} \prod_{v \in \mathscr{V}} \gamma^{\mu_v} \tag{1.4}$$

(repeated indices summed over), $b = \sum_l \beta_l z_l^2$, and $\Pi^e(x^e)$ is the product of external fields corresponding to the legs of G with x^e representing the coordinates of the external vertices $\mathscr{V}^e$ (i.e. those vertices having an attached leg). We suppress the vector and spinor indices on Π^e and the spinor indices on the γ's in p, but it is understood that the fermi fields in Π^e occur in a specific order and that the γ's in p occur in ordered products and traces corresponding to the lines and loops of G. We also break the translation invariance of $pe^{-b/4}$ in (1.3) by setting $x_V = 0$ (which we assume is an external vertex). Then the $(V-1)\times(V-1)$ matrix B defined by

$$b = \sum \beta_l z_l^2 |_{x_V=0} \equiv xBx, \quad x = (x_1, \ldots, x_{V-1})$$

is non-singular.

In general, the expression (1.3) diverges because of UV singularities at $\alpha = 0$ (as well as IR singularities at $\alpha = \infty$, which we deal with below). We define the UV regularized version G_ν of G by making the following replacements (i)–(v):

i) $\beta_{\mathscr{L}}^{d/2} \to \beta_{\mathscr{L}}^{\nu/2}$ where $\nu < d$.
For ν small enough, this replacement removes any UV divergences but on its own does not give an expression "coming from ν dimensions" and so cannot be expected to have the appropriate invariance properties.

ii) For each coordinate x in p or b, $x \to X = (x, \hat{x})$.
Here, $\hat{x}$ is a formal symbol whose calculus we specify below so as to be consistent with the calculus when ν is an integer $> d$, in which case $\hat{x} \in \mathbf{R}^{\nu-d}$ and $X \in \mathbf{R}^\nu$. Note that the arguments of $\Pi^e(x^e)$ are not affected by ii). Let $\mathscr{V}_0 \subset \mathscr{V}^e$ denote those vertices with an attached photon leg, and $\mathscr{V}_1 = \mathscr{V} \backslash \mathscr{V}_0$ those without.

iii) For each Dirac matrix in a fermi factor f_l or associated with a vertex $v \in \mathscr{V}_1$ in (1.4), $\gamma \to \Gamma = (\gamma, \hat{\gamma})$.

iv) Each Kronecker delta in (1.4), $\delta^{\alpha\beta} \to \Delta^{\alpha\beta} = \delta^{\alpha\beta} + \hat{\delta}^{\alpha\beta}$.
The formal symbols $\hat{x}, \hat{\gamma}$ and $\hat{\delta}$ satisfy algebraic rules appropriate to "$(\nu - d)$ dimensions":

$$\{\hat{\gamma}^{\mu_1}, \hat{\gamma}^{\mu_2}\} = -2\hat{\delta}^{\mu_1\mu_2}, \quad \{\gamma^{\mu_1}, \hat{\gamma}^{\mu_2}\} = 0, \tag{1.5}$$

$$\hat{\delta}^{\mu_1\mu_2} = \hat{\delta}^{\mu_2\mu_1}, \quad \hat{\delta}^{\mu_1\mu_2}\hat{x}^{\mu_2} = \hat{x}^{\mu_1}, \quad \hat{\delta}^{\mu_1\mu_2}\hat{\gamma}^{\mu_2} = \hat{\gamma}^{\mu_1}, \tag{1.6}$$

$$\hat{\delta}^{\mu\mu} = \nu - d. \tag{1.7a}$$

We require no further algebraic structure for the $\hat{\gamma}$'s, such as a representation of $\hat{\gamma}^\mu$ as a matrix, a product rule for $\hat{\gamma}$'s or a "trace" on products of $\hat{\gamma}$'s. By Rule v) below, the $\hat{\gamma}$'s will always occur in pairs $\hat{\gamma}^\mu\hat{\gamma}^\mu$ which we can evaluate by (1.5) and (1.7a):

$$\hat{\gamma}^\mu\hat{\gamma}^\mu = d - \nu. \tag{1.7b}$$

Note also that a repeated index μ on $\hat{}$-objects (as in (1.6) or (1.7)) is not actually "summed over."

According to ii)–iv),

$$p \to P = \prod_{l \in \mathscr{L}_f} F_l \prod_{l \in \mathscr{L}_b} \Delta^{\mu_l\nu_l} \prod_{v \in \mathscr{V}_1} \Gamma^{\mu_v} \prod_{v \in \mathscr{V}_0} \gamma^{\mu_v}, \tag{1.8}$$

where

$$F_l = -i\beta_l \not{Z}_l/2 + m,$$
$$\not{Z}_l = \Gamma^\mu Z_l^\mu = \gamma^\mu z_l^\mu + \hat{\gamma}^\mu \hat{z}_l^\mu = \not{z}_l + \hat{\not{z}}_l,$$
$$b \to \mathscr{B} = \sum \beta_l Z_l^2 = \sum \beta_l z_l^2 + \sum \beta_l \hat{z}_l^2 = b + \hat{b}.$$

v) $\int dx \to \int dX = \int dx \int d\hat{x}$, where the computational rule for $\int d\hat{x}$ is Gaussian and follows formally from $\int d\hat{x}\cdot = \int d\hat{x}_1 \cdots d\hat{x}_{V-1} \cdot |_{\hat{x}_V = 0}$:

$$\int d\hat{x} e^{-\hat{b}/4} = (c_1 |B|)^{(d-\nu)/2} \tag{1.9a}$$

$$\int d\hat{x} \prod_{r=1}^{s} \hat{x}_{i_r}^{\mu_r} e^{-\hat{b}/4} = \begin{cases} (c_1|B|)^{(d-\nu)/2} \sum_{\hat{G}} \prod_{l\in\mathscr{L}(\hat{G})} (2B^{-1})_{i_l j_l} \hat{\delta}^{\mu_l \nu_l} & s \text{ even} \\ 0 & s \text{ odd}, \end{cases} \tag{1.9b}$$

where $c_1 = (4\pi)^{1-V}$ and $\hat{G}$ is summed over graphs whose lines join the s $\hat{x}$'s in pairs, with $l \in \mathscr{L}(\hat{G})$ joining $\hat{x}_{i_l}^{\mu_l}$ to $\hat{x}_{j_l}^{\nu_l}$.

The replacements i)–v) and the calculus (1.5)–(1.7), (1.9) define the regularized graph

$$G_\nu = c \int d\alpha \prod_{l\in\mathscr{L}_f} e^{-m^2\alpha_l} \beta_{\mathscr{L}}^{\nu/2} \int dX\, P e^{-\mathscr{B}/4} \Pi^e(x^e). \tag{1.10}$$

We shall be more explicit about the form of G_ν in Sect. 3 and the reader may wish at this point to skip to the example after Corollary 3.2, but for now consider the "leading term" in (1.10), which has the same form as (1.10) but with p in place of P. According to (1.9a) this leading term differs from the value G of (1.3) by virtue of the additional factor

$$\beta_{\mathscr{L}}^{(\nu-d)/2} |B|^{(d-\nu)/2} \equiv U_G(\alpha)^{(d-\nu)/2}. \tag{1.11}$$

The factor U_G is a homogeneous polynomial in α of degree $L - V + 1$ [10] and provides the needed UV regularization at $\alpha = 0$ for ν small enough. Our general strategy in using the dr expression (1.10) will be to integrate out the $\hat{x}$'s when we want bounds but to leave the $\hat{x}$-integrals intact when we want relations such as Ward identities.

Now (1.10) will still have IR singularities as $\alpha \to \infty$ (worsened by the factor (1.11)!). To deal with these we simply insert a cutoff on the photon lines, $\alpha_l \leqq M^{-2I}$, where $M > 1$ is fixed and the IR cutoff $I > -\infty$. We are free to do so in QED because cutoffs and slicing on the photon lines do not disturb the Ward identities. At the end of Sect. 4 we indicate how to remove the IR cutoff, $I \to -\infty$, after renormalization.

Consider the dr version of (1.1) which we write as

$$V_\nu(\Phi^e) = \mathscr{C}_\nu(e^{VI(\Phi + \Phi^e)}), \tag{1.12}$$

where $\mathscr{C}_\nu(F(\Phi + \Phi^e))$ denotes the sum (fps) of connected graphs contributing to $[\mathscr{E}(F(\Phi + \Phi^e))]_0$ with each graph dimensionally regularized. As we show in Corollary 3.2, the graphs contributing to V_ν are finite when $\nu < 2$ and $I > -\infty$. We renormalize V_ν with the tree expansion counterterms appropriate to $d = 4$ dimensions to obtain the renormalized dr effective potential $V_{\mathrm{ren},\nu}(\Phi^e)$. This subtraction scheme does not entail a knowledge of the meromorphic structure of V_ν as a function of ν. Although oversubtracted when $\nu < 4$, $V_{\mathrm{ren},\nu}$ is finite for $\nu \leqq 4$ and is consequently an analytic function of ν for $\mathrm{Re}\,\nu < 4$ (see Theorem 4.2 and Remark 3 following it).

What about the Ward identities? Why do they hold for V_ν or $V_{\mathrm{ren},\nu}$? Although the "expectation" $\mathscr{C}_\nu$ is not given by a genuine integration over fields, it nonetheless satisfies "integration by parts" formulas with respect to ψ and $\bar{\psi}$, such as (see Lemma 3.5):

$$(-i\not\partial_x + m)\mathscr{C}_\nu(\psi(x)e^{V_I(\Phi+\Phi^e)}) = \frac{\delta}{\delta\bar{\psi}^e(x)}\mathscr{C}_\nu(e^{V_I(\Phi+\Phi^e)}). \tag{1.13}$$

These identities correspond to what Breitenlohner and Maison call the "Action Principle" [7].

Choosing $\nu<2$ (and $I>-\infty$) to ensure finiteness of all graphs and counterterms (see Corollary 3.2), we use the integration by parts identities to establish Ward identities for the effective potentials V_ν and $V_{\text{ren},\nu}$ (Corollary 5.4). But $V_{\text{ren},\nu}$ is an analytic function of ν for $\operatorname{Re}\nu<4$ and so the Ward identities for it immediately continue to $\operatorname{Re}\nu \leqq 4$. This guarantees the gauge invariance of the renormalization procedure. For QED_4 it is also possible to make a somewhat more direct version of this statement: with IR and UV cutoffs $I>-\infty$ and $U_p<\infty$ on the photon lines all counterterms are finite when $\nu<4$ except for the mass counterterm to the second order vacuum polarization graph. Consequently, if we analytically continue this one graph to $\nu<4$, we can assert that the other counterterms required to renormalize QED_4 (finite for $\nu<4$ and $-\infty<I\leqq 0\leqq U_p<\infty$) are of gauge invariant form (see Sect. 5).

The main conclusion of this paper is that x-space dr provides an elegant auxiliary regularization that preserves Ward identities and combines clearly with the tree expansion approach to renormalization. We believe that this regularization will prove very useful in the application of the tree expansion to non-abelian gauge theories. Unfortunately, it does not seem possible to implement dr at the functional integral level, and so we are dubious that it will be a useful tool in non-perturbative analyses.

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2. Review of the Tree Expansion

We outline here the tree expansion procedure for renormalizing a field theory as discovered by Gallavotti and Nicolò [2] and developed by Feldman, Hurd, Rosen and Wright [1, 3]. For full details and proofs see these references.

Our description will centre on the example of (Euclidean) QED_d with fields $\Phi=(\Phi_1,\Phi_2,\Phi_3)=(A,\psi,\bar{\psi})$ and Lagrangian $\mathscr{L}=\mathscr{L}_0+\mathscr{L}_I$, where (in Feynman gauge)

$$\mathscr{L}_0 = -\tfrac{1}{2}A\cdot\Delta A + \bar{\psi}(-i\not\partial + m)\psi \quad \text{and} \quad \mathscr{L}_I = e\psi\not{A}\bar{\psi}.$$

In contrast to [1] we shall not Wick order the graphs in the effective potential and so our trees will be slightly different from those of [1].

The unrenormalized effective potential $V(\Phi^e)$ is given by (1.1) as a fps in e whose coefficients may be expressed in terms of connected Feynman graphs with external legs corresponding to the external fields $\Phi^e=(A^e,\psi^e,\bar{\psi}^e)$. Of course, (1.1) is only formal: the fps coefficients are in general divergent. The central task of (perturbative) renormalization theory is to introduce regularizations (which we denote by $N<\infty$) so that the regularized version of (1.1),

$$V_N(\Phi^e) = [\log\mathscr{E}_N(e^{V_I(\Phi+\Phi^e)})]_0, \tag{2.1}$$

has a well-defined (but not necessarily convergent!) fps and to introduce counterterms

$$\delta V_N(\Phi) = -\int \delta\mathscr{L}(\Phi(x))dx, \tag{2.2}$$

which cancel the would-be infinities of V_N so that the renormalized effective potential

$$V_{\text{ren}}(\Phi^e) = \lim_{N\to\infty} [\log \mathscr{E}_N(e^{(V_I+\delta V_N)(\Phi+\Phi^e)})]_0 \tag{2.3}$$

has a well-defined fps. The counterterms δV_N are to be chosen as a fps in e (with finite coefficients when $N<\infty$) and are supposed to have the same form as the terms in the original $\mathscr{L}$ ("local Lagrangian counterterms"). In particular, if $\mathscr{L}$ is invariant under a gauge group, the counterterms $\delta\mathscr{L}$ are required to respect this gauge invariance. This gives rise to the main technical complication in renormalizing a gauge field theory: most convenient regularizations N break gauge invariance and so apparently must δV_N.

The strategy we adopted in [1] to overcome this difficulty was to introduce regularizations convenient for the tree expansion: N, a UV cutoff on the electron propagator; U, a UV cutoff on the photon propagator; I, an IR cutoff on the photon propagator; as well as an auxiliary regularization Λ, on fermi loops (implemented by fictitious spinor fields). We ran the tree expansion with all 4 regularizations in place using counterterms $\delta V_{I,U,\Lambda,N}$ that were gauge *variant* and, because of the need to maintain the loop regularization, could not be chosen so as to renormalize graphs with fictitious field legs correctly. We then took $N\to\infty$. In the $N=\infty$ limit the theory is finite (order by order in perturbation theory) and the Ward identities are recovered. Consequently, the (finite) counterterms $\delta V_{I,U,\Lambda} = \lim_{N\to\infty} \delta V_{I,U,\Lambda,N}$ are gauge *invariant*. We then removed the remaining cutoffs but it was crucial to take $\Lambda\to\infty$ first (followed by $U\to\infty$ and then $I\to-\infty$) in order to control the incorrectly renormalized fictitious field graphs.

In this paper we replace loop regularization Λ by dimensional regularization $\nu<4$ with a considerable reduction in technical difficulties. We describe the tree expansion in this section without ν but include it in subsequent sections.

