Commun. Math. Phys. 132, 537–547 (1990)

Communications in Mathematical Physics

Factorizations for Self-Dual Gauge Fields*

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Abstract. For a particular class of patching matrices on $P_3(\mathbb{C})$, including those for the complex instanton bundles with structure group $\mathrm{Sp}(k,\mathbb{C})$ or $O(2k,\mathbb{C})$, we show that the associated Riemann–Hilbert problem $G(x,\lambda)=G_-(x,\lambda)\cdot G_+^{-1}(x,\lambda)$ can be generically solved in the factored form $G_-=\phi_1\cdot\phi_2\cdot\dots\cdot\phi_n$. If $\Gamma=\Gamma_n$ is the potential generated in the usual way from G_-, and we set $\psi_i=\phi_1\cdot\dots\cdot\phi_i$, with $\psi_n=G_-$, then each ψ_i *also* generates a self-dual gauge potential Γ_i. The potentials are connected via the "dressing transformations"

$$\Gamma_i=\phi_i^{-1}\cdot\Gamma_{i-1}\cdot\phi_i+\phi_i^{-1}D\phi_i$$

of Zakharov–Shabat. The factorization is not unique; it depends on the (arbitrary) ordering of the poles of the patching matrix.

Introduction

In general, it is difficult to solve the Riemann–Hilbert problem associated with Ward's construction of self-dual gauge fields [Wa]. Some time ago, Atiyah and Ward wrote down an upper triangular ansatz for the rank-2 instanton bundles [AW]; this problem was then solved explicitly by Corrigan, et. al. in [CFGY]. For bundles of higher rank, algebraic methods do not (to the author's knowledge) yield upper triangular matrices. Nevertheless, as we show below, for the groups $\mathrm{Sp}(k,\mathbb{C})$ and $O(2k,\mathbb{C})$, patching matrices can be found that allow the Riemann–Hilbert problem to be solved generically in a finite number of steps by means of residues or partial fractions.

To state the main result, let $G:P_3(\mathbb{C})\to\mathrm{Sp}(k,\mathbb{C})$ be a rational map given in homogeneous coordinates by $G(Z)=\Delta_-^{-1}(Z)\cdot\Delta_+^{-1}(Z)\cdot S(Z)$, where Δ_-, Δ_+ are relatively prime homogeneous polynomials of degree n, and S is a matrix of homogeneous polynomials of degree $2n$. Let $V_\pm=\{Z:\Delta_\pm(Z)=0\}$ and $U_\pm=P_3(\mathbb{C})\setminus V_\pm$, and let $\mathscr{P}=U_+\cup U_-$. Let $\mathscr{M}$ be the open subset of the Grassmannian $\mathrm{Gr}(2,4)$ whose points x correspond to projective lines L_x lying in $\mathscr{P}$. The patching matrix G defines a $2k$-dimensional vector bundle $\mathscr{E}$ on $\mathscr{P}$, and we suppose that for some x, $\mathscr{E}|L_x$ is trivial. Then we shall show

* Supported by the General Research Fund of the University of Kansas

Theorem. *For a generic G as above, there exists a Zariski-open subset $\mathscr{U}$ of $\mathscr{M}$ such that for $x \in \mathscr{U}$, $G|L_x$ factors as $G_-(x,\lambda)\cdot G_+^{-1}(x,\lambda)$, with $G_\pm(x,\lambda): U_\pm \cap L_x \to \mathrm{Sp}(k,\mathbb{C})$, and*

$$G_-(x,\lambda) = \phi_1(x,\lambda)\cdot \dots \cdot \phi_n(x,\lambda) \tag{1}$$

with each $\phi_i(x,\lambda)$ of the form $I - A_i(x,\lambda)$. A_i varies algebraically with x, and for each λ in its domain, is a nilpotent of order 2 in $\mathrm{sp}(k,\mathbb{C})$. *An identical result holds for the groups $O(2k,\mathbb{C})$.*

From the construction, it follows immediately that if $\psi_1 = \phi_1, \psi_2 = \phi_1\cdot\phi_2, \dots, \psi_n = \phi_1\cdot\dots\cdot\phi_n = G_-$, the quantities

$$\Gamma_{jA}(x,\lambda) := \psi_j^{-1}\cdot D_A\psi_j, \quad \text{for} \quad j = 1,\dots,n \tag{2}$$

all determine self-dual gauge potentials. (The differential operators D_A are defined below.) They may be "generated" from $\Gamma_{0A} = I$ by a sequence of transformations of the form

$$\Gamma_{jA} = \phi_j^{-1}\cdot\Gamma_{j-1,A}\cdot\phi_j + \phi_j^{-1}\cdot D_A\phi_j. \tag{3}$$

Although our motivation comes from looking at the original monad or ADHM construction [ADHM], the factorization does not depend on the rationality of G. It can be obtained (in general) whenever $G|L_x$ is meromorphic with a suitable pole structure; in particular, G need not originate with the ADHM construction.

