Resonances, Metastable States and Exponential Decay Laws in Perturbation Theory

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Dedicated to Res Jost and Arthur Wightman

Abstract. Resonances which appear as perturbed bound states are discussed in the framework of Balslev-Combes theory. The corresponding metastable states are constructed using the formal perturbation expansion to order $N-1$ for the (nonexistent) perturbed bound states. They are shown to have exponential decay in time governed by the complex resonance energies, up to a background of order $2N$ in the perturbation parameter. The results apply in lowest order $N=1$ to the perturbation of bound states embedded in the continuum and in arbitrary order to cases like the Stark effect.
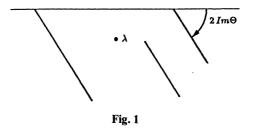
1. Introduction

According to standard textbook wisdom, resonances of quantum systems correspond to metastable states which show exponential decay. The difficulties in making this statement precise are notorious and well explained in [13]. While Gamov's one-particle α-decay model still receives attention (see e.g. [16, 17]), relatively little is known in more general situations. In many cases the Balslev-Combes theory of dilation analytic systems [1, 11] or one of its variants [3, 7, 14, 15] allows an elegant definition of the complex resonance energies, but there is yet no general description of the corresponding metastable states and their time evolution. Notable progress has recently been made by Orth [10], who developed a theory of resonances for N-body Schrödinger operators based on the Mourre estimate rather than dilation analyticity. The scope of our present contribution is more limited. We will use dilation analyticity and perturbation theory in the spirit of Simon [12] to discuss resonances which appear as perturbed bound states. By the same approach we can cover cases like the N-body Stark effect, where some results on exponential decay have previously been given by Herbst [4].

In the Balslev-Combes theory, the resonance energies are the complex eigenvalues of a "dilated" Hamiltonian $H(\Theta)$. For a typical N-body system, $H(\Theta)$ has a spectrum of the form shown in Fig. 1. To understand exponential decay in this setting we must answer the following questions. What is – in good approximation –

a metastable state ψ associated with a given resonance λ? Is there a time interval where

$$(\psi, e^{-iHt}\psi) \approx e^{-i\lambda t} \tag{1}$$

up to a relatively small background term?

Fig. 1

Of course we would like to answer these questions simply in terms of the spectrum shown in Fig. 1, assuming that Im λ is small compared to the separation of λ from the rest of the spectrum. This seems to be impossible – mainly because there is no general effective estimate for the resolvent $(z - H(\Theta))^{-1}$ in terms of the location of z relative to the spectrum. For this reason we turn to the simpler case of a perturbed Hamiltonian

$$H_\kappa = H_0 + \kappa V \ ,$$

dilated into

$$H_\kappa(\Theta) = H_0(\Theta) + \kappa V(\Theta) \ , \tag{2}$$

and we only discuss resonances λ_κ which result from the perturbation of a (real) eigenvalue λ_0 of H_0.
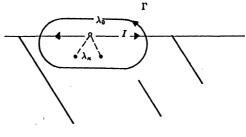

Fig. 2

The unperturbed bound states ψ_0 corresponding to the eigenvalue λ_0 are then obvious candidates for metastable states, and we can measure the background term in powers of the small perturbation parameter κ. As a typical result we will prove

$$(\psi_0, e^{-iH_\kappa t}\psi_0) = e^{-i\lambda_\kappa t} + O(\kappa^2) \tag{3}$$

in the nondegenerate case, uniformly in $0 \leq t < \infty$. Our next question is whether this lowest order approximation for metastable states can be systematically improved. More precisely: is there an expansion

$$\psi_\kappa^N = \psi_0 + \kappa\psi_1 + \ldots + \kappa^{N-1}\psi_{N-1} \tag{4}$$

for metastable states which reduces the background term to higher order in κ? A natural candidate for (4) is the *formal* Rayleigh-Schrödinger (RS-) perturbation expansion for the (nonexistent) perturbed bound states. In fact, if the terms in this expansion are well-defined, we can improve (3) to

$$(\psi_\kappa^N, e^{-itH_\kappa}\psi_\kappa^N) = e^{-i\lambda_\kappa t} + O(\kappa^{2N}) \ . \tag{5}$$

In this way we obtain a physical interpretation of the RS-expansion for the "perturbed bound states" even in cases where no such states exist. We must emphasize, however, that this simple scheme works only in cases like the Stark effect, where λ_0 is a *discrete* eigenvalue of H_0. For embedded bound states the higher order terms in the RS-expansion (4) are not defined in Hilbert space, so that additional approximations (like the truncation of Gamov functions) must be used to construct metastable states. We will not investigate this possibility here.

