Commun. Math. Phys. 131, 571–603 (1990)

Communications in
Mathematical
Physics

Zonal Schrödinger Operators on the n-Sphere: Inverse Spectral Problem and Rigidity

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Abstract. We study the Direct and Inverse Spectral Problems for a class of Schrödinger operators $H = -\Delta + V$ on S_n with *zonal* (axisymmetric) potentials. Spectrum of H is known to consist of clusters of eigenvalues $\{\lambda_{km} = k(k+n-1) + \mu_{km}: m \leqq k\}$. The main result of the work is to derive asymptotic expansion of spectral shifts $\{\mu_{km}\}$ in powers of k^{-1}, and to link coefficients of the expansion to certain transforms of V. As a corollary we solve the Inverse Problem, get explicit formulae for the Weinstein *band-invariants* of cluster distribution measures, and establish *local spectral rigidity* for zonal potential. The latter provides a partial answer to a long standing Spectral Rigidity Hypothesis of V. Guillemin.

The Direct/Inverse Spectral Problems for differential operators ask for connections between the "geometric/dynamical" data (like coefficients of the operator, Riemannian metric, potential, or geometry of the region) on one hand, and its "spectral data" (eigenvalues/eigenfunctions) on the other. The well known examples include

i) M. Kac: "shape of the drum" problem determination (modulo rigid motions) of the boundary of a region from eigenvalues of the Laplacian.
ii) The "shape of the metric" problem ([MS]), which poses a similar question for the Riemannian metric of manifold and the Laplace–Beltrami operator.
iii) Potential (perturbation) problem for Schrödinger operators $H = -\Delta + V(x)$. Here the manifold and metric (hence the Laplacian) are fixed, and one studies the connection between spec(H) and the potential function V.

Our work belongs to the third type.

The classical and best studied example of perturbation problems are the regular Sturm–Liouville (S–L) operators $H = -\partial^2 + V(x)$ on $[0; 1]$, with various types of boundary conditions: 2-point; periodic Floquet, etc. Such operators have simple (multiplicity free) spectrum $\{\lambda_k\}$, and their asymptotics were known since the old wǫrk of G. Borg (see also [Le]),

$$\lambda_k = (\pi k)^2 + b_0 + k^{-1} b_1 + \cdots, \quad \text{as} \quad k \to \infty, \tag{0.1}$$

where $b_0 = \int V dx$, and higher b_j being expressed in terms of the Fourier coefficients of V.

Formula (0.1) solves the Direct Problem. Solution of the Inverse Problem requires first to determine uniqueness or nonuniqueness of the map: $V \to \operatorname{spec}(H_v)$, and to find the ensuing characterization of the *isospectral classes* of potentials, $\operatorname{Iso}(V) = \{V' : \operatorname{spec}(H_{V'}) = \operatorname{spec}(H_V)$, or $\approx \operatorname{spec}(H_V)\}$.

It turns out that S–L problems have typically large (∞–D) isospectral classes both in the periodic/Floquet ([La]; [No]; [MM]) and the "2-point boundary" case ([IMT]; [PT]). So a unique determination of V requires an additional infinite set of data, which, depending on a particular set up, may be the "KdV-flow parameters," "norming constants," etc.

The multidimensional analogue of the periodic S–L Problem is the Schrödinger operator $H = -\Delta + V$ on a compact Riemannian manifold Ω. Multi-D Problems are less studied and understood, since they apparently lack some essential structures of the 1-D theory (e.g. the Hamiltonian/Poisson structure).

Furthermore, they often exhibit very different qualitative features. For instance, it is believed that multi-D Schrödinger operators are *spectrally rigid*, in the sense that their isospectral classes, $\operatorname{Iso}(V)$, consist of finite dimensional families obtained by natural (geometric) symmetries of the Laplacian (i.e. isometries of Ω), rather than hidden KdV-type symmetries.

This conjecture, known as *Spectral Rigidity hypothesis*, was posed by V. Guillemin, and confirmed in a number of cases. The foremost is the case of negatively curved manifolds, where under some additional assumptions it was shown that V is uniquely determined by $\operatorname{spec}(H_V)$, i.e. $\operatorname{Iso}(V) = \{V\}$ (see [*GK*]). Of course, negatively curved manifolds have typically no continuous internal (geometric) symmetry groups.

Another case is the flat torus $\mathbb{T}^n$. Its symmetries consist of all translations and reflections of $\mathbb{T}^n$. It was shown that generic potentials[1] V on $\mathbb{T}^n$ are also spectrally rigid ([ERT]).

The case of the positively curved (and highly symmetric) n-sphere turned out to be the most difficult. So far only partial results were obtained ([We]; [Gui]; [Co]; [Ur]; [Wi]; [Gur4–5]). The present paper continues this line of research, so let us describe the basic results and ideas of the n-sphere spectral theory.

The dominant feature of the n-sphere Laplacian is the regular distributed and highly degenerate set of eigenvalues, $\operatorname{spec}(-\Delta) = \{\lambda_k = k(k+n-1)\}$, the multiplicity of λ_k, $d_k = O(k^{n-1})$. Both features result, of course, from the underlying rotational symmetry group $SO(n+1)$.

Consequently, spectrum of the Schrödinger operator H on S_n breaks into clusters $\Lambda_k = \{\lambda_{km} = \lambda_k + \mu_{km} : m = 1; \ldots d_k\}$ of simple (less degenerate) eigenvalues,

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[1] Obviously, special potentials, like $V = V_1(x_1) + \cdots + V_n(x_n)$, the sum of 1-variable functions, may still possess "large" (infinite) Iso-classes, obtained by the hierarchy of KdV-flows applied to each 1-D constituent $\{V_j(x_j)\}$

and one is interested in associated spectral invariants. Such invariants can be introduced, by analogy with (0.1), as asymptotic coefficients in the expansion of *cluster-distribution measures*,

$$d\nu_k = \frac{1}{d_k}\sum_m \delta(x - \mu_{km}).$$

Namely, A. Weinstein [We] showed that sequence $\{d\nu_k\}$ can be expanded in powers of k^{-1},

$$d\nu_k \sim \beta_0 + k^{-1}\beta_1 + \cdots \tag{0.2}$$

and called the resulting coefficients $\{\beta_0; \beta_1 \ldots\}$ (distributions on $\mathbb{R}$) *band-invariants*. He was able to calculate the first invariant $\beta_0 =$ "distribution density" of the so-called *Radon transform* $\tilde{V}$ of potential V. The higher band-invariants are much harder to calculate explicitly (cf. [Ur1-2]).

Turning to the Inverse Problem and Rigidity, the Weinstein band-invariants provide some valuable information about potential V, but by themselves are insufficient to resolve it. One of the difficulties comes from the fact that knowing the "distribution function of $\tilde{V}$," does not determine $\tilde{V}$ itself, even in the one-variable case, whereas the S_n-Radon transform will generally depend on $2(n-1)$ parameters. Of course, if we knew $\tilde{V}$, the Inverse problem would be easily resolved by the Radon inversion.