The factorization is not unique; it depends (as does the set $\mathscr{U}$) on the (arbitrary) ordering of the n poles of $\Delta_-|L_x$. This is a partial analogue, for self-dual gauge fields, of Uhlenbeck's factorization theorem for harmonic maps [Uh].

In the first section of this paper, we review the ADHM construction and demonstrate the existence of patching matrices having a particular form. Section 2 connects this with Ward's construction and the Riemann–Hilbert problem. In Sect. 3 we show how to factor rational maps into the complex symplectic groups, and use this in Sect. 4 to attack the Riemann–Hilbert problem for self-dual gauge fields. Section 5 briefly mentions some consequences of the preceding results.

1. Algebraic Charts for the ADHM Construction

Assume in what follows that E is an algebraic vector bundle of rank $2k$ on $P_3(\mathbb{C})$, trivial over the generic line, arising from the monad construction of Barth and Horrocks. (See [ADHM, At, Do, OSS] and references quoted therein.) We shall suppose the structure group to be $\mathrm{Sp}(k,\mathbb{C})$, the case of $O(2k,\mathbb{C})$ being essentially identical. Thus E is determined by a map $\mathscr{A}(Z): \mathbb{H} \to (\mathbb{K},\Omega)$, where $\mathbb{H}$ and $\mathbb{K}$ are comolex vector spaces of dimension n and $2n+2k$ respectively, Ω is a non-degenerate symplectic form on $\mathbb{K}$, and $Z \in \mathbb{C}^4\setminus\{0\}$. The requirements on $\mathscr{A}(Z)$ are (1) that it be injective, (2) that the map $\mathscr{B}(Z): \mathbb{K} \to \mathbb{H}^*$ defined by $\mathscr{B}(Z) = \mathscr{A}^t(Z)\Omega$ be surjective, (3) that $\mathscr{B}(Z)\circ\mathscr{A}(Z) = 0$, (4) that $\mathscr{A}(Z)$ be linear in Z and finally (5) that there exist a pair (X, Y) such that $\mathscr{B}(Y)\circ\mathscr{A}(X)$ is an isomorphism. The reality conditions [ADHM] guaranteeing that $E|L_x$ is trivial for $x \in S^4 \subset \mathrm{Gr}(2,4)$ are not important in what follows. We choose and fix bases in $\mathbb{H}$ and $\mathbb{K}$, and take Ω in the specific form

$$\Omega = \begin{bmatrix} \Omega_n & 0 \\ 0 & \Omega_k \end{bmatrix}, \quad \text{where} \quad \Omega_m = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}. \tag{4}$$

The bundle E is defined as $\operatorname{Ker}\mathscr{B}/\operatorname{Im}\mathscr{A}$, and we seek charts on $P_3(\mathbb{C})$ over which E is *algebraically* trivial. Specifically, we require charts on $P_3(\mathbb{C})$ of the form $U_a = P_3(\mathbb{C})\setminus V_a$, where V_a is the zero set of a homogeneous polynomial $\Delta_a(Z)$. In addition, we want $\operatorname{Ker}\mathscr{B}|U_a$ to decompose as the direct sum $(\operatorname{Im}\mathscr{A}|U_a)\oplus F_a$, where the decomposition is algebraic – i.e., given by rational maps. The patching matrices defined on $U_a \cap U_b$ for $\operatorname{Ker}\mathscr{B}\to P_3(\mathbb{C})$, will then be block upper triangular, with the lower right-hand blocks $G_{ab}(Z)$ giving the patching for E over $U_a \cup U_b$. Such charts are readily obtained:

For each Z, the $n\times(2n+2k)$ matrix $\mathscr{B}(Z)$ has rank n, so it contains at least one $n\times n$ invertible submatrix. Running through the different possibilities will give $\binom{2n+2k}{n}$ charts for $\operatorname{Ker}\mathscr{B}$. In particular, let $a=(i_1,\dots,i_n)$ be an n-tuple with $1\leqq i_1\leqq\cdots\leqq i_n\leqq 2n+2k$, let $b_j(Z)$ be the j^{th} column of $\mathscr{B}(Z)$, and let P_a be a non-singular matrix such that $\mathscr{B}_a(Z):=\mathscr{B}(Z)P_a=(b_{i_1}|\cdots|b_{i_n}|*)$. Let $\Delta_a(Z)=\operatorname{Det}(b_{i_1}|\cdots|b_{i_n})(Z)$; this is a homogeneous polynomial of degree n. Let $V_a=\{Z:\Delta_a(Z)=0\}$, and let U_a be the complement of V_a in $P_3(\mathbb{C})$. By the assumptions on $\mathscr{A}(Z)$, the collection $\{U_a\}$ is a Zariski open cover of $P_3(\mathbb{C})$.