2. Preliminaries

In this section we define the framework used throughout this paper.

(a) *Balslev-Combes Theory.* H_κ is a family of selfadjoint operators defined for small $\kappa \geq 0$. $U(\Theta)$ is a strongly continuous one parameter unitary group such that for fixed κ

$$H_\kappa(\Theta) = U(\Theta) H_\kappa U(\Theta)^{-1}$$

extends from real Θ to an analytic family in a strip $|\text{Im } \Theta| < \beta$. (In H_κ and elsewhere we simply drop the variable Θ to indicate that Θ is set equal to zero.) The spectrum of $H_\kappa(\Theta)$ depends only on $\text{Im } \Theta$ and is assumed to lie in the closed lower halfplane for $\text{Im } \Theta > 0$. The relation

$$H_\kappa(\Theta)^* = H_\kappa(\bar{\Theta})$$

holds for real Θ and extends by analyticity to $|\text{Im } \Theta| < \beta$. $U(\Theta)$ itself is defined for complex Θ by the spectral representation

$$U(\Theta) = \int e^{-is\Theta} dE(s)$$

on the natural domain, satisfying $U(\Theta)^* = U(\bar{\Theta})^{-1}$. [We recall that $U(\Theta)$ must be unbounded for $\text{Im } \Theta \neq 0$ if the spectrum of $H_0(\Theta)$ is to depend nontrivially on Θ.] λ_0 is an isolated or embedded eigenvalue of H_0 with eigenprojection P_0, $\dim P_0 = m_0 < \infty$. We assume that λ_0 is separated from the essential spectrum of $H_0(\Theta)$ for $\text{Im } \Theta \neq 0$. Then λ_0 is a discrete eigenvalue of $H_0(\Theta)$. Its eigenprojection $P_0(\Theta)$ is analytic in the *full* strip $|\text{Im } \Theta| < \beta$. In particular,

$$M_0(\Theta) = \text{ran } P_0(\Theta)$$

has constant dimension m_0. The relation

$$P_0(\Theta) = U(\Theta) P_0 U(\Theta)^{-1}$$

holds on the dense domain of $U(\Theta)^{-1}$. Since P_0 has finite rank it follows that M_0 is in the domain of $U(\Theta)$ so that $U(\Theta)$ and $U(\Theta)^{-1}$ act as bounded operators from

M_0 onto $M_0(\Theta)$ and vice-versa. The norms of these maps will in fact enter in our decay estimates. So far we have summarized the Balslev-Combes framework in abstract form.

(b) *Stability*. For fixed Θ with $\text{Im}\,\Theta \neq 0$ we now consider the perturbation of the discrete eigenvalue λ_0 of $H_0(\Theta)$ by the family $H_\kappa(\Theta)$. First we require stability in the following sense [8]:

(i) There is a punctured neighbourhood $W(\Theta)$ of λ_0 (i.e. a complex neighbourhood with the point λ_0 removed) such that the resolvent $R_\kappa(\Theta, z) = (z - H_\kappa(\Theta))^{-1}$ exists and is uniformly bounded for each fixed $z \in W(\Theta)$ and $0 \leq \kappa < \kappa_0(z)$.

(ii) The perturbed spectral projection

$$P_\kappa(\Theta) = (2\pi i)^{-1} \oint_\Gamma dz\, R_\kappa(\Theta, z) \tag{6}$$

satisfies

$$\lim_{\kappa \to 0} \| P_\kappa(\Theta) - P_0(\Theta) \| = 0 \ . \tag{7}$$

Here Γ is an arbitrary loop in $W(\Theta)$ around λ_0. As a consequence of (i), $R_\kappa(\Theta, z)$ is uniformly bounded for $z \in \Gamma$ and $0 \leq \kappa < \kappa_0(\Gamma)$. Equation (7) implies that $\dim P_\kappa(\Theta) = m_0$ for small κ. Therefore λ_0 is the limit as $\kappa \to 0$ of a group of perturbed eigenvalues λ_κ having total algebraic multiplicity m_0. These are the eigenvalues of the reduced operator

$$\tilde{H}_\kappa(\Theta) = P_\kappa(\Theta) H_\kappa(\Theta) P_\kappa(\Theta) \tag{8}$$

