Asymptotic Solutions of the Elastic Wave Equation and Reflected Waves near Boundaries

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Abstract. In the first half of this paper, we construct asymptotic solutions of linear anisotropic elastic equations. In the latter half, we investigate waves reflected by boundaries for plane incident waves in terms of these solutions. Especially, it is examined whether or not the mode-conversion occurs near points where the incident waves hit the boundaries perpendicularly.

Introduction

Let Ω be a domain in $\mathbb{R}^n_x (x = {}^t(x_1, \ldots, x_n), n \geq 2)$ with a C^∞ boundary $\partial\Omega$, and consider the elastic wave equation

$$\left(\partial_t^2 - \sum_{i,j=1}^n a_{ij} \partial_{x_i} \partial_{x_j} \right) u(t,x) = 0 \quad \text{in} \quad \mathbb{R} \times \Omega.$$

Here, $u = {}^t(u_1, \ldots, u_n)$ is the displacement vector, and a_{ij} are real constant $n \times n$-matrices whose (p, q)-components are denoted by a_{ipjq}. We assume that a_{ij} satisfy

$$a_{ipjq} = a_{pijq} = a_{jqip}, \quad i, j, p, q = 1, 2, \ldots, n, \tag{A.1}$$

$$\sum_{i,p,j,q=1}^n a_{ipjq} \varepsilon_{jq} \bar{\varepsilon}_{ip} \geq \delta \sum_{i,p=1}^n |\varepsilon_{ip}|^2 \quad \text{for every Hermitian matrices } (\varepsilon_{ij}), \tag{A.2}$$

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \text{ has eigenvalues of constant multiplicity for}$$

$$\text{any } \xi = {}^t(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n - \{0\}. \tag{A.3}$$

In the isotropic case (i.e. $a_{ipjq} = \mu(\delta_{pq}\delta_{ij} + \delta_{iq}\delta_{jp}) + \lambda\delta_{ip}\delta_{jq}, \lambda + \frac{2}{n}\mu > 0$ and $\mu > 0$), the above assumptions are all satisfied. Let $\{\lambda_l(\xi)\}_{l=1,\ldots,d}$ be the distinct eigenvalues of $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$. Then, $\lambda_l(\xi)$ become positive C^∞ functions ($\xi \neq 0$). We denote by $P_l(\xi)$ the projection into the eigenspace of $\lambda_l(\xi)$.

For each λ_l there are waves of λ_l-mode with the corresponding propagation speed. One of the main purposes in this paper is to examine how the waves are reflected near the boundary when the plane wave of a single mode hits the boundary perpendicularly. In Sect. 2 we make this investigation in terms of asymptotic solutions of the form

$$u^l(t, x; \sigma) = \sum_{j=0}^{\infty} e^{i\sigma(\varphi^l(x)-t)} u_j^l(t, x)(i\sigma)^{-j},$$

where $\varphi^l(x)$ is a C^∞ real-valued function satisfying $\lambda_l(\partial_x \varphi^l(x)) = 1$. In this paper "the wave of λ_l-mode" means the asymptotic solution of this type. The typical plane wave of λ_k-mode is expressed of the form

$$v(t, x; \sigma) = e^{i\sigma(\lambda_k(\omega)^{-1/2}\omega \cdot x - t)} v_0$$

with $\omega \in S^{n-1}$ and $v_0 = P_k(\omega)v_0$, which propagates in the direction ω with the speed $\lambda_l(\omega)^{1/2}$.

We fix the above k, ω, and v_0 ($\neq 0$), and construct the solutions $u^l(t, x; \sigma)$ ($l = 1, \dots, d$) so that $\sum_{l=1}^{d} u^l(t, x; \sigma) + v(t, x; \sigma)$ satisfies the boundary condition. Then, generally, every part u^l does not vanish, that is, the mode-conversion happens. However, in special situations, this conversion does not occur; e.g., when Ω is the half-space and ω is normal to the boundary, all u^l except u^k vanish. This suggests that the waves u^l in general cases with $l \neq k$ vanish on the characteristic curves starting at points on $\partial\Omega$ where the incident wave v hits $\partial\Omega$ perpendicularly. This is correct if $\partial\Omega$ is flat there (cf. Theorem 2.1 and the corollary). But, if $\partial\Omega$ is not flat at those points, this conjecture is not correct (cf. Remark 2.2). We note that the asymptotic solutions are unique (see Theorem 1.2).

Section 1 is devoted to construction of the asymptotic solutions with non-glancing given data on $\partial\Omega$. We carry out this construction for more general operators with the variable coefficients. In the isotropic case, Karal and Keller [2] made such solutions. It seems difficult to apply their methods to our case. The essence of our methods is similar to Lax's [3]. However, the transport equations in our case are of different type, that is, they become first order equations with matrix-valued (not scalar-valued) coefficients. To solve these equations, we introduce the local coordinates associated with the bicharacteristic curves, and reduce them to symmetric differential equations ordinary in the space variables.

The asymptotic solutions in Sect. 1 seem useful for studies of various problems. By a similar idea to ours, Soga [5] has constructed the asymptotic solutions under more restricted assumptions, and used them to solve an inverse scattering problem. But the analysis in [5] is not so precise as to obtain the results in the present paper.

1. Construction of the Asymptotic Solutions

We set $\mathscr{B}^\infty(M) = \{f(x) \in C^\infty(M); \sup_{x \in M} |\partial_x^\alpha f(x)| < \infty \text{ for any } \alpha\}$. Let $a_{ij}(x), b_i(x)$ and $c(x)$ ($i, j = 1, \dots, n$) be $n \times n$-matrices of functions $\in \mathscr{B}^\infty(\mathbb{R}^n)$, and put

$$L(x, \partial_x)u = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{n} b_i(x)\partial_{x_i}u + c(x)u.$$

We assume that the (p, q)-components $a_{ipjq}(x)$ of $a_{ij}(x)$ are real-valued and for each $x \in \mathbb{R}^n$ satisfy the conditions (A.1) $\sim$ (A.3) in introduction. In view of (A.3), we denote by $\lambda_l(x, \xi)$ $(l = 1, \ldots, d)$ the distinct eigenvalues of $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ $(\lambda_1 < \cdots < \lambda_d)$. By (A.1) $\sim$ (A.3), $\lambda_l(x, \xi)$ become positive C^∞ functions homogeneous of order 2 in ξ. Let $P_l(x, \xi)$ be the orthogonal projection into the eigenspace of $\lambda_l(x, \xi)$, which is an $n \times n$-matrix of C^∞ functions homogeneous of order 0 in ξ. $\lambda_l(x, \xi)$ and $P_l(x, \xi)$ are even in ξ (i.e. $\lambda_l(x, \xi) = \lambda_l(x, -\xi)$ and $P_l(x, \xi) = P_l(x, -\xi)$).

Let U be an open set in $\mathbb{R}^n (U \cap \partial\Omega \neq \phi)$, and consider the following equation for each $\lambda_l(x, \xi)$:

$$\begin{cases} \lambda_l(x, \partial_x \varphi^l(x)) = 1 & \text{in} \quad \Omega \cap U, \\ \varphi^l(x') = \varphi(x') & \text{on} \quad \partial\Omega \cap U, \\ \dfrac{\partial \varphi^l}{\partial \nu}(x') < 0 & \text{on} \quad \partial\Omega \cap U, \end{cases} \tag{1.1}$$

where $\varphi(x')$ is a given C^∞ real-valued function and $\nu(x') = {}^t(\nu_1(x'), \ldots, \nu_n(x'))$ is the unit outer vector normal to $\partial\Omega$ at $x' \in \partial\Omega$ (the points on $\partial\Omega$ being denoted by $x' = (x'_1, \ldots, x'_n)$). As is well known, if the (first order) derivatives of $\varphi(x')$ along the boundary are near 0 on $\partial\Omega \cap U$, we can construct a (unique) real-valued solution $\varphi^l(x)$ of (1.1) in a neighborhood of the boundary by the Hamilton–Jacobi methods; we assume that the U is chosen so that $\varphi^l(x)$ is defined on whole $\Omega \cap U$. There exists an implicit function $\tau_l(x', \xi)$ defined on $(\partial\Omega \cap U) \times \Lambda$ (where $\Lambda \ (\subset \mathbb{R}^n)$ is a small neighborhood of $\xi = 0$) such that

$$\lambda_l(x', \tau_l(x', \xi)\nu(x') + \xi) = 1 \quad \text{and} \quad \tau_l(x', \xi) < 0.$$

Let us note that for $x' \in \partial\Omega \cap U$,

$$\begin{aligned} \tau_l(x', 0) &= -\lambda_l(x', \nu(x'))^{-1/2}, \\ \partial_x \varphi^l(x') &= \tau_l(x', \partial_x \varphi^l_{\tan}(x'))\nu(x') + \partial_x \varphi^l_{\tan}(x'), \end{aligned} \tag{1.2}$$

where $\partial_x \varphi^l_{\tan}(x')$ denotes the tangential part (to $\partial\Omega$) of $\partial_x \varphi^l(x')$.