The first step in the tree expansion is to decompose the propagators (1.2) into α-space slices. Fix $M>1$ and for $h=0,\pm 1,\dots$ set

$$C^{(h)}(x_l,y_l) = \int_0^\infty d\alpha_l \chi^{(h)}(\alpha_l)(4\pi\alpha_l)^{-d/2}e^{-z_l^2/4\alpha_l}, \tag{2.4}$$

where

$$\chi^{(h)} = \text{characteristic fn. of } [M^{-2h}, M^{-2h+2}]. \tag{2.5}$$

For $h=0,1,2,\dots$ we set

$$S^{(h)}(x_l,y_l) = \int_0^\infty d\alpha_l \chi_+^{(h)}(\alpha_l)(4\pi\alpha_l)^{-d/2} f_l e^{-z_l^2/4\alpha_l - \alpha_l m^2}, \tag{2.6}$$

where f_l is given after (1.2), $\chi_+^{(h)} = \chi^{(h)}$ for $h>0$, and

$$\chi_+^{(0)} = \text{characteristic fn. of } [1,\infty). \tag{2.7}$$

(For massive particles there is no need to slice up the IR regime $\alpha_l \geqq 1$.) Thus

$$C = \sum_{h=-\infty}^{\infty} C^{(h)} \quad \text{and} \quad S = \sum_{h=0}^{\infty} S^{(h)}. \tag{2.8}$$

The fields corresponding to these slice covariances are denoted $\Phi^{(h)}$ and the corresponding Gaussian expectations $\mathscr{E}^{(h)}$. For example,

$$S^{(h)}(x_l, y_l) = \mathscr{E}^{(h)}(\psi^{(h)}(x_l)\bar{\psi}^{(h)}(y_l)).$$

Corresponding to the decompositions (2.8) we have

$$A = \sum_{h=-\infty}^{\infty} A^{(h)}, \quad \psi = \sum_{h=0}^{\infty} \psi^{(h)} \quad \text{and} \quad \bar{\psi} = \sum_{h=0}^{\infty} \bar{\psi}^{(h)}.$$

The UV- and IR-regularized fields are (we take the UV cutoff to be the same on the photon and electron fields)

$$A^{[I,U]} = \sum_{h=I}^{U} A^{(h)}, \quad \psi^{[0,U]} = \sum_{h=0}^{U} \psi^{(h)}, \quad \text{etc.,} \tag{2.9}$$

and the corresponding cutoff propagators are

$$C^{[I,U]} = \sum_{h=I}^{U} C^{(h)} \quad \text{and} \quad S^{[0,U]} = \sum_{h=0}^{U} S^{(h)}.$$

For the remainder of this section and until the end of Sect. 4 we shall assume that the IR cutoff is fixed at $I = 0$ and we shall write $\Phi^{(-1)}$ for the external field $\Phi^{(e)}$ and $\Phi^{(\leqq k)}$ for $\sum_{h=-1}^{k} \Phi^{(h)}$. We also write $\mathscr{E}^U$ for the Gaussian expectation $\prod_{h=0}^{U} \mathscr{E}^{(h)}$.

The renormalized effective potential with UV cutoff U (and IR cutoff $I = 0$) is given by

$$V^U(\Phi^e) = [\log \mathscr{E}^U(e^{V_I(\Phi^{(\leqq U)})})]_0. \tag{2.10}$$

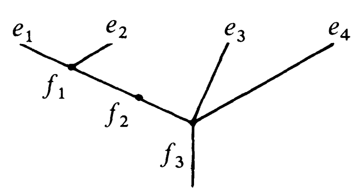
The tree expansion for V^U is obtained by successively integrating out the fields $\Phi^{(U)}, \Phi^{(U-1)}, \dots, \Phi^{(0)}$ and performing a cumulant expansion

$$\log \mathscr{E}^{(h)}(e^W) = \sum_{p=1}^{\infty} \frac{1}{p!} \mathscr{E}_T^{(h)}(\underbrace{W, \dots, W}_{p \text{ arguments}}) \tag{2.11}$$

after each expectation. Here $\mathscr{E}_T^{(h)}$ denotes the truncated or connected expectation.

A *tree* τ is a tree graph (i.e. no closed loops) with a distinguished end-vertex at the bottom (the root), the other end-vertices at the top (called *endpoints*), and each remaining vertex f (called a *fork*) having one line down and the other $p_f \geqq 1$ lines going up. We denote the set of endpoints by $\mathscr{E}(\tau)$ and forks by $\mathscr{F}(\tau)$. The structure of a tree τ determines a natural partial ordering on the set $\mathscr{E}(\tau) \cup \mathscr{F}(\tau)$: $v_1 < v_2$ if v_1 is below v_2. For example the tree

$\tau =$



(2.12)

has 4 endpoints and 3 forks with $f_3 < f_2 < f_1 < e_1, f_1 < e_2, f_3 < e_3$, and $f_3 < e_4$. Each $f \in \mathscr{F}(\tau)$ bears a *scale* lebel h_f such that the scales $h = (h_f)_{f \in \mathscr{F}}$ belong to the set

$$\mathscr{H}(\tau) = \{h \mid 0 \leqq h_{f_1} < h_{f_2} \quad \text{if} \quad f_1 < f_2\}. \tag{2.13}$$

The *root scale* of τ is -1. Given $f \in \mathscr{F}(\tau)$ we let τ_f be the subtree of τ with lowest fork f and root $\pi(f)$, where $\pi(f)$ is the fork of τ immediately below f (if f is the lowest fork of τ then $\pi(f)$ is the root of τ). If e is an endpoint of τ we let τ_e be the trivial tree with single endpoint e and no forks. The *root scale* of τ_v (v a fork or endpoint) is $h_{\pi(v)}$.

The *value* $V^U(\tau, h)$ of a tree is 0 if any $h_f > U$; if every $h_f \leqq U$ the value $V^U(\tau, h)$ is most easily described inductively: if $e \in \mathscr{E}(\tau)$ then $V^U(\tau_e, h) = V_I(\Phi^{(\leqq h_{\pi(e)})})$. If $v_1, \ldots, v_p$ are the forks and endpoints immediately above a fork f, then for $p > 1$

$$V^U(\tau_f, h) = \frac{1}{p!}[Z^{(k,h_f)}\mathscr{E}_T^{(h_f)}(V^U(\tau_{v_1}, h), \ldots, V^U(\tau_{v_p}, h))]_0, \tag{2.14a}$$

where $k = h_{\pi(f)}$, $Z^{(k,h_f)}$ means that the fields $\Phi^{(k+1)} = \cdots = \Phi^{(h_f - 1)} = 0$ and now $[\cdots]_0$ means "drop terms independent of $\Phi^{(\leqq k)}$"; for $p = 1$

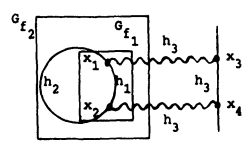
$$V^U(\tau_f, h) = [Z^{(k,h_f)}\mathscr{E}^{(h_f)}(V^U(\tau_{v_1}, h))]_0 - Z^{(k,h_f]}V^U(\tau_{v_1}, h) \tag{2.14b}$$

so that at least one contraction of a pair of fields $\Phi^{(h_f)}$ occurs.

Iteration of the cumulant expansion (2.11) yields the *unrenormalized tree expansion*

$$V^U = \sum_\tau \sum_{h \in \mathscr{H}(\tau)} V^U(\tau, h). \tag{2.15}$$

$V^U(\tau, h)$ can be expressed as a sum over connected graphs whose vertices correspond to endpoints of τ, whose lines are propagators at a specific scale h_f and whose legs correspond to external fields Φ^e. For example, a graph contibuting to $V^U(\tau, h)$ for τ given by (2.12) is

G = 

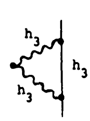
(2.16)

where we write $h_j = h_{f_j}$. Then h_1- and h_2-lines correspond to propagators $S^{(h_1)}$ and $S^{(h_2)}$ and the h_3-lines to two $C^{(h_3)}$'s and one $S^{(h_3)}$. The vertex x_j of G corresponds to the endpoint e_j of τ. The legs attached to the vertices x_3 and x_4 correspond to external fields $\psi^e(x_3)$ and $\bar{\psi}^e(x_4)$.

Let G_f be the subgraph of G whose vertices correspond to endpoints of τ_f and whose lines correspond to propagators formed by contractions of fields at forks $\geqq f$. In the example (2.16) the subgraphs G_{f_1} and G_{f_2} are boxed in. If $\pi(f') = f$ we view $G_{f'}$ as a generalized vertex for the graph G_f and we consider the reduced graph

$$g_f = G_f / \{G_{f'} | \pi(f') = f\}$$

formed by contracting each $G_{f'}$ to a point. In the example,

$g_{f_3} = G_{f_3}/\{G_{f_2}\} =$ 

.

The connectivity requirement on the graphs $G \in \mathscr{G}(\tau)$ *associated with* τ is that each such g_f must be connected.

The value of a graph G is as given in (1.3) except that because each line l is sliced as in (2.5)–(2.8) there is an additional factor

$$\chi^h(\alpha) = \prod_{l \in \mathscr{L}_f} \chi_+^{(h_{f(l)})}(\alpha_l) \prod_{l \in \mathscr{L}_b} \chi^{(h_{f(l)})}(\alpha_l), \tag{2.17}$$

where $f(l)$ is the fork at which the line l is formed. Thus the value of G is

$$G^U = \sum_{h \in \mathscr{H}(\tau)} G^{h,U} \equiv \sum_{h \in \mathscr{H}(\tau)} \int K^{h,U}(x) \Pi^e(x^e) dx$$

$$\equiv \sum_{h \in \mathscr{H}(\tau)} \int d\alpha \chi^h(\alpha) K^U(\alpha, x) \Pi^e(x^e) dx, \tag{2.18a}$$

where $K^U(\alpha, x) = 0$ if any $\alpha_l < M^{-2U}$ and otherwise

$$\mathrm{K}^U(\alpha, x) = c \prod_{l \in \mathscr{L}_f} e^{-m^2 \alpha_l} \beta_{\mathscr{L}}^{d/2} p e^{-b/4} \tag{2.18b}$$

and p and b are given by (1.4). In terms of graphs the unrenormalized tree expansion (2.15) takes the form

$$V^U = \sum_\tau \sum_{h \in \mathscr{H}(\tau)} \sum_{G \in \mathscr{G}(\tau)} G^{h,U}. \tag{2.19}$$

For details of combinatoric factors etc., see [1]. Note that $|\mathscr{E}(\tau)| = |\mathscr{V}(G)| =$ the order of perturbation theory, and that (2.19) is interpreted as a fps in e.

Are the coefficients in this perturbation expansion finite, uniformly in U? We estimate the size of $G^{h,U}$ by the "pinned L^1-norm" of its kernel:

$$\| K^{h,U} \|_0 = \int |K^{h,U}(x)| |_{x_V = 0} dx_1 \cdots dx_{V-1}. \tag{2.20}$$

The following bound is completely elementary, and the equality follows by a summation by parts [1, Lemma 2.1]:

Lemma 2.1.

$$\| K^{h,U} \|_0 \leqq c \prod_{f \in \mathscr{F}(\tau)} M^{D_d(g_f) h_f} = c \prod_{f \in \mathscr{F}(\tau)} M^{D_d(G_f)(h_f - h_{\pi(f)})}, \tag{2.21}$$

where $D_d(G)$ *is the* UV *degree of divergence of a graph* G,

$$D_d(G) = (d-1)L_f + (d-2)L_b - d(V-1) = d\Lambda - L_f - 2L_b \tag{2.22}$$

and L_f, L_b, V *and* Λ *are the number of fermi lines, bose lines, vertices and independent loops of* G, *respectively.*

Remark. We shall use the letter c to denote various constants that are independent of variables such as x, α, h but may depend on G, at worst like c_0^L, where $L = |\mathscr{L}(G)|$.

If every subgraph G_f of G has $D_d(G_f) < 0$, then the sums over $h_f > h_{\pi(f)}$ in (2.18a) converge uniformly in U and the graph G is finite (this is the Dyson–Weinberg Power Counting Theorem). If $D_d(G_f) \geqq 0$ then the subgraph G_f requires renormalization.

Now G_f has the general form

$$G_f^U = \int K_f^U(x) \Pi_f(x) dx,$$

where $x=(x_1,\dots,x_n)$ and

$$\Pi_f(x)=\partial^{q_1}\Phi_{i_1}^{(\leqq k)}(x_1)\cdots\partial^{q_n}\Phi_{i_n}^{(\leqq k)}(x_n),\tag{2.23}$$

where the x-derivatives ∂^{q_j} arise from renormalization operations we are about to define and $k=h_{\pi(f)}$. If Π_f has λ_f^b bose fields $\Phi_1=A$, λ_f^f fermi fields $\Phi_2=\psi$ and $\Phi_3=\bar{\psi}$, and a total of q_f derivatives, then the *dimension* of Π_f is $\frac{d-2}{2}\lambda_f^b+\frac{d-1}{2}\lambda_f^f+q_f$ and the *degree* of G_f is

$$\delta(G_f)=d-\dim\Pi_f=d-\frac{d-2}{2}\lambda_f^b-\frac{d-1}{2}\lambda_f^f-q_f.\tag{2.24}$$

For simplicity we assume that $d=4$ in which case e is dimensionless and $\delta(G_f)=D_4(G_f)-q_f$. We select one of $x_1,\dots,x_n$ as the *localization vertex* x_f of G_f and let

$$x(t_f)=(x_1(t_f),\dots,x_n(t_f)),\quad x_j(t_f)=x_f+t_f(x_j-x_f),\tag{2.25}$$

for $0\leqq t_f\leqq 1$. The *local parts* of Π_f and G_f are

$$L\Pi_f(x)=\begin{cases}0 & \text{if } \delta(G_f)<0\\ \sum_{i=0}^{\delta(G_f)}\frac{1}{i!}\partial_{t_f}^i\Pi_f(x(t_f))|_{t_f=0} & \text{if } \delta(G_f)\geqq 0\end{cases}\tag{2.26a}$$

and

$$LG_f=\int K_f(x)L\Pi_f(x)dx.\tag{2.26b}$$

The *renormalization* of G_f is

$$RG_f=(1-L)G_f.\tag{2.27}$$

Suppose $\mu_f=\delta(G_f)+1>0$. By Taylor's Theorem

$$RG_f=\frac{1}{\mu_f!}\int_0^1 dt_f(1-t_f)^{\mu_f-1}\int K_f(x)(\Delta\cdot\partial)^{\mu_f}\Pi_f(x(t_f))dx,\tag{2.28}$$