If Y lies in $\operatorname{Ker}\mathscr{B}|U_a$, we write

$$P_a^{-1}Y = \begin{bmatrix} \xi_a \\ \eta_a \end{bmatrix}, \quad \text{and} \quad \mathscr{B}_a(Z) = [\alpha_a(Z)|\beta_a(Z)], \tag{5}$$

where α_a is $n\times n$ and ξ_a is $n\times 1$. Since α_a is invertible, $\mathscr{B}(Z)Y=0=\mathscr{B}_a(Z)P_a^{-1}Y$ gives $\xi_a=-\alpha_a^{-1}(Z)\beta_a(Z)\eta_a$, and we can define a chart $\Psi_a: U_a\times\mathbb{C}^{n+2k}\to\operatorname{Ker}\mathscr{B}|U_a$ by

$$\Psi_a(Z,\eta_a) = P_a\begin{bmatrix} -\alpha_a^{-1}(Z)\beta_a(Z) \\ I_{n+2k} \end{bmatrix}\cdot\eta_a. \tag{6}$$

If $Z\in U_a\cap U_b$, then for some η_a and η_b,

$$Y = P_a\begin{bmatrix} -\alpha_a^{-1}(Z)\beta_a(Z) \\ I_{n+2k} \end{bmatrix}\cdot\eta_a = P_b\begin{bmatrix} -\alpha_b^{-1}(Z)\beta_b(Z) \\ I_{n+2k} \end{bmatrix}\cdot\eta_b, \tag{7}$$

and it follows that $\eta_a = K_{ab}(Z)\eta_b$, where

$$K_{ab}(Z) = \tau\circ P_a^{-1}\circ P_b\begin{bmatrix} -\alpha_b^{-1}(Z)\beta_b(Z) \\ I_{n+2k} \end{bmatrix}, \tag{8}$$

τ being the projection onto the last $n+2k$ components. Notice that K_{ab} is $\Delta_b^{-1}(Z)$ times a matrix of homogeneous polynomials of degree n.

Let $A_j(Z)$ be the j^{th} column of $\mathscr{A}(Z)$, so that $\operatorname{Im}\mathscr{A}(Z)=\operatorname{span}\{A_j(Z):1\leqq j\leqq n\}$. The $\{A_j(Z)\}$ are linearly independent in $\mathbb{K}$ for all $Z\neq 0$, and since $\operatorname{Im}\mathscr{A}\subset\operatorname{Ker}\mathscr{B}$, the matrices $A^a(Z):=\tau\circ P_a^{-1}\mathscr{A}(Z)$ are of rank n in U_a. We are looking for algebraic complements to $\operatorname{Im}\mathscr{A}$ in $\operatorname{Ker}\mathscr{B}$; it will be convenient to isolate a subcollection of the $\{U_a\}$ on which these can be found without further refinement of the charts.

Proposition 1. *For $2^n(1+nk)$ of the charts described above, P_a may be chosen so that*
(a) $P_a \in \mathrm{Sp}(k+n, \mathbb{C})$.
(b) *The top $n \times n$ block of $A^a(Z)$ is $-\alpha_a^t(Z)$.*
In particular, the matrix

$$R_a(Z) := \left[\ A^a(Z) \left| \begin{matrix} 0_{n\times 2k} \\ I_{2k} \end{matrix} \right. \right] \tag{9}$$

is invertible in U_a.

Proof. It is readily checked that the following substitutions in $\mathscr{B}(Z)$ are effected by matrices satisfying the above conditions:
1. For $1 \leqq j \leqq n$, $\{-\mathrm{col}(j) \to \mathrm{col}(n+j), \mathrm{col}(n+j) \to \mathrm{col}(j)\}$,
2. For $1 \leqq j \leqq n$, and $1 \leqq m \leqq k$, $\{\mathrm{col}(j) \leftrightarrow \mathrm{col}(2n+m), \mathrm{col}(n+j) \leftrightarrow \mathrm{col}(2n+k+m\}$.
We get 2^n charts from (1). Composing a transformation of type (2) with one of type (1), we can put any of the last $2k$ columns into any one of the first n slots. There are then 2^{n-1} possible replacements for the remaining $n-1$ slots coming from additional transformations of type (1) for a total of $2^n + 2kn\cdot 2^{n-1} = 2^n(1+nk)$ charts. ■

The columns of $R_a(Z)$ then give the desired direct sum decomposition over U_a. To trivialize $\operatorname{Ker}\mathscr{B}|U_a$ and $\operatorname{Ker}\mathscr{B}|U_b$ using this, we should have to divide the first n columns of $R_a(Z)$ and $R_b(Z)$ by, say Z^α and Z^β respectively to obtain objects homogeneous of degree 0; it turns out that the resulting factor of Z^α/Z^β *drops out* of the quotient block, so that E is algebraically trivial over U_a, and we omit this step.

