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The Branching Rules of Conformal Embeddings

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Abstract. After recalling the main properties of a conformal embedding of Lie algebras $g \supset p$, which is defined by the equality of the Sugawara central charges on both sides, we launch a systematic study of their branching rules. The bulk of the paper is devoted to the proof of a general formula in the case $su(mn)_1 \supset su(m)_n \oplus su(n)_m$. At the end we give some applications to the construction of modular invariant partition functions.

1. Introduction

Affine Lie algebras are one of the main tools in the construction of Conformal Field Theories. They underlie some of the most important examples of Rational CFT, namely the Wess–Zumino–Witten models [19]. In these models the Hilbert space is built from unitary representations of an affine Lie algebra $\hat{g}$ at some positive integral level k. Another very useful tool is the coset construction [6]. Given an embedding $g \supset p$ of Lie algebras and an irreducible unitary highest weight representation $L(\Lambda)$ of $\hat{g}$, one obtains a set of representations $U(\Lambda, \lambda)$ of the Virasoro algebra, which intertwines with the action of $\hat{p}$ on $L(\Lambda)$. More precisely,

$$L(\Lambda) = \bigoplus_{\lambda} U(\Lambda, \lambda) \otimes L(\lambda), \tag{1.1}$$

where the sum runs over all representations of $\hat{p}$ of level $\dot{k} = kj, k$ being the level of $L(\Lambda)$, j the index of $g \supset p$ and $U(\Lambda, \lambda)$ the subspace of $\hat{p}$-highest weight vectors with weight λ. The central charge of the Virasoro algebra acting on $U(\Lambda, \lambda)$ is $c(g) - c(p)$ with

$$c(g) = \frac{k \dim g}{k + h(g)}, \tag{1.2}$$

$h(g)$ being the dual Coxeter number and $c(g)$ (respectively $c(p)$) the central charge of the Sugawara representation of the Virasoro algebra acting on $L(\Lambda)$ (respectively

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$L(\lambda)$). Let h and $\dot{h}$ denote Cartan subalgebras of g and p. One can choose them so that $\dot{h} \subset h$. Let $H = \{\tau \in C | \operatorname{Im} \tau > 0\}$ be the upper-half plane. The normalized character χ_Λ of $L(\Lambda)$ is the holomorphic function on $H \times h$:

$$\chi_\Lambda(\tau, z) = q^{-c(g)/24} \operatorname{Tr}_{L(\Lambda)} \exp 2i\pi(\tau L_0 + z), \tag{1.3}$$

where as usual q denotes $\exp 2i\pi\tau$, $0 < |q| < 1$. Suppose that $z \in \dot{h}$, then from (1.1) we get:

$$\chi_\Lambda(\tau, z) = \sum_\lambda b_\lambda^\Lambda(\tau) \chi_\lambda(\tau, z), \tag{1.4}$$

where the branching function b_λ^Λ is

$$b_\lambda^\Lambda(\tau) = q^{(c(g) - c(p))/24} \operatorname{Tr}_{U(\Lambda,\lambda)} q^{L_0}. \tag{1.5}$$

The modular transformation properties of the characters are given by [13]:

$$\chi_\Lambda(\tau + 1, z) = e^{2i\pi(h_\Lambda - c(g)/24)} \chi_\Lambda(\tau, z) \tag{1.6}$$

with

$$h_\Lambda = \frac{(\Lambda + 2\rho | \Lambda)}{2(k + h(g))} \tag{1.7}$$

and

$$\chi_\Lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{ik\pi(z|z)/\tau} \sum_{M \in P_+^k} S(\Lambda, M) \chi_M(\tau, z). \tag{1.8}$$

P_+^k is the set of dominant highest weights of level k, and

$$S(\Lambda, M) = i^{|\Delta_+|} |P/P^*|^{-1/2} (k + h(g))^{-1/2} \sum_{w \in W} \det(w) \exp \frac{-2i\pi}{k + h(g)} (\bar{\Lambda} + \bar{\rho} | w(\bar{M} + \bar{\rho})). \tag{1.9}$$

$|\Delta_+|$ is the number of positive roots of g, P is the weight lattice, P^* its dual, W the Weyl group, and $\bar{\Lambda}$ and $\bar{\rho}$ denote the "finite" parts of Λ and ρ, i.e. $\Lambda = k\Lambda_0 + \bar{\Lambda}, \rho = h(g)\Lambda_0 + \bar{\rho}$, see [14]. Using the Weyl character formula, one can rewrite (1.9) as follows:

$$\frac{S(\Lambda, M)}{S(\Lambda, k\Lambda_0)} = \operatorname{Tr}_{\bar{M}} \exp \frac{-2i\pi}{k + h(g)} (\bar{\Lambda} + \bar{\rho}), \tag{1.10}$$

where $\operatorname{Tr}_{\bar{M}}$ denotes the trace over the finite-dimensional g-module with highest weight $\bar{M}$. One shows that $S(\Lambda, M)$ is a symmetric unitary matrix.

Set $S(\Lambda) = S(\Lambda, k\Lambda_0)$. By the Weyl denominator formula,

$$S(\Lambda) = |P/P^*|^{-1/2} (k + h(g))^{-1/2} \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi(\bar{\Lambda} + \bar{\rho} | \alpha)}{k + h(g)}. \tag{1.11}$$

Hence $S(\Lambda)$ is a positive real number, which appears in the asymptotic behavior of $\chi_\Lambda(\tau, 0)$ as $\tau \to 0$. It turns out to be:

$$\chi_\Lambda(\tau, 0) \sim S(\Lambda) e^{i\pi c(g)/12\tau}. \tag{1.12}$$

Indeed from (1.8), $\chi_\Lambda(\tau,0)=\sum_M S(\Lambda,M)\chi_M(-1/\tau,0)$ and since $\chi_\Lambda(\tau,0)=q^{h_\Lambda-c(g)/24}\cdot(1+o(q))$ and $h_\Lambda \geqq h_{k\Lambda_0}=0$, (1.12) follows.

Returning to (1.4) one easily deduces the transformation law of the branching function:

$$b_\lambda^\Lambda\left(-\frac{1}{\tau}\right)=\sum_{M\in P_+^k,\mu\in\dot{P}_+^k} S(\Lambda,M)\dot{S}^*(\lambda,\mu)b_\mu^M(\tau) \tag{1.13}$$

(from now on dotted quantities refer to the subalgebra p).