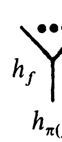
where $\Delta\cdot\partial=\sum_j(x_j-x_f)\cdot\partial_{x_j}$. The factor Δ^{μ_f} together with the Gaussian $e^{-b/4}$ in K_f produces an extra factor $M^{-\mu_f h_f}$ in the power counting:

$$|\Delta^{\mu_f}e^{-b/4}|\leqq cM^{-\mu_f h_f}e^{-b/8}.\tag{2.29}$$

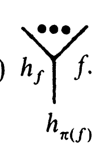
This converts the bad (unrenormalized) power counting factor into a good one:

$$M^{\delta(G_f)(h_f-h_{\pi(f)})}\to M^{-(h_f-h_{\pi(f)})}.$$

These renormalization cancellations are introduced into the tree expansion as follows. The label R attached to a fork f means



$f,R \;=\; \chi(h_f>h_{\pi(f)})(1-L)$



(2.30a)

In order to implement these subtractions with legitimate counterterms in the potential we have to include the local parts for all h_f, not just $h_f > h_{\pi(f)}$. So we must also include the following "useless counterterms:"

$$\underset{h_{\pi(f)}}{h_f \,\Upsilon\, f, C} = -\chi(h_f \leqq h_{\pi(f)}) L \;\underset{h_{\pi(f)}}{h_f \,\Upsilon\, f}. \tag{2.30b}$$

A *renormalized tree* τ is a tree with a label $\rho_f = R$ or C at each fork f whose value $V^U(\tau, \rho, h)$ is defined as in (2.14) but with the modifications (2.30). We let $\mathscr{F}_R = \{f \in \mathscr{F} \mid \rho_f = R\}$ and $\mathscr{F}_C = \mathscr{F} \backslash \mathscr{F}_R$. The appropriate set of scales for a renormalized tree is

$$\mathscr{H}(\tau, \rho) = \{h \mid h_{\pi(f)} < h_f \text{ if } f \in \mathscr{F}_R; 0 \leqq h_f \leqq h_{\pi(f)} \text{ if } f \in \mathscr{F}_C\}. \tag{2.31}$$

The counterterms δV^U are defined as

$$\delta V^U = \sum_{\tau \text{ n.t.}} \sum_{\rho : \rho_F = C} \sum_{h \in \mathscr{H}_C(\tau, \rho)} V^U(\tau, \rho, h), \tag{2.32}$$

where the sums are over non-trivial trees τ, ρ's with $\rho_F = C$ for the lowest fork F, and scales h in the set $\mathscr{H}_C(\tau, \rho)$ defined as in (2.31) except that the root scale $h_{\pi(F)}$ is taken to be U instead of -1. Clearly δV^U is a fps in e whose coefficients are local polynomials in the fields $\Phi^{(\leqq U)}$.

As in (2.10) we define the renormalized effective potential

$$V^U_{\text{ren}}(\Phi^e) = [\log \mathscr{E}^U(e^{V_I + \delta V^U})]_0, \tag{2.33}$$

and as in (2.15) we have:

Theorem 2.2. (*Renormalized Tree Expansion*).

$$V^U_{\text{ren}} = V_I + \sum_{\tau \text{ n.t.}} \sum_{\rho : \rho_F = R} \sum_{h \in \mathscr{H}(\tau, \rho)} V^U(\tau, \rho, h). \tag{2.34a}$$

As in (2.19), $V(\tau, \rho, h)$ can be expanded as a sum of graphs. Each unrenormalized graph $G \in \mathscr{G}(\tau)$ gives rise to a number of renormalized graphs $G_{\text{ren}} \in \mathscr{G}(\tau, \rho)$ according to the choice of i in (2.26a) at a C-fork and to how the derivatives δ_{t_f} act in (2.26a) and (2.28). We write

$$V^U(\tau, \rho, h) = \sum_{G_{\text{ren}} \in \mathscr{G}(\tau, \rho)} G^{h,U}_{\text{ren}}, \tag{2.34b}$$

where the value $G^{h,U}_{\text{ren}}$ is similar to that of $G^{h,U}$ in (2.18) except that R and $-L$ operations are applied to each subgraph as stipulated by ρ and there are resulting integrals over the interpolating parameters $t = (t_f)_{f \in \mathscr{F}(\tau)}$ (see (2.28)):

$$G^{h,U}_{\text{ren}} = \int d\mu(t) \int dx K^{h,U}_{\text{ren}}(x, t) \Pi^e(x^e(t)), \tag{2.35}$$

where $d\mu$ is a positive measure, Π^e is a product of external fields and their derivatives and $x^e(t)$ is the set of external coordinates x^e, appropriately interpolated. For a more detailed description see Appendix B of [1].

The t-interpolated version of (1.11)

$$U_G(\alpha, t)^{-1} = c\beta_{\mathscr{L}}[\int dx e^{-b(t)/4}|_{x_V = 0}]^{2/d} \tag{2.36}$$

satisfies [1, Lemma B.2];

Lemma 2.3.

$$U_G(\alpha,t)^{-1} \leqq c \prod_f M^{2\lambda_f h_f}, \tag{2.37}$$

where $\lambda_f = l_f - v_f + 1 \equiv |\mathscr{L}(g_f)| - |\mathscr{V}(g_f)| + 1$ *is the number of independent loops in the reduced graph* g_f.

As in (2.29) the coordinate differences $\Delta_i(t)$ introduced by R and L operations contribute good power counting factors [3, Lemma 7]:

Lemma 2.4.

$$\prod_i |\Delta_i| e^{-(b(t) + \sum_{l \in \mathscr{L}_f} \alpha_l m^2)/8} \leqq c \prod_f M^{-m_f h_f}, \tag{2.38}$$

where m_f *is the number of* t_f*-derivatives applied at* f.

Equations (2.37) and (2.38) are the two basic ingredients in the bound on a renormalized graph:

Theorem 2.5. *Let* $K^{h,U}_{\text{ren}}$ *be the kernel of graph (2.35) contributing to* $V(\tau,\rho,h)$ *in (2.34b). Then*

$$\| K^{h,U}_{\text{ren}} \|_0 \leqq c \prod_f M^{\delta_f (h_f - h_{\pi(f)})}, \tag{2.39a}$$

where δ_f *is given by (2.24) and satisfies*

$$\begin{aligned} \delta_f &\leqq -1 \quad \text{if } \rho_f = R, \\ 0 \leqq \delta_f &< d \quad \text{if } \rho_f = C. \end{aligned} \tag{2.39b}$$

We can always arrange that in (2.35) the vertex x_V is an external vertex chosen as the localization vertex for the bottom fork F (and hence x_V has no t-dependence). Then we can estimate $G^{h,U}_{\text{ren}}$ in terms of the norm of its kernel:

$$\begin{aligned} |G^{h,U}_{\text{ren}}| &\leqq \int d\mu(t) \int dx_1 \cdots dx_{V-1} |K^{h,U}_{\text{ren}}(x,t)| |_{x_V=0} \sup_{x_1,\ldots,x_{V-1}} \int dx_V |\Pi^e(x^e(t))| \\ &\leqq c \| K^{h,U}_{\text{ren}} \|_0 . \end{aligned} \tag{2.40}$$

By (2.39) the sum over h in (2.34) converges uniformly in U, i.e. the theory is UV-renormalizable. At "marginal" C-forks, i.e. a C-fork f with $\delta_f = 0$, the sum over $h_f \leqq h_{\pi(f)}$ contributes a "logarithmic" factor $h_{\pi(f)}$ and these powers of h can accumulate. This leads to the following bound on a renormalized graph [1, Theorem 2.6]:

Corollary 2.6. *If* $G_{\text{ren}} \in \mathscr{G}(\tau,\rho)$,

$$\sum_{h \in \mathscr{H}(\tau,\rho)} |G^{h,U}_{\text{ren}}| \leqq c_0^L \kappa!, \tag{2.41}$$

where c_0 *is a constant independent of* U *and* G_{ren}, $L = |\mathscr{L}(G_{\text{ren}})|$, *and* κ *is the number of marginal* C*-forks.*

3. Dimensional Regularization of the Effective Potential

We consider the unrenormalized effective potential V^U for QED_d with IR cutoff $I=0$ and UV cutoff U. Expanding (2.10) as a fps in e we obtain a sum over the set $\mathscr{G}(V_I)$ of nontrivial, connected, non-vacuum (i.e. there are external legs) graphs with vertices V_I:

$$V^U = V_I + \sum_{G\in\mathscr{G}(V_I)} G^U,$$

where the fermi lines in G^U are $S^{[0,U]}$ and the photon lines are $C^{[0,U]}$. We let G^U_ν be the dr version of G^U as defined in (1.10) and we write

$$V^U_\nu \equiv \mathscr{C}^U_\nu(e^{V_I}) \equiv V_I + \sum_{G\in\mathscr{G}(V_I)} G^U_\nu, \tag{3.1}$$

where by the symbols on the left we simply mean the fps of graphs on the right. According to (1.10),

$$U^U_\nu = \int K^U_\nu(x)\Pi^e(x^e)dx, \tag{3.2a}$$

where

$$K^U_\nu(x) = c\prod_{l\in\mathscr{L}_b}\int_{M^{-2U}}^{1} d\alpha_l \prod_{l\in\mathscr{L}_f}\int_{M^{-2U}}^{\infty} d\alpha_l e^{-m^2\alpha^l}\beta^{\nu/2}_{\mathscr{L}}\int d\hat{x} P e^{-\mathscr{B}/4}$$

$$\equiv \int d\alpha K^U_\nu(\alpha,x), \tag{3.2b}$$

where $K^U_\nu(\alpha,x)=0$ if $\alpha_l < M^{-2U}$ for any l.

The tree expansion (2.19) decomposes each graph by the insertion of $\chi^h(\alpha)$'s and summation over trees τ and scales h. Since this decomposition in α-space does not interfere with the dr procedure in x-space we can apply dr to (2.19) to obtain the dr tree expansion

$$V^U_\nu = V_I + \sum_{\tau\text{ n.t.}}\sum_{G\in\mathscr{G}(\tau)}\sum_{h\in\mathscr{H}(\tau)} G^{h,U}_\nu, \tag{3.3a}$$

where each vertex of a graph in $\mathscr{G}(\tau)$ is V_I,

$$G^{h,U}_\nu = \int K^{h,U}_\nu(x)\Pi^e(x^e)dx \tag{3.3b}$$

and

$$K^{h,U}_\nu = \int d\alpha\chi^h(\alpha)K^U_\nu(\alpha,x) \tag{3.3c}$$

with $K^U_\nu(\alpha,x)$ defined by (3.2b) and χ^h by (2.17). The main result of this section is that $K^{h,U}_\nu$ is bounded as if it came from "ν dimensions" (see (2.20)–(2.22)):

Theorem 3.1.

$$\|K^{h,U}_\nu\|_0 \leqq c\prod_{f\in\mathscr{F}(\tau)} M^{D_\nu(G_f)(h_f-h_{\pi(f)})}. \tag{3.4}$$

The bound (3.4) immediately yields the UV-regularity of G_ν for sufficiently small ν:

Corollary 3.2. *For* $v < 2$

$$\sum_{h \in \mathscr{H}(\tau)} |G_v^{h,U}| < \infty \tag{3.5}$$

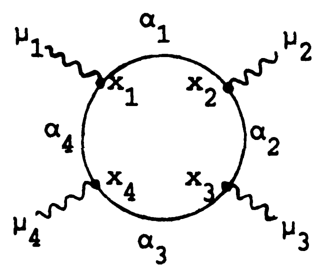
uniformly in the UV *cutoff* U.

Proof. For any graph G,

$$D_v(G) = (v-2)\Lambda - (V-2) - \lambda_f/2,$$

where $\Lambda = L - V + 1$ is the number of independent loops of G, and $\lambda_f = 2(V - L_f)$ is the number of fermi legs. Since $\operatorname{tr}\gamma^\mu = 0$ we must have $V \geqq 2$; hence $D_v(G) < 0$ unless $\Lambda = 0$, $\lambda_f = 0$ and $V = 2$. But this is impossible. Therefore $D_v(G_f) < 0$ in (3.4) and the sums over $h_f > h_{\pi(f)}$ in (3.5) converge uniformly in U. □

Before proving Theorem 3.1 we consider the example of the 4-photon loop (without the scale restrictions $h \in \mathscr{H}(\tau)$),



for which

$$K_v(\alpha, x) = c\left[\prod_{l=1}^{4} e^{-m^2\alpha_l}\alpha_l^{-v/2}\right]\int d\hat{x}\, P e^{-\mathscr{B}/4},$$

where $\mathscr{B} = \sum_{l=1}^{4} \beta_l Z_l^2$, $Z_l = X_{l+1} - X_l$ with $X_5 = X_1$, and

$$P = \operatorname{tr}\prod_{l=1}^{4} F_l \gamma^{\mu_l} \equiv [F_1, F_2, F_3, F_4],$$

where, as in (1.8),

$$F_l = f_l - i\beta_l \hat{\gamma}^\mu \hat{z}_l^\mu/2.$$

We remind the reader that the γ's are Dirac matrices whereas the $\hat{\gamma}$'s are formal symbols satisfying (1.5).

To evaluate $\int d\hat{x}$ we apply the rule (1.9). We write

$$\tfrac{1}{4}\int d\hat{x}\,\beta_l \hat{z}_l^{v_l}\beta_{l'}\hat{z}_{l'}^{v_{l'}} e^{-\hat{b}/4} = (c_1|B|)^{(d-v)/2}\hat{\delta}^{v_l v_{l'}} F_{ll'}, \tag{3.6}$$

where

$$\begin{aligned} F_{ll'} &= \tfrac{1}{2}\beta_l\beta_{l'}[B^{-1}_{l+1,l'+1} - B^{-1}_{l+1,l'} - B^{-1}_{l,l'+1} + B^{-1}_{l,l'}] \\ &= c\beta_l\beta_{l'}|B|^{d/2}\int z_l \cdot z_{l'} e^{-b/4}\,dx. \end{aligned} \tag{3.7}$$

Since

$$(\beta_l\beta_{l'})^{1/2}|z_l \cdot z_{l'}|e^{-b/4} \leqq ce^{-b/8}.$$

we have

$$|F_{ll'}| \leqq c\beta_l^{1/2}\beta_{l'}^{1/2}. \tag{3.8}$$

Using (1.9) and (3.6) we contract the $\hat{z}$'s in pairs to produce factors $F_{ll'}$:

$$\begin{aligned}\int d\hat{x} P e^{-\hat{b}/4} = (c_1|B|)^{(d-v)/2}\{[f_1, f_2, f_3, f_4] - [f_1, f_2, \hat{\gamma}^\mu, \hat{\gamma}^\mu]F_{34} \\ - [f_1, \hat{\gamma}^\mu, f_3, \hat{\gamma}^\mu]F_{24} \cdots + [\hat{\gamma}^\mu, \hat{\gamma}^\mu, \hat{\gamma}^{\mu'}, \hat{\gamma}^{\mu'}]F_{12}F_{34} + \cdots\}.\end{aligned}$$

Using (1.5) we move the members of a pair $\hat{\gamma}^\mu\hat{\gamma}^\mu$ next to each other, e.g.,

$$\hat{\gamma}^\mu f_l = -f_l^- \hat{\gamma}^\mu, \tag{3.9a}$$

where $f_l^\pm = -i\beta_l \not{z}_l/2 \pm m$, and using (1.7) we then eliminate the $\hat{\gamma}$'s by

$$\hat{\gamma}^\mu \hat{\gamma}^\mu = d - v. \tag{3.9b}$$

In this way we find that

$$\begin{aligned}\int d\hat{x} P e^{-\hat{b}/4} = (c_1|B|)^{(d-v)/2}\{p + (d-v)\{[f_1, f_2, 1, 1]F_{34} + [f_1, 1, f_3^-, 1]F_{24} + \cdots\} \\ + (d-v)^2[1,1,1,1]F_{12}F_{34} + \cdots\}.\end{aligned} \tag{3.10}$$

The leading term in (3.10), the polynomial p in x and β, determines the kernel of the unregularized graph G,

$$K(\alpha, x) = c\prod_{l=1}^{4} e^{-m^2\alpha_l}\alpha_l^{-d/2} p e^{-b/4} \equiv pk(\alpha, x).$$

The factor $(c_1|B|)^{(d-v)/2}$ together with $\prod \alpha_l^{(d-v)/2}$ contributes the factor $(c_1 U_G)^{(d-v)/2}$ (see (1.11)). Thus we find that

$$\begin{aligned}K_v &= (c_1 U_G)^{(d-v)/2}\{p + (d-v)[f_1, f_2, 1, 1]F_{34} + \cdots\}k \\ &\equiv (c_1 U_G)^{(d-v)/2}\left\{p + \sum_j c_j(v)p_j\right\}k.\end{aligned} \tag{3.11}$$

Here the coefficients $c_j(v)$ vanish when $v = d$ and the terms $p_j(\alpha, x)$ are polynomials in x that contain a factor f_l, f_l^- or $F_{ll'}$ for each line l. All of the above quantities can be explicitly computed but, as we do not require explicit formulas, we mention only that for the 4-loop example $U_G = \alpha_1 + \cdots + \alpha_4$.