Nevertheless V. Guillemin [Guil] successfully applied β_0 and the known classical heat-invariants to prove spectral rigidity for special classes of potentials: low degree spherical harmonics on S_2.

In our recent works [Gur4-5] we initiated the study of a different class, so-called *zonal potentials* V. The latter refers to axisymmetric functions $\{V\}$, i.e. functions invariant under the group $SO(n)$ of rotations about the "north pole." The corresponding Schrödinger operators possess an additional symmetry: on S_2 they commute with an angular momentum operator $M = i\partial_\theta$, while on S_n the role of M is played by the angular momentum algebra $\mathscr{M} \simeq so(n)$. Hence one can talk about the joint spectrum of H and M. In other words spectral shifts $\{\mu_{km}\}$ of the k^{th} cluster acquire an additional bigraded structure, with index m labeling the angular momentum of the $(km)^{th}$ eigenfunction.

In this context we made in [Gur4] an important improvement over the Weinstein formula (0.2), by replacing asymptotics of cluster-measures $\{d\nu_k\}$ by asymptotics of spectral shifts $\{\mu_{km}\}$,

$$\mu_{km} = \tilde{V}\left(\frac{m}{k}\right) + O(k^{-1}). \tag{0.3}$$

The main advantage of (0.3) is that it yields the Radon transform $\tilde{V}$, as a $[0; 1]$-function[2] (rather than distribution function of its values), and thus directly leads to solution of the Inverse Problem. Then we applied (0.3) to prove the Rigidity hypothesis for generic zonal potentials on S_2.

[2] Let us remark that for zonal functions V the Radon transform $\tilde{V}$ is also zonal (1-variable) function

The first of two papers [Gur4] proved these results in the even potential case, while [Gur5] treated the more difficult odd case.

The purpose of the current work is two-fold, on one hand to extend the S_2-results of [Gur4-5] to higher dimensions, but more significantly to treat the general "even + odd" case.

Let us examine in more detail some difficulties of the S_n-Schrödinger theory.

A general guiding principle in Spectral Theory of differential operators goes under the name of *Correspondence Principle.* It roughly asserts that high energy levels (large eigenvalue) of the Quantum mechanical system (differential operator H) are related to certain geometric/dynamical data of the corresponding classical mechanical system determined by the symbol of H.

In our case the "Quantum object" is the Laplacian Δ, or the Schödinger operator H on a manifold Ω, and the relevant "classical system" consists of the geodesic/Hamiltonian flow on the phase-space $T^*(\Omega)$.

The Correspondence Principle has many specific manifestations from the very general (crude) estimates, like Weyl's "volume counting" principle[3], to fine relations between "asymptotics of large eigenvalue" and "closed geodesics/bicharacteristics of the relevant classical system" (see for instance [Ch], [DG]). The foremost example of the latter is given by the celebrated Selberg trace formula, which in spectral terms provides a link between the eigenvalues of the Laplacian on a hyperbolic surface and the *length spectrum* (the set of length of closed geodesics) of the corresponding geodesic flow.

In the context of Schrödinger theory one usually takes the basic (geodesic) flow of the Laplacian, and then studies the effects of perturbation on both the classical and the quantum levels.

Thus for hyperbolic (negatively curved) manifolds it was shown ([Gui3]; [GK]) that $\operatorname{spec}(H_V)$ yields, among other, the average values of V over all closed geodesics $\{\gamma\}$, i.e. the *Radon transform* $\tilde{V}(\gamma) = \int_\gamma V\,ds$. This data by itself is often enough to establish rigidity (cf. [GK]).

Another illustration of the role of geodesics and the Radon transform is the flat (n-torus) case. Here once again $\operatorname{spec}(H_V)$ was shown to determine spectra of certain reduced (1-D) operators, consequently, a version of the Radon transform on $\mathbb{T}^n$.

Not surprisingly closed geodesics and the Radon transform play the crucial role in the n-sphere theory, although the structure of space $\mathcal{O}$ is very different now[4]. Indeed, the basic methods employed in the n-sphere Schrödinger theory, so called *averaging techniques*, rely heavily on spherical symmetries and the ensuing

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[3] The Weyl principle states that the number of eigenstates of a quantum hamiltonian H (e.g. $H = -\frac{1}{2}\Delta + V$) below the given energy level λ, $N(\lambda; H) = \#\{k : \lambda_k \leq \lambda\}$, is asymptotically equal (estimated) by the phase-space volume $\operatorname{Vol}\{(x;p) : h(x;p) \leq \lambda\}$, where $h(x;p)$ denotes the corresponding classical hamiltonian/energy function on $T^\circ(\Omega)$, or *symbol* of H (e.g. $h = \frac{1}{2}p^2 + V(x)$, for Schrödinger H)

[4] Instead of a discrete family of geodesics/classes $\{\gamma_j\}$ of increasing length as in a typical hyperbolic setting, space $\mathcal{O}$ consists of a single $2(n-1)$-parameter cell of the fixed length ($|\gamma| = 2\pi$), and possesses many other nice structures due to the $SO(n+1)$-symmetry

2π-periodicity on both the classical system, geodesic flow, and the quantum Laplacian.

The basic idea of averaging is to replace perturbation $B = V(x)$, that does not commute with Δ, by an *average operator* $\bar{B}$, that does commute, and such that the pair $H = -\Delta + B$ and $\bar{H} = -\Delta + \bar{B}$ are almost unitary equivalent.

The simplest procedure, due to Weinstein [We], consists in averaging the conjugates $B(t) = e^{-itA}Be^{itA}$, with $A = \sqrt{-\Delta}$, over the period $[0; 2\pi]$. Unfortunately, Weinstein averaging applies only in the even-potential case. The odd case requires substantial modifications, the so-called 2nd *averaging* due to Guillemin ([Gui]), and yet further refinement is needed in the general (even + odd) case [Ur2].

Averaging me `od provides one ingredient of the n-sphere theory, the second involves a suitable form of Symbolic (pseudodifferential) Calculus on S_n. Let us remark that in each case ("even" or "odd") separately, it suffices to work with the standard *principal symbols*, well defined on any (Riemannian) manifold.

The main difficulty in the general (even + odd) case results from the different orders of the "even" and "odd" contributions to spectral shifts, which do not combine. Hence one needs a *complete symbolic Calculus* on S_n (at least to order -3).

There were several attempts to introduce such calculi ([Be]; [Wi]; [Ur1]). The best suited for our purpose (in fact, designed to this end) is the calculus due to A. Uribe [Ur1]. It heavily exploits the representation theory of the group $SO(n+1)$, the notion of *Wick-Berezin symbols* and *coherent states* on S_n. We shall give a brief account of the Uribe calculus in Sect. 2 (step 2°).

The main contribution of the present work is to combine the *Averaging techniques*, the *Uribe Calculus*, along with the author's *Zonal Reduction Method*, the *Arithmetic Rigidity Principle* and some explicit calculations, in the context of the n-sphere Schrödinger theory with general type potentials.