acting on $M_\kappa(\Theta) = \text{ran}\, P_\kappa(\Theta)$. In fact, the spectrum of $\tilde{H}_\kappa(\Theta)$ depends only on the *sign* of $\text{Im}\,\Theta$, due to the relations

$$\begin{aligned} P_\kappa(\Theta_2) &= U(\Theta_2 - \Theta_1) P_\kappa(\Theta_1) U(\Theta_2 - \Theta_1)^{-1} \ , \\ \tilde{H}_\kappa(\Theta_2) &= U(\Theta_2 - \Theta_1) \tilde{H}_\kappa(\Theta_1) U(\Theta_2 - \Theta_1)^{-1} \ , \end{aligned} \tag{9}$$

which hold for small κ if $\text{Im}\,\Theta_1$ and $\text{Im}\,\Theta_2$ have the same sign. By convention we denote with λ_κ the eigenvalues of $\tilde{H}_\kappa(\Theta)$ for $\text{Im}\,\Theta > 0$. These are the resonances corresponding to the unperturbed eigenvalue λ_0. Our assumption on the spectrum of $H_\kappa(\Theta)$ implies that $\text{Im}\,\lambda_\kappa \leq 0$.

(c) RS-*Expansion*. For the purpose of using RS-expansions we restrict $H_\kappa(\Theta)$ to the form (2). More precisely we assume that $V(\Theta)$ is given for $|\text{Im}\,\Theta| < \beta$ as a densely defined, closed operator with $V(\Theta)^* = V(\bar{\Theta})$, such that (2) holds on a core of $H_\kappa(\Theta)$. Then the iterated resolvent equation

$$R_\kappa(\Theta, z) P_0(\Theta) = \sum_{m=0}^{N-1} \kappa^m R_0(\Theta, z) A_m(\Theta, z) + \kappa^N R_\kappa(\Theta, z) A_N(\Theta, z) \ ; \tag{10}$$

$$A_m(\Theta, z) = [V(\Theta) R_0(\Theta, z)]^m P_0(\Theta) \tag{11}$$

is valid for $\text{Im}\,\Theta \neq 0$, $z \in W(\Theta)$ and small κ as long as the individual terms are well defined (cf. [8], Lemma 8.1). We are interested in the case where the *expanded* part – i.e. the terms not involving $R_\kappa(\Theta, z)$ – remains well-defined for $\text{Im}\,\Theta = 0$. Therefore we assume that the finite rank operators $A_m(\Theta, z)$ are analytic in Θ in the full strip

$|\text{Im } \Theta| < \beta$ for $m = 1 \cdots N$ and all z in some punctured neighbourhood W of λ_0, with the properties

$$A_m(\Theta, z) = U(\Theta) A_m(z) U(\Theta)^{-1} \tag{12}$$

on $M_0(\Theta)$, and

$$\sup_{z \in \Gamma} \|A_m(\Theta, z)\| < \infty \tag{13}$$

for any loop $\Gamma \subset W$ around λ_0. The order $N \geq 1$ will be specified in each case. For $N = 1$ our hypothesis only requires that $V(\Theta) P_0(\Theta)$ is analytic in $|\text{Im } \Theta| < \beta$ with the transformation law (12). For $N > 1$, however, it is implied that $R_0(\Theta, z)$ exists for $|\text{Im } \Theta| < \beta$ and $z \in W$, ie. that λ_0 is a *discrete* eigenvalue of H_0.

Our list of assumptions is now complete. Typical examples are:

a) Analytic perturbations of embedded eigenvalues in dilation-analytic systems [12]. Here $V(\Theta)$ is bounded relative to $H_0(\Theta)$ so that our hypothesis is satisfied for $N = 1$.

b) The Stark effect for discrete eigenvalues of Coulomb systems [5, 8]. This is an example where the formal RS-expansion for the (nonexistent) perturbed bound states is well-defined to any finite order, and where our assumptions are valid for arbitrary N.