The factor $U_G^{(d-v)/2}$ provides the desired UV-regularizing factor, rendering K_v integrable at $\alpha = 0$. The additional terms $\sum c_j p_j$, which maintain the "v-dimensional nature" of K_v as required for the Ward identities (see Sect. 5), *make the same power-counting contribution as the leading term p*. For we estimate p by the elementary bound

$$|p|e^{-b/4} \leqq c\prod_l \beta_l^{1/2} e^{-b/8}, \tag{3.12}$$

each f_l in p giving a factor $\beta_l^{1/2}$. p_j satisfies the same bound, each f_l or f_l^- giving a factor $\beta_l^{1/2}$ and each $F_{ll'}$ a factor $(\beta_l\beta_{l'})^{1/2}$ (by (3.8)).

Lemma 3.3. *For the general graph G of* (3.2),

$$\|K_v^U(\alpha, \cdot)\|_0 \leqq c U_G^{-v/2} \prod_{l\in\mathcal{L}_f} \beta_l^{1/2} e^{-m^2\alpha_l}. \tag{3.13}$$

Proof. As in the above example, we insert $F_l = f_l - i\beta_l\hat{\not{z}}_l/2$ into the definition (1.8) of P and integrate the resulting polynomial in $\hat{z}$

$$\int d\hat{x} P e^{-\hat{b}/4} \tag{3.14}$$

by the rule (1.9). The $\hat{\gamma}$'s occur in pairs $\hat{\gamma}^\mu \cdots \hat{\gamma}^\mu$ as a result of the contraction of a pair of $\hat{z}$'s as in (3.6) or of the contribution $\hat{\delta}^{\mu_l \nu_l}$ from a photon line's $\Delta^{\mu_l \nu_l}$. Eliminating the $\hat{\gamma}$'s by (3.9), we express (3.14) as a sum of terms of the form

$$c(\nu)|B|^{(d-\nu)/2} \prod_{l \in \mathscr{L}_f^0} f_l^{\pm} \prod_{(ll') \in \mathscr{P}_f} F_{ll'} \prod_{l \in \mathscr{L}_b^0} \delta^{\mu_l \nu_l} \prod_{v \in \mathscr{V}^0} \gamma^{\mu_v}, \tag{3.15}$$

where $\mathscr{L}_f^0 \subset \mathscr{L}_f$ with $|\mathscr{L}_f \backslash \mathscr{L}_f^0|$ even, $\mathscr{P}_f$ is a partition of $\mathscr{L}_f \backslash \mathscr{L}_f^0$ into pairs (ll') (corresponding to the contracted pairs $\hat{z}_l$ and $\hat{z}_{l'}$), $\mathscr{L}_b^0 \subset \mathscr{L}_b, \mathscr{V}^0 \subset \mathscr{V}$, and the coefficient $c(\nu) = 0$ when $\nu = d$ except in the case of the leading term for which $\mathscr{L}_f^0 = \mathscr{L}_f, \mathscr{L}_b^0 = \mathscr{L}_b$, and $\mathscr{V}^0 = \mathscr{V}$.

Estimating (3.15) as in (3.12) and (3.8), we find that its contribution to (3.2b) is bounded by

$$c(\nu)|\mathscr{B}|^{(d-\nu)/2} \prod_{l \in \mathscr{L}_f} e^{-m^2 \alpha_l} \beta_l^{1/2} \beta_{\mathscr{L}}^{\nu/2} e^{-b/8}.$$

But

$$\| e^{-b/8} \|_1 = c|B|^{-d/2},$$

and so by (1.11) we obtain (3.13). □

Proof of Theorem 3.1. By Lemmas 3.3 and 2.3,

$$\| K_\nu^{h,U} \|_0 \leqq c \int d\alpha \chi^h(\alpha) \prod_f M^{\nu \lambda_f h_f + l_{ff} h_f} \prod_{l \in \mathscr{L}_f} e^{-m^2 \alpha_l}, \tag{3.16}$$

where $\lambda_f = l_f - v_f + 1, l_f = |\mathscr{L}(g_f)|, v_f = |\mathscr{V}(g_f)|$, and $l_{ff} = |\mathscr{L}_f(g_f)|$. Now if $h_{f(l)} > 0$ then

$$\int d\alpha_l \chi^{h_{f(l)}}(\alpha_l) = cM^{-2h_{f(l)}},$$

and if $h_{f(l)} = 0$ and $l \in \mathscr{L}_f$ then

$$\int d\alpha_l \chi^{h_{f(l)}}(\alpha_l) e^{-m^2 \alpha_l} = m^{-2} = cM^{-2h_{f(l)}}.$$

Therefore

$$\| K_\nu^{h,U} \|_0 \leqq c \prod_f M^{(\nu \lambda_f + l_{ff} - 2l_f)h_f} = c \prod_f M^{D_\nu(g_f)h_f},$$

from which (3.4) follows by a summation by parts (Lemma 2.1 of [1]). □

When $\nu < 2$ we can remove the UV cutoff U on V_ν^U: as in (3.1) and (3.3)

$$V_\nu \equiv \mathscr{C}_\nu(e^{V_I}) \equiv V_I + \sum_{G \in \mathscr{G}(V_I)} G_\nu$$

$$= \sum_\tau \sum_{G \in \mathscr{G}(\tau)} \sum_{h \in \mathscr{H}(\tau)} G_\nu^h, \tag{3.17}$$

where the sum over h is convergent by Corollary 3.2.

We next show that for $\nu < 2$ dr expectations satisfy a calculus "appropriate to ν dimensions." This calculus will be needed in Sect. 5 for the proof of Ward identities. First we define the action of $\partial_{\hat{x}_i}, i = 1, \ldots, V-1$, on products

$p(\hat{x})e^{a\cdot\hat{x}}e^{-\hat{x}B\hat{x}/4}$, where p is a polynomial in $\hat{x}=(\hat{x}_1,\ldots,\hat{x}_{V-1})$ and, formally, $a\cdot\hat{x}=\sum_{i=1}^{V-1} a_i^\mu \hat{x}_i^\mu$:

$$\partial_{\hat{x}_i^\mu}\hat{x}_{i'}^{\mu'}=\hat{\delta}^{\mu\mu'}\delta_{ii'},\quad \partial_{\hat{x}_i^\mu}e^{a\cdot\hat{x}}=a_i^\mu e^{a\cdot\hat{x}},\qquad (3.18)$$
$$\partial_{\hat{x}_i^\mu}e^{-\hat{x}B\hat{x}/4}=-\tfrac{1}{2}(B\hat{x}^\mu)_i e^{-\hat{x}B\hat{x}/4},$$

and we extend the action in the obvious way by linearity and the product rule. The Gaussian integration rule (1.9) can be written

$$\int d\hat{x}e^{a\cdot\hat{x}}e^{-\hat{x}B\hat{x}/4}=(c_1|B|)^{(d-\nu)/2}e^{aB^{-1}a}\qquad (3.19)$$

from which it follows easily that

$$\int d\hat{x}\partial_{\hat{x}_i^\mu}(p(\hat{x})e^{-\hat{x}B\hat{x}/4})=0\qquad (3.20)$$

for $i=1,\ldots,V-1$.

We write $D_{X_i}=(\partial_{x_i},\partial_{\hat{x}_i})$ and

$$\not{D}_{X_i}=\Gamma^\mu D_{X_i^\mu}=\not{\partial}_{x_i}+\not{\partial}_{\hat{x}_i}.$$

By (3.20), if $\not{\partial}_{x_i}$ is applied to the vertex x_i of a dr graph G_ν we may replace it by $\not{D}_{X_i}, i=1,\ldots,V-1$. This replacement is possible for $i=V$ as well, provided that we understand that $\partial_{\hat{x}_V}$ means $-\sum_{i=1}^{V-1}\partial_{\hat{x}_i}$

Let b be a scalar boson line of a graph G with mass m_b, endpoints x_1 and x_2, say, and no UV or IR cutoffs; i.e. b contributes the factor

$$(4\pi\alpha_b)^{-d/2}e^{-m_b^2\alpha_b}e^{-(x_1-x_2)^2/4\alpha_b}\equiv k(\alpha_b,x_1-x_2)\qquad (3.21)$$

to the kernel $K(\alpha,x)$ of $G=\int d\alpha\int dxK(\alpha,x)\Pi^e(x^e)$. (There may be cutoffs on the other lines of G to ensure convergence.) Let $\tilde{G}=G/b$ be the graph G with the line b collapsed to a point so that $\tilde{G}$ has vertices $x_2,\ldots,x_V$, and lines $\mathscr{L}(\tilde{G})=\mathscr{L}(G)\backslash\{b\}$ (with x_1 replaced by x_2 when it is the endpoint of a line). If

$$\tilde{G}=\int d\tilde{\alpha}\int d\tilde{x}\tilde{K}(\tilde{\alpha},\tilde{x})\tilde{\Pi}^e(\tilde{x}^e)$$

then since

$$\lim_{\alpha_b\to 0+}k(\alpha_b,x_1-x_2)=\delta(x_1-x_2)\qquad (3.22a)$$

we obtain

$$\lim_{\alpha_b\to 0+}\int dxK(\alpha,x)\Pi^e(x^e)=\int d\tilde{x}\tilde{K}(\tilde{\alpha},\tilde{x})\tilde{\Pi}^e(\tilde{x}^e).\qquad (3.22b)$$

We claim that this same result holds for dr graphs:

Lemma 3.4. *(δ-function Rule) Let b be a scalar boson line (with no cutoffs) of a graph G and let $\tilde{G}=G/b$. If the corresponding* dr *graphs are given by $G_\nu=\int d\alpha\int dxK_\nu(\alpha,x)\Pi^e(x^e)$ and $\tilde{G}_\nu=\int d\tilde{\alpha}\int d\tilde{x}\tilde{K}_\nu(\tilde{\alpha},\tilde{x})\tilde{\Pi}^e(\tilde{x}^e)$, then*

$$\lim_{\alpha_b\to 0+}\int dxK_\nu(\alpha,x)\Pi^e(x^e)=\int d\tilde{x}\tilde{K}_\nu(\tilde{\alpha},\tilde{x})\tilde{\Pi}^e(\tilde{x}^e).\qquad (3.23)$$

Proof. We assume that G has $V > 2$ vertices (otherwise (3.23) is trivial) and we choose x_V to be different from the endpoints x_1 and x_2 of b. Referring to the definition (1.10) we see that (3.23) amounts to

$$\lim_{\alpha_b \to 0+} (4\pi\alpha_b)^{-\nu/2} \int dx \int d\hat{x} P e^{-\mathscr{B}/4} \Pi^e(x^e) = \int d\tilde{x} \int d\tilde{\hat{x}} \tilde{P} e^{-\tilde{\mathscr{B}}/4} \tilde{\Pi}^e(\tilde{x}^e), \tag{3.24}$$

where $\tilde{P} = P \Big|_{X_1 = X_2}$ and $\tilde{\mathscr{B}} = \sum_{l \neq b} \beta_l z_l^2 \Big|_{X_1 = X_2}$. If we expand $P = \sum_m c_m(\alpha) P_m$, where each P_m is a monomial $\prod_r \hat{x}_{i_r}^{\mu_r} \prod_s x_{j_s}^{\lambda_s}$, then $c_m(\alpha)$ is independent of α_b (since b is a boson line) and $\tilde{P} = \sum c_m \tilde{P}_m$, where $\tilde{P}_m = P_m|_{X_1 = X_2}$. Given (3.22), the identity (3.24) thus follows from

$$\lim_{\alpha_b \to 0} (4\pi\alpha_b)^{(d-\nu)/2} \int d\hat{x}\, p(\hat{x}) e^{-\hat{b}/4} = \int d\tilde{\hat{x}}\, \tilde{p}(\tilde{\hat{x}}) e^{-\tilde{\hat{b}}/4}, \tag{3.25}$$

where $p(\hat{x}) = \prod_r \hat{x}_{i_r}^{\mu_r}$. Now by (3.19)

$$(4\pi\alpha_b)^{(d-\nu)/2} \int d\hat{x}\, p(\hat{x}) e^{-\hat{b}/4} = (\tilde{c}_1 \alpha_b |B|)^{(d-\nu)/2} \prod_r \partial_{a_{i_r}^{\mu_r}} e^{aB^{-1}a} \Big|_{a=0}, \tag{3.26a}$$

where $\tilde{c}_1 = (4\pi)^{1-\tilde{V}} = (4\pi)^{2-V}$, and

$$\int d\tilde{\hat{x}}\, \tilde{p}(\tilde{\hat{x}}) e^{-\tilde{\hat{b}}/4} = (\tilde{c}_1 |\tilde{B}|)^{(d-\nu)/2} \prod_r \partial_{\tilde{a}_{\tilde{i}_r}^{\mu_r}} e^{\tilde{a}\tilde{B}^{-1}\tilde{a}} \Big|_{\tilde{a}=0}, \tag{3.26b}$$

where $\tilde{a} = (\tilde{a}_2, \ldots, \tilde{a}_{V-1})$, $\tilde{B}$ is the $(V-2) \times (V-2)$ matrix defined by

$$\tilde{x}\tilde{B}\tilde{x} = xBx|_{x_1 = x_2}, \quad \text{and} \quad \tilde{i} = \begin{cases} i & \text{if } i \neq 1 \\ 2 & \text{if } i = 1 \end{cases}.$$