Let us describe the content of the work. The main result (Theorem 1) gives an asymptotic expansion in k^{-1} of spectral shifts $\{\mu_{km}\}$ of a zonal Schrödinger operator H on S_n, analogous to (0.1) and (0.2),

$$\mu_{km} \sim a\left(\frac{m}{k}\right) + k^{-1}b\left(\frac{m}{k}\right) + k^{-2}c\left(\frac{m}{k}\right) + \cdots, \tag{0.4}$$

with function-coefficients $a(x)$; $b(x)$; $c(x)$, which represent certain transforms of potential V, more precisely, its even and odd parts V_{ev}, V_{od}.

As a corollary we obtain a unique and explicit solution of the Inverse Problem for the joint spectrum of H and the angular momentum algebra. Furthermore, for $\mathrm{spec}(H)$ alone we establish *local spectral rigidity* for generic zonal potential on S_n, generalizing our earlier results [Gur4-5] on S_2, and providing a partial answer to the Rigidity Hypothesis of V. Guillemin.

Finally, we apply (0.4) to obtain in a straightforward way explicit formulae of the first 3 Weinstein band-invariants for zonal Schrödinger operators.

The paper is divided into 3 parts. In Sect. 1 we introduce the basic concepts and state the main result. Section 2 provides the details of the proof, including Averaging procedures, Symbolic calculi and Zonal reduction. The last section contains applications of Theorem 1.

In conclusion let us remark that all results of the paper extend from S_n to other rank 1 symmetric spaces. The problem of extending them to higher ranks seems quite interesting and challenging, but it would require, among other, a new and better understanding of multivariable averaging procedures and the role of symmetries.

Another closely related Problem involves the semiclassical analysis of the S_n-Schrödinger operators with small Plank constant, $H = -(\hbar^2/2)\Delta + V$ (cf. [CD]; [Ka]; [KM]). Many of the techniques, developed in our paper, could be adopted here. We plan to return to this subject elsewhere.

1. The Main Result

We consider the Euclidean space $\mathbb{R}^{n+1} \supset S_n$, equipped with coordinates $\{x=(x_1;x'):\ x_1 \in \mathbb{R},\ x'=(x_2;\ldots x_{n+1}) \in \mathbb{R}^n\}$. Respectively, polar coordinates $(\phi;\theta_1;\ldots\theta_{n-1})$ on S_n will be chosen so that ϕ measures the angle between vector x and the x_1-axis. By zonal potential we mean functions on S_n invariant under all $SO(n)$-rotations about x_1-axis, i.e. $V=V(x_1)$ or $V(\cos\phi)$.

In this section we shall state the main result of the paper on asymptotics of the joint spectrum of the zonal Schrödinger operator $H=-\Delta+V$, and so-called *angular momentum algebra* $\mathscr{M}$, the Lie algebra $so(n)$ of the axisymmetric group $SO(n)$.

The eigenvalues of the unperturbed Laplacian $-\Delta|S_n$ are well known: $\{\lambda_k = k(k+n-1)\}_{k=0}^{\infty}$, and the corresponding eigenspaces $\mathscr{H}_k = \mathscr{H}_k(S_n)$ consist of spherical harmonics of degree k on S_n. The dimension of $\mathscr{H}_k^{(n+1)}$ = multiplicity of $\{\lambda_k\}$, is equal to $d(k) = d_n(k) = \binom{n+k}{n} - \binom{n+k-2}{n} = O(k^{n-1})$.

Perturbation V destroys the underlying rotational symmetry of Δ, so each multiple eigen λ_k splits into a cluster of simple (or less degenerate) eigens $\{\lambda_{km} = k(k+n-1) + \mu_{km} : m = 1;\ldots d(k)\}$. We are interested in asymptotics of spectral shifts $\{\mu_{km}\}$ at large k.

To describe the structure of clusters and the multiplicities of spectral shifts $\{\mu_{km}\}$ for zonal Schrödinger operators, we consider the full orthogonal group $G = SO(n+1)$ acting on S_n and the isotropy subgroup $SO(n)$ of the "northern pole" $x_1 = 1$, i.e. rotations about the x_1-axis.

The action of $SO(n)$ on the "equatorial" $(n-1)$-sphere breaks the function space $L^2(S_{n-1}; d\theta)$ into the direct sum of irreducible components $\bigoplus_{m=0}^{\infty} \mathscr{H}_m$, each $\mathscr{H}_m = \mathscr{H}_m(S_{n-1})$ built of spherical harmonics of degree m on S_{n-1}.

Accordingly the whole space $L^2(S_n)$, on the n-sphere, breaks into the direct sum of subspaces

$$L^2(S_n) = \bigoplus_{m=0}^{\infty} \mathscr{L}_m; \tag{1.1}$$

the m^{th} component $\mathscr{L}_m \simeq L^2[\sin^{n-1}\phi\, d\phi] \otimes \mathscr{H}_m$ being identified with a subspace of functions $f(\phi;\theta)$ on S_n, that *transform according to an irreducible representation*

π^m of $SO(n)$ on $\mathscr{H}_m$ (see e.g. [Vi]). Of course, on S_2 this would correspond to the m^{th} eigensubspace of the angular momentum operator $M = i\partial_\theta$—the infinitesimal generator of the $SO(2)$-action. On the n-sphere, the role of "angular momentum" is played by the Lie algebra $\mathscr{M} = so(n)$, but subspaces $\mathscr{L}_m$ can also be characterized as eigenspaces of the S_{n-1}-Laplacian Δ_{n-1} (so-called *Casimir element* of $\mathscr{M}$),

$$\mathscr{L}_m = \{ f \in L^2(S_n) : \Delta_{n-1}(f) = -m(m+n-2)f \}.$$

Let us observe that zonal Schrödinger operators commute with $SO(n)$. So by analogy with the 2-sphere ([Gur3-4]) we can talk about the *joint eigenfunction expansion* of the pair $\{H; \mathscr{M}\}$. This means a decomposition of $L^2(S_n)$ into the direct sum of eigenspaces E_{km} of H, invariant under $\mathscr{M}$, so that $H|E_{km} = \lambda_{km}$, and $\mathscr{M}$ (respectively $SO(n)$) acts on E_{km} by an irreducible representation π^m. Equivalently, $\{E_{km}\}$ can be characterized as joint eigenspaces of H and Δ_{n-1}.

So the spectrum of a zonal Schrödinger operator H consists of clusters of eigenvalues: $\Lambda_k = \{\lambda_{km} = k(k+n-1) + \mu_{km}\}_{m \leq k}$, the multiplicity $d_k(m)$ of λ_{km} being equal to the degree $d(\pi^m) = \dim \mathscr{H}_m(S_{n-1})$. The angular momentum algebra $\mathscr{M}$ (or its Casimir element) provides the natural labeling of the eigenvalues/spectral shifts $\{\mu_{km}\}$ in the k^{th} cluster.

To state the result we shall introduce an important notion of the *Radon transform* $\mathfrak{R}$ on S_n. The Radon transform maps functions $f(x)$ on S_n into functions (symbols) $\tilde{f}(x, \xi)$ on the cosphere bundle $S^*(S_n)$, or more precisely, on the space $\mathcal{O}$ of all closed geodesics (great circles) $C = C_{x\xi}$. Here ξ denotes a unit cotangent vector at point $x \in S_n$, and $C_{x\xi}$ a unique geodesics through $\{x\}$ in the direction ξ. So $\tilde{f}$ will be considered either as a function of $(x; \xi)$, or of $C = C_{x\xi}$.