Let $Y, W \in E_z$. If $Z \in U_a$, the symplectic form on E is defined by $\omega(Y, W) = \Omega(\Psi_a \cdot Y_a, \Psi_a \cdot W_a)$, where Y_a and W_a are local representatives of the equivalence classes. For the charts described above, we may choose unique local representatives of the form

$$Y_a = R_a(Z)\cdot \begin{bmatrix} 0_n \\ y_a \end{bmatrix} = \begin{bmatrix} 0_n \\ y_a \end{bmatrix}.$$

An easy computation then gives $\omega(Y, W) = y_a^t \cdot \Omega_k \cdot w_a$, and if $Z \in U_b$ as well, we get $\omega(Y, W) = y_b^t \cdot \Omega_k \cdot w_b$. Thus the patching matrix for the quotient given by the lower right-hand block of

$$R_a^{-1} K_{ab} R_b = \begin{bmatrix} * & * \\ 0 & G_{ab} \end{bmatrix} \tag{10}$$

preserves the form Ω_k, and we have

Proposition 2. *The quotient bundle E is algebraically trivial over each of the charts in Proposition 1. In the intersection of two such charts, the patching matrix takes values in* $\mathrm{Sp}(k, \mathbb{C})$ *and has the form*

$$G_{ab}(Z) = \Delta_a^{-1}(Z)\cdot \Delta_b^{-1}(Z)\cdot S_{ab}(Z),$$

where Δ_a and Δ_b are homogeneous polynomials of degree n, and the entries of S_{ab} are homogeneous polynomials of degree 2n.

(The last assertion follows on inspection of $R_a^{-1}K_{ab}R_b$.)

The matrices G_{ab} are not difficult to construct; for example, taking $P_1 = I$, $P_2 = \Omega$, if we write

$$\mathscr{B}(Z) = [\alpha|\rho|\kappa|\tau], \tag{11}$$

where α, ρ, κ and τ have n, n, k and k columns respectively, the lower right block of $R_1^{-1}K_{12}R_2$ is

$$G_{12}(Z) = \begin{bmatrix} -\tau^t\alpha^{t^{-1}}\rho^{-1}\tau & \tau^t\alpha^{t^{-1}}\rho^{-1}\kappa - I_k \\ \kappa^t\alpha^{t^{-1}}\rho^{-1}\tau + I_k & -\kappa^t\alpha^{t^{-1}}\rho^{-1}\kappa \end{bmatrix}. \tag{12}$$

In what follows, we shall only require one pair (U_a, U_b) from the above collection, and we shall take Δ_a and Δ_b to be relatively prime, which holds in the general case.

2. The Relation to Ward's Construction

If $x \in \mathrm{Gr}(2,4)$, let L_x denote the corresponding line in $P_3(\mathbb{C})$. Ward's construction [Wa] begins by restricting both the cover and the patching matrices to projective lines. Using the fact that $E|L_x$ is trivial for generic lines (a consequence of the assumptions on $\mathscr{A}(Z)$ above), the restricted patching matrices on such lines split as $G_{ab}|L_x = G_a(x,\lambda)\cdot G_b^{-1}(x,\lambda)$, with $G_a(x,\lambda)$, $G_b(x,\lambda)$ holomorphic in $\mathfrak{U}_a := U_a \cap L_x$, $\mathfrak{U}_b := U_b \cap L_x$ respectively. Here λ is a complex coordinate on L_x, and x appears parametrically; G_a and G_b can both be taken to depend holomorphically on x. In an affine chart $\cong \mathbb{C}^4$ on $\mathrm{Gr}(2,4)$, one can write x as a 2×2 complex matrix so that $G_{ab}(Z)|L_x$ takes the form $G_{ab}(x\cdot\pi,\pi)$, where $\pi = (\pi_0, \pi_1)$ are homogeneous coordinates on L_x [PR]. Defining the linear operators

$$D_A = \pi_1\partial/\partial x^{A0} - \pi_0\partial/\partial x^{A1} \quad (A = 0, 1), \tag{14}$$

the functional form of G_{ab} now gives $D_A G_{ab}(x\cdot\pi,\pi) = 0$, which leads to

$$G_a^{-1}(D_A G_a) = G_b^{-1}(D_A G_b) \quad \text{in} \quad \mathfrak{U}_a \cup \mathfrak{U}_b. \tag{15}$$

The global quantity defined on L_x by expression (15) is holomorphic and homogeneous of degree 1 in π; it is thus linear in π and can be written as $\Gamma_{A0}(x)\pi_1 - \Gamma_{A1}(x)\pi_0$ for $A = 0$, 1. The potential defined by $\Gamma := \Gamma_{AB}dx^{AB}$ is then self-dual (or anti self-dual, depending on conventions) by virtue of the fact that $[D_A, D_B] = 0$. Given the above, we observe that it is only necessary to split *one* of the $G_{ab}(x\cdot\pi,\pi)$ to obtain Γ. This is the Riemann–Hilbert problem under discussion.

From now on, we take G_{ab} in the form given by Proposition 2 above. A "generic" point x in the 4-dimensional Grassmannian $\mathrm{Gr}(2,4)$ refers to a line $L_x \subset P_3(\mathbb{C})$ such that (1) $L_x \cap V_a \cap V_b = \phi$, (2) L_x is in general position with respect to V_a and V_b (so that it intersects each in n distinct points), and (3) $E|L_x$ is trivial. Thus for generic L_x, we shall have (1) $L_x \subset \mathfrak{U}_a \cup \mathfrak{U}_b$, (2) $\mathfrak{U}_a \cong P_1(\mathbb{C})\backslash\{p_1(x),\ldots,p_n(x)\}$, $\mathfrak{U}_b \cong P_1(\mathbb{C})\backslash\{q_1(x),\ldots,q_n(x)\}$, the deleted points corresponding to the sets $V_a \cap L_x$ and $V_b \cap L_x$ respectively, and (3) $\{p_1(x),\ldots,p_n(x)\} \cap \{q_1(x),\ldots,q_n(x)\} = \phi$.