From (3.26) it is evident that the equality (3.25) follows from

$$\lim_{\alpha_b \to 0} \alpha_b |B| = |\tilde{B}| \tag{3.27}$$

and

$$\lim_{\alpha_b \to 0} B_{ij}^{-1} = \tilde{B}_{\tilde{i}\tilde{j}}^{-1}. \tag{3.28}$$

To prove (3.27), we write

$$(\alpha_b |B|)^{-d/2} = (4\pi)^{(1-V)d/2} \alpha_b^{-d/2} \int e^{-xBx/4} dx_1 \cdots dx_{V-1}$$

and

$$|\tilde{B}|^{-d/2} = (4\pi)^{(2-V)d/2} \int e^{-\tilde{x}\tilde{B}\tilde{x}/4} d\tilde{x}_2 \cdots d\tilde{x}_{V-1}.$$

Thus (3.22a) implies (3.27). As for (3.28) we have

$$\begin{aligned} 2\delta^{\mu\lambda} B_{ij}^{-1} &= \frac{(4\pi\alpha_b)^{-d/2} \int x_i^\mu x_j^\lambda e^{-xBx/4} dx_1 \cdots dx_{V-1}}{(4\pi\alpha_b)^{-d/2} \int e^{-xBx/4} dx_1 \cdots dx_{V-1}} \\ &\to \frac{\int \tilde{x}_{\tilde{i}}^\mu \tilde{x}_{\tilde{j}}^\lambda e^{-\tilde{x}\tilde{B}\tilde{x}/4} d\tilde{x}_2 \cdots d\tilde{x}_{V-1}}{\int e^{-\tilde{x}\tilde{B}\tilde{x}/4} d\tilde{x}_2 \cdots d\tilde{x}_{V-1}} \qquad \text{(by (3.22a))} \\ &= 2\delta^{\mu\lambda} \tilde{B}_{\tilde{i}\tilde{j}}^{-1}. \quad \square \end{aligned}$$

We next establish integration by parts identities for dr expectations

$$\mathscr{C}_\nu(F(x)e^{W(\Phi+\Phi^e)}), \tag{3.29a}$$

where $-\infty < I \leqq 0$, $\nu < 2$, $F(x) = \psi(x)$ or $\bar{\psi}(x)$ or $\bar{\psi}(x)\gamma^\mu\psi(x)$, which we apply for W of the form

$$W(\Phi) = (W_1 + W_2 + W_3)(\Phi) = \int \bar{\psi}(y)(a_1(-i\partial\!\!\!/) + a_2 + a_3 A\!\!\!/(y))\psi(y)dy. \tag{3.29b}$$

As usual, by (3.29a) we mean the sum over connected dr graphs G_ν with vertices $F(x)$, W_1, W_2 and W_3. Although $\mathscr{C}_\nu$ does not arise directly from an integration over fields, it nevertheless enjoys:

Lemma 3.5. (*Integration by Parts*) *Let* $\nu < 2$ *and* $-\infty < I \leqq 0$.

a) $$(-i\partial\!\!\!/_x + m)\mathscr{C}_\nu(\psi(x)e^{W(\Phi+\Phi^e)}) = \frac{\delta}{\delta\bar{\psi}^e(x)}\mathscr{C}_\nu(e^{W(\Phi+\Phi^e)}), \tag{3.30}$$

b) $$\mathscr{C}_\nu(e^{W(\Phi+\Phi^e)}\bar{\psi}(x))(i\overleftarrow{\partial\!\!\!/}_x + m) = \mathscr{C}_\nu(e^{W(\Phi+\Phi^e)})\frac{\overleftarrow{\delta}}{\delta\psi^e(x)}, \tag{3.31}$$

c) $$-i\partial_x\cdot\mathscr{C}_\nu(\bar{\psi}(x)\gamma\psi(x)e^W) = \mathscr{C}_\nu\left(\bar{\psi}(x)\frac{\delta}{\delta\bar{\psi}(x)}e^W - e^W\frac{\overleftarrow{\delta}}{\delta\psi(x)}\psi(x)\right). \tag{3.32}$$

Remarks. 1. The above identities hold if in addition there is a UV cutoff on the photon (but not the electron) lines. Because of the IR cutoff I, $\mathscr{C}_\nu$ does not satisfy a simple integration by parts identity with respect to A.

2. The left Grassmannian derivative $\frac{\delta}{\delta\bar{\psi}^e}$ in (3.30) or the right derivative $\frac{\overleftarrow{\delta}}{\delta\psi^e}$ in (3.31) may be taken inside the expectation and evaluated; e.g.,

$$\frac{\delta}{\delta\bar{\psi}^e(x)}e^{W(\Phi+\Phi^e)} = \frac{\delta}{\delta\bar{\psi}(x)}e^{W(\Phi+\Phi^e)}$$
$$= [a_1(-i\partial\!\!\!/) + a_2 + a_3(A\!\!\!/ + A\!\!\!/^e)(x)](\psi + \psi^e)(x)e^{W(\Phi+\Phi^e)}.$$

3. An identity like (3.30) can be rewritten as

$$\mathscr{C}_\nu(\psi(x)e^W) = \int dy S(x,y)\mathscr{C}_\nu\left(\frac{\delta}{\delta\bar{\psi}^e(y)}e^W\right). \tag{3.33}$$

One might think that (3.33) could be iterated and dr graphs (without photon lines) could be evaluated completely in terms of unregularized propagators S, i.e. dr would have no effect! In fact this is true only for graphs without loops.

4. When the ∂_x's are taken inside the expectations they may, by (3.20), be written as D_X's. Likewise the $\partial\!\!\!/$'s in the W_1 vertices may be written as $D\!\!\!\!/$'s. In this way we are able to establish the α-space dr version (3.36) of the identity $(-i\partial\!\!\!/ + m)S = \delta$, upon which the lemma depends.

5. Moreover, by (3.36), if a graph G has $-i\partial\!\!\!/$ applied to a fermi line l then G_ν is given by the sum of the graph with $-i\partial\!\!\!/$ replaced by $-m$ plus the collapsed graph $\tilde{G}_\nu = G_\nu/l$. Hence the power counting contribution of a W_1 vertex amounts to that of a W_2 vertex and so the graphs involved in the lemma are finite for $\nu < 2$.

Proof. a) Let G_ν be a graph contributing to $\mathscr{C}_\nu(\psi(y)e^W)$. G_ν has the form (1.10):

$$G_\nu = \int d\alpha \int dX d\hat{y} K_\nu(\alpha, X, Y) \Pi^e(x^e),$$

where $Y=(y,\hat{y})$ and X includes the other vertices of G. We let l_1 be the fermi line joining y to another vertex x_1, say, $Z_1 = Y - X_1$, and $\beta_1 = \beta_{l_1} = \alpha_1^{-1}$. We apply $-i\not{\partial}_y + m$ as $-i\not{D}_Y + m$ to the factors depending on Z_1, namely $F_1 = -i\beta_1 \not{Z}_1/2 + m$ and $e^{-B_1} = e^{-\beta_1 Z_1^2/4}$:

$$\begin{aligned}(-i\not{D}_Y + m)F_1 e^{-B_1} &= [-\beta_1 \Gamma^\mu \Gamma^\mu/2 + (i\beta_1 \not{Z}_1/2 + m)(-i\beta_1 \not{Z}_1/2 + m)]e^{-B_1} \\ &= (\beta_1 \nu/2 - \beta_1^2 Z_1^2/4 + m^2)e^{-B_1},\end{aligned} \tag{3.34}$$

since

$$\Gamma^\mu \Gamma^\mu = -\nu \quad \text{and} \quad \not{Z}_1 \not{Z}_1 = -Z_1^2. \tag{3.35}$$

Including the other factors associated with l_1 we thus obtain the basic identity

$$(-i\not{D}_Y + m)e^{-\alpha_1 m^2}\beta_1^{\nu/2}F_1 e^{-\beta_1 Z_1^2/4} = -\frac{\partial}{\partial \alpha_1} e^{-\alpha_1 m^2}\beta_1^{\nu/2} e^{-\beta_1 Z_1^2/4}. \tag{3.36a}$$

Integration over α_1 then gives evaluation at $\alpha_1 = 0$. By the δ-function Rule (3.23) we obtain

$$(-i\not{\partial}_y + m)G_\nu = \tilde{G}_\nu,$$

where $\tilde{G}$ *is the graph obtained from* G by collapsing the line l_1 to a point and setting $x_1 = y$. $\tilde{G}$ is the graph in $\mathscr{C}_\nu\left(\frac{\delta}{\delta\bar{\psi}(y)}e^W\right)$ with the same vertices, legs and lines (save l_1) as G. Since $\frac{\delta}{\delta\bar{\psi}^e}e^W = \frac{\delta}{\delta\bar{\psi}}e^W$ this yields (3.30).

b) The proof is identical to that of a). Suppose $\bar{\psi}(y)$ contracts with $\psi(x_2)$ producing a fermi line factor

$$F_2 = -i\beta_2 \not{Z}_2/2 + m, \quad Z_2 = X_2 - Y.$$

Then, as in (3.36a), we have

$$e^{-\alpha_2 m^2}\beta_2^{\nu/2}F_{l_2}e^{-B_2}(i\overleftarrow{\not{D}}_Y + m) = -\frac{\partial}{\partial\alpha_2}e^{-\alpha_2 m^2}\beta_2^{\nu/2}e^{-B_2}. \tag{3.36b}$$

c) Suppose that $\psi(y)$ contracts with $\bar{\psi}(x_1)$ and $\bar{\psi}(y)$ with $\psi(x_2)$ producing fermi line factors F_1 and F_2 as in a) and b). As in (3.34) we have

$$\begin{aligned}&-iD_Y F_2 \Gamma F_1 e^{-B_1 - B_2} \\ &= [(-\beta_2 \nu/2 - iF_2 \beta_2 \not{Z}_2/2)F_1 + F_2(\beta_1 \nu/2 + i\beta_1 \not{Z}_1 F_1/2)]e^{-B_1 - B_2} \\ &= [(-\beta_2 \nu/2 - F_2(i\beta_2 \not{Z}_2/2 + m))F_1 + F_2(\beta_1 \nu/2 + (i\beta_1 \not{Z}_1/2 + m)F_1)]e^{-B_1 - B_2} \\ &= [(-\beta_2 \nu/2 + \beta_2^2 Z_2^2/4 - m^2)F_1 + F_2(\beta_1 \nu/2 - \beta_1^2 Z_1^2/4 + m^2)]e^{-B_1 - B_2}.\end{aligned}$$

Therefore, as in (3.36a) and (3.36b),

$$\begin{aligned}&-iD_Y e^{-(\alpha_1+\alpha_2)m^2}(\beta_1\beta_2)^{\nu/2}F_2 \Gamma F_1 e^{-B_1 - B_2} \\ &= \left(F_1 \frac{\partial}{\partial\alpha_2} - F_2 \frac{\partial}{\partial\alpha_1}\right)e^{-(\alpha_1+\alpha_2)m^2}(\beta_1\beta_2)^{\nu/2}e^{-B_1 - B_2},\end{aligned} \tag{3.36c}$$

and (3.32) follows. □

Finally we observe that dr respects Euclidean covariance. If $R = e^T \in SO(4)$, where T is a real, antisymmetric 4×4 matrix, let $S(R) = e^{\gamma^\lambda T_{\lambda\sigma}\gamma^\sigma/4}$. Then

$$S(R)^{-1}\gamma^\mu S(R) = R_{\mu\sigma}\gamma^\sigma. \tag{3.37}$$

Let $G_v = \int d\alpha \int dx K_v(\alpha, x)\Pi^e(x^e) \equiv \int d\alpha G_v(\alpha)$ be a graph contributing to $\mathscr{C}_v(e^{W(\Phi+\Phi^e)})$ with W as in (3.29) and $\Pi^e(x^e) = \prod_{k=1}^{s} \Phi^e_{i_k}(x_{j_k})$, where we write $\Phi^e = (\Phi^e_1, \Phi^e_2, \Phi^e_3, \Phi^e_4) = (A^e, \psi^e, \bar{\psi}^e, \partial\psi^e)$, the $\partial\psi^e$ fields coming from the W_1'-vertices. For $a \in \mathbf{R}^4$, define

$$G_v^{R,a}(\alpha) = \int dx K_v(\alpha, x) \prod_{k=1}^{s} (\Phi^e_{i_k})^R(x_{j_k} - a),$$

where

$$(\Phi^e)^R(x_j) = (RA^e, S(R)\psi^e, \bar{\psi}^e S(R)^{-1}, RS(R)\partial\psi^e)(R^{-1}x_j).$$

(In the last component the R acts on ∂ and $S(R)$ on ψ^e.) Then:

Lemma 3.6. (*Euclidean Covariance*) $G_v^{R,a}(\alpha) = G_v(\alpha)$.

Proof. From the fact that $K_v(\alpha, x)$ is a function of differences of the x_j's, and from the relations (3.37), $S(R)^{-1}\hat{\gamma}^\mu S(R) = \hat{\gamma}^\mu$, and

$$S(R)^{-1} f_l(z_l) S(R) = f_l(R^{-1}z_l),$$

where $f_l(z_l) = -i\not{z}_l/2\alpha_l + m$, it is easy to see that

$$G_v^{R,a}(\alpha) = \int dx K_v(\alpha, R^{-1}x) \prod_{k=1}^{s} \Phi^e_{i_k}(R^{-1}x_{j_k}) = G_v(\alpha). \quad \square$$

4. Renormalization of the Effective Potential

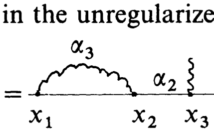
For $I = 0$ and $v < 2$ we let δV_v be the counterterms appropriate for 4 dimensions. (Henceforth we take $d = 4$.) As in (2.32)

$$\delta V_v = \sum_{\tau\, \text{n.t}} \sum_{\rho : \rho_F = C} \sum_{G_{\text{ren}} \in \mathscr{G}(\tau, \rho)} \sum_{h \in \mathscr{H}_C(\tau, \rho)} G^h_{\text{ren}, v}, \tag{4.1}$$

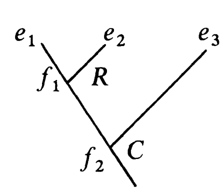
where

$$\mathscr{H}_C(\tau, \rho) = \{h \mid h_{\pi(f)} < h_f \text{ if } \rho_f = R;\ 0 \leqq h_f \leqq h_{\pi(f)} \text{ if } \rho_f = C \text{ and } f > F\},$$

and the value of $G^h_{\text{ren},v}$ is determined from that of G^h_v by applying the $-L$ and R operations of (2.30), where L is given by (2.26) with $\delta(G_f) = 4 - \lambda_b - \frac{3}{2}\lambda_f - q$ (see (2.24)). In the application of L and R to a dr graph the variables $X_1, \ldots, X_n$ are interpolated just as $x_1, \ldots, x_n$ are in the unregularized graph.