By definition, $g\supset p$ is a conformal embedding when $U(\Lambda,\lambda)$ is finite-dimensional, or equivalently when $c(g)=c(p)$. This implies that the level of $\hat{g}$ is one. Conformal embeddings are classified in [4]. In this case $b_\lambda^\Lambda(\tau)=\dim U(\Lambda,\lambda)\equiv b(\Lambda,\lambda)$ is a constant and (1.13) reads:

$$b(\Lambda,\lambda)=\sum_{M,\mu} S(\Lambda,M)\dot{S}^*(\lambda,\mu)b(M,\mu), \tag{1.14}$$

i.e. the rectangular matrix $b(\Lambda,\lambda)$ intertwines with the action of the modular group on the characters of $\hat{g}$ and $\hat{p}$. This matrix also obeys the important identity:

$$S(\Lambda)=\sum_{\lambda\in\dot{P}_+^k} b(\Lambda,\lambda)\dot{S}(\lambda), \tag{1.15}$$

obtained by inserting (1.12) and its analog for p in (1.4).

In this paper we compute $b(\Lambda,\lambda)$ in the case $g=su(mn)_1\supset p=su(m)_n\oplus su(n)_m$, which is conformal for $m,n>1$, the subscripts denoting the levels. The previous equations need to be slightly modified to account for the non-simplicity of p. For example (1.14) becomes:

$$b(\Lambda,\dot{\lambda},\ddot{\lambda})=\sum_{M,\dot{\mu},\ddot{\mu}} S(\Lambda,M)\dot{S}^*(\dot{\lambda},\dot{\mu})\ddot{S}^*(\ddot{\lambda},\ddot{\mu})b(M,\dot{\mu},\ddot{\mu}). \tag{1.16}$$

Here and in the following, single dots refer to $su(m)$ and double dots to $su(n)$. Similarly (1.15) becomes:

$$S(\Lambda)=\sum_{\dot{\lambda}\ddot{\lambda}} b(\Lambda,\dot{\lambda},\ddot{\lambda})\dot{S}(\dot{\lambda})\ddot{S}(\ddot{\lambda}). \tag{1.17}$$

It should be mentioned at this point that some results on the branching coefficients of conformal embeddings have already appeared, see [13] and references therein. For instance the branching rules for the regular conformal embeddings are known. Also in [15] the branching rules of all conformal pairs $g\supset p$ with g exceptional are listed. Of more direct relevance to the present paper is the work of Frenkel [10] who dealt with the case $gl(mn)\supset sl(m)\oplus gl(n)$. The decompositions under $su(mn)\supset su(m)\oplus su(n)$ are also considered in a recent paper [17].

The paper is organized as follows. In Sect. 2 we gather a number of relations among highest weights of $su(m)_n$ and $su(n)_m$, and the matrices $\dot{S}(\dot{\lambda},\dot{\mu})$, $\ddot{S}(\ddot{\lambda},\ddot{\mu})$, before stating the general formula for the branching rules $b(\Lambda,\dot{\lambda},\ddot{\lambda})$. Section 3 deals with the proof of this result. Section 4 contains some applications to the construction of modular invariant partition functions.

2. Weights of $su(m)_n$ and $su(n)_m$

We start by describing $\dot{P}^n_+$ and $\dot{P}^m_+$, i.e. the highest weights of $su(m)_n$ and $su(n)_m$. $\dot{P}^n_+$ is the set of weights

$$\dot{\lambda} = \tilde{k}_0 \dot{\Lambda}_0 + \tilde{k}_1 \dot{\Lambda}_1 + \cdots + \tilde{k}_{m-1} \dot{\Lambda}_{m-1}, \tag{2.1}$$

where $\tilde{k}_i$ are non-negative integers such that

$$\sum_{i=0}^{m-1} \tilde{k}_i = n \tag{2.2}$$

and $\dot{\Lambda}_i = \dot{\Lambda}_0 + \dot{\omega}_i$, $1 \leqq i \leqq m-1$, where $\dot{\omega}_i$ are the fundamental weights of $su(m)$. Instead of $\dot{\lambda}$ it will be more convenient to use

$$\dot{\lambda} + \dot{\rho} = \sum_{i=0}^{m-1} k_i \dot{\Lambda}_i \tag{2.3}$$

with $k_i = \tilde{k}_i + 1$ and $\sum_{i=0}^{m-1} k_i = m + n$.

Due to the cyclic symmetry of the extended Dynkin diagram of $su(m)$, the group Z_m acts on $\dot{P}^n_+$ by

$$\dot{\Lambda}_i \mapsto \dot{\Lambda}_{(i+\sigma) \bmod m}, \quad \sigma \in Z_m. \tag{2.4}$$

Let $\Omega_{m,n} = \dot{P}^n_+ / Z_m$ be the set of orbits under this action. The following observation is crucial:

Lemma 1. *There is a natural bijection between $\Omega_{m,n}$ and $\Omega_{n,m}$.*

Proof. Draw a circle (Fig. 1) and divide it into $m+n$ arcs of equal length. To each partition $\sum_{i=0}^{m-1} k_i = m+n$ there corresponds a "slicing of the pie" into m successive parts with angles $2\pi k_i/(m+n)$, drawn with solid lines. The complementary slicing in broken lines defines a partition of $m+n$ into n successive parts, $\sum_{j=0}^{n-1} l_j = m+n$. □

The careful reader will note that if for instance the k-slicing is ordered counterclockwise, there still remains the freedom to order the ls in the same or in the opposite sense. We choose the latter.

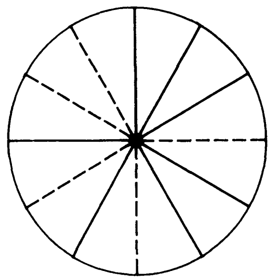

Fig. 1

We shall parametrize this bijection by a map

$$\beta : \dot{P}^n_+ \to \ddot{P}^m_+ \tag{2.5}$$

as follows. Set

$$r_j = \sum_{i=j}^{m} k_i, \quad 1 \leqq j \leqq m, \tag{2.6}$$

where $k_m \equiv k_0$. The sequence $(r_1, r_2, \ldots, r_m)$ is decreasing, $m+n = r_1 > r_2 > \cdots > r_m \geqq 1$. Take the complementary sequence $(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n)$ in $\{1, 2, \ldots, m+n\}$ with $\bar{r}_1 > \bar{r}_2 > \cdots > \bar{r}_n$. Put

$$s_j = m + n + \bar{r}_n - \bar{r}_{n-j+1}, \quad 1 \leqq j \leqq n. \tag{2.7}$$

Then $m+n = s_1 > s_2 > \cdots > s_n \geqq 1$. The map β is defined by

$$(r_1, \ldots, r_m) \mapsto (s_1, \ldots, s_n). \tag{2.8}$$

Thus when $\dot{\lambda}$ runs over an orbit $\dot{\omega} \in \Omega_{m,n}$, $\ddot{\lambda} = \sigma \cdot \beta(\dot{\lambda})$ runs over an orbit $\ddot{\omega} \in \Omega_{n,m}$ if σ runs over Z_n.