We impose the boundary condition $Bu = g$ on $\partial\Omega \cap U$, where

$$Bu = u|_{\partial\Omega} \text{ or } = (N + b)u|_{\partial\Omega},$$

$$N = \sum_{i,j=1}^{n} a_{ij}\nu_i \partial_{x_j}, \quad b = b(x) \in \mathscr{B}^\infty(\partial\Omega).$$

If a function $f(x; \sigma) \in \mathscr{B}^0(M)$ with a parameter $\sigma \in \mathbb{R}$ satisfies $|f(\cdot, \sigma)|_{\mathscr{B}^0(M)} \leq C|\sigma|^m$ for $|\sigma| \geq 1$, we write '$f(x; \sigma) = O(|\sigma|^m)$ in M'. 'The asymptotic solution $u^l(t, x; \sigma) = \sum_{j=0}^{\infty} e^{i\sigma(\varphi^l(x) - t)} u^l_j(t, x)(i\sigma)^{-j}$ (in $\mathbb{R} \times (U \cap \Omega)$ and of λ_l-mode) with the boundary data $g(t, x'; \sigma) = \sum_{j=0}^{\infty} e^{i\sigma(\varphi(x') - t)} g_j(t, x')(i\sigma)^{-j+\varepsilon'}$ means that

$$\begin{cases} (\partial_t^2 - L)\left(\sum_{j=0}^{m} e^{i\sigma(\varphi^l - t)} u^l_j(i\sigma)^{-j} \right) = O(|\sigma|^{-m+m_0}) & \text{in} \quad \mathbb{R} \times (\Omega \cap U), \\ B\left\{ \sum_{j=0}^{m} e^{i\sigma(\varphi^l - t)} u^l_j(i\sigma)^{-j} \right\} - \left\{ \sum_{j=0}^{m} e^{i\sigma(\varphi(x') - t)} g_j(t, x')(i\sigma)^{-j+\varepsilon} \right\} = O(|\sigma|^{-m+m_1}) \\ \hfill \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U) \end{cases}$$

for any positive integer m, where $\varepsilon = 0$ (respectively $= 1$) when $Bu = u|_{\partial\Omega}$ (respectively when $Bu = (N+b)u|_{\partial\Omega}$), and m_0, m_1 are constants independent of m. And we write

$$\begin{cases} (\partial_t^2 - L)u^l(t,x;\sigma) \cong 0 & \text{in} \quad \mathbb{R} \times (\Omega \cap U), \\ Bu^l(t,x';\sigma) \cong g(t,x';\sigma) & \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U). \end{cases}$$

Theorem 1.1. *Assume that the (first order) derivatives of $\varphi(x')$ (in (1.1)) along the boundary are near 0 on $\partial\Omega \cap U$. Then, for any data $g(t,x';\sigma) = \sum\limits_{j=0}^{\infty} e^{i\sigma(\varphi(x')-t)}$ $g_j(t,x')(i\sigma)^{-j+\varepsilon}(g_j(t,x') \in \mathscr{B}^{\infty}(\mathbb{R} \times (\partial\Omega \cap U)); \varepsilon = 0$ (respectively $= 1$) when $Bu = u|_{\partial\Omega}$ (respectively when $Bu = (N+b)u|_{\partial\Omega}$)), we have asymptotic solutions $u^l(t,x;\sigma) = \sum\limits_{j=0}^{\infty} e^{i\sigma(\varphi^l(x)-t)}u_j^l(t,x)(i\sigma)^{-j}$ $(l=1,\ldots,d)$ such that*

(i) $\quad\quad\quad (\partial_t^2 - L)u^l(t,x;\sigma) \cong 0 \quad\quad \text{in} \quad \mathbb{R} \times (\Omega \cap U),$

(ii) $\quad\quad\quad \sum\limits_{l=1}^{d} Bu^l(t,x';\sigma) \cong g(t,x';\sigma) \quad \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U),$

(iii) *if $g(t,x';\sigma) = 0$ for $t < t_0$, then $u^l(t,x;\sigma) = 0$ for $t < t_0$ $(l=1,\ldots,d)$.*

Theorem 1.2. *The asymptotic solutions $u^l(t,x;\sigma) = \sum\limits_{j=0}^{\infty} e^{i\sigma(\varphi^l-t)}u_j^l(i\sigma)^{-j}$ $(l=1,\ldots,d)$ in Theorem 1.1 are unique: If $(\partial_t^2 - L) \sum\limits_{l=1}^{d} u^l \cong 0$ in $\mathbb{R} \times (\Omega \cap U)$ and $B \sum\limits_{l=1}^{d} u^l \cong 0$ on $\mathbb{R} \times (\partial\Omega \cap U)$, then $u_j^l(t,x) = 0$ in $\mathbb{R} \times (\Omega \cap U)$ for $l=1,\ldots,d$ and $j = 0,1,\ldots$.*

Theorem 1.2 will be verified after proving Theorem 1.1.

Let us prove Theorem 1.1. At first, note that the solution $\varphi^l(x)$ of (1.1) exists for every $l\ (=1,\ldots,d)$ since the derivatives of $\varphi(x')$ are near 0.

Hereafter we use the notations $f_x(= \partial_x f)$ and $f_{x_i}(= \partial_{x_i} f)$. We rewrite $(\partial_t^2 - L) \sum\limits_{j=0}^{\infty} e^{i\sigma(\varphi^l-t)}u_j^l(i\sigma)^{-j}$ in the following form:

$$e^{i\sigma(\varphi^l(x)-t)} \sum_{j=0}^{\infty} [(I - \sum a_{pq}\varphi_{x_p}^l \varphi_{x_q}^l)P_l u_j^l + (I - \sum a_{pq}\varphi_{x_p}^l \varphi_{x_q}^l)(I - P_l)u_j^l$$

$$+ (I - P_l)\{(-2\partial_t - H)u_{j-1}^l + (\partial_t^2 - L)u_{j-2}^l\} + P_l(-2\partial_t - H)P_l u_{j-1}^l$$

$$+ P_l\{(-2\partial_t - H)(I - P_l)u_{j-1}^l + (\partial_t^2 - L)u_{j-2}^l\}](i\sigma)^{-j+2},$$

where $u_{-1} = u_{-2} = 0$, $P_l = P_l(x, \varphi_x^l(x))$ and

$$H = \sum_{i,j=1}^{n} (a_{ij}(x) + a_{ji}(x))\varphi_{x_i}^l(x)\partial_{x_j} + (L-c)\varphi^l.$$

In the same way as Lax [3], we determine $(I - P_l)u_j^l$ and next $P_l u_j^l$, and eliminate each term in the above summation together with making u_j^l satisfy the boundary condition. When considering the boundary condition, we use the matrices

$$P(x') = \sum_{l=1}^{d} P_l(x', \varphi_x^l(x')),$$

$$Q(x') = \sum_{l=1}^{d} (N\varphi^l)(x')P_l(x', \varphi_x^l(x')).$$

Since $P(x') = \sum_{l=1}^{d} P_l(x', v(x'))$ and $Q(x') = -\sum_{l=1}^{d} \lambda_l(v(x'))^{1/2} P_l(x', v(x'))$ if the derivatives of $\varphi(x')$ all vanish, $P(x')$ and $Q(x')$ are non-singular when its derivatives are near 0 on $\partial\Omega \cap U$.