For example, for the graph $G =$ 

associated with the tree

$\tau =$ 

the regularized graph is (we omit the integral over α and the factors $c\chi^h(\alpha)e^{-m^2(\alpha_1+\alpha_2)}\beta_{\mathscr{L}}^{\nu/2}$),

$$G_\nu = \int dX \bar{\psi}(x_1)\Gamma^{\mu_1}F_1\Gamma^{\mu_1}F_2\gamma^{\mu_3}A^{\mu_3}(x_3)\psi(x_3)e^{-\mathscr{B}/4},$$

where $F_l = -\beta_l \not{Z}_l/2 + m, Z_1 = X_1 - X_2, Z_2 = X_2 - X_3$, and $\mathscr{B} = (\beta_1 + \beta_3)Z_1^2 + \beta_2 Z_2^2$. Taking x_1 as the localization vertex for both G_{f_1} and $G_{f_2} = G$ so that $X_2(t_1) = X_1 + t_1(X_2 - X_1)$, and noting that $\delta(G_{f_1}) = 1$ and $\delta(G_{f_2}) = 0$, the renormalized graph corresponding to G_ν is given by

$$-\int_0^1 dt_1 \partial_{t_1}^2 \int dX \bar{\psi}(x_1)\Gamma^{\mu_1}F_1\Gamma^{\mu_2}F_2(t_1)\gamma^{\mu_3}A^{\mu_3}(x_1)\psi(x_1)e^{-\mathscr{B}(t_1)/4}, \tag{4.2}$$

where

$$F_2(t_1) = -\beta_2\not{Z}_2(t_1)/2 + m, \quad Z_2(t_1) = X_3 - X_2(t_1),$$

and $\mathscr{B}(t_1) = (\beta_1 + \beta_3)Z_1^2 + \beta_2 Z_2(t_1)^2$.

The t_1-derivatives in (4.2) produce $\hat{x}$ as well as x factors via $\partial_{t_1}Z_2(t_1) = Z_1 = z_1 + \hat{z}_1$. But, just as in the case of the unrenormalized bounds of Theorem 3.1, the $\hat{x}$ factors make the same power counting contributions as the x factors (see Lemma 2.3). In any case, for $\nu < 2$ the counterterms δV_ν are finite (to each order), coming as they do from finite graphs.

For $\nu < 2$, we define the renormalized dr effective potential as in (3.1) by

$$\begin{aligned} V_{\text{ren},\nu} &= \mathscr{C}_\nu(e^{V_I + \delta V_\nu}) \\ &\equiv V_I + \delta V_\nu + \sum_{G \in \mathscr{G}(V_I, \delta V_\nu)} G_\nu, \end{aligned} \tag{4.3}$$

where $\mathscr{G}(V_I, \delta V_\nu)$ is the set of non-trivial connected graphs with vertices corresponding to monomials in V_I or δV_ν and with 2 or more external legs. Applying dr to the renormalized tree expansion (2.34), we obtain:

Theorem 4.1. (*Renormalized Tree Expansion*) *For $I = 0$ and $\nu < 2$*

$$V_{\text{ren},\nu} = V_I + \sum_{\tau \text{ n.t.}} \sum_{\rho: \rho_F = R} \sum_{G_{\text{ren}} \in \mathscr{G}(\tau,\rho)} \sum_{h \in \mathscr{H}(\tau,\rho)} G^h_{\text{ren},\nu}. \tag{4.4}$$

As in the unrenormalized case (3.11),

$$G^h_{\text{ren},\nu} = \int d\alpha \int d\mu(t) \int dx K^h_{\text{ren},\nu}(\alpha, t, x)\Pi^e(x^e(t)),$$

where $t = (t_f)_{f \in \mathscr{F}(\tau)}$ is the set of interpolating parameters, $d\mu(t)$ is a positive measure (see [1. (B.10)]), $\Pi^e(x^e)$ is the product of external fields of G_{ren}, and the kernel $K^h_{\text{ren},\nu}$ has the form

$$K^h_{\text{ren},\nu} = U_G^{(4-\nu)/2}\left[K^h_{\text{ren}} + \sum_j c_j(\nu)K^h_j\right], \tag{4.5}$$

where $U_G(\alpha, t)$ is given by (2.36), K^h_{ren} is the kernel of G^h_{ren} without dr, the kernels K^h_j arise from the contractions of $\hat{z}$'s in pairs, and the coefficients $c_j(\nu) = 0$ when $\nu = 4$. The factor U_G does not involve the renormalization factors Δ_i (see (2.28) and (2.29)) and, except for the dependence on t, depends only on the unrenormalized graph G which produced G_{ren}.

The kernels K_j^h obey the same power counting bounds as K_{ren}^h. More precisely, the bound on K_{ren}^h (or K_j^h) contains an unrenormalized factor U_G^{-2} (see the proof of Lemma 3.3) and renormalization factors $\prod_f M^{-m_f h_f}$ (see Lemma 2.4). The factors

$$U_G^{(4-v)/2} U_G^{-2} = U_G^{-v/2} \tag{4.6}$$

are bounded by Lemma 2.3, and we obtain

$$\begin{aligned}\|K_{\text{ren},v}^h\|_0 &\leqq c\left[1+\sum_j |c_j(v)|\right]\prod_f M^{(v\lambda_f - m_f)h_f} \\ &\leqq c\left[1+\sum_j |c_j(v)|\right]\prod_f M^{\delta_{f,v}(h_f - h_{\pi(f)})}\end{aligned} \tag{4.7a}$$

by summation by parts, where (see (2.22) and (2.24))

$$\delta_{f,v} = D_v(G_f) - q_f = \delta_f + (v-4)\Lambda_f. \tag{4.7b}$$

Here $\Lambda_f = L_f - V_f + 1$ is the number of independent loops of G_f.

When $v < 4$ and $\Lambda_f > 0$ then $\delta_{f,v} < \delta_f$ and we have "over-subtracted for v dimensions." If $f \in \mathscr{F}_C$ the sum of the f-factor in (4.7a) over $0 \leqq h_f \leqq h_{\pi(f)} \equiv k$ will contribute a large exponential factor $M^{-\delta_{f,v}k}$ to $\pi(f)$ when $\delta_{f,v} < 0$. If $\pi(f) \in \mathscr{F}_R$ so that k is summed to ∞, it is possible that such exponential factors might upset the convergence at $k = \infty$. However, the over-subtraction at $\pi(f)$ compensates for any such factors and we have much the same bounds as in the case $v = d$. More precisely, let τ_f be the subtree of τ with lowest fork f and root scale $k = h_{\pi(f)}$, let $\mathscr{H}_f$ be the set $\mathscr{H}(\tau, \rho)$ of (2.31) restricted to the scales $\{h_{f'} | f' \geqq f\}$, and let

$$B_{f,v}(k) = \sum_{h \in \mathscr{H}_f} \prod_{f' \geqq f} M^{\delta_{f',v}(h_{f'} - h_{\pi(f')})}.$$

As in [1, Sect. 2] it is easy to prove by induction on f down the tree that

$$B_{f,v}(k) \leqq 2^{a_f} \lambda_{b_f}(k) M^{(4-v)\Lambda_f k}, \tag{4.8}$$

where

$$\begin{aligned}a_f &= |\{f' \in \mathscr{F}_C | f' \geqq f, \delta_{f'} > 0\}|, \\ b_f &= |\{f' \in \mathscr{F}_C | f' \geqq f, \delta_{f'} = 0\}|,\end{aligned}$$

and

$$\lambda_n(k) = \sum_{i=1}^{\infty} (k+1+i)^n M^{-i/2}.$$

If we estimate $\sum_h G_{\text{ren},v}^h$ in (4.4) by the bounds (2.40), (4.7) and (4.8) with $f = F$ we obtain:

Theorem 4.2. *For* $\operatorname{Re} v \leqq 4$

$$\sum_{h \in \mathscr{H}(\tau,\rho)} |G_{\text{ren},v}^h| \leqq c_0^L \left[1 + \sum_j |c_j(v)|\right] \kappa!, \tag{4.9}$$

where c_0 *is independent of* G_{ren} *and* κ *is the number of marginal C-forks.*

Remarks. 1. The constant $1 + \sum_j |c_j|$ may be quite large ($\sim (L!)^a$ where $a > 0$), corresponding to the number of ways of contracting the $\hat{z}$'s of dr. However, since $c_j(4)=0$, this large number is irrevelant at the physical value $v=4$, where we have the bound on $G^h_{\text{ren}} = G^h_{\text{ren},4}$:

$$\sum_h |G^h_{\text{ren}}| \leqq c_0^L \kappa!. \tag{4.10}$$

2. The factor $\kappa!$ in (4.9) and (4.10) is the expected "renormalon" contribution to G_{ren} [11].
3. The tree representation (4.4) for $V_{\text{ren},v}$ together with the bound (4.9) display $V_{\text{ren},v}$ as an analytic function of v for $\operatorname{Re} v<4$. For each $G^h_{\text{ren},v}$ is obviously an entire function of v, its v-dependence being contained in the factors $U_G(\alpha,t)^{(4-v)/2}, \beta_{\mathscr{L}}^{v/2}$ and the polynomial in v generated by the contraction of $\hat{z}$'s. The convergence of $\sum_h$ uniformly in v for $\operatorname{Re} v \leqq 4$ then implies that the fps $V_{\text{ren},v}$ is analytic for $\operatorname{Re} v<4$. Notice that this conclusion did not entail an analysis of the meromorphic structure of V_v as a function of v, but only the knowledge that the renormalization cancellations in $V_{\text{ren},v}$ remove the divergences in V_v for all $\operatorname{Re} v<4$.

We conclude this section with an outline of how to remove the IR cutoff I. For full details see [1, Sect. 6]. As in (2.9), the photon lines are decomposed into slices with scales $h_f \geqq I$; we refer to the region $h_f<0$ as the "IR region." We decompose the localization operator as $L=L^0+L^+$, where L^0 produces "marginal counterterms" with $\delta_f=0$ and L^+ produces counterterms with $\delta_f>0$. Since we do not wish to introduce marginal counterterms into δV^I_v in the IR region, we generalize the definition (2.30) of the R and C operations as follows: at $f>F$

$$R=\chi(h_f>h_{\pi(f)})(1-L) \tag{4.11a}$$

and

$$\begin{aligned} C &= -\chi(h_f \leqq h_{\pi(f)})[L^+ + \chi(h_f \geqq 0))L^0] + \chi(h_{\pi(f)}<h_f<0)L^0 \\ &\equiv C_- + C_+, \end{aligned} \tag{4.11b}$$

whereas at F (the root scale is $h_{\pi(F)}=I-1$)

$$R=\chi(h_F>h_{\pi(F)})[1-L^+ - \chi(h_F \geqq 0)L^0] \tag{4.11c}$$

and

$$C=C_-. \tag{4.11d}$$

Each $f \in \mathscr{F}(\tau)$ now bears a label $\rho_f = R$, C_- or C_+ and the scales attached to τ, ρ run over the set

$$\mathscr{H}(\tau,\rho)=\{h \,|\, h_{\pi(f)}<h_f \text{ if } \rho_f=R \text{ or } C_+;\ h_f \leqq h_{\pi(f)} \text{ if } \rho_f=C_-\}. \tag{4.12}$$

The value $V^I_v(\tau,\rho,h)$ of a labelled tree is defined as before and may be expressed in terms of renormalized graphs

$$V^I_v(\tau,\rho,h)=\sum_{G_{\text{ren}}\in\mathscr{G}(\tau,\rho)} G^{h,I}_{\text{ren},v}, \tag{4.13}$$

where $G^{h,I}_{\text{ren},v}=0$ if $h_f<I$ for any f. We define the counterterms as in (4.1):

$$\delta V^I_v = \sum_{\tau\,\text{n.t.}} \sum_{\rho:\rho_f=C_-} \sum_{G_{\text{ren}}\in\mathscr{G}(\tau,\rho)} \sum_{h\in\mathscr{H}_C(\tau,\rho)} G^{h,I}_{\text{ren},v}, \tag{4.14}$$

where $\mathscr{H}_C$ is defined as in (4.12) except that the root scale is $h_{\pi(F)} = \infty$. For $\nu < 2$ and $-\infty < I \leqq 0$, the effective potential $V^I_{\text{ren},\nu}$, defined as in (4.3), is given by the following generalization of the tree expansion (4.4):

$$V^I_{\text{ren},\nu} = V_I + \sum_{\tau \text{n.t.}} \sum_{\rho} \sum_{G_{\text{ren}} \in \mathscr{G}(\tau,\rho)} \sum_{h \in \mathscr{H}(\tau,\rho)} G^{h,I}_{\text{ren},\nu}. \tag{4.15}$$

Now for $I > -\infty$ fixed, we know from our UV analysis that the sum over h in (4.15) converges for $\operatorname{Re}\nu \leqq 4$. The issue is whether the convergence is uniform in I. Our strategy for bounding $G^{h,I}_{\text{ren},\nu}$ when $I = 0$ turns out to be inadequate for two reasons: 1) In the IR region, a bound like (2.40) involving $\|K^{h,I}_{\text{ren},\nu}\|_0$ is too crude in the sense that it permits too many integrations of x_j's over all of $\mathbf{R}^4$. 2) The renormalization operation (2.28) can be harmful if $h_f < 0$: a coordinate difference Δ produces a *bad* factor M^{-h_f} which will not be compensated for by a factor $M^{h_{\pi(f)}}$ if the associated ∂ acts on Π^e.

The improved IR strategy of [1, pp. 99–101] was to rewrite the tree expansion by pulling apart the operation $R = 1 - L$ at certain forks $f \in \mathscr{F}_R(\tau)$ to yield a "1-fork" and an "L-fork". This separation is performed, starting at the bottom of τ, for each $f \in \mathscr{F}_R(\tau)$ such that $h_f < 0$ and such that there are some external endpoints above f and only 1-forks below f. Let $\mathscr{F}_{enu}(\tau)$ be the set of 1-forks produced by this separation procedure. By construction, if $f \in \mathscr{F}_{enu}$ and $f' < f$ then $f' \in \mathscr{F}_{enu}$. At a fork $f \in \mathscr{F}_{enu}$ one then takes advantage of the fact that g_f has $v^e_f > 0$ external vertices which need not be integrated over all of $\mathbf{R}^4$. This produces an improvement factor

$$\prod_{f \in \mathscr{F}_{enu}} M^{4h_f(v^e_f - 1)} \tag{4.16}$$

over the bound using the L^1 norm $\|\cdot\|_0$.