At this point the reader may proceed directly to Sect. 3. We continue with a discussion of the RS-expansion in the degenerate case as a preparation of Sect. 4. Inserting (10) into (6) we obtain the expansion of $P_\kappa(\Theta)$ for fixed Θ with $\text{Im } \Theta \neq 0$:

$$P_\kappa(\Theta) P_0(\Theta) = B_\kappa(\Theta) + O(\kappa^N) \ ,$$

where the expanded part $B_\kappa(\Theta)$ is analytic in the full strip $|\text{Im } \Theta| < \beta$. Similar expansions result for

$$P_0(\Theta) P_\kappa(\Theta) = [P_\kappa(\bar{\Theta}) P_0(\bar{\Theta})]^*$$

and for the operator

$$D_\kappa(\Theta) = P_0(\Theta) P_\kappa(\Theta) P_0(\Theta) \tag{14}$$

acting on $M_0(\Theta)$. The RS-expansion of $P_\kappa(\Theta)$ is now obtained from the identity

$$P_\kappa(\Theta) = P_\kappa(\Theta) P_0(\Theta) D_\kappa(\Theta)^{-1} P_0(\Theta) P_\kappa(\Theta) \ , \tag{15}$$

valid for small κ. To justify (15) we note that $D_\kappa(\Theta) \to 1$ as $\kappa \to 0$. Thus $D_\kappa(\Theta)^{-1/2} P_0(\Theta) P_\kappa(\Theta)$ is well-defined as an operator from $M_\kappa(\Theta)$ to $M_0(\Theta)$. By (14) it has a right inverse $P_\kappa(\Theta) P_0(\Theta) D_\kappa(\Theta)^{-1/2}$, and this is in fact the inverse since $\dim M_\kappa(\Theta) = \dim M_0(\Theta)$ for small κ. We write the RS-expansion of $P_\kappa(\Theta)$ as

$$P_\kappa(\Theta) = P_\kappa^N(\Theta) + O(\kappa^N) \ ,$$

noting that the expanded part $P_\kappa^N(\Theta)$ is analytic in Θ in the full strip $|\text{Im } \Theta| < \beta$, satisfying

$$P_\kappa^N(\Theta) = U(\Theta) P_\kappa^N U(\Theta)^{-1} \ .$$

Here $P_\kappa^N = P_\kappa^N(0)$ is the formal perturbative expression to order $N-1$ of the "perturbed total eigenprojection of H_κ" (which need not exist). Our aim is to derive decay estimates under $\exp(-iH_\kappa t)$ for the states in the space

$$M_\kappa^N = \text{ran}(P_\kappa^N P_0) \ , \tag{16}$$

which has dimension m_0 for small κ. This is prepared by the following construction. We define

$$D_\kappa^N = U(\Theta)^{-1} P_0(\Theta) P_\kappa^N(\Theta) P_\kappa(\Theta) P_\kappa^N(\Theta) P_0(\Theta) U(\Theta) \tag{17}$$

as an operator on M_0 for $\operatorname{Im} \Theta > 0$, noting that the right-hand side depends only on the sign of $\operatorname{Im} \Theta$. For $\operatorname{Im} \Theta < 0$ (17) remains valid if D_κ^N is replaced by its adjoint. D_κ^N has an expansion

$$\begin{aligned} D_\kappa^N &= d_\kappa^N + O(\kappa^{2N}) \ , \\ d_\kappa^N &= P_0 (P_\kappa^N)^2 P_0 = (d_\kappa^N)^* \ , \end{aligned} \tag{18}$$

which is derived from the identity

$$P_\kappa^N(\Theta) P_\kappa(\Theta) P_\kappa^N(\Theta) = [P_\kappa^N(\Theta)]^2 + [P_\kappa^N(\Theta) - P_\kappa(\Theta)][P_\kappa(\Theta) - 1]$$
$$\cdot [P_\kappa^N(\Theta) - P_\kappa(\Theta)] \ .$$

By the argument given to justify (15), we see that the map

$$T_\kappa(\Theta) = P_\kappa(\Theta) P_\kappa^N(\Theta) P_0(\Theta) U(\Theta) (D_\kappa^N)^{-1/2} : M_0 \to M_\kappa(\Theta) \tag{19}$$

has the inverse

$$T_\kappa(\Theta)^{-1} = (D_\kappa^N)^{-1/2} U(\Theta)^{-1} P_0(\Theta) P_\kappa^N(\Theta) P_\kappa(\Theta) : M_\kappa(\Theta) \to M_0 \tag{20}$$

for $\operatorname{Im} \Theta > 0$ and small κ. This is used to transform $\tilde{H}_\kappa(\Theta)$ into the equivalent operator

$$h_\kappa = T_\kappa(\Theta)^{-1} \tilde{H}_\kappa(\Theta) T_\kappa(\Theta) \tag{21}$$