Corollary 1. *The orbits of the vacua $n\dot{\Lambda}_0$ and $m\ddot{\Lambda}_0$ correspond to each other.*

There is yet another description of the bijection of Lemma 1, which will be used later on. It is best formulated in terms of Young tableaux, so let us first introduce notations for these. $Y(\lambda)$ will stand for the Young tableau corresponding to a dominant weight λ of $su(N)$, with at most $N-1$ lines. Recall that $Y(\lambda)$ has at most k columns if λ has level k. Let $Y(\lambda)^T$ denote the transposed tableau obtained by exchanging rows and columns. It can be interpreted as a $su(k)_N$ tableau if we erase in it every column with exactly k boxes. We denote by λ^T the corresponding $su(k)$ dominant weight.

Proposition 1. *Let $\dot{\lambda} \in \dot{P}^n_+$ and let $c(\dot{\lambda})$ be the number of columns of $Y(\dot{\lambda})$. Suppose $\sigma = c(\dot{\lambda}) \bmod n$. Then*

$$\sigma \cdot \beta(\dot{\lambda}) = \dot{\lambda}^T. \tag{2.9}$$

Proof. Define $\tilde{k}_m$ by $\sum_{i=1}^{m} \tilde{k}_i = n$, and suppress the tildes from now on in this proof. Let p_j, $j = 0, \ldots, n$ be the coordinates of $\dot{\lambda}^T$:

$$\dot{\lambda}^T = p_0 \ddot{\Lambda}_0 + \cdots + p_{n-1} \ddot{\Lambda}_{n-1} \tag{2.10}$$

which obey the relations, for $j = 1, \ldots, n$:

$$p_n + \cdots + p_j = \sup\{i \mid k_{m-1} + \cdots k_i \geqq j\}, \tag{2.11}$$

where $p_n \equiv p_0$. Consider, when $i = 0, \ldots, m-1$:

$$\sum_{j=1}^{i} k_j + i. \tag{2.12}$$

These are m increasing numbers between 0 and $m+n-1$. On the other hand, when $q = 1, \ldots, n$,

$$q + \sup\left\{i \,\middle|\, \sum_{j=1}^{i} k_j < q\right\} \tag{2.13}$$

are n increasing numbers between 0 and $m+n-1$. We claim that these two sequences are complementary. Indeed for $q \in \{1, \ldots, n\}$ let i_0 be such that

$$\sum_{j=1}^{i_0} k_j < q < \sum_{j=1}^{i_0+1} k_j + 1. \tag{2.14}$$

Then

$$\sum_{j=1}^{i_0} k_j + i_0 < q + \sup\left\{i \,\middle|\, \sum_{j=1}^{i} k_j < q\right\} = q + i_0 < \sum_{j=1}^{i_0+1} k_j + i_0 + 1. \tag{2.15}$$

Therefore we obtain the formula

$$\bar{r}_q = m + n - q - \sup\left\{i \,\middle|\, \sum_{j=1}^{i} k_j < q\right\} \tag{2.16}$$

which, according to (2.7) implies

$$s_q = m + n - q + 1 - \sup\left\{i \,\middle|\, \sum_{j=i}^{m} k_j \geqq 1\right\} + \sup\left\{i \,\middle|\, \sum_{j=i}^{m} k_j \geqq q\right\}. \tag{2.17}$$

We now observe, that if $q - k_m \geqq 1$,

$$\sup\left\{i \,\middle|\, \sum_{j=i}^{m} k_j \geqq q\right\} = p_n + \cdots + p_{q-k_m}, \tag{2.18}$$

whereas if $q - k_m \leqq 0$ (implying $k_m \geqq 1$):

$$\sup\left\{i \,\middle|\, \sum_{j=i}^{m} k_j \geqq q\right\} = m \tag{2.19}$$

and $p_{q+n-k_m} = 0$, since

$$p_n + \cdots + p_{n+1-k_m} = \sup\{i \mid k_m + \cdots + k_i \geqq n+1\} = 0. \tag{2.20}$$

Set $s_{n+1} = 0$ and

$$t_q = s_q - s_{q+1} - 1, \tag{2.21}$$

which are the coordinates of $\beta(\dot{\lambda})$:

$$\beta(\dot{\lambda}) = t_0 \ddot{\Lambda}_0 + \cdots + t_{n-1} \ddot{\Lambda}_{n-1}, \tag{2.22}$$

where $t_0 \equiv t_n$. Assume that $0 < k_m < n$ (the other possibilities are trivial). Take q such that $q \leqq n-1$ and suppose that $q - k_m \geqq 1$. From (2.18):

$$\begin{aligned} t_q &= \sup\left\{i \,\middle|\, \sum_{j=i}^{m} k_j \geqq q\right\} - \sup\left\{i \,\middle|\, \sum_{j=i}^{m} k_j \geqq q+1\right\} \\ &= p_n + \cdots + p_{q-k_m} - p_n - \cdots - p_{q+1-k_m} = p_{q-k_m}, \end{aligned} \tag{2.23}$$

and the proposition is proved in this case. The other cases can be dealt with similarly. □

We turn now to a study of the relations between the different S matrices entering (1.16). We start with $S(\Lambda, M)$. The set of level 1 highest weights of $su(mn)$ are in 1-1 correspondence with the elements of Z_{mn}: $\Lambda_0, \Lambda_1, \ldots, \Lambda_{mn-1}$, and we have:

$$S(\Lambda, M) = (mn)^{-1/2} \exp\left(\frac{2i\pi}{mn} \Lambda M\right), \tag{2.24}$$

where we identify the weights Λ, M with the corresponding elements of Z_{mn}.

Let $\dot{\lambda} \in \dot{P}^n_+$. Put $\phi_i(\dot{\lambda}) = r_i - m^{-1} \sum_{j=1}^{m} r_j$, where the r_i are given by (2.6). Then [13]:

$$\dot{S}(\dot{\lambda}, \dot{\mu}) = N_{m,n} \det\left(\exp -\frac{2i\pi\phi_j(\dot{\lambda})\phi_k(\dot{\mu})}{m+n}\right)_{1 \leqq j,k \leqq m}, \tag{2.25}$$

where

$$N_{m,n} = i^{m(m-1)/2} m^{-1/2} (m+n)^{-(m-1)/2}. \tag{2.26}$$

The next lemma gives the relation between $\dot{S}$ and $\ddot{S}$.