It is obvious that the part $(I - \sum a_{pq}\varphi_{x_p}^l \varphi_{x_q}^l)P_l u_j^i$ is equal to $(I - \lambda_l I)P_l u_j^i = 0$. To eliminate the other parts, we consider the equations

$$P_l(2\partial_t + H)P_l u_j^i + P_l f_j^i = 0 \quad \text{in} \quad \mathbb{R} \times (\Omega \cap U), \tag{1.3}$$

$$P_l u_j^i = \begin{cases} P_l P^{-1}\left(-\sum_{i=1}^{d}(I - P_i)u_j^i + g_j\right) & \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U) \quad \text{when} \quad Bu = u|_{\partial\Omega}, \\ P_l Q^{-1}\left(-\sum_{i=1}^{d}(N\varphi^i)(I - P_i)u_j^i - \sum_{i=1}^{d} Bu_{j-1}^i + g_j\right) \\ \qquad\qquad \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U) \quad \text{when} \quad Bu = (N + b)u|_{\partial\Omega}, \end{cases} \tag{1.4}$$

$$(I - \sum a_{pq}\varphi_{x_p}^l \varphi_{x_q}^l)(I - P_l)u_j^i + (I - P_l)h_{j-1}^i = 0 \quad \text{in} \quad \mathbb{R} \times (\Omega \cap U), \tag{1.5}$$

where $f_j^i = (2\partial_t + H)(I - P_l)u_j^i - (\partial_t^2 - L)u_{j-1}^i$ and $h_{j-1}^i = (-2\partial_t - H)u_{j-1}^i + (\partial_t^2 - L)u_{j-2}^i$. Note that f_j^i and h_{j-1}^i are defined from only u_{j-2}^i, u_{j-1}^i and $(I - P_l)u_j^i$. At first put $(I - P_l)u_0^i = 0$; i.e. take u_0^i in the eigenspace $P_l\mathbb{R}^n$. Then $f_0^i = 0$. If Eqs. (1.3) and (1.4) for $j = 0$ are solvable, u_0^i is determined. Noting that $(I - P_l) = \sum_{i \neq l} P_i$, we see that (1.5) is of the form $\sum_{i \neq l}\{(1 - \lambda_i)P_i u_1^i + P_i h_0^i\}$. Therefore, setting $P_i u_1^i = -(1 - \lambda_i)^{-1} P_i h_0^i$ (for $i \neq l$) determines $(I - P_l)u_1^i$. Repeating this procedure inductively we can determine all u_j^i $(j = 0, 1, \dots)$. Furthermore it is seen that the asymptotic solution $\sum_{l=1}^{d} u^l(t, x; \sigma)$ with these $u_j^l(t, x)$ satisfy the required boundary conditions: In fact, when $Bu = u|_{\partial\Omega}$, (1.4) yields that

$$\sum_{l=1}^{d} \sum_{j=0}^{m} e^{i\sigma(\varphi^l - t)}u_j^l(i\sigma)^{-j} = e^{i\sigma(\varphi - t)} \sum_{j=0}^{m} \sum_{l=1}^{d} \left\{P_l u_j^l + \sum_{l=1}^{d}(I - P_l)u_j^l\right\}(i\sigma)^{-j}$$

$$= \sum_{j=0}^{m} e^{i\sigma(\varphi - t)} g_j(i\sigma)^{-j} \quad \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U);$$

when $Bu = (N + b)u|_{\partial\Omega}$,

$$B \sum_{l=1}^{d} \sum_{j=0}^{m} e^{i\sigma(\varphi^l - t)}u_j^l(i\sigma)^{-j} = e^{i\sigma(\varphi - t)} \sum_{j=0}^{m} \left\{\sum_{l=1}^{d}(N\varphi^l)P_l u_j^l + \right.$$

$$\sum_{l=1}^{d}(N\varphi^l)(I - P_l)u_j^l + \sum_{l=1}^{d} Bu_{j-1}^l\bigg\}(i\sigma)^{-j+1} + e^{i\sigma(\varphi - t)} \sum_{l=1}^{d} Bu_m^l(i\sigma)^{-m}$$

$$= \sum_{j=0}^{m} e^{i\sigma(\varphi - t)} g_j(i\sigma)^{-j+1} + e^{i\sigma(\varphi - t)} \sum_{l=1}^{d} Bu_m^l(i\sigma)^{-m} \quad \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U).$$

Hence we have only to show that (1.3) and (1.4) are solvable for each $l = 1, \dots, d$. Fix $\tilde{x} \in \partial\Omega \cap U$ and take bases $\tilde{e}_i$ $(i = 1, \dots, \tilde{d})$ in $P_l(\tilde{x}; v(\tilde{x}))\mathbb{R}^n$. Then

$e_i'(x) = P_l(x, \varphi_x^l(x))\tilde{e}_i$ $(i = 1, \ldots, \tilde{d})$ are linearly independent if the (first order) derivatives of $\varphi(x')$ are near 0 and x moves near $\tilde{x}$. We assume that $e_i'(x)$ are linearly independent for $x \in \Omega \cap U$. Setting

$$e_1 = |e_1'|^{-1}e_1', \, e_i = \left| e_i' - \sum_{j=1}^{i-1} ({}^te_i'e_j)e_j \right|^{-1} \left(e_i' - \sum_{j=1}^{i-1} ({}^te_i'e_j)e_j \right) \quad (i = 2, \ldots, \tilde{d}),$$

we obtain orthonormal bases $e_i(x)$ $(i = 1, \ldots, \tilde{d})$ in $P_l(x, \varphi_x^l(x))\mathbb{R}^n$, which belong to $\mathscr{B}^\infty(\Omega \cap U)$. We put

$$e(x) = (e_1(x), \ldots, e_{\tilde{d}}(x)), \tag{1.6}$$
$$v_j(t, x) = {}^te(x)u_j^l(t, x) \quad (j = 0, 1, \ldots).$$

Let $(q(s; y), p(s; y))$ be the bicharacteristic curves defined by

$$\begin{cases} \dfrac{dq}{ds} = \partial_\xi \lambda_l(q, p), \quad \dfrac{dp}{ds} = -\partial_x \lambda_l(q, p) \\[2mm] q(0; y) = x'(y), \quad p(0; y) = \varphi_x^l(x'(y)), \end{cases} \tag{1.7}$$

where $x'(y)$ is a system of local coordinates of $\partial\Omega \cap U$ $(y \in \mathbb{R}^{n-1})$. Then the mapping: $(s, y) \to q(s; y)$ becomes diffeomorphic if the (first order) derivatives of $\varphi(x')$ are near 0 and $|s|$ are small enough. We assume that $x = q(s; y)$ becomes a system of the local coordinates of $\Omega \cap U$. By using the variable (s, y) and $w(s, t, y) = v_j(t, q(s; y))$, Eqs. (1.3) and (1.4) are transformed as follows:

$$\left\{ 2\partial_t + a_0(s, y)\partial_s + \sum_{i=1}^{n-1} a_i(s, y)\partial_{y_i} + b_0(s, y) \right\} w(s, t, y) = f(s, t, y) \quad \text{in} \quad (0, s_0) \times \mathbb{R} \times \tilde{U},$$
$$\tag{1.8}$$

$$w(0, t, y) = h(t, y) \quad \text{on} \quad \mathbb{R} \times \tilde{U},$$

where $\Omega \cap U \subset \{q(s; y) : (s, y) \in (0, s_0) \times \tilde{U}\}$ and

$$a_0 = {}^te \sum_{i,j=1}^n (a_{ij} + a_{ji})\varphi_{x_i}^l \frac{\partial s}{\partial x_j} e, \tag{1.9}$$

$$a_i = {}^te \sum_{p,j=1}^n (a_{pj} + a_{jp})\varphi_{x_p}^l \frac{\partial y_i}{\partial x_j} e \quad (i = 1, \ldots, n-1), \tag{1.10}$$

$$b_0 = {}^te(L\varphi^l - c\varphi^l)e + {}^te \sum_{i,j=1}^n (a_{ij} + a_{ji})\varphi_{x_i}^l \frac{\partial s}{\partial x_i} \partial_s e$$
$$+ {}^te \sum_{i=1}^{n-1} \sum_{p,j=1}^n (a_{pj} + a_{jp})\varphi_{x_p}^l \frac{\partial y_i}{\partial x_j} \partial_{y_i} e.$$

Solving this equation with $f = -{}^tef_j^l$ and $h = {}^teP^{-1}\left(-\sum_{i=1}^d (I - P_i)u_j^i + g_j \right)$
(respectively $= {}^teQ^{-1}\left(-\sum_{i=1}^d (N\varphi^i)(I - P_i)u_j^i - \sum_{i=1}^d Bu_{j-1}^i + g_j \right)$ when $Bu = u|_{\partial\Omega}$ (respectively when $Bu = (N + b)u|_{\partial\Omega}$), we get the required solution $P_l u_j^l = ew$ for each l $(= 1, \ldots, d)$.

Now, let us examine (1.8). At first, note that every a_i $(i = 1, \ldots, n-1)$ defined

by (1.10) vanishes: (1.8) is reduced to

$$(2\partial_t + a_0\partial_s + b_0)w = f \quad \text{in} \quad (0, s_0) \times \mathbb{R} \times \tilde{U}, \tag{1.11}$$

which is seen from

Lemma 1.3. *If ξ and $\eta \in \mathbb{R}^n$ ($\xi \neq 0$) satisfy $\eta \cdot \partial_\xi \lambda_l(x, \xi) = 0$ (x and l being fixed arbitrarily), then we have*

$$P_l(x, \xi) \sum_{i,j=1}^n (a_{ij}(x) + a_{ji}(x))\xi_i\eta_j P_l(x, \xi) = 0.$$

For the local coordinates $x = q(s; y)$ we have $(\partial_x y_i) \cdot (\partial_\xi \lambda_l(x, \varphi_x^l)) = (\partial_x y_i) \cdot (\partial_s q) = 0$ ($i = 1, \ldots, n-1$); using the above lemma with $\xi = \varphi_x^l$ and $\eta = \partial_x y_i$, we obtain

$P_l(x, \varphi_x^l) \sum_{k,j=1}^n (a_{kj} + a_{jk})\varphi_{x_k}^l \dfrac{\partial y_i}{\partial x_j} P_l(x, \varphi_x^l) = 0$, which yields that $a_i = 0$ ($i = 1, \ldots, n-1$).