By definition $\tilde{f}$ is obtained by averaging f over great circles $C = C_{x\xi}$,

$$\mathfrak{R}: f \to \tilde{f}(x, \xi) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_C f \, ds. \tag{1.2}$$

Obviously, the Radon transform takes even functions into even and odd into zero. For zonal functions f, $\tilde{f}$ also turns out to be "zonal," in the sense that $\tilde{f}^{\text{th}}$ depends on a single variable. We shall demonstrate it below and find explicit formula of the reduced Radon transform on zonal functions.

The cosphere bundle $S^*(S_n)$ consists of all pairs of unit orthogonal vectors $(x; \xi)$ in $\mathbb{R}^{n+1}$, equivalently complex vectors $z = x + i\xi \in \mathbb{C}^{n+1}$, satisfying $z \cdot z = 0$, $|z|^2 = 2$. Space $\mathcal{O}$ of closed geodesics $\{C = C_{x\xi} = C_z\}$ is obtained by taking the quotient of $S^*(S_n)$ modulo the natural $SO(2)$-action.

We denote the components of $(x; \xi)$ by $\begin{pmatrix} x_1 & \xi_1 \\ x' & \xi' \end{pmatrix}$, where $x_1; \xi_1 \in \mathbb{R}$ and $x', \xi' \in \mathbb{R}^n$. Then

$$x_1 \xi_1 + x' \cdot \xi' = 0; \quad \text{and} \quad {x_1}^2 + |x'|^2 = {\xi_1}^2 + |\xi'|^2 = 1.$$

By definition (1.2),

$$\tilde{f}(x, \xi) = \frac{1}{2\pi} \int f(x \cos\theta + \xi \sin\theta) \, d\theta = \frac{1}{2\pi} \int f(x_1 \cos\theta + \xi_1 \sin) \, d\theta,$$

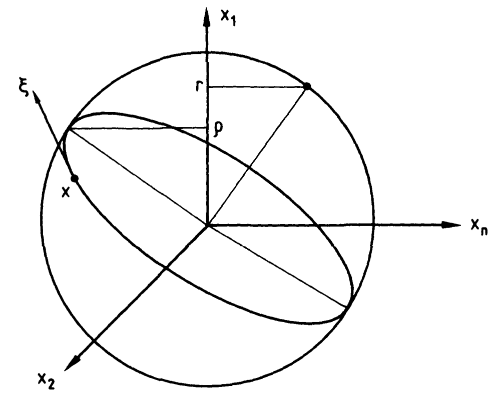

Fig. 1. The "highest-point" and the "polar-sphere" parameters: $\rho = \rho(C)$ and $r = r(C)$

and the latter reduces to

$$\tilde{f}(\rho) = \frac{2}{\pi}\int_0^1 f(\rho t)\frac{dt}{\sqrt{1-t^2}} = \frac{2}{\pi}\int_0^\rho f(t)\frac{dt}{\sqrt{\rho^2-t^2}}, \tag{1.3}$$

where parameter

$$\rho = \sqrt{x_1{}^2 + \xi_1{}^2} \tag{1.4}$$

measures the highest (relative to the x_1-axis) point of the great circle $C_{x\xi}$ (Fig. 1).

It will be often convenient to consider $\tilde{f}$ as a function of another parameter:

$$r = \sqrt{1-\rho^2} = \sqrt{1 - x_1{}^2 - \xi_1{}^2}, \tag{1.5}$$

the x_1-coordinate of the "polar sphere" of $C_{x\xi}$ (Fig. 1). In terms of the r-parameter the *Reduced Radon transform* takes the form

$$\mathfrak{R}: f \to \tilde{f}(r) = \frac{2}{\pi}\int_0^{\sqrt{1-r^2}} f(t)\frac{dt}{\sqrt{1-t^2-r^2}}, \tag{1.6}$$

as defined in [Gur4-5].

Two other differential operators on [0; 1] will appear in the formulation of Theorem 1 along with $\mathfrak{R}$: the *Gegenbauer/Legendre operator*

$$\mathfrak{L} = \frac{d}{dx}(1-x^2)\frac{d}{dx} + \left(\frac{\alpha x}{1-x^2}\right)^2 - \alpha, \quad \text{with} \quad \alpha = \frac{n-2}{2}, \tag{1.7}$$

and the operator $x(d/dx)$. Operator $\mathfrak{L}$ typically arises in the reduction of the n-sphere Laplacian Δ_n on the subspace of *zonal functions* ([Sz], ch. 4).

Now we can state our main result. Let $H = -\Delta + V$ be a Schrödinger operator on S_n with a zonal potential $V = V(\cos\phi)$, ("quantum spherical pendulum" in the terminology of [CD]). We decompose V into the even and odd parts: $V = V_{\text{ev}} + V_{\text{od}}$, and denote by $\mathcal{M}$ the angular momentum algebra $so(n)$, acting on $L^2(S_n)$.

Theorem 1. *Spectral shifts $\{\mu_{km}\}$ of the joint eigenvalue Problem of H and $\mathcal{M}$ (or H and Δ_{n-1}) admit the following asymptotic expansion in powers of k^{-1}, as $k \to \infty$,*

$$\mu_{km} = a\left(\frac{m}{k}\right) + k^{-1} b\left(\frac{m}{k}\right) + k^{-2} c\left(\frac{m}{k}\right) + O(k^{-3}), \tag{1.8}$$

uniformly in $|m| \leqq k$. The function-coefficients $\{a; b; c\}$ of (1.8) are expressed in terms of the reduced Radon transform $\mathfrak{R}$ (1.6), the Gegenbauer/Legendre operator $\mathfrak{L}$ (1.7) and the operator $x(d/dx)$, applied to potential V, its even and odd parts V_{ev}, V_{od}, as functions on $[0; 1]$. Namely,

$$\begin{aligned}
a(r) &= \mathfrak{R}(V_{\mathrm{ev}}) = \tilde{V}_{\mathrm{ev}}(r);\\
b(r) &= -\tfrac{1}{4}\mathfrak{R}\mathfrak{L}(V_{\mathrm{ev}});\\
c(r) &= \tfrac{1}{32}\mathfrak{R}\{\mathfrak{L}^2 + 2(n+1)\mathfrak{L}\}(V_{\mathrm{ev}})\\
&\quad + \tfrac{1}{2}\{\mathfrak{R}(V_{\mathrm{ev}}{}^2 + V_{\mathrm{od}}{}^2) - (\mathfrak{R}V_{\mathrm{ev}})^2\}\\
&\quad - \frac{r^2}{2(1-r^2)}\mathfrak{R}\left(x\frac{d}{dx}(V_{\mathrm{ev}}{}^2 + V_{\mathrm{od}}{}^2)\right).
\end{aligned} \tag{1.9}$$

Special cases of Theorem 1 for even and odd zonal potentials V on S_2 were established in our earlier works [Gur4-5], namely

$$\mu_{km} = \tilde{V}\left(\frac{m}{k}\right) + O(k^{-1}), \quad \text{for even } V,$$

and

$$\mu_{km} = k^{-2} U\left(\frac{m}{k}\right) + O(k^{-3}), \quad \text{for odd } V,$$

with $U(x)$ obtained from V^2 by the transform $\mathfrak{I} = \frac{1}{2}(\mathfrak{R} - (r^2/1 - r^2)\mathfrak{R}x(d/dx))$.