Lemma 2. *Let $\dot{\lambda} \in \dot{P}^n_+$, $\sigma \in Z_n$ and*

$$\delta(\dot{\lambda}, \sigma) = \left(\sum_{j=1}^{m} r_j\right) + m\sigma - \tfrac{1}{2} m(m+1). \tag{2.27}$$

Then, for all $\dot{\lambda}, \dot{\mu} \in \dot{P}^n_+$, $\sigma, \nu \in Z_n$ one has

$$\dot{S}(\dot{\lambda}, \dot{\mu}) = \left(\frac{n}{m}\right)^{1/2} \exp\left(\frac{2i\pi}{mn} \delta(\dot{\lambda}, \sigma)\delta(\dot{\mu}, \nu)\right) \ddot{S}^*(\sigma\beta(\dot{\lambda}), \nu\beta(\dot{\mu})). \tag{2.28}$$

In particular,

$$\dot{S}(\dot{\lambda}) = \left(\frac{n}{m}\right)^{1/2} \ddot{S}(\sigma\beta(\dot{\lambda})) \tag{2.29}$$

for all $\sigma \in Z_n$.

Proof. The main ingredient in deriving (2.28) is Laplace's formula from the theory of determinants [11]. Let A be an $N \times N$ matrix. We choose a $k \times k$ submatrix of A by fixing two sequences of indices, $1 \leqq i_1 < i_2 < \cdots < i_k \leqq N$ and $1 \leqq j_1 < j_2 < \cdots < j_k \leqq N$ with $k \leqq N$. Let $1 \leqq \bar{i}_i < \cdots < \bar{i}_{N-k} \leqq N$ and $1 \leqq \bar{j}_1 < \cdots < \bar{j}_{N-k} \leqq N$ be the complementary sequences. Let $B(i,j)$ denote the minor of A^{-1} defined by (i) and (j), and $A(\bar{i}, \bar{j})$ the minor of A defined by $(\bar{i})$ and $(\bar{j})$. Then

$$B(i,j) = (\det A)^{-1} (-1)^{\Sigma i_l + \Sigma j_l} A(\bar{i}, \bar{j}). \tag{2.30}$$

The rest is a matter of (lengthy) computation, expanding the determinant in (2.25) until one gets a minor of A^{-1}, where

$$(A)_{jk} = (m+n)^{-1/2} \exp\left(\frac{2i\pi jk}{m+n}\right), \quad 1 \leqq j, k \leqq m+n. \tag{2.31}$$

At this point one applies (2.30) and then recasts the new minor obtained in the form of a $\ddot{S}(\ddot{\lambda}, \ddot{\mu})$. Another useful formula is

$$\det A = (-i)^{(m+n-1)(m+n+2)/2}. \tag{2.32}$$

Equation (2.29) follows from Corollary 1. □

In the next section we will need the conformal dimensions corresponding to the various highest weights we have discussed. For $\Lambda \in \{0, \ldots, mn-1\}$,

$$h_\Lambda = \frac{\Lambda(mn - \Lambda)}{2mn}, \tag{2.33}$$

and for $\dot{\lambda} \in \dot{P}^n_+$, $\sigma \in Z_n$ we have

$$h_{\dot{\lambda}} + h_{\sigma\beta(\dot{\lambda})} = \frac{1}{2mn}\delta(\dot{\lambda}, \sigma)(mn - \delta(\dot{\lambda}, \sigma)) \bmod Z. \tag{2.34}$$

3. The Branching Rules

Now we are in a position to state our main result, the proof of which will occupy most of this section. The notations are those of Sect. 2.

Theorem 1. *Let $\Lambda \in Z_{mn}$ denote a level one highest weight of su(mn). Let $\dot{\lambda} \in \dot{P}^n_+, \ddot{\lambda} \in \ddot{P}^m_+$. Then the multiplicity $b(\Lambda, \dot{\lambda}, \ddot{\lambda})$ of $L(\dot{\lambda}) \otimes L(\ddot{\lambda})$ in $L(\Lambda)$ has the value:*

$b(\Lambda, \dot{\lambda}, \ddot{\lambda}) = 1$, *if* $\ddot{\lambda} = \sigma\beta(\dot{\lambda})$, $\sigma \in Z_n$ *and* $\Lambda = \delta(\dot{\lambda}, \sigma) \bmod mn$,
$b(\Lambda, \dot{\lambda}, \ddot{\lambda}) = 0$ *otherwise.*

As a first consequence, we obtain a special case of a general conjecture in [13]:

Corollary 2. *If $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) \neq 0$ and $b(M, \dot{\mu}, \ddot{\mu}) \neq 0$, then $S(\Lambda, M)\dot{S}^*(\dot{\lambda}, \dot{\mu})\ddot{S}^*(\ddot{\lambda}, \ddot{\mu}) \geqq 0$.*

Proof. Use Eq. (2.28).

The next lemma is the most technical and we have relegated its proof after all the others. It gives a vanishing/non-vanishing criterion for $b(\Lambda, \dot{\lambda}, \ddot{\lambda})$ which cannot be obtained from modular considerations.

Lemma 3. *Let $\dot{Q}$ be the root lattice of su(m), $\dot{\Lambda}_i, 0 \leqq i \leqq m-1$ its fundamental weights and $\dot{Q}_i = (\dot{Q} + \dot{\Lambda}_i) \cap \dot{P}^n_+$. Let $\Lambda \in Z_{mn}$ and $\dot{\lambda} \in \dot{Q}_{\Lambda \bmod m}$. Then there exists $\ddot{\lambda} \in \ddot{P}^m_+, \sigma \in Z_n$ with $\ddot{\lambda} = \sigma\beta(\dot{\lambda})$ such that $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) \neq 0$. Furthermore, $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) = 0$ if $\dot{\lambda} \notin \dot{Q}_{\Lambda \bmod m}$.*

Lemma 4. *Let $i \in \{0, 1, \ldots, m-1\}$, then*

$$\sum_{\dot{\lambda} \in \dot{Q}_i} (\dot{S}(\dot{\lambda}))^2 = \frac{1}{m}. \tag{3.1}$$

Proof. See [13].

