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Inequivalent Quantizations for Non-Linear σ Model

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Abstract. We compute the homotopy groups Π_0 and Π_1 of the classical configuration space of an $O(3)$ invariant field theory on $\mathbb{X} \times \mathbb{R}$, where $\mathbb{X}$ is a compact two dimensional manifold for arbitrary genus g and $\mathbb{R}$ denotes the time coordinate. We also present the finite dimensional, unitary, irreducible, inequivalent representations of the appropriate fundamental groups and comment on some of their implications.

1. Introduction

In the recent past the $2+1$ dimensional $O(3)$ sigma model with Hopf term added to the action has been considered in the context of high temperature superconductivity [1] and fractional statistics conceivable for a $2+1$ dimensional system [2]. If the two dimensional space is taken to be S^2 (equivalently, if the σ model field $\mathbf{n}(\mathbf{x}, t) \to$ the north pole as $|\mathbf{x}| \to \infty$), the coefficient of the Hopf term in the action is arbitrary which in turn reflects the possibility of fractional statistics [2]. Recently two of us (T. R. G. and R. S.) considered the case where $\mathbf{n}(\mathbf{x}, t)$ satisfies periodic boundary conditions (equivalently the two dimensional space is a torus). They showed that there are two possible inequivalent quantizations and suggested that only the usual Bose–Einstein and Fermi–Dirac statistics are possible [3].

In the present paper we generalize the analysis to include the case where the two dimensional space is an arbitrary compact, orientable surface of genus $g \geqq 1$.

When the genus $g \geqq 1$, the fundamental group, Π_1, of the configuration space of the σ model field theory is non-abelian and is different for each g. This therefore provides examples to analyze the effects of Π_1 on the quantization procedure [4], which is interesting in itself. A "physical" situation where a genus g surface could be relevant may be conceived as follows:

Imagine a macroscopic lattice system with an $O(3)$ invariant Hamiltonian and whose boundary is a $4g$ sided polygon. For large enough g this may be considered

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as an approximation to the actual physical boundary. Various types of boundary conditions can be considered for such a system. Among these there is a class of boundary conditions such that the basic variables assume the same values on pairs of sides of the polygonal boundary. In a continuum limit, preserving the $O(3)$ symmetry and the boundary conditions such a system can be approximated by an $O(3)$ invariant σ model with basic variables defined on a genus g surface. Physical results derived from such a model, which are insensitive to the genus could then be considered to be valid for a large class of boundary conditions. Results which are sensitive to the genus could be used to discriminate between different boundaries and boundary conditions.

Let us briefly consider the ambiguities in quantization introduced by a nontrivial fundamental group of the classical configuration space.

Let Γ be a classical configuration space which is a differentiable manifold admitting a suitable measure. One can construct a Hilbert space of complex valued, square integrable "functions" on Γ. If $\Pi_1(\Gamma)$ is trivial then one may consider only single valued functions and the Hilbert space of square integrable functions then constrains candidate quantum mechanical state spaces. However if the $\Pi_1(\Gamma)$ is nontrivial then there is a possibility of defining multivalued functions on Γ (as functions obtained from singled valued functions on the universal covering space of Γ) and constructing the corresponding Hilbert spaces. These multivalued functions are classified by unitary, one dimensional representations of $\Pi_1(\Gamma)$. More generally instead of "functions" one may consider sections of finite rank complex vector bundles with structure group containing $\Pi_1(\Gamma)$ as a subgroup. Hilbert spaces of such sections may now be classified by unitary, finite dimensional representations of $\Pi_1(\Gamma)$. Since in general there exist several unitarily inequivalent representations one gets several inequivalent candidate state spaces for a quantum system associated with Γ. Which of these is to be chosen in a given situation is the ambiguity in quantization referred to above. Note that even if one chooses a particular representation of $\Pi_1(\Gamma)$, depending upon the algebra of physical observables one may get further inequivalent physical state spaces, but at present we concentrate only on inequivalence implied by inequivalent representations of $\Pi_1(\Gamma)$. For simple illustrative examples of these "Π_1 effects" see Isham, in ref. [4].

In the present work we take Γ to be the configuration space of an $O(3)$ invariant sigma model defined on (a genus g surface) $\times\, \mathbb{R}$ (time coordinate), compute its Π_0 and Π_1 groups and classify the finite dimensional, unitary representations of Π_1.

The paper is organized as follows:

In Sect. 2 we present an explicit description of the configuration space Γ and compute the $\Pi_0(\Gamma)$—the set of homotopically disjoint sectors of Γ (which signals the possibility of the existence of topological solitons in a model).

In Sect. 3 we compute the Π_1 of these disjoint sectors, using some homotopy invariant integrals.

In Sect. 4 we present the unitary, finite dimensional, inequivalent, irreducible representations of Π_1.

Section V summarizes the conclusions and includes a brief discussion of the results.

An appendix is included to prove some of the results used in the main body.

2. Computation of $\Pi_0(\Gamma)$

Let $\mathbb{X}$ be a compact Riemann surface of genus g greater than 0 and let Γ be the space of smooth maps from $\mathbb{X}$ to the sphere, S^2 such that a fixed point of $\mathbb{X}$ goes to the north pole of S^2 (say). We take the sphere as the unit sphere in $\mathbb{R}^3$ described by a three dimensional vector $\mathbf{n}$ satisfying $\mathbf{n}\cdot\mathbf{n}=1$. Thus,

$$\Gamma = \{\mathbf{n}(\mathbb{X})/\mathbf{n}:\mathbb{X}\to S^2;\mathbf{n}\cdot\mathbf{n}=1\}. \tag{2.1}$$

The homotopy classes of these maps will form a set Π_0. We wish to determine Π_0. It is convenient to choose a model for $\mathbb{X}$ just as we have chosen the unit sphere in $\mathbb{R}^3$ as a model for S^2. Recall that a genus g surface can be represented as a $4g$ sided polygon in $\mathbb{R}^2$ with pairs of sides identified [5]. Equivalently we choose as a model for $\mathbb{X}$ the unit disc in $\mathbb{R}^2$ with $4g$ boundary arcs identified pair wise (Fig. 1). Introducing the (r,θ) coordinates in the unit disc, the $4g$ arcs can be described explicitly as the set of points (see Fig. 1):

$$(r,\theta)=(1,\theta);\quad \theta_i\leqq\theta\leqq\theta_{i+1}\quad \forall i=0,1,\dots,4g-1;$$

where (2.2)

$$\theta_i=\frac{2\pi i}{4g},\quad\text{and}\quad \theta_{4g}=2\pi.$$

Label the arcs, $r=1$, as: (for $i=1\cdots g$)

$$\begin{aligned} a_i&=[\theta_{4i-4},\theta_{4i-3}], & b_i&=[\theta_{4i-3},\theta_{4i-2}]\\ a_i^{-1}&=[\theta_{4i-2},\theta_{4i-1}], & b_i^{-1}&=[\theta_{4i-1},\theta_{4i}]. \end{aligned} \tag{2.3}$$

The points $(1,\theta_i)$, $i=1\cdots 4g$ are identified as the base point P_0 on $\mathbb{X}$ and all the maps $\mathbf{n}$ are required to take P_0 to the north pole, $\hat{\mathbf{n}}$, of the S^2. Also the arcs a_i, a_i^{-1}, b_i, b_i^{-1} are identified $\forall i=1\cdots g$.

The maps can now be specified more explicitly as maps $\mathbf{n}(r,\theta)$ defined for $0\leqq r\leqq 1$, $0\leqq\theta\leqq 2\pi$, such that,

i) $\quad \mathbf{n}(1,\theta_i)=\hat{\mathbf{n}} \qquad$ for $\qquad i=0,1,\dots,4g-1;$

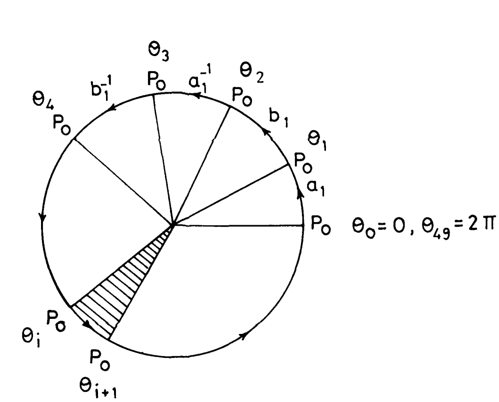

Fig. 1. A model for a genus g surface. The arcs $a_i a_i^{-1}$, $b_i b_i^{-1}$ are identified and so are the points labeled P_0

$$\text{ii) } \mathbf{n}(1,\theta_i+\vartheta)=\mathbf{n}(1,\theta_{i+3}-\vartheta) \quad \text{for} \quad \vartheta\in[0,\pi/2g], \qquad (2.4)$$

$$\text{and for} \quad i=4j \quad \text{or} \quad 4j+1 \quad \text{with} \quad j=0,1,\dots,g-1.$$

Theorem 2.1. *Any* $\mathbf{n}:\mathbb{X}\to S^2$ *is homotopic to a map* $\mathbf{n}':\mathbb{X}\to S^2$ *which can be viewed as a map from* S^2 *to* S^2.