Proof of Lemma 1.3. Applying ∂_{ξ_p} to the equality

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j P_l(x, \xi) = \lambda_l(x, \xi)P_l(x, \xi),$$

we obtain the equality $\displaystyle\sum_{i=1}^n (a_{ip} + a_{pi})\xi_i P_l + \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \partial_{\xi_p} P_l = (\partial_{\xi_p}\lambda_l)P_l + \lambda_l \partial_{\xi_p} P_l$.
Multiply this equality by η_p and sum it up in $p = 1, \ldots, n$. Then we have

$$P_l \sum_{i,p=1}^n (a_{ip} + a_{pi})\xi_i\eta_p P_l$$

$$= \left(\sum_{p=1}^n \eta_p\partial_{\xi_p}\lambda_l\right)P_l + \lambda_l P_l \sum_{p=1}^n \eta_p\partial_{\xi_p} P_l - P_l \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \sum_{p=1}^n \eta_p\partial_{\xi_p} P_l = 0.$$

The proof is complete.

Next, let us note that $a_0(s, y)$ is of the special form:

Lemma 1.4. $a_0(s, y)$ *(defined by (1.9)) is a symmetric $\tilde{d} \times \tilde{d}$-matrix of real-valued functions, and satisfies*

$$|a_0(s, y) - I|_{\mathscr{B}^0((0, s_0) \times \tilde{U})} \leq C|\partial_y\varphi(x'(y))|_{\mathscr{B}^0(\tilde{U})}.$$

Proof. Let $(q(s; y), p(s; y))$ be the solution of (1.7). Then, $p(s; y) = \varphi_x^l(q(s; y))$ and $\lambda_l(q(s; y), p(s; y)) = 1$. By the Euler equality, we have $2 = \partial_\xi\lambda \cdot p = \partial_s q \cdot p$. Differentiating this in y_i yields that $\partial_s\partial_{y_i}q \cdot p + \partial_s q \cdot \partial_{y_i}p = 0$. In the same way, we obtain $\partial_{y_i}q \cdot \partial_s p - \partial_s q \cdot \partial_{y_i}p = 0$ from the equality $\lambda_l(q, p) = 1$. Therefore, $\partial_s(\partial_{y_i}q \cdot p) = 0$. Combining this euality with (1.2), we have

$$|\partial_{y_i}q \cdot p| \leq C_1|\partial_y\varphi(x'(y))|_{\mathscr{B}^0(\tilde{U})}. \tag{1.12}$$

Let $\mu(s, y)$ be the unit vector orthogonal to every $\partial_{y_i}q(s; y)$ ($i = 1, \ldots, n-1$) with $v \cdot \mu > 0$. $\gamma = {}^t\!\left(\dfrac{\partial s}{\partial x_1}, \ldots, \dfrac{\partial s}{\partial x_n}\right)$ satisfies $\gamma \cdot \partial_{y_i}q = 0$ ($x = q(s; y)$), which implies that $\gamma = -|\gamma|\mu$. From this and the equality $p = (p \cdot \mu)\mu + (p - (p \cdot \mu)\mu)$, we have

$$\gamma = -|\gamma|(p \cdot \mu)^{-1}p + |\gamma|(p \cdot \mu)^{-1}(p - (p \cdot \mu)\mu)$$
$$\equiv -|\gamma|(p \cdot \mu)^{-1}p + \tilde{\gamma}. \tag{1.13}$$

Since $\tilde{\gamma}$ is expressed by linear combination of $\partial_{y_1} q, \ldots, \partial_{y_{n-1}} q$ and satisfies $\tilde{\gamma} \cdot \partial_{y_i} q = |\gamma|(p \cdot \mu)^{-1} \partial_{y_i} q \cdot p$, (1.12) yields that

$$|\tilde{\gamma}| \leq C_2 |\partial_y \varphi(x'(y))|_{\mathscr{B}^0(\tilde{U})}. \qquad (1.14)$$

Insert (1.13) into $1 = \gamma \cdot \partial_s q$. Then $1 = -|\gamma|(p \cdot \mu)^{-1} p \cdot \partial_s q + \tilde{\gamma} \cdot \partial_s q = -2|\gamma|(p \cdot \mu)^{-1} + \tilde{\gamma} \cdot \partial_s q$. Therefore, using (1.13) again, we have

$$\gamma = 2^{-1} p - 2^{-1}(\tilde{\gamma} \cdot \partial_s q) p + \tilde{\gamma}.$$

Mixing this with (1.14), we get

$$\left| \frac{\partial s}{\partial x_i} - \frac{1}{2} \varphi_{x_i}^l \right| \leq C_3 |\partial_y \varphi(x'(y))|_{\mathscr{B}^0(\tilde{U})},$$

which proves the lemma. The proof is complete.

Thus it suffices to analyze the following problem:

$$\begin{cases} (2\partial_t + \alpha(s)\partial_s + \beta(s))w(s,t) = f(s,t) & \text{in} \quad I \times \mathbb{R} \quad (I = (0,s_0)), \\ w(0,t) = h(t) & \text{on } \mathbb{R}, \end{cases} \qquad (1.15)$$

where α and β are $\tilde{d} \times \tilde{d}$-matrices of functions in $\mathscr{B}^\infty(I)$ and α is real-valued and symmetric. As is easily seen, all the requirements are derived from

Lemma 1.5. (i) *If* $|\det \alpha| \geq \alpha_0$ (> 0), *there exists a unique solution* $w(s,t)$ *of* (1.15) *in* $\mathscr{B}^\infty(I \times \mathbb{R})$ *for any* $f(s,t) \in \mathscr{B}^\infty(I \times \mathbb{R})$ *and* $h(t) \in \mathscr{B}^\infty(\mathbb{R})$.

(ii) *Let* $2^{-1}|\zeta|^2 \leq {}^t\zeta \alpha \zeta$ *for* $\zeta \in \mathbb{R}^{\tilde{d}}$. *Then, if the support of the data* (f,h) *in* (1.15) *is contained in* $\{t_0 \leq t\}$, *so is the support of the solution* w *of* (1.15).

Proof. (i) From the assumption, $\alpha^{-1}(s)$ becomes a symmetric matrix of functions $\in \mathscr{B}^\infty(I)$. Equation (1.15) is equivalent to

$$\begin{cases} (\partial_s + 2\alpha^{-1}\partial_t + \alpha^{-1}\beta)w = f & \text{in} \quad I \times \mathbb{R}, \\ w|_{s=0} = h & \text{on } \mathbb{R}. \end{cases}$$

This is a well-known Cauchy problem (cf. Friedrichs [1] or Chap. 5 of Mizohata [4]). Therefore (i) in the lemma is obtained.

(ii) Introduce the variables $s' = s$ and $t' = 2^{-1}t - s$. Then (1.15) is transformed into the problem

$$\begin{cases} (\partial_{s'} + (\alpha^{-1} - I)\partial_{t'} + \alpha^{-1}\beta)w'(s',t') = f'(s',t') & \text{in} \quad I \times \mathbb{R}, \\ w'|_{s'=0} = h(t') & \text{on } \mathbb{R}. \end{cases} \qquad (1.16)$$

It is seen from the assumption that $\alpha^{-1} - I$ is a symmetric matrix satisfying $|{}^t\zeta(\alpha^{-1} - I)\zeta| \leq |\zeta|^2$ for $\zeta \in \mathbb{R}^{\tilde{d}}$. This implies that every eigenvalue of this matrix is in $[-1, 1]$. Therefore (1.16) is not only solvable but also has finite propagation speed less than 1 (cf. Note 2 in Sect. 12 of Chap. 6 of [4]). Hence, in the original Eq. (1.15), the solution $w(s,t)$ exists and is equal to 0 on the cone $D = \{(s,t) \in I \times \mathbb{R}; |2^{-1}(t - t_1) - s + s_1| \leq s_1 - s, s < s_1\}$ if the data vanish on D, which proves (ii) in the lemma.