Lemma 5. *Let $\Lambda \in Z_{mn}$ and $\dot{\lambda} \in \dot{Q}_{\Lambda \bmod m}$. There exists a unique $\ddot{\lambda} \in \ddot{P}^m_+$ such that $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) \neq 0$. In this case $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) = 1$ and $\ddot{\lambda} = \sigma\beta(\dot{\lambda})$ for some $\sigma \in Z_n$.*

Proof. We start from Eq. (1.17) and use Lemma 3:

$$S(\Lambda) = \sum b(\Lambda, \dot{\lambda}, \ddot{\lambda})\dot{S}(\dot{\lambda})\ddot{S}(\ddot{\lambda}), \tag{3.2}$$

where now summation is over $\dot{\lambda} \in \dot{Q}_{\Lambda \bmod m}$ and $\ddot{\lambda} \in \ddot{P}_+^m$ such that $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) \neq 0$. For each $\dot{\lambda} \in \dot{Q}_{\Lambda \bmod m}$, we choose one of those $\ddot{\lambda}$ for which $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) \neq 0$, and write it as $\sigma\beta(\dot{\lambda})$. Equations (2.29) and (3.1) imply:

$$S(\Lambda) \geqq \left(\frac{m}{n}\right)^{1/2} \sum b(\Lambda, \dot{\lambda}, \sigma\beta(\dot{\lambda}))(\dot{S}(\dot{\lambda}))^2$$

$$\geqq \left(\frac{m}{n}\right)^{1/2} \sum (\dot{S}(\dot{\lambda}))^2 = (mn)^{-1/2}. \tag{3.3}$$

However by (2.24) we have $S(\Lambda) = (mn)^{-1/2}$. The lemma follows. □

Proof of the Theorem. The preceding lemmas imply all the assertions in the theorem except the fact that $\Lambda = \delta(\dot{\lambda}, \sigma) \bmod mn$, a condition which determines σ, which in turn locates the weight $\ddot{\lambda}$ being coupled to Λ and $\dot{\lambda}$. We prove this in two steps.

1. *Classical Case.* Each representation $L(\Lambda)$ of $\widehat{su}(mn)$ is graded by the eigenvalues of L_0. The subspace with the lowest eigenvalue h_Λ is nothing but the finite-dimensional representation $L(\bar{\Lambda})$ of $su(mn)$ whose tableau $Y(\Lambda)$ is a single column with Λ boxes. $Y(\Lambda)$ decomposes into a sum of finite-dimensional representations of $su(m) \oplus su(n)$ as follows [18]:

$$Y(\Lambda) = \bigoplus_{|Y(\dot{\lambda})| = \Lambda} Y(\dot{\lambda}) \otimes Y(\dot{\lambda})^T, \tag{3.4}$$

where $|Y(\dot{\lambda})|$ is the number of boxes of the tableau $Y(\dot{\lambda})$. We will say that $b(\Lambda, \dot{\lambda}, \ddot{\lambda})$ is classical if it is non-zero and $|Y(\dot{\lambda})| = \Lambda$. According to (3.4) this implies that $\ddot{\lambda} = \dot{\lambda}^T$. By Proposition 1, $\dot{\lambda}^T = \sigma\beta(\dot{\lambda})$ with $\sigma = c(\dot{\lambda}) =$ number of columns of $Y(\dot{\lambda}) \bmod n$. Also, one finds that

$$\delta(\dot{\lambda}, \sigma) = |Y(\dot{\lambda})| + m(\sigma - c(\dot{\lambda})) + mn, \tag{3.5}$$

so we have shown that $\delta(\dot{\lambda}, \sigma) = \Lambda \bmod mn$ when $b(\Lambda, \dot{\lambda}, \ddot{\lambda})$ is classical.

2. *The General Case* is reduced to the classical case by means of automorphisms. The center Z_n of $su(n)$ is embedded in the center Z_{mn} of $su(mn)$ by the map

$$\nu \mapsto m\nu \tag{3.6}$$

in the additive notation. Since the elements of Z_n and Z_{mn} act as automorphisms of the corresponding algebras, this implies that

$$b(\Lambda + m\nu, \dot{\lambda}, \nu \cdot \ddot{\lambda}) = b(\Lambda, \dot{\lambda}, \ddot{\lambda}). \tag{3.7}$$

Suppose now that $b(\Lambda, \dot{\lambda}, \ddot{\lambda}) \neq 0$. There exists $\nu \in Z_n$ such that $b(\Lambda + m\nu, \dot{\lambda}, \nu \cdot \ddot{\lambda})$ is classical. This is because by Lemma 3, we know that $\dot{\lambda} \in \dot{Q}_{\Lambda \bmod m}$, which is the same as $|Y(\dot{\lambda})| = \Lambda \bmod m$. Rewriting the latter explicitly as $|Y(\dot{\lambda})| = \Lambda + m\nu$ proves our claim. But now we have

$$\Lambda + m\nu = \delta(\dot{\lambda}, \nu\sigma) = \delta(\dot{\lambda}, \sigma) + m\nu \bmod mn, \tag{3.8}$$

and the proof is complete. □

Proof of Lemma 3. All required properties of $b(\Lambda, \dot{\lambda}, \ddot{\lambda})$ will be obtained from a study of the conformal pair

$$so(2mn)_1 \supset su(m)_n \oplus su(n)_m \oplus u(1) \tag{3.9}$$

which in turn is the isometry embedding of the symmetric space

$$SU(m+n)/SU(m) \times SU(n) \times U(1). \tag{3.10}$$

The link of (3.9) with

$$su(mn)_1 \supset su(m)_n \oplus su(n)_m \tag{3.11}$$

is provided by

$$so(2mn) \supset su(mn) \oplus u(1) \tag{3.12}$$

which is also conformal, with known branching rules (see below). For short, set $g = so(2mn)$, $h = su(m+n)$, $p = su(m) \oplus su(n) \oplus u(1)$. One has the decomposition

$$h = p \oplus V \tag{3.13}$$

in which V is a p-module and (symmetric space property):

$$[V, V] \subset p. \tag{3.14}$$

In [1] it was explained how the decomposition of the spinor representation $s \oplus t$ (sum of the two half-spins) of $\hat{g}$ as a $\hat{p}$-module may be obtained by considering the affine root system of h. However [1] only covered the case when h and p have the same rank, h is simple and p is semisimple. Let us first extend the method given there to a reductive p, i.e. allow a $u(1)$ in it. What we want to show is that no terms will cancel each other in $\chi_s - \chi_t$. Let $\bar{s}$ and $\bar{t}$ denote the finite-dimensional spinor representations. It is known that considered as p-modules, they do not have any common weight. Assume now that s and t do have a common weight, so that

$$\lambda - q\delta - \sum k_i \alpha_i = \mu - r\delta - \sum l_i \alpha_i, \tag{3.15}$$

where λ and μ are respectively weights of $\bar{s}$ and $\bar{t}, q, r$ are non-negative integers, α_i are the simple roots of p and $k_i, l_i \in Z$. Therefore $q = r$ and $\lambda - \mu \in Q_0$, the root lattice of p. Hence $\bar{s}$ and $\bar{t}$ contain respectively irreducible components $\bar{s}_1$ and $\bar{t}_1$ in the same congruence class. But these must have a common weight, a contradiction.