Proof. Let $\mathbf{n}$ be a given map from $\mathbb{X}$ to S^2. Let $\mathbf{n}_i$ denote its restriction to the i^{th} sector (Fig. 1). Now $\mathbf{n}_i(1,\theta)$ gives a map from S^1 to S^2 since each of the arcs is homeomorphic to a circle. Since $\Pi_1(S^2)$ is trivial this map is homotopic to a constant map. Furthermore this constant may be chosen as $\hat{\mathbf{n}}$ since at θ_i and θ_{i+1} $\mathbf{n}(1,\theta)=\hat{\mathbf{n}}$. This implies that there exists a 1-parameter family of maps from S^1 to S^2, $\mathbf{m}_i(\sigma,\theta)$, such that

$$\begin{aligned} &\text{(i)} \quad \mathbf{m}_i(\sigma,\theta_i)=\mathbf{m}_i(\sigma,\theta_{i+1})=\hat{\mathbf{n}} \quad \forall\sigma\in[0,1],\\ &\text{(ii)} \quad \mathbf{m}_i(0,\theta)=\mathbf{n}_i(1,\theta),\\ &\text{(iii)} \quad \mathbf{m}_i(1,\theta)=\hat{\mathbf{n}} \quad \text{for} \quad \theta_i\leqq\theta\leqq\theta_{i+1}. \end{aligned} \qquad (2.5)$$

Note that (i) implies that for all $\sigma\in[0,1]\mathbf{m}_i(\sigma,\theta)$ is a map from S^1 to S^2.

It is straightforward to see that given a family of maps $\mathbf{m}_i(\sigma,\theta)$, for every $0\leqq\varepsilon\leqq 1$ there exists another family $\mathbf{m}_i'(\sigma,\theta)$ such that $\mathbf{m}_i'(\sigma,\theta)$ satisfies conditions (i) and (ii) of Eq. (2.5) and satisfies,

$$\text{(iii)}' \quad \mathbf{m}_i'(\sigma,\theta)=\hat{\mathbf{n}} \quad \forall 1-\varepsilon\leqq\sigma\leqq 1. \qquad (2.6)$$

Therefore without loss of generality we take $\mathbf{m}_i(\sigma,\theta)$ to satisfy Eq. (2.6).

Define:

$$\mathbf{n}_i'(r,\theta,\sigma)=\begin{cases}\mathbf{n}_i(r/f(\sigma),\theta) & \text{for} \quad 0\leqq r\leqq f(\sigma)\\ \mathbf{m}_i\left(\dfrac{r}{f(\sigma)}-1,\theta\right) & \text{for} \quad f(\sigma)\leqq r\leqq 1\end{cases}, \qquad (2.7)$$

where $f(\sigma)$ is a smooth function satisfying,

$$1/2\leqq f(\sigma)\leqq 1, \quad f(0)=1, \quad f(1)=1/2. \qquad (2.8)$$

The conditions on $f(\sigma)$ ensure that $\mathbf{n}'(\mathbf{r},\theta,\sigma)$ is well defined for each of its domains of definitions and is continuous at $r=f(\sigma)$ by virtue of condition (ii) of Eq. (2.5). Furthermore,

$$\begin{aligned} &\mathbf{n}_i'(r,\theta,0)=\mathbf{n}_i(r,\theta) \qquad \text{for} \quad 0\leqq r\leqq 1 \quad \text{and},\\ &\mathbf{n}_i'(r,\theta,1)=\begin{cases}\mathbf{n}_i(2r,\theta) & \text{for} \quad 0\leqq r\leqq 1/2\\ \mathbf{m}_i(2r-1,\theta) & \text{for} \quad 1/2\leqq r\leqq 1\end{cases}. \end{aligned} \qquad (2.9)$$

Hence $\mathbf{n}_i'(\mathbf{r},\theta,1)=\hat{\mathbf{n}}$ for $1-\varepsilon/2\leqq r\leqq 1$ and every $\mathbf{n}_i$ is homotopic to a map is constant outside $r<1-\varepsilon/2$ sub sector. This can be repeated for all the sectors. All these $\mathbf{n}_i$, $i=1,\dots,4g$, maps match smoothly across the sector boundaries. Therefore for every map $\mathbf{n}:\mathbb{X}\to S^2$, there exists $\mathbf{n}':\mathbb{X}\to S^2$ which is homotopic to $\mathbf{n}$ and $\mathbf{n}'(1,\theta)=\hat{\mathbf{n}}$ $\forall 0\leqq\theta\leqq 2\pi$. But this can be viewed as a map from $S^2\to S^2$, proving the theorem. ■

Since homotopy relation is an equivalence relation it follows that if $\mathbf{n}_1$ and $\mathbf{n}_2$ are homotopic then so are $\mathbf{n}'_1$ and $\mathbf{n}'_2$ viewed as maps from S^2 to S^2. Thus we have a one to one correspondence between homotopy classes in Γ and those in $\Pi_2(S^2)$. This correspondence is also onto since any map from S^2 to S^2 (domain S^2 viewed as a sub-disc of Fig. 1) can be trivially extended to a map from $\mathbb{X}$ to S^2. Hence $\Gamma = \bigcup_{n\in\mathbb{Z}} \Gamma$, where Γ_n denotes the n^{th} homotopy class of Γ. The one-to-one and onto correspondence can also be used to make $\Pi_0(\Gamma) \equiv \{\Gamma_n / n \in \mathbb{Z}\}$ a group, isomorphic to $\Pi_2(S^2)$.

Thus we get $\Pi_0(\Gamma) \cong \mathbb{Z}$.

Incidentally this result can also be proved using the facts that a) $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ and b) S^2 is the 3-skeleton of $\mathbb{C}P^\infty$, where $K(\mathbb{Z}, 2)$ is an Eilenberg–MacLane space [6].

3. Computation of $\Pi_1(\Gamma_n)$

In Sect. 2 we saw that the configuration space Γ splits into topologically disjoint sectors Γ_n, $n\in\mathbb{Z}$. The quantization on Γ can be then analyzed by looking at each of the Γ_n's one at a time. Thus we need to compute $\Pi_1(\Gamma_n)$, $\forall n\in\mathbb{Z}$. It turns out that $\Pi_1(\Gamma_n)$ is isomorphic to $\Pi_1(\Gamma_0)$ $\forall n\in\mathbb{Z}$. This result is established in appendix. (Section D)

Thus it is sufficient to compute $\Pi_1(\Gamma_0)$. For this computation we need further machinery. Given a map $\mathbf{n}: M \to S^2$, where M is some manifold of dimension $k \geqq 2$, define a closed two form ω as

$$\omega = \alpha \varepsilon^{abc} n^a dn^b \wedge dn^c, \quad \alpha = 1/8\pi. \tag{3.1}$$

Let S be a two dimensional submanifold of M, without boundary, i.e. $\partial S = \phi$. Define,

$$I_S \equiv \int_S \omega(n). \tag{3.2}$$

As shown in the appendix, the integral is a homotopy invariant which gives the

Result. *If* $\mathbf{n}$ *is homotopic to* $\mathbf{n}'$ *then* $I_S[\mathbf{n}] = I_S[\mathbf{n}']$.