Theorem 1 has numerous applications. On one hand it exhibits a multiscale structure of spec(H) with 3 levels: the highest $= O(k^2)$, measuring the location of clusters, intermediate $= O(1)$, measuring the cluster size, and the lowest $= O(k^{-1})$ (or $O(k^{-3})$), which shows individual spectral shifts within clusters. Each spectral shift is in turn expanded into the power series of k^{-1}.

Theorem 1 confirms the well known results on the cluster size of Λ_k ([We], [Gui2]): $\mu_{km} = O(1)$ for even/generic V, and $\mu_{km} = O(k^{-2})$ for odd V.

In Sect. 3 we shall use Theorem 1 to prove a variety of spectral results for zonal Schrödinger operators on S_n, including solution of the Inverse Problem for the joint $(H; \mathcal{M})$-spectrum; explicit formulae for the Weinstein *band invariants* of cluster distribution measures (cf. [We], [Gui], [Ur]), and *local Spectral rigidity* for generic zonal potentials on S_n.

2. Proof of the Main Result

The proof will consists of three major steps:

1° *Averaging methods.*

2° *Symbolic Calculi on S_n: estimates and asymptotics of spectral shifts.*

3° *Zonal reduction procedure.*

The key idea of the 3rd step is to transform the multidimensional Schrödinger eigenvalue Problem into a series of 1-D associated Gegenbauer–Legendre Problems.

Step 1°. Averaging Methods. It is convenient to write the Schrödinger operator as $H = -\Delta + \alpha^2 + V$, with $\alpha = (n - 1/2)$. By a proper scalar adjustment, $V \to V +$ Const, one can always assume H to be positive. Then $\sqrt{H}$ exists and is equal to the sum of two pseudodifferential operators (ψDO's) $A + B$ of orders 1 and -1 respectively,

$$A = \sqrt{-\Delta + \alpha^2}; \quad B = \sqrt{H} - A \approx \tfrac{1}{2}A^{-1}V + \cdots.$$

A ψdo B can in turn be expanded into a series of decreasing order terms $B_1 + B_2 + B_3 + \cdots$[5]. To compute the lower order terms of B we write

$$H = -\Delta + V = (A + B)^2 = A^2 + 2AB + [B; A] + B^2.$$

Assuming $B = B_1 + B_2 + B_3 + \cdots$ and inserting this expansion in $H = (A + B)^2$, one gets

$$(A + B)^2 = A^2 + 2AB + [B; A] + B^2 = A^2 + V; \quad \text{and} \quad B^2 = B_1{}^2 + 2B_1B_2 + \cdots. \tag{2.1}$$

Dividing both sides of (2.1) by $2A$ on the left, and introducing operations $\mathrm{ad}_A(\ldots) = [\ldots; A]$ and $T = \frac{1}{2}A^{-1}\,\mathrm{ad}_A$, we rewrite the latter as

$$A + (I + T)[B_1 + B_2 + \cdots] + \tfrac{1}{2}A^{-1}(B_1{}^2 + 2B_1B_2 + \cdots) = A + \tfrac{1}{2}A^{-1}V. \tag{2.2}$$

It follows by the standard Product/Commutator rules of ψDO Calculus (to be explained below), that operation T lowers the order by 1. So all terms of (2.2) can be arranged in decreasing orders and the first few of them are computed explicitly:

$$\begin{aligned} B_1 &= \tfrac{1}{2}A^{-1}V && (\text{order } -1),\\ B_2 &= -TB_1 = -\tfrac{1}{4}A^{-2}\,\mathrm{ad}_A(V) && (\text{order } -2),\\ B_3 &= \tfrac{1}{8}(A^{-3}\,\mathrm{ad}_A{}^2(V) - A^{-2}VA^{-1}V) && (\text{order } -3). \end{aligned}$$

Now we shall outline three averaging methods for perturbations $A + B$, known in the literature (see [We], [Gui2], [Ur2]).

Let us observe that operator $(A - \alpha)$ has integral spectrum $\{0; 1; \ldots k; \ldots\}$, hence generates a periodic unitary group $\{e^{itA}: 0 \leqq t \leqq 2\pi\}$.

We want to replace potential V of H, or perturbation B of A, by an operator W, that commutes (or almost commutes) with Δ, and such that $\mathrm{spec}(A + W)$ approximates $\mathrm{spec}(A + B)$.

1° *Weinstein's average* is defined as

$$B \to \bar{B} = \frac{1}{2\pi} \int_0^{2\pi} B(t)\,dt, \tag{2.3}$$

[5] Let us remark that $B \sim \sum B_j$, is not a standard expansion of the ψdo calculus. In fact, expansions to lower orders are generally not well defined on manifolds. In the next section we shall introduce a form of "nonstandard calculus" on S_n due to A. Uribe, that will allow such expansions

where $B(t)$ denotes the conjugates of B by the unitary group of A, $B(t) = e^{-itA}Be^{itA}$.

It is easy to see that $\bar{B}$ commutes with A. Weinstein [We] has shown that operators $A+B$ and $A+\bar{B}$ are "almost unitarily equivalent," modulo "small remainder." Precisely, there exists a skew symmetric operator Q and unitary $U = e^Q$, so that

$$A + B = U^{-1}(A + \bar{B})U + R, \tag{2.4}$$

where the remainder R is of lower order than B.

This allowed to reduce spectral asymptotics of operator $A + B$ (respectively H) to those of $A + \bar{B}$, and consequently to get the leading asymptotics of the cluster distribution measures,

$$d\nu_k(\lambda) = \frac{1}{d_k}\sum_m \delta(\lambda - \mu_{km}), \quad \text{as} \quad k \to \infty,$$

the so-called Weinstein's *band-invariant* β_0 [We].

Furthermore, in case of zonal potentials on S_2 [Gur4] we use the Weinstein average to get the leading asymptotics of spectral shifts $\{\mu_{km}\}$, which eventually led to solution of the Inverse Problem and the proof of Spectral Rigidity for even zonal potentials.

Unfortunately, Weinstein's averaging applies only in the even case.