acting on M_0, which is independent of Θ for $\operatorname{Im} \Theta > 0$. By construction, the eigenvalues of h_κ are the resonances λ_κ. We will prove below that

$$h_\kappa = t_\kappa^* H_\kappa t_\kappa + O(\kappa^{2N}) \ , \tag{22}$$

on M_0, where

$$t_\kappa = P_\kappa^N P_0 (d_\kappa^N)^{-1/2} \tag{23}$$

maps M_0 isometrically onto M_κ^N for small κ. This shows in particular that

$$h_\kappa^* - h_\kappa = O(\kappa^{2N}) \ , \tag{24}$$

which gives the estimate

$$\operatorname{Im} \lambda_\kappa = O(\kappa^{2N}) \tag{25}$$

for the width of the resonances. To prove (22) we first remark that $M_\kappa^N(\Theta)$ $= \operatorname{ran} P_\kappa^N(\Theta) P_0(\Theta) \subset D(H_\kappa(\Theta))$ for $|\operatorname{Im} \Theta| < \beta$. Indeed, $z - H_\kappa(\Theta)$ can be applied to each term in the expanded part of (10), with the result

$$(z - H_\kappa(\Theta)) R_0(\Theta, z) A_m(\Theta, z) = A_m(\Theta, z) - \kappa A_{m+1}(\Theta, z)$$

for $z \in W$ and $m = 0 \cdots N-1$. From the remainder in (10) we also see that

$$H_\kappa(\Theta)(P_\kappa^N(\Theta) - P_\kappa(\Theta)) P_0(\Theta) = O(\kappa^N)$$

if $\operatorname{Im} \Theta \neq 0$. Now we write out (21) and use the decomposition

$$\tilde{H}_\kappa(\Theta) = H_\kappa(\Theta) - (1 - P_\kappa(\Theta)) H_\kappa(\Theta)(1 - P_\kappa(\Theta)) \ .$$

The contribution of the last term to h_κ is of order κ^{2N}, since

$$H_\kappa(\Theta)(1 - P_\kappa(\Theta))P_\kappa^N(\Theta)P_0(\Theta) = (1 - P_\kappa(\Theta))H_\kappa(\Theta)(P_\kappa^N(\Theta) - P_\kappa(\Theta))P_0(\Theta)$$

and

$$P_\kappa^N(\Theta)(1 - P_\kappa(\Theta)) = (P_\kappa^N(\Theta) - P_\kappa(\Theta))(1 - P_\kappa(\Theta))$$

are both of order κ^N. Using (18) we thus find

$$h_\kappa = (d_\kappa^N)^{-1/2} U(\Theta)^{-1} P_0(\Theta) P_\kappa^N(\Theta) H_\kappa(\Theta) P_\kappa^N(\Theta) P_0(\Theta) U(\Theta)(d_\kappa^N)^{-1/2}$$
$$+ O(\kappa^{2N}) \ ,$$

where we can now set $\Theta = 0$.

3. The Nondegenerate Case in Lowest Order

In this section we discuss the simplest case $m_0 = N = 1$ to give the essence of the argument.

Theorem 1. *Let λ_0 be a simple (discrete or embedded) eigenvalue of H_0 with normalized eigenvector ψ_0. Let $g \in C_0^\infty(R)$ be supported sufficiently close to λ_0 with $g = 1$ in some open interval containing λ_0. Then*

$$(\psi_0, e^{-iH_\kappa t} g(H_\kappa)\psi_0) = a(\kappa)e^{-i\lambda_\kappa t} + b(\kappa, t) \tag{26}$$

for small κ and $0 \le t < \infty$, where λ_κ is the resonance eigenvalue and

$$|b(\kappa, t)| \le \kappa^2 c_m (1 + t)^{-m} \tag{27}$$

for any $m \ge 0$. $a(\kappa)$ is given for (arbitrary) Θ in $0 < \mathrm{Im}\,\Theta < \beta$ by

$$a(\kappa) = (U(\bar\Theta)\psi_0, P_\kappa(\Theta)U(\Theta)\psi_0)$$
$$= 1 + O(\kappa^2) \ . \tag{28}$$