Now let us consider Γ_0 = space of maps which are connected to the constant maps. Consider a closed path in Γ_0. Such paths can be viewed as maps $\mathbf{n}: M \to S^2$, where $M = \mathbb{X} \times S^1$. M can be modelled as in Fig. 2, and the map $\mathbf{n}: M \to S^2$ can be specified as a map $\mathbf{n}(r, \theta, t)$ such that for each $t\in[0,1]$, $\mathbf{n}(r,\theta,t)$ is a map from $\mathbb{X}$ to S^2 and $\mathbf{n}(r,\theta,t)$ satisfies,

$$\text{(i)} \quad \mathbf{n}(r,\theta,0) = \mathbf{n}(r,\theta,1) = \hat{\mathbf{n}}.$$

$$\text{(ii)} \quad \mathbf{n}(1,\theta_i,t) = \hat{\mathbf{n}} \quad \forall i = 0,1,\ldots,4g-1; \quad \text{and} \quad t\in[0,1]. \tag{3.3}$$

These conditions imply that each surface S_α, $\alpha = 1,\ldots,2g$, is topologically a torus (Fig. 2). Furthermore the surfaces S_α and $S_{\alpha^{-1}}$ have identical values of $\mathbf{n}$ and $\omega(\mathbf{n})$. For each $\alpha = 1,\ldots,2g$ define,

$$I_\alpha[\mathbf{n}] = \int_{S_\alpha} \omega(\mathbf{n}). \tag{3.4}$$

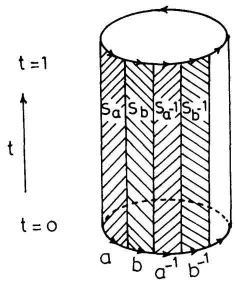

Fig. 2. A model for $M = \mathbf{X} \times S^1, \mathbf{X}$ being represented as in Fig. 1 while S^1 is labeled by t with $t = 0$ and 1 being identified

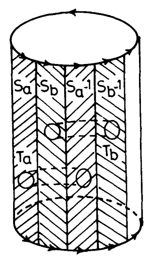

Fig. 3. Tubes T_a and T_b used in describing X_a, Y_a configuration

Since these are homotopy invariants we get that if $\mathbf{n}_1 \sim \mathbf{n}_2$ then

$$I_\alpha[\mathbf{n}_1] = I_\alpha[\mathbf{n}_2] \quad \forall \alpha = 1, \ldots, 2g. \tag{3.5}$$

For each $\alpha = 1, \ldots, 2g$ define configurations X_α such that X_α is $\hat{\mathbf{n}}$ outside a small tube beginning at S_α and ending at $S_{\alpha^{-1}}$ such that $I_\alpha[X_\alpha] = 1$ (see Fig. 3). Since X_α restricted to $S_{\alpha'}$ is $\hat{\mathbf{n}}$ $\forall \alpha' \neq \alpha$ we have,

$$I_{\alpha'}[X_\alpha] = \delta_{\alpha'\alpha}. \tag{3.6}$$

Now on the space of maps $\mathbf{n}(r, \theta, t)$ define a composition law as

$$\begin{aligned} \mathbf{n}_1 \circ \mathbf{n}_2(r, \theta, t) &= \mathbf{n}_1(r, \theta, 2t) \qquad 0 \leqq t \leqq 1/2 \\ &= \mathbf{n}_2(r, \theta, 2t - 1) \quad 1/2 \leqq t \leqq 1. \end{aligned} \tag{3.7}$$

Clearly $\mathbf{n}_1 \circ \mathbf{n}_2$ is well defined. An example is shown in Fig. 4. It is easy to see that

$$I_\alpha[n_1 \circ n_2] = \int_{S_\alpha} \omega[n_1 \circ n_2] = I_\alpha[n_1] + I_\alpha[n_2]. \tag{3.8}$$

Theorem 3.1. *Let* $\mathbf{n}$ *be a configuration such that* $I_\alpha[\mathbf{n}] = 0$ *for all* $\alpha = 1, \ldots, 2g$. *Then* $\mathbf{n}$ *is homotopic to a map which can be viewed as a map from* S^3 to S^2.

Proof. Let $\mathbf{n}_\alpha$ be the restriction on $\mathbf{n}$ to S_α. $\mathbf{n}_\alpha$'s are maps from torus T^2 to S^2 which may be viewed as a map from S^2 to S^2 by Theorem 2.1. Furthermore since $I_\alpha[\mathbf{n}] = 0$ $\forall \alpha$, each of $\mathbf{n}_\alpha$'s can be reduced to a constant map. Since all S_α have one point in common—namely the fixed point P_0 on $\mathbf{X}$, all these constants must be equal to

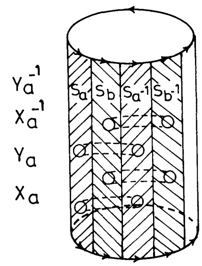

Fig. 4. Configuration $X_a \circ Y_a \circ X_a^{-1} \circ Y_a^{-1}$ as an illustration of the homotopy composition law

$\hat{\mathbf{n}}$. Thus the given map $\mathbf{n}$ can be homotopically reduced to a map which is constant on all surfaces, S_α's. But then it can be viewed as a map from S^3 to S^2. ■

Now let $\mathbf{n}: M \to S^2$ such that $I_\alpha[\mathbf{n}] = m_\alpha$. Define

$$\tilde{\mathbf{n}} = \mathbf{n} \circ (X_1^{-1})^{m_1} \circ (X_2^{-1})^{m_2} \circ (X_3^{-1})^{m_3} \circ \dots \circ (X_{2g}^{-1})^{m_{2g}}. \tag{3.10}$$

Clearly $I_\alpha[\tilde{\mathbf{n}}] = 0 \ \forall \alpha = 1, \dots, 2g$. Hence $\tilde{\mathbf{n}}$ is homotopic to a map that can be viewed as a map from S^3 to S^2. Homotopy classes of maps from S^3 to S^2 constitute the group $\Pi_3(S^2)$ which by Hopf theorem is isomorphic to $\mathbb{Z}$ [7]. The homotopy classes of $\tilde{\mathbf{n}}$ maps can then be characterized by a homotopy invariant, $H[\mathbf{n}]$, defined as: $H[\mathbf{n}] = I_m[\tilde{\mathbf{n}}]$, with $I_m[\mathbf{n}]$ defined in Eq. (A.11) of the appendix.

Theorem 3.2. *Two maps* $\mathbf{n}_i, \mathbf{n}_2$ *are homotopic to each other* iff

$$I_\alpha[\mathbf{n}_1] = I_\alpha[\mathbf{n}_2] \quad \forall \alpha = 1, \dots, 2g$$

and

$$H[\mathbf{n}_1] = H[\mathbf{n}_2].$$

Proof. Since each of the I_α's and H are homotopy invariants it follows that $\mathbf{n}_1 \sim \mathbf{n}_2 \Rightarrow$ all invariants are equal. Conversely let $I_\alpha[\mathbf{n}_i]$, and $H[\mathbf{n}_i]$ be equal for $i = 1, 2$ and $\alpha = 1, \dots, 2g$.

$$H[\mathbf{n}_1] = H[\mathbf{n}_2] \Rightarrow \tilde{\mathbf{n}}_1 \sim \tilde{\mathbf{n}}_2,$$

i.e.

$$\mathbf{n}_1 \circ (X_1^{-1})^{m_1} \circ\circ\circ (X_{2g}^{-1})^{m_{2g}} \sim \mathbf{n}_2 \circ (X_1^{-1})^{m_1} \circ\circ\circ (X_{2g}^{-1})^{m_{2g}}$$

$\Rightarrow \mathbf{n}_1 \sim \mathbf{n}_2$, since $X_i^{-1} \circ X_i$ is homotopic to a constant map. ■

Now let Z be a configuration which is $\hat{\mathbf{n}}$ outside a ball in M (so that $I_\alpha[Z] = 0 \, \forall \alpha$'s) and such that $H[Z] = 1$. If $H[\mathbf{n}] \equiv m_{\mathrm{H}}$ then $\tilde{\mathbf{n}} \sim (Z)^{m_{\mathrm{H}}}$. As a by-product we see that any $\mathbf{n}: M \to S^2$ can be expressed as:

$$\mathbf{n} \sim (Z)^{m_{\mathrm{H}}} \circ (X_{2g})^{m_{2g}} \circ\circ\circ (X_1)^{m_1}. \tag{3.11}$$

Hence $\Pi_1(\Gamma_0)$ is generated by $2g + 1$ generators $X_1, X_2, \dots, X_{2g}$, and Z. The group composition law is specified if we specify the relations among the generators.

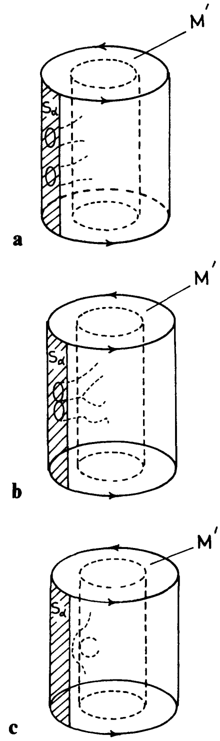

Fig. 5 a–c. **a** M' connected to S_a by two tubes. **b** The tubes brought closer to touch each other without disturbing the configuration inside M'. **c** The tubes fused together and detached from S_a to form a handle on M'

From Theorem 3.2 proved above we see that,

$$Z \circ X_\alpha \sim X_\alpha \circ Z \quad \text{and},$$
$$X_\alpha \circ X_\beta \circ X_\alpha^{-1} \circ X_\beta^{-1} \sim (Z)^{k(\alpha,\beta)} \quad \text{for some integers} \quad k(\alpha,\beta). \tag{3.12}$$

Computation of $\Pi_1(\Gamma_0)$ will be accomplished if we obtain $k(\alpha, \beta)$.

The computation of $k(\alpha, \beta)$ is facilitated by the following observation.