Proof of Theorem 1.2. Inserting $u = \sum_{j=0}^{\infty} \sum_{l=1}^{d} e^{i\sigma(\varphi^l - t)} u_j^l (i\sigma)^{-j}$ into $(\partial_t^2 - L)u \cong 0$ and

$Bu \cong 0$, we have

$$\sum_{l=1}^{d} e^{i\sigma\varphi^l} \left\{ \left(I - \sum a_{pq}\varphi^l_{x_p}\varphi^l_{x_q} \right) u^l_j - (2\partial_t + H)u^l_{j-1} + (\partial^2_t - L)u^l_{j-2} \right\}$$

$$= 0 \quad \text{in} \quad \mathbb{R} \times (\Omega \cap U), \qquad (1.17)$$

$$\sum_{l=1}^{d} u^l_j = 0 \quad \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U) \quad \text{when} \quad Bu = u|_{\partial\Omega}, \qquad (1.18)$$

$$\sum_{l=1}^{d} \left\{ \sum_{p,q=1}^{d} v_p(\varphi^l_{x_q}a_{pq}u^l_j + a_{pq}\partial_{x_a}u^l_{j-1}) + bu^l_{j-1} \right\} = 0$$

$$\text{on} \quad \mathbb{R} \times (\partial\Omega \cap U) \quad \text{when} \quad Bu = (N+b)u|_{\partial\Omega}, \quad (1.19)$$

for any $j = 0, 1, \dots$, and any large σ.

Take any pair φ^i and φ^j $(1 \le i < j \le d)$. Then, for any open set $W \subset \Omega \cap U$ there exists a point x in W such that $\varphi^i(x) \neq \varphi^j(x)$; because, if not, $\varphi^i_x = \varphi^j_x$ in W and so $1 = \lambda_i(x, \varphi^i_x) = \lambda_i(x, \varphi^j_x) < \lambda_j(x, \varphi^j_x) = 1$, which is a contradiction. This fact implies that the set $\tilde{\Omega} = \{x : \varphi^i(x) \neq \varphi^j(x) \text{ for every } i \text{ and } j \text{ with } i \neq j\}$ is dense in $\Omega \cap U$. On the other hand, if $\varphi^1, \dots, \varphi^d$ are different each other, each term (hereafter denoted by $e^{i\sigma\varphi^l}\Phi^l$) in the summation of (1.17) must be equal to 0. In fact, differentiating $\sum_{l=1}^{d} e^{i\sigma\varphi^l}\Phi^l = 0$ $(d-1)$-times in σ, we have $(\Phi^1, \dots, \Phi^d)J = 0$, where J is the $d \times d$ matrix whose (l, k)-component is $(i\varphi^l)^{k-1}e^{i\sigma\varphi^l}$. This yields that $(\Phi^1, \dots, \Phi^d) = 0$ since $\det J \neq 0$. Therefore each term $e^{i\sigma\varphi^l}\Phi^l$ vanishes in $\mathbb{R} \times \tilde{\Omega}$ and so in $\mathbb{R} \times (\Omega \cap U)$ (since u_j are assumed smooth). Hence, rewriting the Φ^l, we obtain from (1.17)

$$(I - \sum a_{pq}\varphi^l_{x_p}\varphi^l_{x_q})(I - P_l)u^l_j + (I - P_l)\{(-2\partial_t - H)u^l_{j-1} + (\partial^2_t - L)u^l_{j-2}\}$$

$$- P_l(2\partial_t + H)P_l u^l_{j-1} + P_l\{(-2\partial_t - H)(I - P_l)u^l_{j-1} + (\partial^2_t - L)u^l_{j-2}\}(i\sigma)^{-j+2} = 0$$

$$\text{in} \quad \mathbb{R} \times (\Omega \cap U) \quad (l = 1, \dots, d). \quad (1.20)$$

Equation (1.20) with $j = 0$ means that $(I - P_l)u^l_0 = 0$ in $\mathbb{R} \times (\Omega \cap U)$ for $l = 1, \dots, d$. Combining this with (1.18) and (1.19), we have $\sum_{l=1}^{d} P_l u^l_0 = 0$ and $\sum_{l=1}^{d} \sum_{p,q=1}^{d} v_p\varphi^l_{x_q}a_{pq}P_l u^l_0 = 0$ on $\mathbb{R} \times (\partial\Omega \cap U)$ respectively. These yield that $P_l u^l_0 = 0$ on $\mathbb{R} \times (\partial\Omega \cap U)$ for $l = 1, \dots, d$ since the following estimates are derived:

$$\sum_{l=1}^{d} |P_l v^l|^2 \le C \left| \sum_{l=1}^{d} P_l v^l \right|^2, \qquad v^l \in \mathbb{R}^n,$$

$$\sum_{l=1}^{d} |P_l v^l|^2 \le C \left| \sum_{l=1}^{d} \sum_{p,q=1}^{d} v_p\varphi^l_{x_q}a_{pq}P_l v^l \right|^2, \quad v^l \in \mathbb{R}^n,$$

which are seen from the facts that $P_l(= P_l(\varphi^l_x))$ is near $P_l(v)$ (cf. (1.2)) and that $P_1(v), \dots, P_d(v)$ are the orthogonal projections. Applying P_l to (1.20) with $j = 1$, we get $P_l(2\partial_t + H)P_l u^l_0 = 0$ in $\mathbb{R} \times (\Omega \cap U)$. Therefore, considering Eq. (1.8) (or (1.11)) as we did for (1.3), we obtain $P_l u^l_0 = 0$ in $\mathbb{R} \times (\Omega \cap U)$; hence $u^l_0 = 0$ in $\mathbb{R} \times (\Omega \cap U)$.

Multiplying (1.20) (with $j = 1$) by $(I - P_l)$ yields that $(I - \sum a_{pq}\varphi^l_{x_p}\varphi^l_{x_q})(I - P_l)u^l_1 = 0$

in $\mathbb{R} \times (\Omega \cap U)$, which implies that $(I - P_l)u_1^l = 0$ in $\mathbb{R} \times (\Omega \cap U)$. By the same procedures as for u_0^l, we have $u_1^l = 0$ in $\mathbb{R} \times (\Omega \cap U)$. Continuing these processes inductively, we obtain $u_j^l = 0$ in $\mathbb{R} \times (\Omega \cap U)$ for all $j = 0, 1, \ldots$ and $l = 1, \ldots, d$, which proves the theorem.

2. Reflected Waves for Plane Incident Waves

In this section, we assume that a_{ij} are constant matrices satisfying (A.1), (A.2) and (A.3) stated in the introduction and that $L = \sum_{i,j=1}^{n} a_{ij} \partial_{x_i} \partial_{x_j}$. Furthermore we suppose that the boundary operator is of the form $Bu = u|_{\partial\Omega}$ or $= Nu|_{\partial\Omega}$ ($N = \sum_{i,j=1}^{n} v_i a_{ij} \partial_{x_j}$). We use the same notations that in Sect. 1. When they do not depend on the variable x, they are abbreviated in the following way: $\lambda_k(\xi) (= \lambda_k(x, \xi))$, $P_k(\xi) (= P_k(x, \xi))$, etc.

If $v_0 \in \mathbb{R}^n$ satisfies $v_0 = P_k(\omega)v_0$ ($\omega \in S^{n-1}$), the function

$$v(t, x; \sigma) = e^{i\sigma(\lambda_k(\omega)^{-1/2}\omega \cdot x - t)}v_0$$

is one of the solutions of the equation $(\partial_t^2 - L)v = 0$ (where σ is a real parameter with $|\sigma| \geq 1$). We call the solutions of this type the plane waves of λ_k-mode proceeding in the direction ω. Hereafter we fix the k and the ω arbitrarily. Set

$$M = \{x \in \partial\Omega : v(x) = \omega\},$$

and take a sufficiently small neighborhood U of M so that the discussions later are correct.

Since it can be assumed that the gradient of $\varphi(x') = \lambda_k(\omega)^{-1/2}\omega \cdot x'$ is small enough on $U \cap \partial\Omega$, for each $l (= 1, \ldots, d)$ there exists the solution $\varphi^l(x)$ of (1.1) with $\varphi(x') = \lambda_k(\omega)^{-1/2}\omega \cdot x'$. By the procedures stated in Sect. 1, we can construct an asymptotic solution $u(t, x; \sigma) = \sum_{l=1}^{d} u^l(t, x; \sigma) = \sum_{l=1}^{d} \left\{ \sum_{j=0}^{\infty} e^{i\sigma(\varphi^l(x) - t)} u_j^l(x)(i\sigma)^{-j} \right\}$ such that

$$\begin{cases} (\partial_t^2 - L)(u + v) \cong 0 & \text{in} \quad \mathbb{R} \times (\Omega \cap U), \\ B(u + v) \cong 0 & \text{on} \quad \mathbb{R} \times (\partial\Omega \cap U). \end{cases} \quad (2.1)$$

Note that in this case the amplitude functions $u_j^l(x)$ do not depend on t, and that u_j^l are determined uniquely (cf. Theorem 1.2).