Proof. We fix Θ with $0 < \mathrm{Im}\,\Theta < \beta$. By hypothesis there exists an open interval $I \ni \lambda_0$ with endpoints in the set $W(\Theta)$ defined in Sect. 2. Let $g \in C_0^\infty(I)$. Then

$$F(t) = (\psi_0, e^{-iH_\kappa t} g(H_\kappa)\psi_0)$$
$$= \lim_{\varepsilon\ 0} (2\pi i)^{-1} \int_I dz\, e^{-izt} g(z)(\psi_0, [R_\kappa(z - i\varepsilon) - R_\kappa(z + i\varepsilon)]\psi_0) \ .$$

This can be expressed in terms of $H_\kappa(\Theta)$ by

$$F(t) = f(\bar\Theta, t) - f(\Theta, t) \ , \tag{29}$$

$$f(\Theta, t) = (2\pi i)^{-1} \int_I dz\, e^{-izt} g(z)(\psi_0(\bar\Theta), R_\kappa(\Theta, z)\psi_0(\Theta)) \ ,$$

where $\psi_0(\Theta) = U(\Theta)\psi_0$. Here we have assumed for simplicity that $\mathrm{Im}\,\lambda_\kappa < 0$ so that I is contained in the resolvent set of $H_\kappa(\Theta)$ for small κ. If λ_κ is real, then the integral (29) must be modified by a detour around λ_0 in the upper halfplane. This is of no consequence for our estimates and will be ignored.

Next we choose a loop $\Gamma \subset W(\Theta)$ around I as shown in Fig. 2. For z inside Γ we decompose the resolvent into singular and regular parts:

$$R_\kappa(\Theta, z) = P_\kappa(\Theta)(z - \lambda_\kappa)^{-1} + \hat{R}_\kappa(\Theta, z) \ ,$$

$$\hat{R}_\kappa(\Theta, z) = (2\pi i)^{-1} \oint_\Gamma d\xi \, R_\kappa(\Theta, \xi)(\xi - z)^{-1} \ , \tag{30}$$

noting that

$$\hat{R}_\kappa(\Theta, z) P_\kappa(\Theta) = P_\kappa(\Theta) \hat{R}_\kappa(\Theta, z) = 0 \ . \tag{31}$$

(a) Contribution of the regular part. Using (31) we can write the contribution of $\hat{R}_\kappa(\Theta, z)$ to $f(\Theta, t)$ as

$$(u_\kappa(\bar{\Theta}), (2\pi i)^{-1} \int_I dz \, e^{-izt} g(z) \hat{R}_\kappa(\Theta, z) u_\kappa(\Theta)) \ ,$$

where

$$u_\kappa(\Theta) = [P_0(\Theta) - P_\kappa(\Theta)] \psi_0(\Theta)$$

is of order κ. By partial integration the last integral is seen to be bounded by $a_m t^{-m}$ for any $m \geq 0$ since $g \in C_0^\infty(I)$. The constants a_m involve norms of derivatives (i.e. powers) of $\hat{R}_\kappa(\Theta, z)$. In fact $\hat{R}_\kappa(\Theta, z)$ is uniformly bounded for small κ and $z \in I$: this follows from (30) since $\Gamma \subset W(\Theta)$. As a result, the contribution of the regular part to $f(\Theta, t)$ is bounded by $\kappa^2 c_m (1 + t)^{-m}$ for small κ and any $m \geq 0$. The contribution to $f(\bar{\Theta}, t)$ is estimated in the same way.

(b) Contribution of the singular part. The singular part in (30) gives rise to the term

$$\overline{a(\kappa)} (2\pi i)^{-1} \int_I dz \, e^{-izt} g(z)(z - \overline{\lambda_\kappa})^{-1} - a(\kappa)(2\pi i)^{-1} \int_I dz \, e^{-izt} g(z)(z - \lambda_\kappa)^{-1} \tag{32}$$

in $F(t)$, where $a(\kappa)$ is defined by (28). We now use the fact that $g = 1$ on some open interval $I_0 \ni \lambda_0$ to deform the path I in both integrals to the lower halfplane, as shown in Fig. 3:
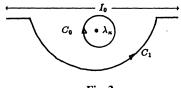