Let M' be a region in M as shown in Fig. (5a). Let M' be connected by two tubes to a surface S_α, say, as shown in the figure. Let **n** be a configuration which is the north pole $\hat{\mathbf{n}}$ outside M' and the connecting tubes and satisfying,

$$\int_{S_\alpha} \omega[\mathbf{n}] = 0. \tag{3.13}$$

The two tubes can be brought closer to touch each other in a continuous manner (Fig. 5a–c). Thus we get another homotopically equivalent configuration $\mathbf{n}'$ which equals $\hat{\mathbf{n}}$ outside M' and the tubes, and matches **n** inside M'. On S_a itself $\mathbf{n}'$ can be made equal to $\hat{\mathbf{n}}$ due to Eq. (3.13). Clearly the same can be done inside the tubes. But then the tubes can be fused and detached from S_a to form a handle on M'. In these continuous processes the configuration inside M' is unaffected. If there are several pairs of tubes connecting M' to several surfaces S_α's then the same procedure can be repeated independently for each pair. Thus the original configuration $\mathbf{n}$ is homotopically equivalent to a configuration $\mathbf{n}'$ which is confined to M' with several handles attached and matching with $\mathbf{n}$ inside M'.

It is convenient to label the configurations X_α's as,

$$\left.\begin{array}{l} X_a \equiv X_{2a-1} \\ Y_a \equiv X_{2a} \end{array}\right\} \quad a=1,\ldots,g. \tag{3.14}$$

Applying this reasoning to the configurations $X_a \circ Y_b \circ X_a^{-1} \circ Y_b^{-1}$ we see that $X_a \circ Y_b \circ X_a^{-1} \circ Y_b^{-1}$ is homotopic to a configuration confined to two closed tubes (coming from X_a, X_a^{-1} and Y_b, Y_b^{-1}) which are linked if $a=b$ unlinked if $a \neq b$. The closed tubes coming from $X_a \circ X_b \circ X_a^{-1} \circ X_b^{-1}$ and $Y_a \circ Y_b \circ Y_a^{-1} \circ Y_b^{-1}$ are always unlinked.

In the appendix, (Section C) using these observations, $k(\alpha, \beta)$ is computed. The result is:

$$X_a \circ X_b \circ X_a^{-1} \circ X_b^{-1} \sim (Z)^0, \quad Y_a \circ Y_b \circ Y_a^{-1} \circ Y_b^{-1} \sim (Z)^0$$

$$X_a \circ Y_b \circ X_a^{-1} \circ Y_b^{-1} \sim \begin{cases} (Z)^2, & \text{if} \quad a=b \quad \text{and} \\ (Z)^0, & \text{if} \quad a \neq b \end{cases} \tag{3.15}$$

Denoting the homotopy classes of X_α and Z by the same symbols, Eq. (3.15) translates into the specification of group multiplication relations for the generators of $\Pi_1(\Gamma_0)$.

Thus to summarize, the fundamental group $\Pi_1(\Gamma_0)$ is an infinite group generated by $2g+1$ generators Z and X_a, Y_a, $a=1,2,\ldots,g$, satisfying the group multiplication:

$$X_a \cdot X_b = X_b \cdot X_a, \quad Y_a \cdot Y_b = Y_b \cdot Y_a,$$

$$X_a \cdot Z = Z \cdot X_a, \quad Y_a \cdot Z = Z \cdot Y_a,$$

$$X_a \cdot Y_b = \begin{cases} Y_b \cdot X_a, & \text{if} \quad a \neq b, \\ Z^2 \cdot Y_b X_a, & \text{if} \quad a=b\,. \end{cases} \tag{3.16}$$

4. Representations of $\Pi_1(\Gamma_0)$

In the previous section we computed the group $\Pi_1(\Gamma_0)$ for generic values of the genus g. If we denote by G the infinite group generated by X, Y, Z satisfying

$$X \cdot Z = Z \cdot X, \quad Y \cdot Z = Z \cdot Y, \quad X \cdot Y = Z^2 \cdot Y \cdot X, \tag{4.1}$$

then

$$\tilde{G} \equiv G \times G \times \cdots \times G \quad (g \text{ factors}). \tag{4.2}$$

is homomorphic to $\Pi_1(\Gamma_0)$.

Consequently any finite dimensional, unitary, irreducible representation $\mathbf{R}$ of $\Pi_1(\Gamma_0)$ is expressible as

$$\mathbf{R}[\Pi_1(\Gamma_0)] \sim \mathbf{R}_1(G) \otimes \mathbf{R}_2(G) \otimes \cdots \otimes \mathbf{R}_g(G), \tag{4.3}$$

where $\mathbf{R}_i(G)$ are various finite dimensional, unitary, irreducible representations of G satisfying $\mathbf{R}_i(Z) = \mathbf{R}_j(Z)\,\forall i,j$. It follows that $\mathbf{R}[\Pi_1(\Gamma_0)]$ is equivalent to $\mathbf{R}'[\Pi_1(\Gamma_0)]$ iff the corresponding $\mathbf{R}_i(G)$ and $\mathbf{R}'_i(G)$ are equivalent for all $i = 1, \dots, g$. It suffices then to classify finite dimensional, unitary, irreducible representations of G up to unitary equivalence. This mathematical problem (and some of its generalizations) has been solved in detail in ref. [8]. We summarize the results below.

(i) Since Z commutes with X and Y and therefore with all elements of G, in any unitary, irreducible representation of G, $Z = e^{i\theta}I$, where I denotes a unit matrix of order n. For finite dimensional representations taking determinant of both sides of Eq. (4.1) we see that $\xi \equiv e^{2i\theta}$ must be an n^{th} root of unity for some $n(= \dim \mathbf{R}(G))$.

(ii) If $A, B, \xi I$ are unitary representatives of X, Y, Z^2 respectively then so are $A' = \mu A$, $B' = \nu B$, $Z^2 = \xi I$, provided μ, ν are phases, i.e. $\mu = e^{i\alpha}$, $\nu = e^{i\beta}$.

(iii) It is easy to see that for every n and ξ an n^{th} root of unity, A^n, B^n commute with the whole group G and thus are multiplis of the identity matrix—the multiples being phases. In view of (ii) above, we may choose A, B to satisfy $A^n = B^n = I$.

The unitary $k \times k$ matrices A, B, Z^2 satisfying

$$Z^2 = \xi \cdot I, \quad \xi^n = 1, \quad A^n = B^n = I, \tag{4.4}$$

generate a finite group of matrices containing n^3 elements.

Now applying the standard theory of representations for finite groups one gets the

Result. *Given $n \geqq 1$, for every $q = 1, 2, \dots, n$ there are $(q,n)^2$ representations of A, B, Z^2 matrices, each of dimension $n/(q,n)$; where (q,n) denotes the greatest common divisor of q and n.*

Taking $q = n$ we see that there are n^2, one dimensional representations and for every q such that q and n are relatively prime there is an n dimensional representation. In particular if n is a prime then there are $(n-1)$ representations of dimension n each, apart from the n^2 1-dimensional representations. When n is a prime it is easy to describe these representations explicitly as:

a) The n^2 one dimensional representations are given as:

$$A = (\xi_0)^k; \quad B = (\xi_o)^l; \quad Z^2 = 1;$$

where

$$\xi_0 = e^{2\pi i/n}, \quad \text{and} \quad k, l = 1, 2, \dots, n.$$

b) The n dimensional representations are given as:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}, \quad B = \begin{bmatrix} \xi & 0 & 0 & \cdots & 0 \\ 0 & \xi^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi^n \end{bmatrix}_{n \times n}$$

$$Z^2 = \xi \cdot I_{n \times n} \quad \text{with} \quad \xi = (\xi_0)^k, \quad k = 1, 2, \dots, m-1.$$

For the case when n is not a prime, see ref. [8]

(iv) Let us specialize to the 1-dimensional representations.

It is obvious that $X = e^{i\alpha}$, $Y = e^{i\beta}$, $Z^2 = 1 \equiv e^{2i\theta}$ gives a representation $\forall \alpha, \beta \in [0, 2\pi)$, $\theta = 0$ or π. Furthermore all these representations are inequivalent. The one dimensional representations described above when n is a prime integer are just subsets of these. Thus the set of all inequivalent, unitary representations of G are parameterized by $\alpha, \beta \in [0, 2\pi)$ and $\theta = 0, \pi$.

The 1-dimensional representations of $\Pi_1(\Gamma_0) \sim G \times G \times \cdots \times G$ are similarly given by $X_i = \exp(i\alpha_i)$, $Y_i = \exp(i\beta_i)$, $Z^2 = 1$. Thus the set of these representations is parametrized by $2g$ parameters $\alpha_i, \beta_i \in [0, 2\pi)$ $\forall i = 1, 2, \ldots, g$; and $\theta = 0, \pi$.