We choose local coordinates $x'(y)$ of $\partial\Omega \cap U$ defined on $\tilde{U} \subset \mathbb{R}^{n-1}$ ($\partial\Omega \cap U = \{x'(y) : y \in \tilde{U}\}$). Let $(q^l(s; y), p^l(s; y))$ be the bicharacteristic curve associated with $\varphi^l(x)$ (defined by (1.7)). With these notations we have

Theorem 2.1. (i) *Every principal amplitude function u_0^l of u^l with $l \neq k$ satisfies*

$$u_0^l(q^l(s; y)) = 0 \quad \text{when} \quad x'(y) \in M \quad \text{and} \quad q^l(s; y) \in \Omega \cap U.$$

(ii) *If the derivatives of $v(x')$ vanishes of order $m + 1$ (≥ 1) on M (i.e. $\partial_y^\alpha v(x'(y)) = 0$ when $1 \leq |\alpha| \leq m + 1$ and $x'(y) \in M$), then the $u_j^l(x)$ with $l \neq k$ satisfies*

$$\partial_x^\alpha u_j^l(q^l(s; y)) = 0 \quad \text{when} \quad |\alpha| \leq m + 1 - 2j, \, x'(y) \in M \quad \text{and} \quad q^l(s; y) \in \Omega \cap U.$$

From the above theorem, the following corollary follows immediately:

Corollary. *If $\partial\Omega$ is flat on M (i.e. $\partial_y^\alpha v(x'(y)) = 0$ for every $\alpha \neq 0$ when $x'(y) \in M$), then $u^l(t, x; \sigma) = O(|\sigma|^{-\infty})$ on $\{x = q^l(s; y) : q^l(s; y) \in \Omega \cap U, x'(y) \in M\}$ for $l \neq k$.*

In the above corollary, if $\partial\Omega$ is not flat on M, the conclusion is not necessarily correct. Let us give an example showing it. Let a_{ij} be isotropic (i.e. $a_{ipjq} = \mu(\delta_{pq}\delta_{ij} + \delta_{iq}\delta_{jp}) + \lambda\delta_{iq}\delta_{jp}$, $\lambda + \frac{2}{n}\mu > 0$ and $\mu > 0$). Then, all the assumptions (A.1) $\sim$ (A.3) are satisfied, and the eigenvalues $\{\lambda_l(\xi)\}_{l=1,\ldots,d}$ become of the forms: $\lambda_1(\xi) = \mu|\xi|^2$ and $\lambda_2(\xi) = (\lambda + 2\mu)|\xi|^2 (d = 2)$. Let $Bu = u|_{\partial\Omega}$, and assume that $\Omega = \{x = {}^t(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} ; \psi(y) < z\}$. We suppose that $\psi(0) = 0$, $\psi_y(0) = 0$ and $\psi_y(y) \neq 0$ for all $y \neq 0$ in a neighborhood of $y = 0$, and consider Eq. (2.1) near the origin. In these situations, we obtain

Remark 2.2. (i) Assume that the Gaussian curvature of $\partial\Omega$ does not vanish at the origin, and choose the incident wave $v(t, x; \sigma) = e^{i\sigma(\lambda_k(\omega)^{-1/2}\omega \cdot x - t)}v_0$ as follows:

$$\omega = {}^t(0, \ldots, 0, -1), \quad k = 1 \quad \text{and} \quad v_0 = {}^t(v_{01}, \ldots, v_{0n-1}, 0) \neq 0$$

(i.e. v is of $\lambda_1(\omega)$-mode). Then the $\lambda_2(\varphi_x^2)$-mode part $u^2(t, x; \sigma) = \sum_{j=0}^{\infty} e^{i\sigma(\varphi^2(x) - t)}u_j^2(x)(i\sigma)^{-j}$ satisfies

$$u_1^2(0) \neq 0.$$

(ii) Assume that the mean curvature of $\partial\Omega$ does not vanish at the origin, and choose the incident wave as follows:

$$\omega = {}^t(0, \ldots, 0, -1), \quad k = 2 \quad \text{and} \quad v_0 = {}^t(0, \ldots, 0, v_{0n}) \neq 0$$

(i.e. v is of $\lambda_2(\omega)$-mode). Then the function $u_1^1(x)$ of the $u^1(t, x; \sigma)$ satisfies

$$u_1^1(0) \neq 0.$$

Let us note in the above remark that the incident wave hits $\partial\Omega$ perpendicularly only at $x = 0$ (i.e. $M = \{0\}$), and that the properties $u_1^l(q^l(s; 0)) \neq 0$ ($l = 1, 2$) follow for any small $s > 0$ in the respective cases. The above remark will be provided at the end of this section.

From now on, we shall prove Theorem 2.1. At first let us check the statement (i). As was discussed in Sect. 1, $(I - P_l)u_0^l = 0$, and $P_l u_0^l$ satisfies

$$P_l H P_l u_0^l = 0 \quad \text{in } \Omega \cap U,$$

$$P_l u_0^l = \begin{cases} -P_l P^{-1}v_0 & \text{on } \partial\Omega \cap U \quad \text{when} \quad Bu = u|_{\partial\Omega}, \\ -P_l Q^{-1}v_{-1} & \text{on } \partial\Omega \cap U \quad \text{when} \quad Bu = Nu|_{\partial\Omega}, \end{cases}$$

where $v_{-1} = \lambda_k(\omega)^{-1/2} \sum_{i,j=1}^{n} a_{ij}v_i\omega_j v_0$. Furthermore, this is equivalent to the following equation for $w_0(s, y) = {}^t e(q^l(s; y)) u_0^l(q^l(s; y))$ ($e(x)$ being the matrix in (1.6)):

$$\begin{cases} (a_0(s, y)\partial_s + b_0(s, y))w_0(s, y) = 0 & \text{in } I \times \tilde{U}, \\ w_0(0, y) = \begin{cases} -{}^t e(x'(y))P^{-1}v_0 & \text{on } \tilde{U} \quad \text{when} \quad Bu = u|_{\partial\Omega}, \\ -{}^t e(x'(y))Q^{-1}v_{-1} & \text{on } \tilde{U} \quad \text{when} \quad Bu = Nu|_{\partial\Omega}. \end{cases} \end{cases} \qquad (2.2)$$

We have $P^{-1}v_0 = \left\{ \sum_{l=1}^{d} P_l(w) \right\}^{-1} v_0 = v_0$ and $Q^{-1}v_{-1} = \left\{ -\sum_{l=1}^{d} \lambda_l(\omega)^{1/2} P_l(\omega) \right\}^{-1} \times$

$\lambda_k(\omega)^{1/2}v_0 = -v_0$ when $x'(y) \in M$. Therefore, if $l \neq k$, the data $-{}^t e(x'(y))P^{-1}v_0$ and $-{}^t e(x'(y))Q^{-1}v_{-1}$ vanish when $x'(y) \in M$. Hence, for these y and l we have $w_0(s, y) = 0$ on $0 \leq s \leq s_0$ (by the uniqueness of the solutions of (2.2)), which yields that $P_l u_0^l(q^l(s; y)) = 0$ when $l \neq k$ and $x'(y) \in M$. Thus (i) is proved.

Next let us show (ii) in the theorem. To begin with, we note

Lemma 2.3. *Assume that*

$$\partial_y^\alpha v(x'(y)) = 0 \quad when \quad 1 \leqq |\alpha| \leqq m + 1 \ (m \geqq 0) \quad and \quad x'(y) \in M.$$

Then the function $\varphi(x') = \lambda_k(\omega)^{-1/2} \omega \cdot x' \ (x' \in \partial\Omega \cap U)$ *satisfies*

$$\partial_y^\alpha \varphi(x'(y)) = 0 \quad when \quad 1 \leqq |\alpha| \leqq m + 2 \quad and \quad x'(y) \in M.$$

We choose a system of orthonormal coordinates ${}^t(y, z)$ in $\mathbb{R}^n$ such that $z = -\omega \cdot x$ and that the origin $x = 0$ is represented by $z = 0, y = 0$. Note that correctness of Lemma 2.3 does not depend on choice of the coordinates $x'(y)$. Let $\Omega \cap U$ be expressed by $z > \psi(y)$. Then $x'(y) = {}^t(y, \psi(y))$ become local coordinates of $\partial\Omega \cap U$. It is obvious that

$$\omega \cdot x'(y) = -\psi(y),$$

$$v(x'(y)) = (1 + |\psi_y(y)|^2)^{-1/2} \, {}^t(\psi_y(y), -1). \tag{2.3}$$

From these equalities, Lemma 2.3 follows immediately. Hereafter, we introduce these coordinates ${}^t(y, z)$.