Fig. 3

From the second integral in (32) we pick up the residue $a(\kappa) e^{-i\lambda_\kappa t}$ at the point $z = \lambda_\kappa$. The remainder is given by (32) with both integrals taken along the path C_1 (where $g(z) = 1$ for $\text{Im}\, z < 0$). Using the identity

$$P_0(\Theta) P_\kappa(\Theta) P_0(\Theta) = [P_0(\Theta)]^2 + [P_0(\Theta) - P_\kappa(\Theta)][P_\kappa(\Theta) - 1][P_0(\Theta) - P_\kappa(\Theta)]$$

and the fact that $(\psi_0(\bar{\Theta}), \psi_0(\Theta)) = (\psi_0, \psi_0) = 1$, we see that $a(\kappa) = 1 + O(\kappa^2)$. Thus we can write the remainder in the form

$$(\operatorname{Im}\lambda_\kappa)\pi^{-1}\int_{C_1} dz\, e^{-izt} g(z)(z-\overline{\lambda_\kappa})^{-1}(z-\lambda_\kappa)^{-1}$$

$$+O(\kappa^2)\int_{C_1} dz\, e^{-izt} g(z)(z-\overline{\lambda_\kappa})^{-1}+O(\kappa^2)\int_{C_1} dz\, e^{-izt} g(z)(z-\lambda_\kappa)^{-1}\ .$$

For $0\leq t<\infty$ all three integrals have bounds of the form $a_m t^{-m}$ with arbitrary $m\geq 0$. The proof is completed by noting that $\operatorname{Im}\lambda_\kappa=O(\kappa^2)$. $\square$

Discussion. With Theorem 1 we can construct the metastable states

$$\phi_\kappa=g(H_\kappa)\psi_0\,\|g(H_\kappa)\psi_0\|^{-1}\ .$$

Since $|g|^2$ also satisfies the conditions imposed on g we obtain the decay estimate

$$(\phi_\kappa,e^{-iH_\kappa t}\phi_\kappa)=(1-b(\kappa,0))e^{-i\lambda_\kappa t}+b(\kappa,t) \tag{33}$$

with $|b(\kappa,t)|\leq\kappa^2 c_m(1+t)^{-m}$ for small κ, $0\leq t<\infty$ and any $m\geq 0$. To discuss the decay of ψ_0 itself we choose $0\leq g\leq 1$. By Theorem 1 we have

$$(\psi_0,e^{-iH_\kappa t}g(H_\kappa)\psi_0)=e^{-i\lambda_\kappa t}+O(\kappa^2) \tag{34}$$

uniformly in $0\leq t<\infty$. Setting $t=0$ we find

$$(\psi_0,(1-g(H_\kappa))\psi_0)=\|(1-g(H_\kappa))^{1/2}\psi_0\|^2=O(\kappa^2)\ .$$

Inserting this into (34) we arrive at the result

$$(\psi_0,e^{-iH_\kappa t}\psi_0)=e^{-i\lambda_\kappa t}+O(\kappa^2)\ . \tag{35}$$

4. The General Case

Here we extend the decay estimate to the general case within the framework of Sect. 2, where m_0 and N are now arbitrary.

Theorem 2. *If g is chosen as in Theorem 1, then, as an operator relation on the unperturbed eigenspace M_0,*

$$P_0 P_\kappa^N e^{-iH_\kappa t}g(H_\kappa)P_\kappa^N P_0=(D_\kappa^N)^{1/2}e^{-ih_\kappa t}(D_\kappa^N)^{1/2}+B(\kappa,t) \tag{36}$$

for small κ and $0\leq t<\infty$, where

$$\|B(\kappa,t)\|\leq\kappa^{2N}c_m(1+t)^{-m} \tag{37}$$

for any $m\geq 0$ and corresponding constants c_m.

Proof. As in the proof of Theorem 1 we have

$$F(t)=P_0 P_\kappa^N e^{-iH_\kappa t}g(H_\kappa)P_\kappa^N P_0$$

$$=f(\overline{\Theta},t)-f(\Theta,t)\ ,$$

$$f(\Theta,t)=(2\pi i)^{-1}\int_I dz\, e^{-izt}g(z)U(\Theta)^{-1}P_0(\Theta)P_\kappa^N(\Theta)R_\kappa(\Theta,z)P_\kappa^N(\Theta)$$

$$\cdot P_0(\Theta)U(\Theta)\ ,$$

for $\operatorname{Im}\Theta > 0$, where $U(\Theta)$ is the bounded operator $M_0 \to M_0(\Theta)$. The resolvent $R_\kappa(\Theta, z)$ has the singular part

$$P_\kappa(\Theta) R_\kappa(\Theta, z) P_\kappa(\Theta) \tag{38}$$

and the regular part $\hat{R}_\kappa(\Theta, z)$ given by (30). The contribution of the regular part to $(F(t)$ is estimated as before, using that