Restricted to L_x, the functions Δ_a, Δ_b and the entries of S_{ab} become homogeneous polynomials in the components of x and π. Assuming the point corresponding to $\pi=(0,1)$ does not coincide with one of the $p_i(x)$, we set $\lambda=\pi_1/\pi_0$, $\Lambda=(1,\lambda)$ and conclude that $G(x\cdot\pi,\pi)$ which is homogeneous of degree zero in π, can be written as

$$G(x\cdot\Lambda,\Lambda)=G(x,\lambda)=\prod_1^n[\lambda-p_i(x)]^{-1}\cdot\prod_1^n[\lambda-q_j(x)]^{-1}\cdot\tilde{S}(x,\lambda), \tag{16}$$

where we have dropped the indices on the matrices and absorbed a rational function of x into the original $S(x,\lambda)$. Strictly speaking, the (x,λ) are local coordinates on the flag manifold $\mathscr{F}_{12}$ consisting of all ordered pairs {(line in $P_3(\mathbb{C})$, point on the line)}; see Wells [We]. The Riemann–Hilbert problem is then to find a Zariski-open set $\mathscr{U}\subset\mathrm{Gr}(2,4)$ such that $x\in\mathscr{U}\Rightarrow G(x,\lambda)$ factors as $G_-(x,\lambda)\cdot G_+^{-1}(x,\lambda)$, with $G_-(x,\lambda)$ (respectively $G_+(x,\lambda)$) holomorphic in $\hat{\mathfrak{U}}_a|\mathscr{U}$ (respectively $\hat{\mathfrak{U}}_b|\mathscr{U}$), where $\hat{\mathfrak{U}}_i=\mathscr{F}_{12}|_{\mathbb{C}^4}\backslash\{(x,\pi):\Delta_i(x\cdot\pi,\pi)=0\}$.

3. Factoring Maps into Sp(k, $\mathbb{C}$)

Suppose D is a closed disk centered at p in the complex λ plane and that $G:D\backslash\{p\}\to\mathrm{Sp}(k,\mathbb{C})$ is holomorphic with a simple pole at $\lambda=p$. Then $G(\lambda)=(\lambda-p)^{-1}G_{-1}+G_0+(\lambda-p)H(\lambda)$, with H holomorphic in D. Write G_{-1} and G_0 in block form:

$$G_m=\begin{bmatrix}\alpha_m & \beta_m\\ \gamma_m & \delta_m\end{bmatrix},\quad\text{where the entries are } k\times k \text{ blocks,} \tag{17}$$

and define $\chi=\gamma_{-1}^t\alpha_0-\alpha_{-1}^t\gamma_0$, and $\hat{\chi}=\delta_{-1}^t\beta_0-\beta_{-1}^t\delta_0$. Finally, suppose that χ, α_{-1}, and δ_{-1} are invertible. Then we have

Lemma 3. *Under the assumptions stated,*
(a) *The following expressions for the* $2k\times 2k$ *matrix* A *are identical*:

$$A=\begin{bmatrix}-\alpha_{-1}\chi^{-1}\gamma_{-1}^t & \alpha_{-1}\chi^{-1}\alpha_{-1}^t\\ -\gamma_{-1}\chi^{-1}\gamma_{-1}^t & \gamma_{-1}\chi^{-1}\alpha_{-1}^t\end{bmatrix}=\begin{bmatrix}-\beta_{-1}\hat{\chi}^{-1}\delta_{-1}^t & \beta_{-1}\hat{\chi}^{-1}\beta_{-1}^t\\ -\delta_{-1}\hat{\chi}^{-1}\delta_{-1}^t & \delta_{-1}\hat{\chi}^{-1}\beta_{-1}^t\end{bmatrix}. \tag{18}$$

(b) $A\in\mathrm{sp}(k,\mathbb{C})$; $A^2=0$; $I+(\lambda-p)^{-1}A=\exp[(\lambda-p)^{-1}A]\in\mathrm{Sp}(k,\mathbb{C})$ *for* $\lambda\neq p$.
(c) $[I+(\lambda-p)^{-1}A]\cdot G(\lambda)$ *is a holomorphic map from* D *to* $\mathrm{Sp}(k,\mathbb{C})$.