Can this strategy be combined with the estimation techniques of this paper which rely critically on the cancellation (4.6) between the dr factor $U_G^{(4-\nu)/2}$ and the factor U_G^{-2} arising from the bound on $\|K^h_{\text{ren}}\|_0$? Yes, but we do not wish to repeat the entire analysis of [1] here; we shall demonstrate only how the factor (4.16) is extracted in a way consistent with the cancellation (4.6).

We can always arrange that given $f \in \mathscr{F}_{enu}$ there are v^e_f external vertices x^e_f in g_f which are independent of t; e.g., choose these vertices as localization vertices of the graphs $G_{f'}, f' > f$. Then they are independent of $t_{f'}, f' > f$, and, since $f \in \mathscr{F}_{enu}$, they do not acquire any dependence on $t_{f'}, f' \leqq f$. Let $\tilde{x} = \{x^e_f | f \in \mathscr{F}_{enu}\}$. By construction, $x_V \in \tilde{x}$. Instead of the bound (2.40) we use

$$\begin{aligned} |G^{h,I}_{\text{ren},\nu}| &\leqq \int d\mu(t) \int dx_1 \cdots dx_{V-1} e^{-\tilde{x}^2/4} |K^{h,I}_{\text{ren},\nu}(x,t)| \,|_{x_V = 0} \\ &\quad \cdot \sup_{x_1,\ldots,x_{V-1}} \int dx_V e^{\tilde{x}^2/4} |\Pi^e(x^e(t))| \\ &\leqq c \sup_t \int dx_1 \cdots dx_{V-1} e^{-\tilde{x}^2/4} |K^{h,I}_{\text{ren},\nu}| \,|_{x_V = 0}. \end{aligned} \tag{4.17}$$

Equation (4.17) holds since each $x_j \in \tilde{x}$ occurs as an argument of Π^e without any t-dependence.

We estimate the norm of the kernel in (4.17) by the same procedure as before.

In addition to the usual factors from fermi lines and renormalization operations, and the dr factor $U_G^{(4-\nu)/2}$, we obtain the modified factor

$$U_{\tilde{G}}^{-2} = c\beta^2 \int dx\, e^{-(b(t)+\tilde{x}^2)/4}|_{x_V=0}, \tag{4.18}$$

where $\tilde{G}$ is the graph G augmented by additional lines (of strength $\alpha_l = 1$) pinning the vertices of $\tilde{x}$ to 0. Instead of (4.6) we have

$$U_G^{(4-\nu)/2} U_{\tilde{G}}^{-2} \leqq U_{\tilde{G}}^{-\nu/2}. \tag{4.19}$$

$U_{\tilde{G}}^{-2}$ obeys a better IR bound than U_G^{-2}. The proof of the bound (2.37) on U_G^{-2} is based on dropping lines from $b(t)$ to leave a tree T connecting G; each $l \in T$ then contributes a factor $\alpha_l^2 \cong M^{-4h_{f(l)}}$ to the integral in (2.36) – a bad factor in the IR region. By dropping lines from T and replacing them by the additional pinning lines of $\tilde{G}$ to form a tree connecting $\tilde{G}$, we obtain the improved bound

$$U_{\tilde{G}}^{-\nu/2} \leqq c \prod_{f \in \mathscr{F}_{enu}} M^{\nu h_f(v_f^e - 1)} \prod_{f \in \mathscr{F}} M^{\nu\lambda_f h_f}. \tag{4.20}$$

In this way we can extract the improvement factor (4.16) without disturbing the bounds needed for the UV analysis. Now to carry out the IR analysis of [1] we need the full improvement factor, i.e. with $\nu = 4$. Accordingly, we first take $\nu \to 4$ (with UV convergence guaranteed by Theorem 4.2), and then we take $I \to -\infty$ (with IR convergence guaranteed by Theorems 6.5–6.7 of [1]).

5. Ward Identities

For fixed $\nu < 2$ and IR cutoff $I \leqq 0$ we investigate here the form of the renormalized effective potential $V^I_{\text{ren},\nu}$ of (4.3) and the counterterms δV^I_ν of (4.1). (We shall omit the superscript I if there is no confusion.) By Corollary 3.2 we know that the order e^n contribution $\delta V_{\nu,n}$ to δV_ν is finite, and, by Lemma 3.6, that it is Euclidean invariant. A priori we do not know that δV_ν respects gauge invariance. Thus $\delta V_{\nu,n}$ consists of finite, local, Euclidean invariant terms of dimension $\leqq 4$; i.e., it has the form

$$\delta V_{\nu,n}(\Phi) = -\int [\bar{\psi}(-a_n i\partial\!\!\!/ + b_n + c_n A\!\!\!/)\psi + d_n F^2]dx + \delta W_n(A), \tag{5.1a}$$

where

$$F^2 = F_{\mu\lambda}F_{\mu\lambda},\ F_{\mu\lambda} = \partial_\mu A_\lambda - \partial_\lambda A_\mu,$$

and

$$\delta W_n(A) = -\int [e_n(\partial \cdot A)^2 + f_n A^2 + g_n A^4]dx. \tag{5.1b}$$

We say that $\delta V_{\nu,\leqq n} = \sum_{k=2}^{n} \delta V_{\nu,k}$ has *gauge invariant form* if $\delta W_k = 0$ for $k \leqq n$ and if

$$c_2 = 0, \quad c_k = e a_{k-1} \quad \text{for} \quad k = 3, \ldots, n. \tag{5.2}$$

Equation (5.2) is the "$Z_1 = Z_2$" condition.

We let $a_{\leqq n} = \sum_{k=2}^{n} a_k$, etc., $\alpha_n = 1 + a_{\leqq n}$, and $\delta_n = \alpha_n/\alpha_{n-1}$. Note that

$$\delta_n = 1 + a_n + O(e^{n+1}). \tag{5.3}$$

Let L_n be the part of the localization operator (2.26) which produces n^{th} order terms, L_n^0 the part of L_n which produces marginal terms, etc. We introduce the effective potential at scale -1,

$$V_{-1} \equiv \mathscr{C}_v^{[0,\infty)}(e^{(V_I+\delta V_v)(\Phi^{[0,\infty)}+\Phi^e)}), \tag{5.4}$$

obtained by regularizing the graphs contributing to $\mathscr{E}^{[0,\infty)}$; the effective potential renormalized up to order n,

$$V_n \equiv \mathscr{C}_v(e^{V_I+\delta V_{v,\leqq n}}); \tag{5.5a}$$

and the effective potential at scale -1, renormalized up to order n,

$$V_{-1,n} \equiv \mathscr{C}_v^{[0,\infty)}(e^{V_I+\delta V_{v,\leqq n}}). \tag{5.5b}$$

V_{-1} has a tree expansion like (4.15) where the root scale is $h_{\pi(F)} = -1$. If we apply L^0 to this expansion, the graphs with $\rho_F = C_-$ drop out because $L^0 C_- = 0$ (see (4.11b)), and we obtain

$$L^0 V_{-1} = L^0 V_I = -e\int \bar{\psi}^e A\!\!\!/^e \psi^e dx. \tag{5.6a}$$

Applying L^+ to (4.15) we obtain

$$L^+ V_{\text{ren},v=0}. \tag{5.6b}$$

The relations (5.6) are the renormalization conditions of the tree expansion: dimensionless parameters are fixed at scale -1 (corresponding roughly to external momentum of order 1) and parameters with nonzero dimension (such as mass) are fixed at scale $I-1$ (corresponding to zero external momentum in the limit $I \to -\infty$).

If we apply L_{n+1}^+ (with $n>0$) to (4.3) (with general I) we obtain

$$L_{n+1}^+ V_{\text{ren},v} = L_{n+1}^+ \delta V_v + L_{n+1}^+ \sum_{G \in \mathscr{G}(V_I, \delta V_{v,\leqq n})} G_v, \tag{5.7a}$$

since vertices in $\delta V_{v,>n}$ cannot contribute to order $n+1$. Similarly if we apply L_{n+1}^0 to V_{-1} we obtain

$$L_{n+1}^0 V_{-1} = L_{n+1}^0 \delta V_v + L_{n+1}^0 V_{-1,n}. \tag{5.7b}$$

A comparison of (5.6) and (5.7) yields:

Lemma 5.1. *For $v<2$, $-\infty < I \leqq 0$, and $n>0$*

$$\delta V_{v,n+1} = -L_{n+1}^+ V_n - L_{n+1}^0 V_{-1,n}, \tag{5.8}$$

$$L^+ V_n = -L_{n+1}^+ \delta V_v + O(e^{n+2}), \tag{5.9a}$$

$$L^0 V_{-1,n} = -e\int \bar{\psi}^e A\!\!\!/^e \psi^e dx - L_{n+1}^0 \delta V_v + O(e^{n+2}). \tag{5.9b}$$

The following Ward identity is a first order version of Lemma 4.2 of [1]:

Theorem 5.2. *Suppose $v<2$ and $-\infty < I \leqq 0$. If $\delta V_{v,\leqq n}$ has gauge invariant form then V_n and $V_{-1,n}$ each satisfy the identity*

$$\delta_n \partial_x \cdot \frac{\delta V_n}{\delta A^e(x)} + ie\int dy \left[V_n \frac{\overleftarrow{\delta}}{\delta \psi^e(y)} S(y,x)(S^{-1}\psi^e)(x) - (\bar{\psi}^e \overleftarrow{S}^{-1})(x) S(x,y) \frac{\delta}{\delta \bar{\psi}^e(y)} V_n \right]$$
$$= -e\partial_x \bar{\psi}^e(x)\gamma\psi^e(x), \tag{5.10}$$

where $\bar{\psi}^e \overleftarrow{S}^{-1} = \bar{\psi}^e(i\overleftarrow{\not\partial} + m) = i\partial\cdot\bar{\psi}^e\gamma + m\bar{\psi}^e$.

Before proving the theorem we note that it implies:

Corollary 5.3. *Suppose* $v < 2$ *and* $-\infty < I \leqq 0$. *If* $\delta V_{v,\leqq n}$ *has gauge invariant form then so does* $\delta V_{v,n+1}$. *Consequently*, δV_v *has gauge invariant form.*

Proof. Consider the quadratic term

$$Q_n = \int Q_{\mu_1\mu_2}(x_1 - x_2)A_{\mu_1}(x_1)A_{\mu_2}(x_2)dx_1dx_2$$

in V_n. By (5.9a) and (5.1b)

$$L^+Q_n = \int dx \int dy Q_{\mu_1\mu_2}(y)A_{\mu_1}(x)A_{\mu_2}(x) = f_{n+1}\int A^2 dx. \tag{5.11}$$

Setting $\psi^e = \bar{\psi}^e = 0$ in (5.10), we obtain

$$\delta_n\partial_x\cdot\frac{\delta V_n(A^e,0,0)}{\delta A^e(x)} = 0,$$

which implies

$$\int Q_{\mu_1\mu_2}(y-x)\partial_{\mu_2}A_{\mu_1}(y)dy = 0. \tag{5.12}$$

Choosing $A_{\mu_1}(y) = y_{\mu_1}$ we conclude from (5.11) and (5.12) that $f_{n+1} = 0$.

Similarly by considering the quadratic term in $V_{-1,n}(A,0,0)$, applying L^0 and choosing $A_{\mu_2}(y) = y_{\mu_2}y_\sigma y_\lambda$ we conclude from (5.12) that $e_{n+1} = 0$. Applying L^0 to the quartic term in $V_{-1,n}(A,0,0)$ yields the conclusion $g_{n+1} = 0$. Hence $\delta W_{n+1} = 0$.

We next consider the bilinear term

$$B_n = \int \bar{\psi}(x_1)B(x_1 - x_2)\psi(x_2)dx_1dx_2$$

and trilinear term

$$T_n = \int \bar{\psi}(x_1)T_\mu(x_1 - x_2, x_2 - x_3)\psi(x_2)A_\mu(x_3)dx_1dx_2dx_3$$

in $V_{-1,n}$. By (5.9b) and (5.1a)

$$L^0B_n = \int \bar{\psi}(x)B(-y)y\cdot\partial\psi(x)dxdy = -a_{n+1}\int \bar{\psi}i\not\partial\psi dx + O(e^{n+2}) \tag{5.13}$$

and

$$\begin{aligned} L^0T_n &= \int \bar{\psi}(x)T_\mu(y,z)\psi(x)A_\mu(x)dxdydz \\ &= (-e + c_{n+1})\int \bar{\psi}\not{A}\psi dx + O(e^{n+2}). \end{aligned} \tag{5.14}$$

As for the positive dimension part L^+B_n of B_n, we cannot invoke (5.9a) which is a normalization condition at scale $I-1$, but at least we can say by Euclidean invariance (Lemma 3.6) that for some constant b

$$L^+B_n = \int \bar{\psi}(x)B(y)\psi(x)dydx = b\int \bar{\psi}(x)\psi(x)dx. \tag{5.15}$$

We pick out the bilinear terms in (5.10) for $V_{-1,n}$ and set $\bar{\psi}^e = 1$ and $\psi^e(x_2) = (x_2 - x)_\sigma$. The term on the right gives $-e\gamma^\sigma$. The first term on the left gives

$$\begin{aligned} \delta_n\int \partial_{x_\mu}T_\mu(x_1 - x_2, x_2 - x)(x_2 - x)_\sigma dx_1dx_2 &= \delta_n\int T_\sigma(y,z)dydz \\ &= \delta_n(-e + c_{n+1})\gamma^\sigma + O(e^{n+2}) \end{aligned}$$

by integration by parts and (5.14). The second term on the left gives

$$ie\int dy[\int dx_1 B(x_1 - y)S(y,x)(-i\gamma^\sigma) - m\int dx_2 S(x,y)B_n(y - x_2)(x_2 - x)_\sigma]. \tag{5.16}$$

Using (see (5.15) and (5.13))

$$\int dx_1 B(x_1) = b, \quad \int dx_2 B(-x_2)(x_2)_\sigma = -a_{n+1} i\gamma^\sigma,$$
$$\int dy S(y,x) = \int dy S(x,y) = m^{-1}, \quad \int dy S(x,y)(y-x)_\sigma = -im^{-2}\gamma^\sigma,$$

we compute that

$$\begin{aligned}(5.16) &= ie[-bm^{-1}i\gamma^\sigma + a_{n+1}i\gamma^\sigma + im^{-1}\gamma^\sigma b] \\ &= -ea_{n+1}\gamma^\sigma = O(e^{n+2}).\end{aligned}$$

Hence we conclude from (5.10) that

$$\delta_n(-e + c_{n+1})\gamma^\sigma + O(e^{n+2}) = -e\gamma^\sigma,$$

and from (5.3) we deduce that $c_{n+1} = ea_n$. Thus $\delta V_{v,n+1}$ has gauge invariant form. □

Remark. The above proof corrects an oversight in the proof of Theorem 7.2 of [1], namely a failure to control the positive dimension part $L^+ V_{-1,n}$.