Next, the affinization of $u(1)$ is the Heisenberg algebra, so its irreducible characters are inverse of Dedekind's η function. The root system, Weyl group, etc. of p are defined to be those of its semisimple part. All the other arguments work without modifications. Hence the decomposition of $s \oplus t$ of $\hat{g}$ into irreducible $\hat{p}$-modules is

$$s \oplus t = \bigoplus_{w \in W/W_0} L(w(\rho) - \rho_0), \tag{3.16}$$

where from now on quantities with a subscript 0 refer to p, W is the affine Weyl group of h, ρ is the affine Weyl vector of h. In (3.16) it is understood that W/W_0 is a system of coset representatives such that $w(\rho) - \rho_0$ is a dominant weight of $\hat{p}$. It breaks into

$$W/W_0 = (S^{m+n}/S^m \times S^n) \times (Q/Q_0), \tag{3.17}$$

where S^m is the symmetric group on m objects. The root lattice Q of h is spanned by simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{m+n-1}$. We take $\{\alpha_1, \ldots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \ldots, \alpha_{m+n-1}\}$ as a basis of Q_0. Therefore $Q/Q_0 = Z\alpha_m$.

As a generator of the $u(1)$ subalgebra of p, we can take the fundamental weight ω_m dual to α_m. Remember that weights of $su(m)$ are of the form

$$\lambda = \sum_{j=1}^{m} \lambda_j \varepsilon_j, \tag{3.18}$$

where $\{\varepsilon_i\}$ is an orthonormal basis of C^m and $\sum_{j=1}^{m} \lambda_j = 0$. S^m acts as permutations of the ε_i. The fundamental weights are chosen to be

$$\dot{\omega}_i = \sum_{j=1}^{i} \varepsilon_j - \frac{i}{m} \sum_{j=1}^{m} \varepsilon_j, \tag{3.19}$$

$$\ddot{\omega}_i = \sum_{j=1}^{i} \varepsilon_{m+n+1-j} - \frac{i}{n} \sum_{j=1}^{n} \varepsilon_{m+j}. \tag{3.20}$$

One has the relation:

$$\alpha_m = -\dot{\omega}_{m-1} + \ddot{\omega}_{n-1} + \left(\frac{m+n}{mn}\right)\omega_m \tag{3.21}$$

obtained by choosing $\varepsilon_i, 1 \leqq i \leqq m+n$ to span the weight space of $su(m+n)$. (Remember: single dots refer to $su(m)$, double dots to $su(n)$.)

In order to get a dominant weight $w(\rho) - \rho_0$, we have to use as representatives of Q/Q_0 in W/W_0, not the translations $t(k\alpha_m)$ by multiples of α_m, but the powers of

$$T = w^{(m)} w^{(n)} t(\alpha_m) \tag{3.22}$$

with $w^{(m)}$ and $w^{(n)}$ some fixed elements of S^m and S^n. See [2] for details. Then we get a more explicit form of (3.16):

$$s \oplus t = \bigoplus_{w \in C_{m,n}} \bigoplus_{k \in Z} L(\sigma_m^{-k} \sigma_n^k (w(\rho) - \rho_0)) \otimes F(h_k), \tag{3.23}$$

$C_{m,n} = S^{m+n}/S^m \times S^n$. σ_m and σ_n are generators of Z_m and Z_n acting as in (2.4). $F(h_k)$ is an irreducible Fock space representation of the $u(1)$ Heisenberg algebra with conformal weight h_k. In deriving (3.23) we used

$$T(\lambda) = \sigma_m^{-1} \sigma_n(\lambda) \bmod C\omega_m, \tag{3.24}$$

which is a consequence of (3.21) and (3.22).

Now the Weyl vector (half-sum of the positive roots) of $su(m+n)$ is

$$\rho = \frac{1}{2} \sum_{j=1}^{m+n} (m+n+1-2j)\varepsilon_j. \tag{3.25}$$

A weight λ of $su(m)$ is (strictly) dominant iff it is of the form (3.18) and $\lambda_j > \lambda_{j+1}$, $j = 1, \ldots, m-1$. Thus we see that a suitable choice for $C_{m,n}$ consists of permutations which bring the sequence $(1, 2, \ldots, m+n)$ into $(\mu_1, \ldots, \mu_m \bar{\mu}_1, \ldots, \bar{\mu}_n)$, where (μ_i) is increasing, $(\bar{\mu}_i)$ is decreasing, and they are complementary sequences in

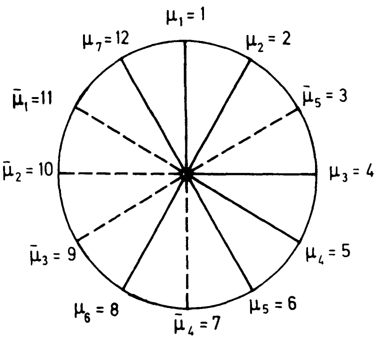

Fig. 2

$\{1, 2, \ldots, m+n)$. Using the same kind of diagram as in the proof of Lemma 1 (see Fig. 2) we number consecutively $m+n$ lines on the circle, m of which are the solid lines μ_i, the other n lines $\bar{\mu}_i$ being broken.

We set for $w \in C_{m,n}$

$$w(\rho) = \sum_{k=1}^{m} \rho_k \varepsilon_k + \sum_{k=1}^{n} \bar{\rho}_k \varepsilon_{k+m} \tag{3.26}$$

with

$$\begin{aligned} \rho_k &= (m+n+1)/2 - \mu_k, \\ \bar{\rho}_k &= (m+n+1)/2 - \bar{\mu}_k. \end{aligned} \tag{3.27}$$

Let

$$N = \sum_{k=1}^{m} \rho_k = -\sum_{k=1}^{n} \bar{\rho}_k. \tag{3.28}$$

Then one has

$$w(\rho) = \dot{\lambda} + \dot{\rho} + \ddot{\lambda} + \ddot{\rho} + N\left(\frac{m+n}{mn}\right)\omega_m, \tag{3.29}$$

where

$$\dot{\lambda} + \dot{\rho} = \sum_{k=1}^{m} \rho_k \varepsilon_k - \frac{N}{m} \sum_{k=1}^{m} \varepsilon_k, \tag{3.30}$$

$$\ddot{\lambda} + \ddot{\rho} = \sum_{k=1}^{n} \bar{\rho}_k \varepsilon_{k+m} + \frac{N}{n} \sum_{k=1}^{n} \varepsilon_{k+m}. \tag{3.31}$$

Define

$$\begin{aligned} r_k &= \rho_k + m + n - \rho_1, \\ s_k &= \bar{\rho}_k + m + n - \bar{\rho}_1. \end{aligned} \tag{3.32}$$

With the help of Fig. 2 it is now easy to see that $\beta(r_k) = (s_k)$ up to a possible rotation by some $\sigma \in Z_n$. Thus we have shown that $\ddot{\lambda} = \sigma\beta(\dot{\lambda})$ in (3.29).