Proof. Writing out the left-hand side of (c), we see that A must satisfy the (apparently) overdetermined system of equations

$$AG_{-1}=0,\quad AG_0+G_{-1}=0. \tag{19}$$

The system turns out to be consistent provided that (a) holds; as shown below, this is a consequence of the identities on the Laurent coefficients resulting from the requirement that $G(\lambda)\in\mathrm{Sp}(k,\mathbb{C})$. The inverses of α_{-1} and δ_{-1} are required here. Once (a) is established, (c) follows directly. Condition (b) is immediate from the identities below and the form of A. We remark that invertibility of the matrices required is generic. To verify (a) write

$$G(\lambda)=\begin{bmatrix}\alpha(\lambda) & \beta(\lambda)\\ \gamma(\lambda) & \delta(\lambda)\end{bmatrix}.$$

Then $G^t(\lambda)\Omega_k G(\lambda)=\Omega_k$ is equivalent to

$$\gamma^t\alpha=\alpha^t\gamma,\quad \delta^t\beta=\beta^t\delta,\quad \alpha^t\delta-\gamma^t\beta=I_k,$$

which gives conditions on the components of G_m:

$$\gamma^t_{-1}\alpha_{-1}=\alpha^t_{-1}\gamma_{-1},\quad \gamma^t_{-1}\alpha_0+\gamma^t_0\alpha_{-1}=\alpha^t_{-1}\gamma_0+\alpha^t_0\gamma_{-1}, \tag{20.1}$$

$$\delta^t_{-1}\beta_{-1}=\beta^t_{-1}\delta_{-1},\quad \delta^t_{-1}\beta_0+\delta^t_0\beta_{-1}=\beta^t_{-1}\delta_0+\beta^t_0\delta_{-1}, \tag{20.2}$$

$$\alpha^t_{-1}\delta_{-1}=\gamma^t_{-1}\beta_{-1},\quad \alpha^t_{-1}\delta_0+\alpha^t_0\delta_{-1}=\gamma^t_{-1}\beta_0+\gamma^t_0\beta_{-1}. \tag{20.3}$$

Writing A as $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, (19) gives 8 equations, which break up naturally into 2 sets:

$$\text{(I)}\quad \begin{aligned} a\alpha_{-1}+b\gamma_{-1}&=0\\ a\alpha_0+b\gamma_0&=-\alpha_{-1}\\ c\alpha_{-1}+d\gamma_{-1}&=0\\ c\alpha_0+d\gamma_0&=-\gamma_{-1}\end{aligned},\qquad \text{(II)}\quad \begin{aligned} a\beta_{-1}+b\delta_{-1}&=0\\ a\beta_0+b\delta_0&=-\beta_{-1}\\ a\beta_{-1}+b\delta_{-1}&=0\\ a\beta_0+b\delta_0&=-\delta_{-1}\end{aligned}.$$

If χ is invertible, then the first version of A given in (18) above can be formed, and it is easily checked that the $a,\ldots,d$ so determined satisfy (I) above. We must check that (II) is satisfied as well. Suppose α_{-1} and δ_{-1} are also invertible. Then (20.3) shows that γ_{-1} and β_{-1} are invertible, and (20.1) gives

$$\delta_{-1}=\alpha^{t^{-1}}_{-1}\gamma^t_{-1}\beta_{-1}=\gamma_{-1}\alpha^{-1}_{-1}\beta_{-1},\quad \alpha_{-1}=\beta_{-1}\delta^{-1}_{-1}\gamma_{-1}. \tag{20.4}$$

We now claim that

$$\chi\alpha^{-1}_{-1}\beta_{-1}=\gamma^t_{-1}\delta^{t^{-1}}_{-1}\hat{\chi}. \tag{20.5}$$

Writing out the left-hand side of (20.5), we get

$$[\alpha^t_0\gamma_{-1}-\gamma^t_0\alpha_{-1}]\alpha^{-1}_{-1}\beta_{-1}=\alpha^t_0\gamma_{-1}\alpha^{-1}_{-1}\beta_{-1}-\gamma^t_0\beta_{-1}=\alpha^t_0\delta_{-1}-\gamma^t_0\beta_{-1},$$

where we have used (20.4) and $\chi=\chi^t$. Similarly, the right-hand side of (20.5) gives $\gamma^t_0\beta_{-1}-\alpha^t_0\delta_{-1}$, and the two expressions are identical by virtue of (20.3). This shows that $\hat{\chi}$ is invertible and allows us to write down the second expression for A, which involves $\hat{\chi}^{-1}$. It is then easily checked, using (20.4) and (20.5), that the two expressions are identical and the rest of the proof follows. ■

Setting $G_-(\lambda):=I-(\lambda-p)^{-1}A$, and $G_+^{-1}(\lambda):=[I+(\lambda-p)^{-1}A]\cdot G(\lambda)$ (recall that $A^2=0$), we have solved the Riemann–Hilbert problem for $G(\lambda)$ on any positively oriented contour in $D\backslash\{p\}$ equivalent to ∂D. Moreover, $G_-(\lambda)$ is the *unique* solution with $G_-(\infty)=I$. (See Novikov, et. al. [NMPZ] for a general discussion.) Finally, we observe that both of $G_\pm(\lambda)$ take values in $\mathrm{Sp}(k,\mathbb{C})$ in their respective domains.