2° In the odd case V. Guillemin [Gui2] developed the so-called *Return operator approach.* The return operator W defined as

$$W = e^{2\pi(A+B)} - I,$$

can be expanded into the series[6]

$$W_1 + W_2 + W_3 + \cdots = 2\pi i\bar{B} - 2\pi^2\bar{B}^2 + 2\pi i\bar{\bar{B}} + \cdots. \tag{2.5}$$

The first 3 terms of (2.5) are expressed by the 1st (Weinstein) average $\bar{B}$ and the 2nd (Guillemin) average

$$B \to \bar{\bar{B}} = \frac{1}{4\pi i}\int_0^{2\pi}\int_0^t [B(t); B(s)]\, ds\, dt. \tag{2.6}$$

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[6] Expansion (2.5) with operators (2.3), (2.6) are derived from the standard *time-ordered perturbation series* for the unitary group $e^{it(A+B)}$,

$$e^{it(A+B)} = e^{itA} + i\int_0^t e^{i(t-s)A}Be^{isA}\, ds - \int_0^t ds \int_0^s d\tau\, e^{i(t-s)A}Be^{i(s-\tau)A}Be^{i\tau A} + \cdots.$$

The first two terms of the series can be recast in the form

$$e^{it(A+B)} = e^{itA}\left\{I + i\int_0^t B(s)\, ds - \int_0^t\int_0^s B(s)B(\tau)\, d\tau\, ds + \cdots\right\}.$$

Then rewriting the double integral as

$$\frac{1}{2}\left(\int_0^t B(s)\, ds\right)^2 - \frac{1}{2}\int_0^t\int_0^s [B(s); B(\tau)]\, d\tau\, ds,$$

and substituting $t = 2\pi$ (using 2π-periodicity of $\{e^{itA}\}$), we get (2.5)

Let us note that the eigenvalues of W are simply related to spectral shifts $\{\mu_{km}\}$ of $A+B$, namely

$$\lambda_{km}(W) = e^{2\pi i \mu_{km}} - 1.$$

As in the even case it can be shown that operator $\bar{\bar{B}}$ commutes with A, modulo l.o.t., and the pair $A+B$ and $A+\bar{\bar{B}}$ are almost unitary equivalent. Thus one gets the leading asymptotics of cluster distribution measures for odd potentials (band-invariant β_2, [Gui2]; [Ur2]). Furthermore, using the 2nd averaging method, we derived the leading asymptotics of individual spectral shifts $\{\mu_{km}\}$, and applied them to the Inverse Problem and spectral rigidity for odd zonal potentials on S_2 [Gur5].

3° In the general (even + odd) case A. Uribe [Ur2] refined further the Weinstein averaging by subsequent application of (2.3) to remainders R (2.4). We shall adopt this procedure in our derivation, so let us briefly outline the basic steps.

An intertwining operator Q in (2.4) can be constructed explicitly

$$Q = \frac{1}{2\pi i} \int_0^{2\pi} \int_0^t B(t)\, ds\, dt, \tag{2.6}$$

and is shown to satisfy the commutation relation

$$[A; Q] = \bar{B} - B. \tag{2.8}$$

Exponentiating (2.8) one finds (see [We]; [Ur1])

$$e^Q(A+B) = (A+\bar{B})e^Q + R,$$

where the remainder R is expanded into the series

$$R = \tfrac{1}{2}[B+\bar{B}; Q] + \cdots.$$

If perturbation B had order $-m$, then order $(R) = -2m-1$. Successive application of the Weinstein averaging (2.3) to remainders, $R \to \bar{R} = R^{\mathrm{av}}$, yields a ψdo $W = \bar{B} + \tfrac{1}{2}[B+\bar{B}; Q]^{\mathrm{av}} + \cdots$, which commutes with A, and such that $A+W$ is unitary equivalent to $A+B$, modulo ∞-smoothing operators.

For our purposes we shall need only the first correction R^{av} to $\bar{B}$. A straightforward calculation by means of (2.8) then gives[7]

$$R^{\mathrm{av}} = \tfrac{1}{2}[B+\bar{B}; Q]^{\mathrm{av}} = [\bar{B}; Q] + \frac{i}{4\pi} \int_0^{2\pi} \int_0^t [B(t); B(s)]\, ds\, dt. \tag{2.9}$$

Notice that each of two terms in the right-hand side of (2.10) by itself does not commute with A, but their sum R^{av} does.

[7] To check (2.10) we use the differentiation formula for $Q(t) = e^{-itA} Q e^{itA}$: $Q'(t) = i[Q; A] = i(B - \bar{B})(t)$, whence follows $Q(t) = Q + i\int_0^t (B-\bar{B})(s)\, ds$. Then substituting $Q(t)$ into the average commutator $[B; Q]^{\mathrm{av}} = \frac{1}{2\pi} \int_0^{2\pi} [B(t); Q(t)]\, dt$, yields $[B; Q]^{\mathrm{av}} = 2[\bar{B}; Q] + \frac{1}{2\pi} \int_0^{2\pi} \int_0^t [B(t); B(s)]\, ds\, dt$, while $Q^{\mathrm{av}} = -i\pi\bar{B}$, implies $[\bar{B}; Q]^{\mathrm{av}} = 0$

Thus the average operator to be studied is

$$W=\bar{B}+[\bar{B};Q]+\frac{i}{4\pi}\int_0^{2\pi}\int_0^t [B(t);B(s)]\,ds\,dt. \tag{2.10}$$

Let us observe that W incorporates both the Weinstein average $\bar{B}$ (of order -1) and Guillemin's $\bar{\bar{B}}$ (of order -3).

In what follows it will be more convenient to replace operators $\{B(t)\}$, as well as their averages $\bar{B}$, Q, etc., by conjugates and averages of the multiplication operator V. Remembering, that $B=\frac{1}{2}A^{-1}V$ and A commutes with the averaging, it follows

$$B(t)=\tfrac{1}{2}A^{-1}V(t);\ \bar{B}=\tfrac{1}{2}A^{-1}\bar{V};\ Q=\tfrac{1}{2}A^{-1}\tilde{Q},$$

where $\tilde{Q}$ is defined similar to Q (2.6), with $V(t)$ in place of $B(t)$,

$$\tilde{Q}=\frac{1}{2\pi i}\int_0^{2\pi}\int_0^t V(t)\,ds\,dt.$$

After a somewhat tedious calculation involving the differentiation formulae: $(d/dt)V(t)=i[V(t);A]$, commutation relations $[A^{-1};\tilde{Q}]=A^{-2}(V-\bar{V})$, and integration by parts we end up with the following expression of the complete (Uribe) average,

$$W=\tfrac{1}{2}A^{-1}\bar{V}+\tfrac{1}{4}A^{-3}(\bar{V}^2-\bar{V}^2)+\tfrac{1}{4}A^{-2}\left([\bar{V};\tilde{Q}]+\frac{1}{4\pi i}\iint[V(t);V(s)]\,ds\,dt\right). \tag{2.11}$$

We need to show that the spectrum of operator $A+B$ is approximated by the spectrum of its average $A+W$. The following results will play the crucial role here.

Lemma 1. *Operators $A+B$ and $A+W$ are almost unitary equivalent,* i.e. $U^{-1}(A+W)U=A+B+R_1$, *with the remainder R_1 of order -4. The remainder R_1 is estimated in terms of A as*

$$|R|=(R^*R)^{1/2}\leqq \mathrm{Const}\,A^{-4}, \tag{2.12}$$

in the sense of comparison of selfadjoint operators.