Let $\tau_l(y, \eta)$ be the function τ_l in (1.2) represented by the variables (y, η) $(\eta(\in \mathbb{R}^{n-1})$ moving near 0):

$$\lambda_l(\tau_l(y, \eta)v(x'(y)) + \eta \cdot \partial_y x'(y)) = 1 \quad \left(\eta \cdot \partial_y = \sum_{i=1}^{n-1} \eta_i \partial_{y_i} \right), \quad \tau_l(y, \eta) < 0, \tag{2.4}$$

Lemma 2.4. *Let* $\tau_l(y, \eta)$ *be the function in (2.4). If* $v(x'(y))$ *satisfies*

$$\partial_y^\alpha v(x'(y)) = 0 \quad when \quad 1 \leqq |\alpha| \leqq m + 1 \ (m \geqq 0) \quad and \quad x'(y) \in M,$$

then we have for any β

$$\partial_y^\alpha \partial_\eta^\beta \tau_l(y, \eta) = 0 \quad when \quad 1 \leqq |\alpha| \leqq m + 1 \quad and \quad x'(y) \in M.$$

Proof. From (2.3), the assumption for $v(x'(y))$ is equivalent to

$$\partial_y^\alpha \psi(y) = 0 \quad when \quad 1 \leqq |\alpha| \leqq m + 2 \quad and \quad x'(y) \in M.$$

Therefore we have

$$\partial_y^\alpha x'(y) = {}^t(0, \partial_y^\alpha \psi(y)) = 0 \quad when \quad 2 \leqq |\alpha| \leqq m + 2 \quad and \quad x'(y) \in M.$$

Applying ∂_{y_i} to Eq. (2.4) yields that $\partial_\xi \lambda_l \cdot ((\partial_{y_i}\tau_l)v + \tau_l \partial_{y_i}v + \eta \cdot \partial_y \partial_{y_i}x') = 0$. Therefore it follows that $(\partial_\xi \lambda_l \cdot v)\partial_{y_i}\tau_l = 0$ when $x'(y) \in M$. Since $\partial_\xi \lambda_l \cdot v \neq 0$ (assuming that $|\eta|$ is small enough), we obtain $\partial_{y_i}\tau_l(y, \eta) = 0$ when $x'(y) \in M$. Repeating this procedure inductively, we get the lemma. The proof is complete.

Lemma 2.5. *Let* $\varphi(x')$ *and* $\varphi^l(x)$ *be the functions in (1.1), and assume that*

$$\partial_y^\alpha \varphi(x'(y)) = 0 \quad when \quad 2 \leqq |\alpha| \leqq m + 2 \quad and \quad x'(y) \in M,$$
$$\partial_y^\alpha v(x'(y)) = 0 \quad when \quad 1 \leqq |\alpha| \leqq m + 1 \quad and \quad x'(y) \in M.$$

Then the projections $P_l(\xi)$ $(l = 1, \ldots, d)$ satisfy

$$\partial_y^\alpha \{P_l(\varphi_x^l(q^l(s; y)))\} = 0 \quad when \quad 1 \leq |\alpha| \leq m + 1, \quad q^l(s; y) \in \Omega \cap U \quad and \quad x'(y) \in M.$$

Proof. Let us note that $\varphi_x^l(q^l(s; y)) = p^l(s; y) = p^l(0; y)$ since $\partial_x \lambda_l = 0$. This yields that

$$p^l(s; y) = \tau_l(y, A^{-1}\varphi_y)v + (A^{-1}\varphi_y) \cdot \partial_y x'(y), \tag{2.5}$$

where $A = A(y)$ is the $(n-1) \times (n-1)$ matrix whose (i, j)-component is $x'_{y_i} \cdot x'_{y_j} = \delta_{ij} + \psi_{y_i}(y)\psi_{y_j}(y)$. We see that $\partial_y^\alpha A = 0$ when $1 \leq |\alpha| \leq 2m + 2$ and $x'(y) \in M$; therefore, $\partial_y^\alpha A^{-1} = 0$ when $1 \leq |\alpha| \leq 2m + 2$ and $x'(y) \in M$. Hence, applying ∂_y^α to the equality (2.5), by Lemma 2.4 we have

$$\partial_y^\alpha p^l(s; y) = 0 \quad when \quad 1 \leq |\alpha| \leq m + 1, \quad q^l(s, y) \in \Omega \cap U \quad and \quad x'(y) \in M,$$

which proves the lemma.

From the construction of e in (1.6), it follows that $e(q^l(s; y))$ does not depend on s; we denote it by $e(y)$. Furthermore, by Lemma 2.5, this $e(y)$ satisfies

$$\partial_y^\alpha e(y) = 0 \quad when \quad 1 \leq |\alpha| \leq m + 1 \quad and \quad x'(y) \in M, \tag{2.6}$$

if $\partial_y^\alpha \varphi(x'(y)) = 0$ when $2 \leq |\alpha| \leq m + 2$ and $x'(y) \in M$ and $\partial_y^\alpha v(x'(y)) = 0$ when $1 \leq |\alpha| \leq m + 1$ and $x'(y) \in M$. Let us note that the first condition '$\partial_y^\alpha \varphi(x'(y)) = 0$' follows from the second one '$\partial_y^\alpha v(x'(y)) = 0$' if $\varphi(x') = \lambda_k(\omega)^{-1/2}\omega \cdot x'$ (cf. Lemma 2.3). Hereafter we put $\varphi(x') = \lambda_k(\omega)^{-1/2}\omega \cdot x'$, and assume that $\partial_y^\alpha v(x'(y)) = 0$ when $1 \leq |\alpha| \leq m + 1$ and $x'(y) \in M$.

As was shown in the proof of Lemma 2.5, the following equality holds:

$$\partial_y^\alpha \{\varphi_x^l(q^l(s; y))\} = \partial_y^\alpha p^l(0; y) = 0 \quad when \quad 1 \leq |\alpha| \leq m + 1 \quad and \quad x'(y) \in M. \tag{2.7}$$

In view of the definition of (q^l, p^l) (cf. (1.7)), we see that $\partial_s^\beta \partial_y^\alpha (\partial_s q^l, \partial_{y_1} q^l, \ldots, \partial_{y_{n-1}} q^l)(s, y) = 0$ when either '$\beta \geq 2$' or '$1 \leq \beta + |\alpha|, |\alpha| \leq m$ and $x'(y) \in M$', which means that

$$\partial_s^\beta \partial_y^\alpha \frac{\partial s}{\partial x_i}(s, y) = \partial_s^\beta \partial_y^\alpha \frac{\partial y_j}{\partial x_i}(s, y) = 0 \quad if \ either \ '\beta \geq 2'$$
$$or \quad '1 \leq \beta + |\alpha|, |\alpha| \leq m \quad and \quad x'(y) \in M'. \tag{2.8}$$

Mixing (2.6), (2.7) and (2.8) yields that

$$\partial_s^\beta \partial_y^\alpha a_i(s, y) = 0 \quad (i = 0, \ldots, n-1) \quad when \quad 1 \leq |\alpha| \leq m \quad and \quad x'(y) \in M,$$
$$\partial_s^\beta \partial_y^\alpha b_0(s, y) = 0 \quad for \quad \beta \geq 0 \quad when \quad 0 \leq |\alpha| \leq m \quad and \quad x'(y) \in M. \tag{2.9}$$

As was seen in Sect. 1, $w_j(s, y) = {}^t e u_j^l(q^l(s; y))$ satisfies

$$\begin{cases} (a_0(s, y)\partial_s + b_0(s, y))w_j(s, y) = f_j(s, y) & in \quad I \times \tilde{U}, \\ w_j(0, y) = g_j(y) & on \quad \tilde{U}, \end{cases} \tag{2.10}$$

where $f_j = -{}^t e\{H(I - P_l)u_j^l + Lu_{j-1}^l\}$, $g_j = {}^t e P^{-1}\left(-\sum_{i=1}^{d}(I - P_i)u_j^i - \delta_{0j}v_0\right)$ when $Bu = u|_{\partial\Omega}$ and $g_j = {}^t e Q^{-1}\left(-\sum_{i=1}^{d}(N\varphi^i)(I - P_i)u_j^i - \sum_{i=1}^{d} Nu_{j-1}^i - \delta_{0j}v_{-1}\right)$ when $Bu = Nu|_{\partial\Omega}$. Note that $a_0 - I = b_0 = 0$ when $x'(y) \in M$ (cf. Lemma 1.4 and (2.9)); furthermore, $g_0 = 0$ when $l \neq k$ and $x'(y) \in M$, and $f_0 = 0$. Therefore it is seen from (2.10) that $w_0(s, y) = \partial_s w_0(s, y) = 0$ when $x'(y) \in M$ and $l \neq k$. From the definitions of $P(y) (= P(x'(y)))$, $Q(y) (= Q(x'(y)))$ and Lemma 2.5, we have $\partial_y^\alpha P(y) = \partial_y^\alpha Q(y) = 0$ when

$1 \leqq |\alpha| \leqq m + 1$ and $x'(y) \in M$, which yields that

$$\partial_y^\alpha P^{-1}(y) = \partial_y^\alpha Q^{-1}(y) = 0 \quad \text{when} \quad 1 \leqq |\alpha| \leqq m + 1 \quad \text{and} \quad x'(y) \in M.$$

Combining this with (2.6) and Lemma 2.5, we obtain

$$\partial_y^\alpha g_0(s, y) = 0 \quad \text{when} \quad 1 \leqq |\alpha| \leqq m + 1 \quad \text{and} \quad x'(y) \in M. \tag{2.11}$$

Differentiating Eq. (2.10) (with $j = 0$) in y_i and fixing y so that $x'(y) \in M$, by (2.9) and (2.11) we have $\partial_{y_i} w_0(s, y) = 0$ for $i = 1, \ldots, n - 1$ if $l \neq k$. Repeating this procedure yields that for any $\beta = 1, 2, \ldots,$

$$\partial_s^\beta \partial_y^\alpha w_0(s, y) = 0 \quad \text{when} \quad l \neq k, \ 1 \leqq |\alpha| \leqq m + 1, \ x'(y) \in M.$$

Therefore we obtain the statement (ii) in Theorem 2.1 for $j = 0$.