4. The Riemann–Hilbert Problem for Self-Dual Yang-Mills Fields

Returning now to $G(x, \lambda)$, choose a simple, positively oriented contour $\mathscr{C}_x$ on L_x surrounding the n points of $V_a \cap L_x$. Order these points as $\{p_1(x), p_2(x), \ldots, p_n(x)\}$. Choose contours $\mathscr{C}_i$ to surround only $\{p_1(x), p_2(x), \ldots, p_i(x)\}$, with $\mathscr{C}_n = \mathscr{C}_x$. Let D_i be the closure of $\operatorname{int} \mathscr{C}_i$. The following construction works for a generic x (see the remarks below): On $D_1 \setminus \{p_1(x)\}$ apply the lemma to get

$$G(x, \lambda) = (I - (\lambda - p_1(x))^{-1} A_1(x)) \cdot G_1(x, \lambda), \tag{21}$$

with $G_1(x, \lambda)$ holomorphic in D_1, hence in $D_2 \setminus \{p_2(x)\}$, and taking values in $\mathrm{Sp}(k, \mathbb{C})$. Continue with $G_{i-1}(x, \lambda)$ in $D_i \setminus \{p_i(x)\}$ to obtain

$$G_{i-1}(x, \lambda) = (I - (\lambda - p_i(x))^{-1} A_i(x)) \cdot G_i(x, \lambda) \tag{22}$$

or, equivalently,

$$G(x, \lambda) = \left(\prod_{j=1}^{i} [I - (\lambda - p_j(x))^{-1} A_j(x)] \right) \cdot G_i(x, \lambda) \tag{23}$$

arriving at the factorization

$$G(x, \lambda) = \left(\prod_{j=1}^{n} [I - (\lambda - p_j(x))^{-1} A_j(x)] \right) \cdot G_n(x, \lambda) = G_-(x, \lambda) \cdot G_+^{-1}(x, \lambda) \tag{24}$$

valid on the original curve $\mathscr{C}_x$.

Remarks. At each stage of the factorization, one must restrict the domain of x further by cutting out the Zariski-closed subsets of $\mathrm{Gr}(2,4)$ in which the $k \times k$ determinants of the required terms made from the Laurent expansion of G vanish. The end result of this process is the set $\mathscr{U}$ referred to above. Evidently, G_- is holomorphic in $\hat{\mathfrak{U}}_a | \mathscr{U}$. Since the A_i are nilpotents of order 2, $G_-^{-1} = \prod_{j=n}^{1} (I + (\lambda - p_j(x))^{-1} A_j(x))$, so that $G_-^{-1} \cdot G = G_+^{-1}$ is holomorphic in $\hat{\mathfrak{U}}_b | \mathscr{U}$, and the problem is solved.

Even for the case $k = 1$ ($Sl(2, \mathbb{C})$), the stated conditions are not necessary, only sufficient. This can be seen from the following example for $n = 2$, $k = 1$. Take

$$\mathscr{A}^t(Z) = \begin{bmatrix} Z_0 & Z_2 & Z_1 & Z_3 & 0 & 0 \\ Z_2 & Z_0 & Z_3 & Z_1 & Z_2 & \varepsilon Z_3 \end{bmatrix}, \quad \text{with} \quad \varepsilon \neq 0.$$

Using (12), we find

$$G_{12}(Z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{Z_0 Z_1 + Z_2 Z_3}{(Z_1^2 - Z_3^2)(Z_0^2 - Z_2^2)} \begin{bmatrix} Z_2^2 & \varepsilon Z_2 Z_3 \\ \varepsilon Z_2 Z_3 & \varepsilon^2 Z_3^2 \end{bmatrix}.$$

Using the standard coordinates $x = \begin{bmatrix} y & -\tilde{z} \\ z & \tilde{y} \end{bmatrix}$, one finds, after a routine computation at the pole $p_1(x) = z/(1 - \tilde{y})$, that

$$\chi(p_1(x)) = \frac{2(\varepsilon - 1)}{1 + z\tilde{z} + y\tilde{y} - (y + \tilde{y})}$$

which evidently vanishes when $\varepsilon = 1$. On the other hand, for $\varepsilon = 1$, we find immediately that

$$G_{12}(Z)|L_0 = \begin{bmatrix} \lambda^{-1} & 0 \\ 2 & \lambda \end{bmatrix} \sim I_2$$

so that E is, in fact, trivial over the generic line. We mention that if $\varepsilon \neq 1$, there are no difficulties, and the construction goes through as advertised.

The restriction to "generic G" in the statement of the theorem eliminates the possibility that one or more of the determinants may vanish identically and ensures that Δ_a and Δ_b are relatively prime.

In this procedure, one introduces "artificial" singularities into the gauge potential; this happens as well in the Atiyah–Ward construction [At]. The Riemann–Hilbert problem may be solvable for some of these x with a different ordering of the singular points, or, since we only have a sufficient condition, it might be solvable in a different form. In addition, requiring that L_x be in general position with respect to the pair (V_a, V_b) cuts out another set of the Grassmannian on part of which the problem might be solvable.