(v) For non-one-dimensional representations of G also we get a family of representations parameterized by two continuous parameters α and β and by a discrete parameter taking a finite number of values (the value being dependent on whether n is a prime or not). The parameters enter as shown in (ii) above. However, depending upon the dimension k of the representation, $e^{i\alpha}$, $e^{i\beta}$ give equivalent representations if they are k^{th} roots of unity. Similarly for non-one-dimensional representations of $\Pi_1(\Gamma_0)$ there will be g times as many parameters.

Since there are non-denumerably many inequivalent representations of the fundamental group, there is a corresponding non-countable infinity of inequivalent quantizations $\forall g \geqq 1$.

5. Summary and Discussion

The results derived above can be summarized as follows.

1. For a generic $2+1$ dimensional $O(3)$ invariant non-linear sigma model on a (genus g compact surface) $\times \mathbb{R}^1$, the classical configuration space has countably infinite number of homotopically disjoint sectors, labeled by elements of the $\Pi_0 \sim \mathbb{Z}$ group, for all $g \geqq 0$. The fundamental groups Π_1's of all these sectors are isomorphic to $\Pi_1(\Gamma_0)$.

2. The fundamental groups fall into two classes:

i) $g = 0$: $\Pi_1(\Gamma_0)$ is abelian. All its unitary, irreducible representations are one dimensional and these can be parametrized by a single θ. In ref. [1] this has been interpreted—albeit in a different framework—to suggest the existence of fractional statistics (or θ-statistics).

ii) $g > 0$: $\Pi_1(\Gamma_0)$ is non-abelian. It has 2g parameter family of inequivalent, unitary, one dimensional representations. However, in addition, it has infinitely many non-one-dimensional, unitary, irreducible, inequivalent representations allowing for the possibility of "exotic statistics" [9, 10].

In both cases there are infinitely many inequivalent quantizations.

These mathematical facts raise several questions. Two of these are singled out below.

a) The issue of statistics for a quantum system is a subtle one. From the definition of statistics adopted in ref. [10], we see that a quantum system may have several inequivalent quantizations and yet fewer types of statistics—some of which may be "exotic." For the system discussed in the present work one may try to follow two approaches. One may regarded the quantum field theory as a single system with several inequivalent quantizations and analyze inequivalent quantizations and statistics for a collection of such systems. This will amount to a generalization of the procedure discussed in ref. [10]. Alternatively one may attempt to interpret the quantum field theory itself as a collection of several subsystems and try to specify their possible statistics. This approach may be viewed as raising the question: under what conditions can a given quantum theory be interpreted as a collection of subsystems obeying some (perhaps exotic) statistics?

b) If in some specific dynamical model, one introduces a motion of statistics for the states—in some semi-classical approximation, say—a la Wilczek–Zee in [2], then it seems that only the value of the Z generator is relevant for determining the statistics. If true, then for all $g \geqq 1$ we may have only the usual two statistics, although infinitely may inequivalent quantizations, for quantizations involving one dimensional representations. In view of the interpretation of the genus in terms of a class of boundary conditions for a finite size lattice system, the genus zero case appears to be a "set of measure zero" within this class of boundary conditions. At least for systems with single component wave functions, the system may "largely" exhibit the Bose–Einstein or the Fermi–Dirac statistics. One way to test these heuristic arguments is to study a finite size lattice system with a probability distribution for the class of boundary conditions and analyze its continuum limit in detail.

Both of these questions will get stronger motivation if one can conceive of a physical system where the mathematical results will become directly relevant and testable.

Appendix

A. Homotopy Invariants

Let M be a 3 dimensional manifold and let $\mathbf{n}: M \to S^2$ be a smooth map, i.e. a set of 3 maps $n^i: M \to \mathbb{R}$ such that $\sum_i n^i \cdot n^i = 1$.

Define a two form

$$\omega = \alpha \varepsilon^{abc} n^a dn^b \wedge dn^c, \quad \text{where} \quad \alpha = 1/8\pi. \tag{A.1}$$

Since only two of the dn^a's are independent it follows that

$$d\omega = \alpha \varepsilon^{abc} dn^a \wedge dn^b \wedge dn^c = 0. \tag{A.2}$$

For a two dimensional surface S consider the integeral

$$I_S[\mathbf{n}] \equiv \int_S \omega(\mathbf{n}). \tag{A.3}$$

Under a smooth variation $\delta\mathbf{n}$ of $\mathbf{n}$ maintaining the constraint $\mathbf{n}\cdot\mathbf{n} = 1$, the integral varies, as,

$$\delta I_S[\mathbf{n}] = \int_S \omega(\delta\mathbf{n}) = \int_S \delta\omega(\mathbf{n}). \tag{A.4}$$

Now,

$$\delta\omega(\mathbf{n}) = 3\alpha\varepsilon^{abc}\delta n^a dn^b \wedge dn^c + 2\alpha\varepsilon^{abc} d(n^a \delta n^b dn^c). \tag{A.5}$$

Lemma.

$$\varepsilon^{abc}\delta n^a dn^b \wedge dn^c = 0. \tag{A.6}$$

Proof.

$$\varepsilon^{abc}\delta n^a dn^b \wedge dn^c = 2(\delta n^1 dn^2 \wedge dn^3 + \delta n^2 n^3 \wedge dn^1 + \delta n^3 dn^1 \wedge dn^2), \tag{A.7}$$

$n^a a^a = 1 \Rightarrow \delta n^i n^i = 0$ and $dn^i n^i = 0$, and not all n^i can be zero simultaneously. Let $n^3 \neq 0$ then,

$$\delta n^3 = -(\delta n^1 n^1 + \delta n^2 n^2)/n^3 \quad \text{and} \quad dn^3 = -(dn^1 n^1 + dn^2 n^2)/n^3 \tag{A.8}$$

$$\therefore \text{ left-hand side} = -(2/n^3)[\delta n^1 dn^2 \wedge (dn^1 n^1 + dn^2 n^2) + dn^2(dn^1 n^1 + dn^2 n^2) \wedge dn^1 + (\delta n^1 n^1 + \delta n^2 n^2) dn^1 \wedge dn^2] = 0. \quad \blacksquare \tag{A.9}$$

Therefore

$$\delta I_S[\mathbf{n}] = \int_S 2\alpha\varepsilon^{abc} d(n^a \delta n^b dn^c) = \int_{\partial S} 2\alpha\varepsilon^{abc} n^a \delta n^b dn^c. \tag{A.10}$$

Thus we get the

Result.

$$\delta I_S[\mathbf{n}] = 0$$

if (i) $\partial S = \phi$

or (ii) *if the variations of* $\delta\mathbf{n}$ *vanish on* ∂S.

These are precisely the conditions we need in Sect. 2 and hence $I_S[\mathbf{n}]$ are homotopy invariants.

Since $d\omega = 0$, locally there exists a 1-form A such that $\omega = dA$ and A is defined up to an exact 1-form.

Let Σ be a 3 dimensional submanifold of M, define

$$I_\Sigma[\mathbf{n}] \equiv \int_\Sigma A \wedge \omega. \tag{A.11}$$

Theorem A.1. *$I_\Sigma[\mathbf{n}]$ is a homotopy invariant provided either,*

$$\text{(i)} \quad \partial\Sigma = \phi \quad \textit{or} \quad \text{(ii)} \quad \omega|_{\partial\Sigma} = 0 \quad \text{and} \quad \delta\mathbf{n}|_{\partial\Sigma} = 0.$$

Proof. It is easy to see that $I_\Sigma(\mathbf{n})$ is well defined when either of the conditions (i) and (ii) is satisfied.

Consider,

$$\delta I_\Sigma(\mathbf{n}) \equiv \int \delta(A \wedge \omega) = \int \delta A \wedge \omega + A \wedge \delta\omega.$$

But

$$\delta\omega = d(2\alpha\varepsilon^{abc} n^a \delta n^b dn^c) \equiv d\gamma \quad \text{from Eqs. (A.5, 6)}$$

$$\therefore \quad d\delta A \equiv \delta\omega = d\gamma$$

$$\therefore \quad \delta A = \gamma + \beta, \quad \text{where } \beta \text{ is a closed 1-form.}$$

$$\therefore \quad \delta A = \gamma + df. \quad \text{(locally)}$$

Thus locally we get,

$$\delta A \wedge \omega + A \wedge \delta\omega = d(\gamma \wedge A + f\omega) + 2\gamma \wedge \omega.$$

From the explicit expressions for γ and ω it is easy to see that $\gamma \wedge \omega = 0$.

$$\therefore \quad \delta I_\Sigma(\mathbf{n}) = \int_\Sigma d(\gamma \wedge A + f\omega) = \int_{\partial\Sigma} (\gamma \wedge A + f\omega)$$

$$= 0 \quad \text{if} \quad \text{(i)}\ \partial\Sigma = \phi \quad \text{or} \quad \text{(ii)}\ \omega = 0 = \delta\mathbf{n} \quad \text{on } \partial\Sigma \text{ since } \delta\mathbf{n} = o \Leftrightarrow \gamma = 0.$$

Again these are precisely the situations encountered in Sect. 2 and thus $I_\Sigma[\mathbf{n}]$ is also a homotopy invariant. ■

Note. $A \wedge \omega = A \wedge dA$ looks like a Chern–Simon 3-form for an Abelian gauge theory. In general, integrals of Chern–Simon forms are *not* topological invariants since $\delta(A \wedge dA)$ is not exact. However in the present case $\gamma \wedge \omega = 0$ ensures that $\delta(A \wedge dA)$ is exact and hence the integrals are topological invariants.