In the same way, noting that $\partial_s^\beta \partial_y^\alpha f_j$ and $\partial_y^\alpha g_j = 0$ for $|\alpha| \leqq m + 1 - 2j$ if $\partial_s^\beta \partial_y^\alpha u_{j-1}^l = 0$ for $|\alpha| \leqq m + 3 - 2j$ and any β, by induction we get (ii) in the theorem for $j \geqq 1$. The proof is complete.

Proof of Remark 2.2. Let us prove statement (i). For $\xi, \eta \in \mathbb{R}^n$ we denote the matrix by $\xi \otimes \eta$ whose (i, j)-component is $\xi_i \eta_j$. It is seen that H and $P_2(\xi)$ are of the forms

$$H = 2\mu(\varphi_x^2 \cdot \partial_x) I + (\mu + \lambda)\varphi_x^2 \otimes \partial_x + (\mu + \lambda)\partial_x \otimes \varphi_x^2 + L\varphi^2,$$

$$P_2(\xi) = (|\xi|^{-1}\xi) \otimes (|\xi|^{-1}\xi), \tag{2.12}$$

where the (i, j)-component of $\partial_x \otimes \varphi_x^2$ means $\varphi_{x_j}^2 \partial_{x_i}$. The equations of $q^2(s; y)$ and $p^2(s; y)$ $(= \varphi_x^2(q^2(s; y)))$ (cf. (1.7)) become of the forms

$$\begin{cases} \dfrac{dq^2}{ds} = 2(\lambda + 2\mu)p^2, \quad \dfrac{dp^2}{ds} = 0, \\ q^2(0; y) = {}^t(y, \psi(y)), \quad p^2(0; y) = \varphi_x^2(y, \psi(y)). \end{cases}$$

From the equalities that $\varphi^2(y, \psi(y)) = -\mu^{-1/2}\psi(y)$ and $|\varphi_x^2|^2 = (\lambda + 2\mu)^{-1}$, it is seen that

$$\varphi_x^2(y, \psi(y)) = -\mu^{-1/2} \sum_{i=1}^{n-1} (A^{-1}\psi_y)_i \, {}^t(0, \ldots, \overset{i}{1}, \ldots, 0, \psi_{y_i})$$

$$+ \{(\lambda + 2\mu)^{-1} - \mu^{-1\,t}\psi_y A^{-1}\psi_y\}^{1/2}(1 + |\psi_x|^2)^{-1/2\,t}(-{}^t\psi_y, 1), \tag{2.13}$$

where A is the matrix in (2.5) and $(A^{-1}\psi_y)_i$ means i-component of the vector $A^{-1}\psi_y$. Note that $A - I = \partial_{y_i} A = 0$ ($i = 1, \ldots, n - 1$) when $y = 0$; hence, $A^{-1} - I = \partial_{y_i} A^{-1} = 0$ ($i = 1, \ldots, n - 1$) when $y = 0$.

Take the coordinates y so that $\partial_{y_i}\partial_{y_j}\psi(0) = \delta_{ij}\partial_{y_i}^2\psi(0)$. Then, (2.13) yields that

$$\varphi_x^2(0) = (\lambda + 2\mu)^{-1/2\,t}(0, 1),$$

$$\partial_{y_i}\{\varphi_x^2(y, \psi(y))\}|_{y=0} = -(\mu^{-1/2} + (\lambda + 2\mu)^{-1/2})\psi_{y_i y_i}(0)$$

$$\cdot {}^t(0, \ldots, \overset{i}{1}, \ldots, 0) \quad (i = 1, \ldots, n - 1).$$

Therefore, in view of (2.12) we have

$$P_2(\varphi_x^2(0)) = K_n,$$

$$\partial_{y_i} P_2(\varphi_x^2(y, \psi(y)))|_{y=0} = -(\mu^{-1/2} + (\lambda + 2\mu)^{-1/2})\psi_{y_i y_i}(0)K_i$$

$$(i = 1, \ldots, n - 1), \tag{2.14}$$

$$\text{where } K_i = \begin{pmatrix} & & & 0 \\ & & & \vdots \\ & 0 & & 1 \\ & & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix} i \quad (i = 1, \ldots, n).$$ In the same way we get

$$P_2(\varphi_x^1(0)) = K_n,$$
$$\partial_{y_i} P_2(\varphi_x^1(y, \psi(y)))|_{y=0} = -2\mu^{-1/2}\psi_{y_i y_i}(0) K_i \quad (i = 1, \ldots, n-1). \tag{2.15}$$

Differentiating the equality $P(y)P(y)^{-1} = I(P(y) = P_1(\varphi_x^1(y, \psi(y))) + P_2(\varphi_x^2(y, \psi(y))))$, and using (2.14) and (2.15), we have

$$\partial_{y_i} P^{-1}(0) = ((\lambda + 2\mu)^{-1/2} - \mu^{-1/2})\psi_{y_i y_i}(0) K_i \quad (i = 1, \ldots, n-1). \tag{2.16}$$

Since $e(y) = (\lambda + 2\mu)^{1/2} \varphi_x^2(y, \psi(y))$, it follows from (2.13) that $e(0) = {}^t(0, \ldots, 0, 1)$. Therefore, noting that $v_0 = {}^t(v_{10}, \ldots, v_{0n-1}, 0)$ and $P(0)^{-1} = 1$, from Eq. (2.2) of w_0 we obtain $w_0(s, 0) = 0$ for any s. This yields that

$$u_0^2(0) = \partial_{x_n} u_0^2(0) = 0, \tag{2.17}$$

since $u_0^2(q^2(s; y)) = e(y)w_0(s, y)$ and $\partial_s \{u_0^2(q^2)\} = \sum_{i=1}^{n-1} u_{0x_i}^2 \dfrac{dq_i^2}{ds} + u_{0x_n}^2 \dfrac{dq_n^2}{ds} = (\lambda + 2\mu)^{-1} u_{0x_n}^2(0)$ for $y = 0$. Furthermore, applying ∂_{y_i} to $u_0^2(y, \psi(y))$ ($= P_2 u_0^2$) $= -P_2(\varphi_x^2(y, \psi(y)))P(y)^{-1}v_0$ (cf. (1.4)) and using (2.15) and (2.16), we have

$$\partial_{x_i} u_0^2(0) = 2\mu^{-1/2}\psi_{y_i y_i}(0)v_{0i}{}^t(0, \ldots, 0, 1) \quad (i = 1, \ldots, n-1). \tag{2.18}$$

From (1.5) it follows that $(I - P_2)u_1^2 = (1 - \lambda_1(\varphi_x^2))^{-1}(I - P_2)Hu_0^2$. It is seen that

$$(I - P_2)H = 2\mu(\lambda + 2\mu)^{-1/2}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\partial_{x_n} + (\mu + \lambda)(\lambda + 2\mu)^{-1/2}\sum_{i=1}^{n-1}\begin{pmatrix} & & & 0 \\ & & & \vdots \\ 0 & & 1 \\ & & & \vdots \\ & & & 0 \end{pmatrix} i$$

$\partial_{x_i} + \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}L\varphi^2$ when $x = 0$. Therefore, by (2.17) and (2.18), we obtain

$$(I - P_2)u_1^2(0) = 2\mu^{-1/2}(\lambda + 2\mu)^{1/2}{}^t(v_{01}\psi_{y_1 y_1}(0), \ldots, v_{0n-1}\psi_{y_{n-1}y_{n-1}}(0), 0).$$

The above equality shows that $(I - P_2)u_1^2(0) \neq 0$ since at least one of $v_{0i}\psi_{y_i y_i}(0)$ ($i = 1, \ldots, n-1$) does not vanish from the assumptions.

Next, let us consider the statement (ii). In the same way as in the case of (i), we see that

$$(I - P_1)u_1^1(0) = \mu^{1/2}((\lambda + 2\mu)^{-1/2} + \mu^{-1/2})v_{0n}\left(\sum_{i=1}^{n-1}\psi_{y_i y_i}(0)\right){}^t(0, \ldots, 0, 1),$$

which proves that $u_1^1(0) \neq 0$.

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