Although $G_-(x, \lambda)$ is certainly unique, the factorization is not; it depends on the (arbitrary) ordering of the singular points. Indeed, for certain values of x, the factorization exists for one such ordering and fails for another.

5. Backlund Transformations Associated with the Factorization

At this point, we may choose to *forget* that we know where to put the contour $\mathscr{C}_x$, and observe that for each i, the partial factorization given in (23) above permits the construction of a *sequence* of self-dual gauge potentials $\Gamma_{iAB}dx^{AB}$ via

$$\Gamma_{iA}(x, \pi) = G_i^{-1}(x, \pi) \cdot D_A G_i(x, \pi), \tag{25}$$

with

$$G_i(x, \pi) = \prod_{j=1}^{i} (I - (\lambda - p_j(x))^{-1} A_j(x)) \tag{26}$$

obtained by solving the Riemann–Hilbert problem on the contour $\mathscr{C}_i$. The potentials are related by the Bäcklund transformations (or "dressing transformations" of Zakharov & Shabat) [ZS, Ch, PSW, Cr, MCN]:

$$\Gamma_{iA} = G_i^{-1} \cdot \Gamma_{i-1A} \cdot G_i + G_i^{-1} D_A G_i. \tag{27}$$

For $i < n$, these hold for x in supersets of $\mathscr{U}$. The potential Γ_i may be regarded as having been obtained by i successive such transformations applied to the trivial solution $\Gamma_0 = I$. Of course, these transformations are not well-defined on isomorphism classes of bundles; they do not, for example, respect topological invariants like Chern classes.

To relate this to the standard treatments of the Riemann problem [NMPZ], set $\phi_j = I - (\lambda - p_j(x))^{-1} A_j(x)$ and consider the situation at the i^{th} stage of the induction. On the contour $\mathscr{C}_i$, we need to solve the problem $G_{i-1} = \phi_i \cdot G_i$; equivalently, we have the *singular* solution $G = \phi_1 \phi_2 \cdot \cdots \cdot \phi_{i-1} \cdot G_{i-1}(x, \lambda)$ to the "Riemann problem with zeros" for G on $\mathscr{C}_i$. If a regular solution exists, it differs from this by the interpolation of a factorization of I; this is exactly what we get, since

$$G=(\phi_1\cdot\dots\cdot\phi_{i-1}\cdot\phi_i)\cdot(\phi_i^{-1}\cdot G_{i-1}).$$

As mentioned earlier, an analogous result holds for the even dimensional orthogonal groups $O(2k,\mathbb{C})$. Here one takes the quadratic form

$$Q=\begin{bmatrix} Q_n & 0 \\ 0 & Q_k \end{bmatrix}, \quad \text{where} \quad Q_m=\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$$

on $\mathbb{K}$ and requires that $\operatorname{Im}\mathscr{A}(Z)$ be totally null with respect to Q. Everything goes through with the obvious modifications.

As was also mentioned, the factorization does not depend on $G_{ab}(Z)$ having come from the monad construction. It is only necessary that G have the correct form (meromorphic with a finite number of simple poles that can be isolated from the rest on some open subset of the Grassmannian). These factorizations are quite similar to those obtained recently [Uh] for harmonic maps.

Finally, we mention a simple relation between the nilpotents $A_i(x)$ and the gauge potential Γ. In the standard coordinates $x=\begin{bmatrix} y & -\tilde{z} \\ z & \tilde{y} \end{bmatrix}$, the normalization $G_-(x,\infty)=I$ gives $\Gamma_y(x)=\Gamma_z(x)=0$. Sufficiently far from $\lambda=0$, we can, for each x and i, write $[\lambda-p_i(x)]^{-1}=\lambda^{-1}+\lambda^{-2}p_i(x)+O(\lambda^{-2})$, so that

$$G_-(x,\lambda)=I+\lambda^{-1}\left(\sum_1^n A_i(x)\right)+\lambda^{-2}H(x,\lambda^{-1}).$$

Now (15) gives

$$\begin{aligned}(\partial_y+\lambda^{-1}\partial_{\tilde{z}})G_- &=\lambda^{-1}G_-\cdot\Gamma_{\tilde{z}},\\ (\partial_z-\lambda^{-1}\partial_{\tilde{y}})G_- &=-\lambda^{-1}G_-\cdot\Gamma_{\tilde{y}},\end{aligned} \tag{28}$$

and, equating the lowest order coefficients we get

$$\partial_y\left(\sum_1^n A_i(x)\right)=\Gamma_{\tilde{z}}, \quad \text{and} \quad \partial_z\left(\sum_1^n A_i(x)\right)=-\Gamma_{\tilde{y}} \tag{29}$$

as the infinitesimal version of the factorization. The quantity $Q_1(x)=\left(\sum_1^n A_i(x)\right)$ is the first of an infinite number of conserved "charges" for the self-dual Yang-Mills fields [Ch, Ta].

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Communicated by S.-T. Yau

Received November 2, 1989