In fact if Σ is topologically S^3 and if $\mathbf{n}|_\Sigma$ is a map from S^3 to S^2 (i.e. $\mathbf{n}|_{\partial\Sigma}$ = the north pole $\hat{\mathbf{n}}$ of S^2), then $I_\Sigma(\mathbf{n})$ can be normalized to integers and it essentially computes the Hopf invariant [7].

B. Generators of $\Pi_1(\Gamma_0)$

In Sect. 3 we introduced configurations X_α's and Z which generate $\Pi_1(\Gamma_0)$. These can be constructed explicitly as follows. Recall that X_α is a configuration defined so that,

(i) $X_\alpha: M \to S^2$,
(ii) X_i outside and on the boundary of the solid tube shown in Fig. 3 goes to the north pole, $\hat{\mathbf{n}}$.
(iii) $X_\alpha|_{S_\alpha \text{or} S_\alpha - 1}$ satisfies $I_{S_\alpha}(X_\alpha) = 1$.

X_α can be constructed as follows:
Let the tube be described by coordinates (ρ, φ, z) such that

$$0 \leqq \rho \leqq \Delta, \quad 0 \leqq \varphi \leqq 2\pi, \quad 0 \leqq z \leqq 1. \tag{B.1}$$

Clearly z = constant defines a disc, D, of radius Δ, Fig. 6.
Let $\mathbf{n} = \mathbf{n}(\rho, \varphi)$, i.e. $\mathbf{n}$ is independent of z, such that

$$\mathbf{n}(\rho = 1, \varphi) = \hat{\mathbf{n}} \quad \text{and} \quad \int_{\text{disc at } z} \omega(\mathbf{n}) = 1. \tag{B.2}$$

Thus X_α is constructed if we construct a map $\mathbf{n}$: disc $\to S^2$ such that $\int_{\text{disc}} \omega = 1$. The S^2 can be described by angles Θ and ϕ and therefore $\mathbf{n}$: disc $\to S^2$ can be

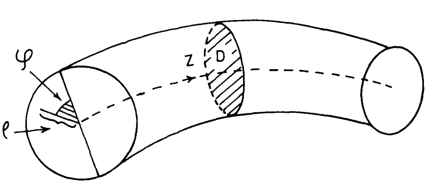

Fig. 6. (ρ, φ, z) coordinates for a tube. D is a disc at $z = z$.

described in terms of $\Theta = \Theta(\rho, \varphi)$, $\phi = \phi(\rho, \varphi)$, with

$$n^1 = \sin\Theta \cdot \cos\phi, \quad n^2 = \sin\Theta \cdot \sin\phi, \quad n^3 = \cos\Theta. \tag{B.3}$$

Thus gives

$$\omega = 2\alpha \cdot \sin\Theta \cdot d\Theta \wedge d\phi. \tag{B.4}$$

Define,

$$\Theta(\rho, \varphi) = \pi(1 - \rho/\Delta) \quad \text{and} \quad \phi(\rho, \varphi) = -\varphi \quad \text{for} \quad \rho \neq 0, \Delta. \tag{B.5}$$

Then

$$\omega = 2\alpha\pi \cdot \sin(\pi\rho) \cdot d\rho \wedge d\varphi \quad \text{and} \quad \int_{\text{disc}} \omega = 8\pi\alpha = 1 \quad \text{since} \quad \alpha = 1/8\pi. \tag{B.6}$$

This establishes explicitly the existence of X_α configurations.

The Z configuration was defined to be non-constant inside a ball contained in M such that $H[Z] = 1$. Parameterizing the ball as S^3, Z can simply be taken to be the Hopf map from S^3 to S^2 [11].

C. Basic Algebraic Relations for $\Pi_1(\Gamma_0)$

In this section we calculate the basic algebraic relations among the generators.

Recall that in Sect. 3 we established that, $X_\alpha \cdot X_\beta \cdot X_\alpha^{-1} \cdot X_\beta^{-1} \sim Z^{k(\alpha,\beta)}$ and that the left-hand side configuration was homotopic to a configuration involving either two linked or unlinked tubes such that $\mathbf{n} = \hat{\mathbf{n}}$ outside and on the tubes and the integrals of ω over discs within each tube is ± 1.

Let T_1 and T_2 be two closed tubes either linked or unlinked but non-intersecting, as shown in Fig. 7a, b.

Let ω be the restriction of ω to the tubes T_i, $i = 1, 2$. Since ω is zero outside the tubes we have $\omega = \omega_1 + \omega_2$. Let A_i, $i = 1, 2$ be two 1-forms so that locally $\omega_i = dA_i$, for $i = 1, 2$.

Define $A = A_1 + A_2$. Clearly $dA = \omega_1 + \omega_2 = \omega$. Now,

$$\int A \wedge \omega = \int (A_1 + A_2) \wedge \omega_1 + \int (A_1 + A_2) \wedge \omega_2. \tag{C.1}$$

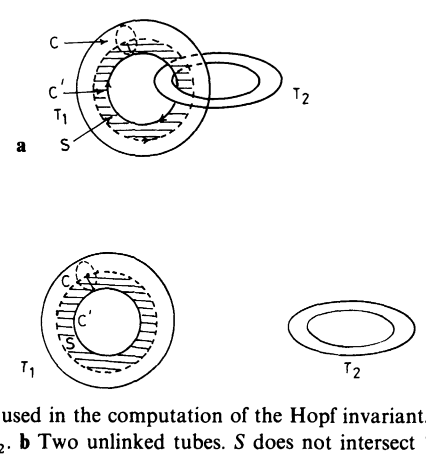

Fig. 7a. Two linked tubes used in the computation of the Hopf invariant. The surface S, bounded by C, necessarily intersects T_2. **b** Two unlinked tubes. S does not intersect T_2 giving the integral of ω_2 over S to be zero

Lemma.

$$\int_{T_1} A_i \wedge \omega_i = 0 \quad \text{for} \quad i = 1, 2. \tag{C.2}$$

Proof. Inside a tube we can always choose the (ρ, φ, z) co-ordinates and write,

$$\omega = \omega_{\rho\varphi}(\rho, \varphi)\cdot d\rho \wedge d\varphi \quad \text{and} \quad A = \hat{A}(\rho, \varphi) + \beta(z), \tag{C.3}$$

where,

$$\hat{A}(\rho, \varphi) = \hat{A}_\rho(\rho, \varphi)d\rho + \hat{A}_\varphi(\rho, \varphi)d\varphi, \quad d\hat{A} = \omega; \quad \text{and} \quad \beta(z) = \beta_z(z)dz, \quad d\beta = 0. \tag{C.4}$$

Clearly

$$A \wedge \omega = \hat{A} \wedge \omega + \beta \wedge \omega = \beta \wedge \omega. \tag{C.5}$$

Thus,

$$\int_T A \wedge \omega = \int \beta_z(z)\cdot\omega_{\rho\varphi}(\rho, \varphi)\cdot dz \wedge d\rho \wedge d\varphi$$

$$= \int_0^1 \beta_z(z)dz \quad \text{since} \int \omega_{\rho\varphi}(\rho, \varphi)d\rho \wedge d\varphi = 1. \tag{C.6}$$

To determine $\int \beta_z dz$ consider a surface S as shown in Fig. 7a. S is bounded by the curves C and C' given by, $C: \rho = \hat{\rho}, \varphi = \hat{\varphi}$ and $C': \rho = \Delta, \varphi = \hat{\varphi}$ (say). Since $\rho = \Delta$ is identified to a point, C' is actually a trivial curve and therefore S has only one boundary component namely the curve C.

Now $\omega|_S = 0$ since $d\rho \wedge d\varphi|_S = 0$. Therefore,

$$\int_S \omega = 0 = \int_S dA = \oint_C A = \oint_C \hat{A} + \oint_C \beta. \tag{C.7}$$

But $\hat{A}|_C = 0$ since $d\rho|_C = d\varphi|_C = 0$ and $\oint_C \beta = \int_0^1 \beta_z(z)dz$.