The proof is essentially outlined above and consists of applying twice the Weinstein average, first to perturbation B, then to remainder $R=\frac{1}{2}[B+\bar{B};Q]+\cdots$ (see [Ur2], for details). Estimate (2.12) follows from the general properties of ψdo's of order -4, in fact any ψdo B of oder $-m$ on S_n satisfies, $|B|\leqq \mathrm{Const}\,A^{-m}$ (see [We]).

As a consequence we can approximate shifts $\{\mu_{km}\}$ of operator $A+B$ by the shifts $\{\bar{\mu}_{km}\}$ of $A+W$, to order -4,

$$|\mu_{km}-\bar{\mu}_{km}|\leqq \mathrm{Const}\,k^{-4}.$$

Our goal now is to expand μ_{km} (respectively $\bar{\mu}_{km}$) into the power series in k^{-1},

$$\mu_{km}\sim k^{-1}(a_{km}+b_{km}k^{-1}+c_{km}k^{-2}+\cdots), \tag{2.13}$$

and to compute the first 3 coefficients $\{a_{km};b_{km};c_{km}\}$. This goal will be accomplished via Symbolic Calculi on S_n to be described in the next step.

Step 2°: Symbolic Calculi on S_n; Estimates and Asymptotics of Spectral Shifts. We shall start with the standard Symbolic Calculus on S_n and find the principal symbols of operators A, B, as well as their averages: $\bar{B}$, $\bar{\bar{B}}$, Q and W, introduced in step 1°.

Given any *ψdo* B of order m on S_n one can associate to B its *principal symbol* $\sigma_B(x;\xi)$, a function on the phase space (cotangent bundle) $T^\circ(S_n)$, homogeneous of degree m in the momentum variable ξ.

Thus symbol of operator $A=\sqrt{-\Delta}$ is $\sigma_A=|\xi|$ (norm of ξ in the natural Riemannian metric on S_n). Symbol of perturbation $B=\sqrt{H}-A\approx\frac{1}{2}A^{-1}V+\cdots$ is obtained by the basic *Product Rule* of the *ψdo* calculus: $\sigma_{AB}=\sigma_A\sigma_B$ (for principal symbols!). So $\sigma_B=(V/2|\xi|)$.

To find symbols of averages we shall use two other basic Rules of Symbolic calculus.

(i) *Conjugation with a unitary group* $\{e^{itA}\}$, *generated by a self adjoint ψdo* A (known as Egorov's Theorem): symbol of a *ψdo* $B(t)=e^{-itA}Be^{itA}$ is obtained by composing σ_B with the Hamiltonian flow of $\sigma_A=a(x;\xi)$,

$$\sigma_{B(t)}=\sigma_B\circ\exp(t\Xi),$$

where $\Xi=\Xi_a=\partial_x a\cdot\partial_\xi-\partial_\xi a\cdot\partial_x$—the Hamiltonian vector field of A.

(ii) *Commutator Rule*: the commutator $[A;B]$ of two *ψdo*'s has symbol $\sigma_{[A;B]}=i\{\sigma_A;\sigma_B\}$—the Poisson bracket of two symbols[8].

In our case, symbol $a=|\xi|$, generates the geodesic flow on $T^*(S_n)$. If $V(t)$ denotes $V\circ\exp(t\Xi)$, i.e. V evaluated along the geodesics (great circle) $C=C_{x,\xi}$, as a function of the flow parameter, then

$$\sigma_{B(t)}=\frac{1}{2|\xi|}V(t);$$
$$\sigma_{\bar{B}}=\frac{1}{2|\xi|}\int V(t)\,dt;\quad \sigma_Q=\frac{1}{2|\xi|}\int(2\pi-t)V(t)\,dt.$$

We denote symbol of the operator $\tilde{Q}=\frac{1}{2\pi^i}\int(2\pi-t)V(t)\,dt$, by $\tau(V)$ to indicate its dependence on V,

$$\tau(V)=\frac{1}{2\pi}\int(2\pi-t)V(t)\,dt. \tag{2.14}$$

By some abuse of notation here and henceforth we use $V(t)$ for 3 different (but closely related) objects: the conjugate of the multiplication operator, $e^{-itA}Ve^{itA}$; its symbol, function V composed with the geodesic flow; and finally V evaluated

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[8] Two rules are obviously related: (ii) essentially means that the map $A\to\sigma_A$ is a homomorphism of the Lie algebra of operators on $L^2(S_n)$ into the Poisson algebra of functions on $T^*(S_n)$, while (i) is obtained by exponentiating the Lie derivatives of operators/symbols: $\mathrm{ad}_A(B)=[A;B]$ (respectively $i\{\sigma_A;\sigma_B\}$), to the corresponding Lie group of inner automorphisms, $B\to e^{-itA}Be^{itA}$ (respectively $\sigma_B\to\sigma_B\circ\exp t\Xi_A$), generated by hermitian elements A

along the circle $C_{x\xi}$ (as a function of the flow parameter t). We hope that in each case the meaning of "$V(t)$" would be clear from the context.

Finally, to get symbol of R^{av} (a ψdo of order -3), we apply the above Rules of symbolic manipulation to (2.11). This yields ([Ur2]):

$$\sigma_{R^{av}} = \tfrac{1}{4}((V^2)^{av} - (V^{av})^2) + \tfrac{1}{4}\{\tau(V); V^{av}\} - \frac{1}{8\pi}\int_0^{2\pi}\int_0^t \{V(t); V(s)\}\, ds\, dt. \quad (2.15)$$

In special cases of even or odd V, spectral shifts $\{\mu_{km}\}$ of the operator $A + B$ are reduced to eigenvalues of an appropriate average ψdo, $\bar{B}$ or $\bar{\bar{B}}$.

The main difficulty of the general (even + odd) case is that the "even" and "odd" contributions do not combine, because of different orders of two averages. As the reader might have noticed the 1st average $\bar{B}$, represents a ψdo of order -1, so spectral shifts $\{\mu_{km}\}$ of the even Schrödinger operator $H = (A + B)^2$ are $O(1)$. On the other hand for odd V, the 2nd average $\bar{\bar{B}}$ has order -3, hence $\mu_{km}(H) = O(k^{-2})$.

In each case (even or odd) separately the leading asymptotics of spectral shifts (and the related band-invariants) can be deduced from the appropriate *principal symbols* of operators $\bar{B}$ and $\bar{\bar{B}}$ (cf. [Gur4-5]).

In general case, however, we need a *complete symbolic expansion* to order -3. The standard Symbolic calculus (see, e.g. [Ta]) does not provide such expansions on manifolds, different from $\mathbb{R}^n$, as lower order symbols are ill defined. Several attempts were made in the literature to construct *Complete Symbolic Calculi* (cf. [Wi]; [Be]; [Ur1]).

Here we shall adopt the form of Symbolic Calculus on S_n introduced by A. Uribe [Ur1], and based on the representation theory of the Lie group $SO(n+1)$. So let us briefly review the basic concepts and results of the Uribe calculus, referring to [Ur1] for more details.