$$P_\kappa^N(\Theta) \hat{R}_\kappa(\Theta, z) P_\kappa^N(\Theta) = [P_\kappa^N(\Theta) - P_\kappa(\Theta)] \hat{R}_\kappa(\Theta, z)[P_\kappa^N(\Theta) - P_\kappa(\Theta)]$$

$$= O(\kappa^{2N}) \ .$$

As a result this contribution has a bound of the form (37) for any $m \geq 0$. Using (21) we can write the contribution of the singular part (38) to $f(\Theta, t)$ as

$$(D_\kappa^N)^{1/2}(2\pi i)^{-1} \int_I dz\, e^{-izt} g(z)(z - h_\kappa)^{-1}(D_\kappa^N)^{1/2} \ .$$

The contribution to $f(\bar{\Theta}, t)$ has the same form, with the difference that D_κ^N and h_κ are replaced by their adjoints. In both terms we deform the path I as shown in Fig. 3, where the loop C_0 now encloses the full spectrum of h_κ, i.e. all the resonance eigenvalues λ_κ corresponding to λ_0. This loop gives rise to the term

$$(D_\kappa^N)^{1/2} e^{-ih_\kappa t}(D_\kappa^N)^{1/2}$$

in $F(t)$, which comes from $f(\Theta, t)$. The remainder is given by

$$(D_\kappa^{N*})^{1/2}(2\pi i)^{-1} \int_{C_1} dz\, e^{-izt} g(z)(z - h_\kappa^*)^{-1}(D_\kappa^{N*})^{1/2}$$

$$- (D_\kappa^N)^{1/2}(2\pi i)^{-1} \int_{C_1} dz\, e^{-izt} g(z)(z - h_\kappa)^{-1}(D_\kappa^N)^{1/2} \ . \tag{39}$$

Using (18) we can replace D_κ^N and its adjoint by d_κ^N – committing an error of the form (37). Then we combine the two terms of (39) to

$$(d_\kappa^N)^{1/2}(2\pi i)^{-1} \int_{C_1} dz\, e^{-izt} g(z)(z - h_\kappa^*)^{-1}(h_\kappa^* - h_\kappa)(z - h_\kappa)^{-1}(d_\kappa^N)^{1/2} \ ,$$

which by (24) has again a bound of the form (37). $\square$

Discussion. Metastable states with exponential decay laws are obtained from the eigenvectors of h_κ. Let

$$\phi_\kappa = g(H_\kappa) P_\kappa^N (D_\kappa^N)^{-1/2} \psi_\kappa$$

be normalized to 1, with $h_\kappa \psi_\kappa = \lambda_\kappa \psi_\kappa$. Then

$$(\phi_\kappa, e^{-iH_\kappa t}\phi_\kappa) = (1 - b(\kappa, 0))e^{-i\lambda_\kappa t} + b(\kappa, t) \tag{40}$$

for small κ and $0 \leq t < \infty$, where

$$|b(\kappa, t)| \leq \kappa^{2N} c_m(1 + t)^{-m}$$

for any $m \geq 0$. In the example of the Stark effect N is arbitrary and h_κ has a complete

set of eigenvectors [6]. In general, the Jordan normal form of h_κ must be used to exhibit the t-dependence of $\exp(-ih_\kappa t)$. In analogy to (35) we can also find decay laws for the states in M_κ^N with a time-independent background estimate. From (36) and (18) we obtain

$$t_\kappa^* e^{-iH_\kappa t} g(H_\kappa) t_\kappa = e^{-ih_\kappa t} + O(\kappa^{2N}) \ ,$$

where t_κ is given by (23). Choosing $0 \le g \le 1$ and setting $t = 0$ we find

$$\left\| (1 - g(H_\kappa))^{1/2} t_\kappa \right\|^2 = O(\kappa^{2N})$$

and therefore

$$(t_\kappa \phi, e^{-iH_\kappa t} t_\kappa \psi) = (\phi, e^{-ih_\kappa t} \psi) + O(\kappa^{2N}) \tag{41}$$

uniformly for states $\phi, \psi \in M_0$ and $0 \le t < \infty$. In the nondegenerate case this is the result quoted in (5): then $t_\kappa \psi$ is the formal "perturbed eigenstate to order $N-1$". As a consequence of (41) we also note that $\left\| \exp(-ih_\kappa t) \right\| \le 1 + O(\kappa^{2N})$. In fact we see no reason to believe that $\exp(-ih_\kappa t)$ should be a contraction.

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