Therefore

$$\int_T A \wedge \omega = 0. \quad \blacksquare$$

It follows then that

$$\int_{T_1+T_2} A \wedge \omega = \int_{T_1} A_2 \wedge \omega_1 + \int_{T_2} A_1 \wedge \omega_2. \tag{C.8}$$

Consider

$$\int_{T_1} A_2 \wedge \omega_1 = \int_{T_1} A_2(\rho, \varphi, z) \wedge [\omega_{1\rho\varphi}(\rho, \varphi)\cdot d\rho \wedge d\varphi]$$

$$= \int_{T_1} A_{2z}(\rho, \varphi, z)\cdot\omega_{1\rho\varphi}(\rho, \varphi)d\rho \wedge d\varphi \wedge dz$$

$$= \int_{\text{disc}} \omega_{1\rho\varphi}(\rho, \varphi)d\rho \wedge d\varphi \cdot \int_0^1 A_{2z}(\rho, \varphi, z)dz. \tag{C.9}$$

Note that the integral of $A_z dz$ is over a closed curve defined by $\rho = \text{constant}$, $\varphi = \text{constant}$. If S' is a surface such that $\partial S'$ is the closed curve, (e.g. the surface S

itself) then the line integral of $A_{2z}dz$ is the same as the surface integral of $d(A_{2z}dz)$ over S'. But this is the same as the integral of ω_2 on S'.

If T_1 and T_2 are unlinked then S' can be chosen so as not to intersect with T_2 and then since ω_2 is zero outside T_2, $\int_{S'} \omega_2 = 0$. However if T_1 and T_2 are linked then S' necessarily intersects T_2 giving $\int_{S'} \omega_2 = \pm 1$. Therefore,

$$\int_{T_1} A_2 \wedge \omega_1 = \int_{\text{disc 1}} \omega_1 \cdot \int_{\text{disc 2}} \omega_2 = \pm 1 \quad \text{or} \quad 0. \tag{C.10}$$

Similarly $\int_{T_2} A_1 \wedge \omega_2 =$ the same expression as above $= \pm 1$ or 0.

Thus

$$\int_{T_1+T_2} A \wedge \omega = \begin{cases} \pm 2 & \text{if the tubes are linked} \\ 0 & \text{otherwise.} \end{cases} \tag{C.11}$$

Thus

$$k(\alpha, \beta) = \begin{cases} \pm 2 & \text{if } X_\alpha \text{ and } X_\beta \text{ tubes are linked} \\ 0 & \text{otherwise.} \end{cases} \tag{C.12}$$

For the linked tubes corresponding to the X_a, Y_a generators the integrals of ω on the respective discs is $+1$ giving us the algebraic relations in Sect. 3.

D. Computation of $\Pi_1(\Gamma_N)$, $N \in \mathbb{Z}$

The method of computation of $\Pi_1(\Gamma_N)$ follows closely the method of computation of $\Pi_1(\Gamma_0)$ with some modifications.

Consider Γ_N, the space of maps from $\mathbb{X}$ to S^2 for which $I_{\mathbb{X}}[\mathbf{n}] \equiv \int_{\mathbb{X}} \omega[\mathbf{n}] = N$. Closed paths in Γ_N can be viewed as maps $\mathbf{n}: M \to S^2$, where $M = \mathbb{X} \times S^1$. M can be modeled as in Fig. 2, and the map $\mathbf{n}: M \to S^2$ can be specified as a map $\mathbf{n}(r, \theta, t)$ such that for each $t \in [0, 1]$, $\mathbf{n}(r, \theta, t)$ is a map from $\mathbb{X}$ to S^2 belonging to Γ_N and it satisfies,

$$\text{(i)} \quad \mathbf{n}(r, \theta, 0) = \mathbf{n}(r, \theta, 1) = \mathbf{n}^*,$$

$$\text{(ii)} \quad \mathbf{n}(1, \theta_i, t) = \hat{\mathbf{n}} \quad \forall i = 0, 1, \ldots, 4g-1; \quad \text{and} \quad t \in [0, 1]. \tag{D.1}$$

Here $\mathbf{n}^*$ is the base point for the closed loops in Γ_N. $\mathbf{n}^*$ is chosen once and for all throughout the computation of the fundamental group. We choose $\mathbf{n}^*$ to be non-trivial inside a disc $r < r_0 (< 1)$ and require that $I_{\mathbb{X}}[\mathbf{n}^*] = N$. Note that all closed paths in Γ_N, based at $\mathbf{n}^*$, satisfy the condition that the integral of ω over a surface $t = \text{constant}$ is N.

These conditions imply that each surface S_α, $\alpha = 1, \ldots, 2g$, is topologically a torus (Fig. 2). Furthermore the surfaces S_α and $S_{\alpha^{-1}}$ have identical values of $\mathbf{n}$ and $\omega(\mathbf{n})$. For each $\alpha = 1, \ldots, 2g$ define,

$$I_\alpha[\mathbf{n}] = \int_{S_\alpha} \omega(\mathbf{n}). \tag{D.2}$$

Since these are homotopy invariants we get that if $\mathbf{n}_1 \sim \mathbf{n}_2$ then

$$I_\alpha[\mathbf{n}_1] = I_\alpha[\mathbf{n}_2] \; \forall \alpha = 1, \ldots, 2g. \tag{D.3}$$

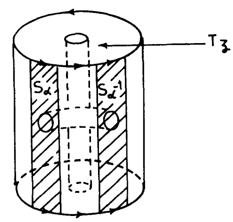

Fig. 8. Tubes T_α and $T_{\mathfrak{z}}$ used in describing the configuration $\underline{X}_\alpha$

Let $\mathfrak{z}: M \to S^2$ be a map such that

$$\mathfrak{z}(r, \theta, t) = \mathbf{n}^*(r, \theta) \quad \forall t \in [0, 1]. \tag{D.4}$$

Since $\mathbf{n}^*(r, \theta)$ is non-trivial only in a disc $r < r_0$, $\mathfrak{z}(r, \theta, t)$ is non-trivial only in a tube $T_{\mathfrak{z}}$ connecting $t = 0$ and $t = 1$ surfaces.

For each $\alpha = 1, \ldots, 2g$ let T_a be a small tube connecting the surfaces S_α and $S_{\alpha^{-1}}$ and not intersecting the tube $T_{\mathfrak{z}}$. Define $\underline{X}_\alpha$ to be a configuration which is (see Fig. 8):

(a) $\hat{\mathbf{n}}$ outside $T_\alpha \cup T_{\mathfrak{z}}$,

(b) matches with $\mathfrak{z}(r, \theta, t)$ inside $T_{\mathfrak{z}}$ and (D.5)

(c) satisfies $I_{\alpha'}[\underline{X}_\alpha] = \delta_{\alpha,\alpha'}$.

Also it follows that $I_{t=\text{constant}}[\underline{X}_\alpha] = N \ \forall t \in [0, 1]$.

Now on the space of maps $\mathbf{n}(r, \theta, t)$ define a composition law as

$$\begin{aligned} \mathbf{n}_1 \circ \mathbf{n}_2(r, \theta, t) &= \mathbf{n}_1(r, \theta, 2t) \qquad 0 \leqq t \leqq 1/2 \\ &= \mathbf{n}(r, \theta, 2t - 1) \quad 1/2 \leqq t \leqq 1. \end{aligned} \tag{D.6}$$

Clearly $\mathbf{n}_1 \circ \mathbf{n}_2$ is well defined.

It is easy to see that

$$I_\alpha[n_1 \circ n_2] = \int_{S_\alpha} \omega[n_1 \circ n_2] = I_\alpha[n_1] + I_\alpha[n_2]. \tag{D.7}$$

Theorem D.1. *Let* $\mathbf{n}$ *be a configuration such that* $I_\alpha[\mathbf{n}] = 0$ *for all* $\alpha = 1, \ldots, 2g$. *Then* $\mathbf{n}$ *is homotopic to a map which can be viewed as a map from* $S^2 \times S^1$ *to* S^2.