It is also fairly easy to show that every $\dot{\lambda}\in\dot{P}^n_+$ arises in (3.29) as w runs through $C_{m,n}$. Indeed suppose that w is such that $\mu_1=1$, say. Given $\dot{\lambda}+\dot{\rho}$, or rather a sequence (r_k), the rest of the μ_i are determined using (3.27) and (3.32):

$$\mu_i=m+n+\mu_1-r_i, \quad 2\leqq i\leqq m. \tag{3.33}$$

This proves the existence of $w\in C_{m,n}$ such that $w(\rho)$ gives rise to any given $\dot{\lambda}+\dot{\rho}$ in (3.29).

Observe, that since $h_s=h_t=mn/8$, with (2.34) we can determine h_k in (3.23):

$$h_k=\frac{(\delta(\dot{\lambda},\sigma)-mn/2)^2}{2mn} \bmod Z. \tag{3.34}$$

We now want to compare the branching rules of (3.11) and (3.9). The latter are [13, 3]:

$$s\oplus t=\bigoplus_{\Lambda\in Z} L(\Lambda \bmod mn)\otimes F(h_\Lambda), \tag{3.35}$$

where $h_\Lambda=(\Lambda-mn/2)^2/2mn$. Both right-hand sides of (3.23) and (3.35) are periodic with period mn, up to differences in h_k and h_Λ by integers. It is therefore enough to compare (3.23) and (3.35) over two basic periods.

The embedding (3.11) is integral so $L(\dot{\lambda})\otimes L(\ddot{\lambda})$ occurs in $L(\Lambda)$ implies $\dot{\lambda}\in\dot{Q}_{\Lambda \bmod m}$ and $\ddot{\lambda}\in\ddot{Q}_{\Lambda \bmod n}$.

Finally, every pair $(\dot{\lambda},\ddot{\lambda})\in\dot{Q}_{\Lambda \bmod m}\times\dot{Q}_{\Lambda \bmod n}$ with $\ddot{\lambda}=\sigma\beta(\dot{\lambda})$ occurs in $L(\Lambda)$, as otherwise it would be missing in the decomposition of $s\oplus t$, which as we showed before contains all of them. All assertions of Lemma 3 follow from these considerations. □

4. Modular Invariants

The use of conformal embeddings in the construction of modular invariant partition functions goes back to [8]. Fix a conformal embedding $g\supset p$, and take a modular invariant partition function:

$$Z=\sum_{\Lambda,M}\chi_\Lambda\Omega_{\Lambda,M}\chi^*_M \tag{4.1}$$

built from the level one characters χ_Λ of $\hat{g}$. An obvious way to obtain a modular invariant for $\hat{p}$ is to simply substitute in (4.1) the decomposition of the characters $\chi_\Lambda=\sum_\lambda b^\Lambda_\lambda\chi_\lambda$. Even when Z is the trivial invariant, $\Omega=$ identity, it is possible in this way to get interesting invariants, such as the E_6 and E_8 in the ADE classification [9] of $p=su(2)$ models, out of $g=sp(4)$, G_2, respectively.

Another kind of construction is as follows. Suppose now that the conformal subalgebra consists of two commuting components, $p=p_1\oplus p_2$. Let Ω be as above, a g invariant, and M be a p_1 invariant, then:

$$N_{\ddot{\lambda}\ddot{\mu}}=b(\Lambda,\dot{\lambda},\ddot{\lambda})\Omega_{\Lambda,M}M_{\dot{\lambda},\dot{\mu}}b(M,\dot{\lambda},\ddot{\mu}) \tag{4.2}$$

(where Einstein's summation convention is enforced) is a p_2 invariant, as can be seen at once from the intertwining property of the branching coefficients, Eq. (1.16).

However, N usually does not qualify directly as a "physical" invariant, the reason being that the corresponding theory in general does not have a unique vacuum: $N_{0,0} > 1$, where 0 labels the representation with conformal dimension zero. So one needs either to subtract from N a known invariant, or divide it by some positive integer, or a combination of both. The E_7 invariant is obtained in this way from $E_8 \supset su(3)_6 \oplus su(2)_{16}$.

Below we will consider mainly this second construction, starting from an $su(mn)$ invariant Ω and an $su(m)_n$ invariant M, ending with an $su(n)_m$ invariant N. The modular invariants of $su(r)_1$ (in the following we put $r = mn$) have been classified [12], and the general matrix Ω depends on the divisors of r: let Δ, Δ' be such that $\Delta\Delta' = r$ or $2r$, if r is odd or even, respectively, and also assume that r, Δ and Δ' all have the same parity. Then:

$$\Omega_{x,y} = \delta_{x-y=0 \bmod \Delta}\delta_{x+y=0 \bmod \Delta'}, \tag{4.3}$$

where $x, y \in \{0, \ldots, r-1\}$.

Before giving examples a few preliminary remarks are needed. The classification of modular invariants is complete only in the two cases mentioned above, namely $su(2)$ all levels and $su(n)$ level one. However many invariants are known for $su(n)$ at higher levels, as well as for other Lie algebras. Let us empirically divide the invariants for $su(n)_m$ into four classes, neglecting the possible effects of conjugations [12]. Class $\mathscr{A}$ consists of the trivial invariant which exists at all levels. Class $\mathscr{C}$ exists only at the levels $m = n$ and $m = n \pm 2$, when $su(n)$ can be conformally embedded in $so(n^2 - 1)$ or $su(p)$ with $p = n(n \pm 1)/2$. Class $\mathscr{D}$ consists of invariants related to the cyclic group Z_n or its subgroups. In the more general context of rational CFT, such invariants associated with cyclic groups have been studied in [16], and here we will be able obtain concrete and non-trivial cases using our method. All type $\mathscr{D}$ invariants listed below are obtained by setting $M =$ identity in (4.2). Therefore they have the property that $N_{\ddot{\lambda},\ddot{\mu}} \neq 0$ if $\ddot{\mu} = \sigma(\ddot{\lambda})$, where $\sigma \in Z_n$. Finally class $\mathscr{E}$ contains exceptional invariants.

In the sequel we shall present a few sample applications of formula (4.2). A more comprehensive analysis is in progress [5].

4.1 Invariants at Level 2. Our first examples of invariants belong to the case $m = 2$, i.e. will be $su(n)_2$ invariants obtained from Ω and $su(2)_n$ invariants. Note first that the branching coefficients in this case are easy to compute explicitly. From Theorem 1 we find the decomposition[1]:

$$\chi_\Lambda = \bigoplus_{j:j=\Lambda \bmod 2} \chi((n-j)\dot{\Lambda}_0 + j\dot{\Lambda}_1)\chi(\ddot{\Lambda}_{(\Lambda+j)/2 \bmod n} + \ddot{\Lambda}_{(\Lambda-j)/2 \bmod n}). \tag{4.4}$$

When M is trivial one recovers by the method indicated above invariants of type $\mathscr{A}$ and $\mathscr{D}$ by varying Ω.