Proof of Theorem 5.2. We discuss only the case of V_n since the proof of (5.10) for $V_{-1,n}$ is identical. Like V_n, $V_{-1,n}$ has its fermi fields integrated out at all scales and so we may "integrate by parts" with respect to ψ, $\bar{\psi}$. This is the key step in the proof.

Since $\delta V_{v,\leqq n}$ has gauge invariant form

$$\begin{aligned} I_n(\Phi) &\equiv (V_I + \delta V_{v,\leqq n})(\Phi) \\ &= -\int [\bar{\psi}(a_{\leqq n}(-i\partial\!\!\!/) + b_{\leqq n} + e\alpha_{n-1} A\!\!\!/)\psi + d_{\leqq n}F^2]dx. \end{aligned} \tag{5.17}$$

We generate formulas for functional derivatives of $V_n(\Phi^e) = \mathscr{C}_v(e^{I_n(\Phi+\Phi^e)})$ from the corresponding formulas for functional derivatives of the effective potential $[\log \mathscr{E}^U(e^{I_n})]_0$ in which the graphs are not dr (although I_n as given by (5.17) is still v-dependent) and a UV cutoff U is imposed for finiteness. (This cutoff is then removed in the formulas for V_n.) In terms of the notation

$$\langle F(x)\rangle_v = \mathscr{C}_v(F(x)e^{I_n(\Phi+\Phi^e)})$$

we have, for example,

$$\begin{aligned} \frac{\delta V_n}{\delta A^e_\mu(x)} &= \left\langle \frac{\delta}{\delta A^e_\mu(x)} I_n(\Phi + \Phi^e)\right\rangle_v \\ &= -\langle e\alpha_{n-1}(\bar{\psi} + \bar{\psi}^e)(x)\gamma^\mu(\psi + \psi^e)(x) \\ &\quad + 4d_{\leqq n}(\partial_{x^\mu}\partial_x\cdot(A + A^e)(x) - \partial_x^2(A_\mu + A^e_\mu)(x))\rangle_v, \end{aligned} \tag{5.18}$$

where, according to (3.20), the derivatives on the fields A may be taken as ∂_x or D_X. When we apply the derivative ∂_{x^μ} to (5.18), the second term on the right is eliminated. Thus

$$\delta_n \partial\cdot\frac{\delta V_n}{\delta A^e} = -e\alpha_n\partial\cdot\langle(\bar{\psi} + \bar{\psi}^e)\gamma(\psi + \psi^e)\rangle_v. \tag{5.19}$$

Next we compute that

$$V_n \frac{\overleftarrow{\delta}}{\delta\psi^e}\psi^e - \bar{\psi}^e \frac{\delta}{\delta\bar{\psi}^e} V_n$$
$$= \langle -ia_{\leqq n}[\bar{\psi}^e \not{\partial}(\psi + \psi^e) + (\bar{\psi} + \bar{\psi}^e)\overleftarrow{\not{\partial}}\psi^e]$$
$$+ b_{\leqq n}(\bar{\psi}^e\psi - \bar{\psi}\psi^e) + e\alpha_{n-1}[\bar{\psi}^e(\not{A} + \not{A}^e)\psi - \bar{\psi}(\not{A} + \not{A}^e)\psi^e]\rangle_\nu$$
$$= -ia_{\leqq n}\partial\cdot\langle(\bar{\psi} + \bar{\psi}^e)\gamma(\psi + \psi^e)\rangle_\nu + \left\langle \bar{\psi}\frac{\delta}{\delta\bar{\psi}} I_n - I_n \frac{\overleftarrow{\delta}}{\delta\psi}\psi \right\rangle_\nu . \tag{5.20}$$

From (5.19) and (5.20)

$$\delta_n \partial\cdot\frac{\delta V_n}{\delta A^e} + ieV_n\frac{\overleftarrow{\delta}}{\delta\psi^e}\psi^e - ie\bar{\psi}^e\frac{\delta}{\delta\bar{\psi}^e}V_n$$
$$= -e\partial\cdot\langle(\bar{\psi} + \bar{\psi}^e)\gamma(\psi + \psi^e)\rangle_\nu + ie\left\langle \bar{\psi}\frac{\delta}{\delta\bar{\psi}} I_n - I_n \frac{\overleftarrow{\delta}}{\delta\psi}\psi \right\rangle_\nu .$$

Applying Lemma 3.5a)–c) to the first term on the right to "integrate the $\bar{\psi}$ and ψ by parts," we obtain

$$\delta_n\partial_x\cdot\frac{\delta V_n}{\delta A^e(x)} + ieV_n\frac{\overleftarrow{\delta}}{\delta\psi^e(x)}\psi^e(x) - ie\bar{\psi}^e(x)\frac{\delta}{\delta\bar{\psi}^e(x)}V_n$$
$$= -e\partial_x\cdot\left\{\bar{\psi}^e\gamma\psi^e(x) + \int dy\left[\bar{\psi}^e(x)\gamma S(x,y)\frac{\delta}{\delta\bar{\psi}^e(y)}V_n + V_n\frac{\overleftarrow{\delta}}{\delta\psi^e(y)}S(y,x)\gamma\psi^e(x)\right]\right\}. \tag{5.21}$$

Equation (5.10) follows from (5.21) and the identities

$$\not{\partial}_x S(x,y) = -imS(x,y) + i\delta(x-y)$$

and

$$S(x,y)\overleftarrow{\not{\partial}}_x = imS(y,x) - i\delta(x-y). \quad \square$$

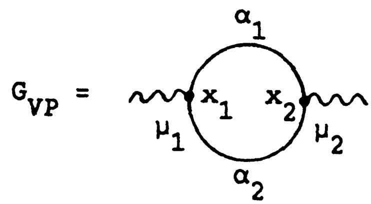
The coefficient $\delta_n \neq 1$ occurs in the Ward identity (5.10) because the effective potential V_n has been renormalized up to order n only. Clearly the fully renormalized effective potential $V_{\text{ren},\nu}$ satisfies, for $\nu < 2$,

$$\partial\cdot\frac{\delta V_{\text{ren},\nu}}{\delta A^e} + ie\left[V_{\text{ren},\nu}\frac{\overleftarrow{\delta}}{\delta\psi^e}S(-i\not{\partial} + m)\psi^e - \bar{\psi}^e(i\overleftarrow{\not{\partial}} + m)S\frac{\delta}{\delta\bar{\psi}^e}V_{\text{ren},\nu}\right]$$
$$= -e\partial\cdot\bar{\psi}^e\gamma\psi^e. \tag{5.22}$$

But by Remark 3 following Theorem 4.2, $V_{\text{ren},\nu}$ is an analytic function of ν for $\operatorname{Re}\nu < 4$. Hence (5.22) analytically continues from $\nu < 2$ to $\operatorname{Re}\nu < 4$ (the identity (5.10) does not continue since δ_n and V_n are not defined for $\operatorname{Re}\nu \geqq 2$):

Corollary 5.4 (*Ward Identity*). *For* $\operatorname{Re}\nu \leqq 4$ *and* $-\infty \leqq I \leqq 0$, $V_{ren,\nu}$ *satisfies* (5.22).

It is amusing to examine the Ward identity for some simple graphs contributing to $V_{\text{ren},\nu}$ in order to see the role of the "extra terms" in a dr graph. The vacuum polarization graph



had dr kernel

$$K_\nu^{\mu_1\mu_2}(\alpha,x)=c\beta_1^2\beta_2^2 e^{-m^2\lambda}\lambda^{2-\nu/2}\,\mathrm{tr}[f_1\gamma^{\mu_1}f_2\gamma^{\mu_2}+(4-\nu)\gamma^{\mu_1}\gamma^{\mu_2}F_{12}]e^{-b/4},$$

where $\lambda=U_G=\alpha_1+\alpha_2$, $f_1=-i\beta_1\not{z}/2+m$, $f_2=i\beta_2\not{z}/2+m$, $z=x_2-x_1$, $F_{12}=\dfrac{1}{2\lambda}$ and $b=(\beta_1+\beta_2)z^2$. Evaluating the traces, we compute that

$$\begin{aligned}K_\nu^{\mu_1\mu_2}(\alpha,x)=c\beta_1^2\beta_2^2 e^{-m^2\lambda}\lambda^{2-\nu/2}[&\beta_1\beta_2(2z^{\mu_1}z^{\mu_2}-\delta^{\mu_1\mu_2}z^2)-4m^2\delta^{\mu_1\mu_2}\\&+2(4-\nu)\lambda^{-1}\delta^{\mu_1\mu_2}]e^{-b/4}.\end{aligned}\tag{5.23}$$

Using

$$\int e^{-b/4}dz=c(\beta_1+\beta_2)^{-2}\quad\text{and}\quad\int z^2e^{-b/4}dz=c(\beta_1+\beta_2)^{-3},$$

it is easy to see that all terms in (5.23) have norm $\|\cdot\|_0$ bounded by $c\lambda^{-1-\nu/2}e^{-m^2\lambda}$ and so are integrable with respect to α when $\nu<2$.

The "extra term" $2(4-\nu)\lambda^{-1}\delta^{\mu_1\mu_2}$ in (5.23) is required for the Ward identity

$$\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\,\partial_{z^{\mu_2}}K_\nu^{\mu_1\mu_2}(\alpha,x)=0.\tag{5.24}$$

To see this we compute that

$$\begin{aligned}\partial_{z^{\mu_2}}K_\nu^{\mu_1\mu_2}&=cz^{\mu_1}e^{-m^2\lambda}\lambda^{2-\nu/2}\beta_1^2\beta_2^2\left\{4\beta_1\beta_2-(\beta_1+\beta_2)\left[\frac{\beta_1\beta_2z^2}{4}-m^2+\frac{4-\nu}{2\lambda}\right]\right\}e^{-b/4}\\&=-cz^{\mu_1}(\partial_{\alpha_1}\beta_2+\partial_{\alpha_2}\beta_1)e^{-m^2\lambda}\lambda^{2-\nu/2}\beta_1^2\beta_2^2e^{-b/4}.\end{aligned}\tag{5.25}$$

Equation (5.25) is a sum of perfect α-derivatives and integrates to 0, verifying (5.24). As a consequence of (5.24) the corresponding mass counterterm

$$L^+G_{VP,\nu}=\int d\alpha\int dx\,K_\nu^{\mu_1\mu_2}A_{\mu_1}(x_2)A_{\mu_2}(x_2)=0\tag{5.26}$$

for $\nu<2$.

The validity of the Ward identity (5.22) at $\nu=4$ and $I=-\infty$ is our guarantee that the renormalization has been carried out in a gauge invariant way. Note that for the Ward identity to hold, the free photon measure need not be gauge invariant, but $\delta V_\nu(\Phi+\Phi^e)$ must not contain the gauge variant terms (5.1b). Strictly speaking, for this assertion to make sense, we must keep $\nu<2$ and $I>-\infty$ so that the terms in δV_ν, as defined in (4.1), are all finite.

For QED_4 it is possible to make such an assertion for $\nu<4$ provided we keep an UV cutoff $U_p<\infty$ on the photon lines (but not on the electron lines) as well as an IR cutoff $I>-\infty$. To analyze the situation we apply the power counting of Theorem 3.1 to the terms in (3.17) except that we bound the β's of the photon lines by $\beta_l\leqq M^{2U_p}$. The degree of divergence $\tilde{D}_\nu(G)$ of a graph then contains no contributions from photon lines (compare with (2.22)):

$$\tilde{D}_\nu(G) = (\nu - 1)L_f - \nu(V - 1) = \nu - V - \frac{\nu - 1}{2}\lambda_f. \tag{5.27}$$

As in Theorem 3.1 we have the bound on the kernel of G_ν^{h,U_p},

$$\| K_\nu^{h,U_p} \|_0 \leqq c(U_p) \prod_f M^{\tilde{D}_\nu(G_f)(h_f - h_{\pi(f)})}, \tag{5.28}$$

where the constant $c(U_p)$ depends on U_p.

For $\nu < 4$, $\tilde{D}_\nu(G) < 0$ except for the vacuum polarization graph for which $\tilde{D}_\nu(G_{VP}) = \nu - 2$. It follows from (5.28) that for $\nu < 4$,

$$\sum_{h \in \mathscr{H}(\tau)} |G_\nu^{h,U_p}| < \infty \tag{5.29}$$

(with a bound depending on I and U_p) unless G contains the subgraph $G_f = G_{VP}$. Accordingly, we (partially) renormalize G_{VP}, replacing it by

$$\tilde{R}G_{VP,\nu} \equiv (I - L^+)G_{VP,\nu} = \int d\alpha \int dx\, K_\nu^{\mu_1\mu_2}(\alpha, x)(A_{\mu_1}(x_1) - A_{\mu_1}(x_2))A_{\mu_2}(x_2). \tag{5.30}$$

For $\nu < 2$, $\tilde{R}G_{VP,\nu} = G_{VP,\nu}$ by (5.26), but the advantage of the representation (5.30) is that it extends (as an analytic function of ν) from $\nu < 2$ to $2 \leqq \nu < 4$.

We similarly extend (4.1) and (4.3): $\delta\tilde{V}_\nu^{U_p}$ is defined as in (4.1) except that in the sum over G_{ren} we exclude the VP mass graph (5.26); $\tilde{V}_{\text{ren},\nu}^{U_p}$ is defined as in (4.3),

$$\tilde{V}_{\text{ren},\nu}^{U_p} = V_I + \delta\tilde{V}_\nu^{U_p} + \sum_{G \in \mathscr{G}(V_I, \delta\tilde{V}_\nu^{U_p})} \tilde{R}G_\nu^{U_p}, \tag{5.31}$$

where $\tilde{R}$ renormalizes every 2[nd] order VP subgraph of G as in (5.30), and only those subgraphs. By (5.26),

$$\delta\tilde{V}_\nu^{U_p} = \delta V_\nu^{U_p} \tag{5.32a}$$

and

$$\tilde{V}_{\text{ren},\nu}^{U_p} = V_{\text{ren},\nu}^{U_p} \tag{5.32b}$$

for $\nu < 2$, but, by the power counting (5.27)–(5.29), $\delta\tilde{V}_\nu^{U_p}$ and $\tilde{V}_{\text{ren},\nu}^{U_p}$ extend analytically to $\nu < 4$. Now we know that the fully renormalized tree expansion representation for $V_{\text{ren},\nu}^{U_p}$ extends to $\nu \leqq 4$. Hence the equality (5.32b) extends to $\nu < 4$.

We conclude that for $\nu < 4$, $-\infty < I \leqq 0$, and $0 \leqq U_p < \infty$ $V_{\text{ren},\nu}^{U_p}$ is given by (5.31) where the (finite) counterterms $\delta\tilde{V}_\nu^{U_p}$ are of gauge invariant form by virtue of the Ward identity argument of Corollary 5.3.

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