Proof. Let $\mathbf{n}_\alpha$ be the restriction of $\mathbf{n}$ to S_α. $\mathbf{n}_\alpha$'s are maps from torus T^2 to S^2 which may be viewed as a map from S^2 to S^2 by Theorem 2.1. Furthermore since $I_\alpha[\mathbf{n}] = 0 \forall \alpha$, each of $\mathbf{n}_\alpha$'s can be reduced to a constant map. Since all S_α have one point in common—namely the fixed point P_0 on $\mathbf{X}$, all these constants must be equal to $\hat{\mathbf{n}}$. Thus the given map $\mathbf{n}$ can be homotopically reduced to a map which is constant on all surfaces, S_α's. But then it can be viewed as a map from $S^2 \times S^1$ to S^2. Note that unlike the case of Γ_0 sector, here, on the $t = 0$ and 1 surfaces $\mathbf{n}(r, \theta \cdot t)$ is non-trivial and hence the map can not be viewed as a map from S^3 to S^2. ∎

Now let $\mathbf{n}: M \to S^2$ such that $I_\alpha[\mathbf{n}] = m_\alpha$. Define

$$\tilde{\mathbf{n}} = \mathbf{n} \circ (\underline{X}_1^{-1})^{m_1} \circ (\underline{X}_2^{-1})^{m_2} \circ (\underline{X}_3^{-1})^{m_3} \circ \cdots \circ (\underline{X}_{2g}^{-1})^{m_{2g}}. \tag{D.8}$$

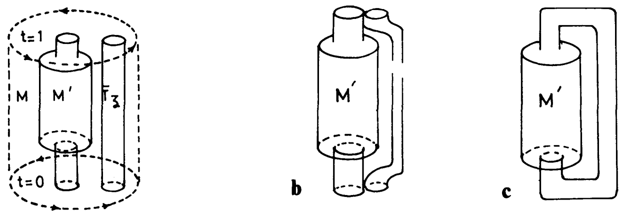

Fig. 9a. M' connected to M by two small tubes. **b** The tube $\bar{T}_3$ brought closer to touch the other tubes without disturbing the configuration inside M'. **c** The tubes fused together and detached from the $t=0$ and $t=1$ surfaces of M to form a handle on M'

Clearly $I_\alpha[\tilde{\mathbf{n}}]=0\,\forall\alpha=1,\ldots,2g$. Hence $\tilde{\mathbf{n}}$ is homotopic to a map that can be viewed as a map from $S^2\times S^1$ to S^2. To exploit the Hopf theorem we have to go through one more step. For each $\mathbf{n}: M\to S^2$ (i.e. a path in Γ_N based at $\mathbf{n}^*$) we associate a map $\tilde{\tilde{\mathbf{n}}}: M\to S^2$ which is concentrated in a ball in M and hence can be viewed as a map from S^3 to S^2. Note that $\tilde{\tilde{\mathbf{n}}}$ is a path in Γ_0. Now the Hopf theorem can be applied to the $\tilde{\tilde{\mathbf{n}}}$ map. The map $\tilde{\tilde{\mathbf{n}}}$ is constructed as follows.

For a given map $\mathbf{n}$, construct $\tilde{\mathbf{n}}$ as in Eq. (D.8). $\tilde{\mathbf{n}}$ can be viewed as a map from $S^2\times S^1$ to S^2. It is concentrated in M' and two small tubes connecting M' to the $t=0$ and $t=1$ surfaces of M (see Fig. 9a).

Let $\bar{T}_3$ be a tube in M which does not intersect M' and joins the surfaces $t=0$ and $t=1$, (Fig. 9a). Let $\bar{\mathbf{n}}^*$ be a map from $\mathbf{X}$ to S^2 such that $I_{\mathbf{X}}[\bar{\mathbf{n}}^*]=-N$ and it is concentrated in the disc obtained by intersecting of $\bar{T}_3$ and the $t=0$ surface. The fact that $\Pi_0(\Gamma)$ is a group (isomorphic to $\mathbb{Z}$) guarantees the existence of such a map. Define $\bar{\mathfrak{z}}(r,\theta\cdot t)=\bar{\mathbf{n}}^*(r,\theta)\,\forall t\in[0,1]$, and construct a map $\tilde{\mathbf{n}}': M\to S^2$ satisfying the following properties:

$$\begin{aligned}&\text{a) } \tilde{\mathbf{n}}'=\tilde{\mathbf{n}} \quad \text{inside } M' \text{ and the tubes connecting } M' \\ &\qquad\qquad\ \ \text{to } t=0 \text{ and } t=1 \text{ surfaces,} \\ &\text{b) } \tilde{\mathbf{n}}'=\bar{\mathfrak{z}} \quad \text{inside } \bar{T}_3 \quad \text{and} \quad \text{c) } \tilde{\mathbf{n}}'=\hat{\mathbf{n}} \quad \text{elsewhere.}\end{aligned} \tag{D.9}$$

Clearly,

$$I_\alpha[\tilde{\mathbf{n}}']=0\,\forall\alpha=1,\ldots,2g \quad \text{and} \quad I_{t=0}[\tilde{\mathbf{n}}']=I_{t=1}[\tilde{\mathbf{n}}']=0. \tag{D.10}$$

Now following the sequence of steps shown in Figs. 9a, b, c, we obtain a map which is concentrated in M' with a handle attached. Clearly this can be viewed as map from S^3. to S^2 and this is our desired map $\tilde{\tilde{\mathbf{n}}}$.

Now by Hopf theorem the homotopy classes of $\tilde{\tilde{\mathbf{n}}}$ maps can be labeled by integers and can be characterized by a homotopy invariant, $I_{\mathrm{M}}[\tilde{\tilde{\mathbf{n}}}]$, defined in Eq. (A.11). For uniformity in notation we define $H[\mathbf{n}]=I_{\mathrm{M}}[\tilde{\tilde{\mathbf{n}}}]$.

Theorem D.2. *Two maps* $\mathbf{n}_1,\mathbf{n}_2$ *are homotopic to each other* iff

$$I_\alpha[\mathbf{n}_1]=I_\alpha(\mathbf{n}_2] \quad \forall\alpha=1,\ldots,3g \quad \text{and} \quad H[\mathbf{n}_1]=H[\mathbf{n}_2].$$

Proof. Since each of the I_α's and H are homotopy invariants it follows that $\mathbf{n}_1\sim\mathbf{n}_2\Rightarrow$ all invariants are equal. Conversely let $I_\alpha[\mathbf{n}_i]$, and $H[\mathbf{n}_i]$ be equal for

$i = 1, 2$ and $\alpha = 1, \ldots, 2g$.

$$H[\mathbf{n}_1] = H[\mathbf{n}_2] \Rightarrow \tilde{\tilde{\mathbf{n}}}_1 \sim \tilde{\tilde{\mathbf{n}}}_2,$$

i.e.

$$\tilde{\mathbf{n}}_1 \sim \tilde{\mathbf{n}}_2$$

i.e.

$$\mathbf{n}_1 \circ (\underline{X}_1^{-1})^{m_1} \circ\circ\circ (\underline{X}_{2g}^{-1})^{m_{2g}} \sim \mathbf{n}_2 \circ (\underline{X}_1^{-1})^{m_1} \circ\circ\circ (\underline{X}_{2g}^{-1})^{m_{2g}}$$

$\Rightarrow \mathbf{n}_1 \sim \mathbf{n}_2$, since $\underline{X}_i^{-1} \circ \underline{X}_i$ is homotopic to a constant map. ∎

Choose a ball B is M such that it does not intersect the tube T_3. Now let $\underline{Z}$ be a configuration which is $\hat{\mathbf{n}}$ outside the ball and the tube T_3 (so that $I_\alpha[\underline{Z}] = 0\ \forall \alpha$'s) and such that $H[\underline{Z}] = 1$. It is easy to see that if $H[\mathbf{n}] \equiv m_{\mathrm{H}}$ then $\tilde{\mathbf{n}} \sim (\underline{Z})^{m_{\mathrm{H}}}$. As a by-product we see that any $\mathbf{n}: M \to S^2$ (in the Γ_N sector) can be expressed as:

$$\mathbf{n} \sim (\underline{Z})^{m_{\mathrm{H}}} \circ (\underline{X}_{2g})^{m_{2g}} \circ\circ\circ (\underline{X}_1)^{m_1}. \tag{D.11}$$

Hence $\Pi_1(\Gamma_N)$ is generated by $2g+1$ generators $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{2g}$, and $\underline{Z}$. The group composition law is specified if we specify the relations among the generators. From the theorem proved above we see that,

$$\begin{aligned} &\underline{Z} \circ \underline{X}_\alpha \sim \underline{X}_\alpha \circ \underline{Z} \quad \text{and}, \\ &\underline{X}_\alpha \circ \underline{X}_\beta \circ \underline{X}_\alpha^{-1} \circ \underline{X}_\beta^{-1} \sim (\underline{Z})^{\underline{k}(\alpha,\beta)} \quad \text{for some integers } \underline{k}(\alpha,\beta). \end{aligned} \tag{D.12}$$

Computation of $\Pi_1(\Gamma_N)$ will be accomplished if we obtain, $\underline{k}(\alpha, \beta)$. Note that the generators $\underline{X}_\alpha$'s and $\underline{Z}$ differ from the X_α's and Z in the Γ_0 sector only by the presence of the additional tube T_3 in their definitions. The observation made in Sect. 3, which facilitated the computation of $k(\alpha, \beta)$ for the Γ_0 sector applies in the present case as well. Repeating the arguments from Sect. 3 it is straightforward to see that $\underline{k}(\alpha, \beta)$'s are the same as the corresponding $k(\alpha, \beta)$'s. This establishes that the fundamental groups of Γ_0 and Γ_N sectors are isomorphic for all $N \in \mathbb{Z}$.

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