For $n = 3, 5, 6$ there are two possibilities for Ω and one gets the trivial invariant along with the $\mathscr{D}$ invariant belonging to the series found in [7, 2], which from now

[1] One of the authors would like to take this opportunity to correct [3], in which Eq. (3.38) should be replaced by (4.4). Everything else should remain unchanged

on will be denoted $\mathscr{D}_0$. Recall that these invariants for $su(n)_m$ have the following shape:

$$\sum_{\ddot{\lambda}\in\ddot{P}_+^m} \chi_{\ddot{\lambda}}\chi^*_{\sigma(\ddot{\lambda})\cdot\ddot{\lambda}} \tag{4.5}$$

if m, n are coprime, where $\sigma(\ddot{\lambda})\in Z_n$ and:

$$\sum_{\ddot{\lambda}\in\ddot{P}_+^m\cap\ddot{Q}} n^{-1}\sum_{\sigma\in Z_n}|\chi_{\sigma\ddot{\lambda}}|^2 \tag{4.6}$$

if $n|m$, where $\ddot{Q}$ is the root lattice of $su(n)$.

Already for $n=4$ something noteworthy happens. There are three possibilities for Ω. Two of them, $(\Delta,\Delta')=(8,2)$ and $(2,8)$ give respectively the trivial and the $\mathscr{D}_0$ invariant, but $(4,4)$ produces another kind of $\mathscr{D}$ invariant:

$$|\chi_{2000}+\chi_{0020}|^2+|\chi_{0002}+\chi_{0200}|^2+2|\chi_{1010}|^2+2|\chi_{0101}|^2, \tag{4.7}$$

where the characters are labeled by the Dynkin integers $k_i, i=0,\ldots,3$. This is obviously an invariant associated with a Z_2 subgroup of Z_4. It is interesting to note that this invariant could also be obtained using the conformal embedding $su(6)_1\supset su(4)_2$ by expanding the trivial $su(6)$ invariant.

Of course it is possible to obtain exceptional invariants by setting M to be one of the E invariants of $su(2)$. Here we will limit ourselves to the case when M is the E_6 invariant of $su(2)_{10}$. If Ω is the identity one obtains[2]:

$$\begin{aligned}\sum_{j=0}^{4} &|\chi(2\ddot{\Lambda}_j)+\chi(\ddot{\Lambda}_{3+j}+\ddot{\Lambda}_{7+j})|^2+|\chi(\ddot{\Lambda}_{2+j}+\ddot{\Lambda}_{9+j})+\chi(\ddot{\Lambda}_{4+j}+\ddot{\Lambda}_{7+j})|^2\\ &+|\chi(\ddot{\Lambda}_{2+j}+\ddot{\Lambda}_{8+j})+\chi(2\ddot{\Lambda}_{5+j})|^2,\end{aligned} \tag{4.8}$$

where we have set $\ddot{\Lambda}_{i+10}=\ddot{\Lambda}_i$. Let $N_{\ddot{\lambda},\ddot{\mu}}$ stand for the matrix elements of this invariant. By setting now $(\Delta,\Delta')=(4,10)$ one gets another invariant with matrix elements $N_{\ddot{\lambda},\nu(\ddot{\mu})}$, where

$$\nu(\ddot{\Lambda}_i)=\ddot{\Lambda}_{10-i}. \tag{4.9}$$

4.2 Invariants at Level 3. We will study the case $su(5)_3$. Assume first that M is the identity. There are four possibilities to consider for (Δ,Δ'):$(15,1),(1,15),(5,3)$ and $(3,5)$. One obtains for N the trivial $su(5)_3$ invariant in the cases $(15,1)$ and $(5,3)$, and the $\mathscr{D}_0$ invariant for $(3,5)$ and $(1,15)$.

Next we take M to correspond to a $\mathscr{C}$ type invariant. There is a conformal embedding $su(6)_1\supset su(3)_5$ from which one gets the matrix M of an $su(3)_5$ invariant by expanding the trivial $su(6)$ invariant. Similarly one gets the matrix N of an $su(5)_3$ invariant using the conformal embedding $su(10)_1\supset su(5)_3$. We have observed that N can be obtained from M by using our method based on $su(15)_1\supset su(3)_5\oplus su(5)_3$, Ω being the trivial $su(15)$ invariant.

4.3 Invariants at Level 4. Here we will study the case $su(4)_4$. The possibilities for Δ,Δ' are $(16,2)$, which corresponds to $\Omega=$ identity, $(2,16)$, $(4,8)$, $(8,4)$. Suppose first we start with M being the trivial invariant. For $(8,4)$ we obtain an invariant

[2] In [17] the same invariant was computed, but given there with a misprint in the third term

associated with a Z_2 subgroup:

$$|\chi_{4000}+\chi_{0040}|^2+|\chi_{0400}+\chi_{0004}|^2+|\chi_{1012}+\chi_{1210}|^2+|\chi_{2101}+\chi_{0121}|^2 \\ +|\chi_{2002}+\chi_{0220}|^2+|\chi_{0022}+\chi_{2200}|^2+|\chi_{3010}+\chi_{1030}|^2+|\chi_{0103}+\chi_{0301}|^2 \\ +2|\chi_{2020}|^2+2|\chi_{0202}|^2|+2|\chi_{1111}|^2, \tag{4.10}$$

while for (2, 16) or (4, 8) we get a "twisted" version:

$$|\chi_{4000}+\chi_{0040}|^2+|\chi_{0400}+\chi_{0004}|^2+|\chi_{1012}+\chi_{1210}|^2+|\chi_{2101}+\chi_{0121}|^2 \\ +(\chi_{2002}+\chi_{0220})(\chi_{0022}+\chi_{2200})^*+(\chi_{3010}+\chi_{1030})(\chi_{0103}+\chi_{0301})^*+\text{c.c.} \\ +2|\chi_{2020}|^2+2|\chi_{0202}|^2+|2|\chi_{1111}|^2, \tag{4.11}$$

Remarkably the $\mathscr{D}_0$ invariant is not produced by this method. Also there is a $\mathscr{C}$ invariant obtained from the conformal embedding $so(15)_1 \supset su(4)_4$:

$$|\chi_{4000}+\chi_{0040}+\chi_{1012}+\chi_{1210}|^2 \\ +|\chi_{0400}+\chi_{0004}+\chi_{2101}+\chi_{0121}|^2+4|\chi_{1111}|^2. \tag{4.12}$$

This one has a peculiar behavior: if we take this invariant for M in (4.2) then we always find $N = M$, no matter what value of Ω is chosen.

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