The Uribe calculus was specifically tailored for the algebra $\mathfrak{U}$ of operators $\{B\}$, that commute with the Laplacian (or $A = \sqrt{-\Delta}$), and can be formally expanded into powers of A^{-1}:

$$B \approx B_0 + A^{-1}B_1 + A^{-2}B_2 + \cdots, \quad \text{with operator-coefficients } [B_j; A] = 0. \quad (2.16)$$

Symbols of operators B in $\mathfrak{U}$ will be defined on the manifold $\mathcal{O}$ of *closed bicharacteristics*, trajectories of the Hamiltonian flow $\{\exp(t\Xi)\}$ of $\sigma_A = |\xi|$ on the cosphere bundle $S^*(S_n)$. As we mentioned in Sect. 1 the cosphere bundle is identified with the set of complex vectors $\{z = x + iy \in \mathbb{C}^{n+1}\}$, satisfying

$$z \cdot z = |x|^2 - |y|^2 + 2ix \cdot y = 0; \quad |z|^2 = |x|^2 + |y|^2 = 2,$$

equivalently, the set of pairs $\{x; y\}$ of orthogonal unit vectors in $\mathbb{R}^{n+1}$, with x running over the sphere, and y representing a unit covector at $\{x\}$.

The Hamiltonian flow, $\{\exp(t\Xi)\}$, then amounts to multiplication of complex vectors z with complex scalars: $z \to e^{it}z$; $(0 \leqq t \leqq 2\pi)$. So $\mathcal{O}$ represents a quotient space $S^*(S_n)/\mathbb{T}$, or, phrased differently, the space of complex rays (projective variety) of the algebraic set $\{z : z \cdot z = 0\}$.

We would like to associate to each operator-coefficient $B = B_j$ its symbol, a function $\chi = \chi_B(z)$ on $\mathcal{O}$.

The key element in the construction is the notion of a *coherent system* (*or coherent states* [Be]) on a homogeneous space X of the Lie group G. We denote the action of G on X by $g: x \to x^g$, and assume that X has a G-invariant measure dx. In our case group $G = SO(n+1)$ acts by orthogonal rotations, $z \to z^g$, on $\mathcal{O} = S^*/\mathbb{T}$.

We take a representation λ of G on an inner product space $\mathscr{V} = \mathscr{V}_\lambda$ of $\dim \mathscr{V}_\lambda = d_\lambda$, and define a *coherent states* as a continuous family of vectors $\{v(x) = v^\lambda(x)\} \subset \mathscr{V}$, labeled by points $z \in X$, with the property:

$$\lambda_g(v(z)) = v(z^g); \quad \text{for all } z \in X \text{ and } g \in G.$$

A coherent states defines a map of $\mathscr{V}$ into a function space on X,

$$\mathscr{F}: v \to \tilde{v}(z) = \langle v; v(z) \rangle,$$

and thus allows to embed λ into the regular representation of G on $L^2(X; dz)$[9]. Furthermore, each operator $A = A^\lambda$ on $\mathscr{V}_\lambda$ is determined by its *reproducing kernel*, $K_A(z; y) = d_\lambda \langle Av(z); v(y) \rangle$, so that

$$\mathscr{F}: Av \to \int_X K_A(z; y) \tilde{v}(y)\, dy.$$

The *Wick–Berezin symbol* of A is obtained by restricting the reproducing kernel on the diagonal,

$$\sigma_A(z; \lambda) \overset{\text{def}}{=} d_\lambda \langle Av^\lambda(z); v^\lambda(z) \rangle = K_A(z; z).$$

One can show that each operator A on $\mathscr{V}$ is uniquely determined by its Wick symbol, and the usual trace formula of Symbolic calculus holds,

$$\operatorname{tr} A = \int \sigma_A(z; \lambda)\, dz.$$

In applications to the n-sphere Calculus one does not fix any particular λ, but considers a sequence of representations $\{\lambda = \lambda^k\}$, and a sequence of operators $\{A_\lambda\}$ in spaces $\mathscr{V}_\lambda$, each equipped with an appropriate coherent states, based on a given homogeneous space X. Then *symbol* of $\{A_\lambda\}$ is understood as a sequence of functions $\{\sigma(z; \lambda) = \sigma_{A_\lambda}(z; \lambda)\}$ on $X \times \{\lambda^k\}_1^\infty$.

In our case the family $\{\lambda^k\}$ consists of all irreducible components of the regular representation of $SO(n+1)$ on $L^2(S_n)$. All of them are labeled by integers $\{k \geqq 0\}$, and each λ^k acts on the space $\mathscr{H}_k$ of spherical harmonics of degree k on S_n.

To define Wick–Berezin symbols we need a family of coherent states in each space $\mathscr{H}_k$. The natural choice are complex spherical harmonics

$$\{Y_{kz}(s) = \text{Const}(s \cdot z)^k;\, s \in S_n,\, z \in \mathbb{C}^{n+1}\},$$

properly normalized, $\| Y_{kz} \|_{L^2(S_n)} = 1$.

One easily verifies that polynomial functions $Y_{kz}(s) = (s \cdot z)^k$ on $\mathbb{R}^{n+1}$ are

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[9] Thus one can view a coherent state and the map $\mathscr{F}$ as a noncommutative version of the system of exponentials $\{e^{i\xi \cdot x}: \xi \in \mathbb{R}^n\}$ and the related Fourier transform on $\mathbb{R}^n$

harmonic[10], iff $z \cdot z = 0$. We normalize z to be $|z|^2 = 2$, and thus obtain a coherent states $\{Y_{kz}\}$ based on space $X = S^*(S_n)$.

Each operator B of the commutator algebra $\mathfrak{U}$ of Δ is well known to respect the decomposition of L^2 into the sum of spherical harmonics. So we associate to any such B a sequence of finite-D blocks $\{B_k\}$, acting on irreducible subspaces $\{\mathscr{H}_k\}$, $B_k = B|\mathscr{H}_k$. The corresponding Wick–Berezin symbols: $\sigma(z;k) = \langle B_k(Y_{kz}); Y_{kz}\rangle$, are then well defined functions on the set $\mathcal{O} = S^*(S_n)/\mathbb{T}$.

The sequence of coherent states $\{Y_{kz}\}_k$ has many interesting features relating it to the averaging and the Radon transform[11] on S_n. Precisely, any ψdo B of order m on S_n after averaging, yields a ψdo $\bar{B}$ in $\mathfrak{U}$, and the corresponding sequence of Wick symbols converges as $k \to \infty$ to the limiting symbol,

$\sigma_{\bar{B}}(z;k) \to \mathfrak{R}(\sigma_B)$—the Radon transform of the classical principal symbol of B.

More generally, A. Uribe has shown that symbols of operators $B \in \mathfrak{U}$ admit an expansion in powers of k^{-1},

$$\sigma_B(z;k) \sim \chi_0(z) + \chi_1(z)k^{-1} + \cdots + \chi_m(z)k^{-m} + \cdots \tag{2.17}$$

whose leading coefficient χ_0 coincides with the classical principal symbol of B. Here complex vector $z = x + i\xi$ representing an element C_z in $\mathcal{O}$, is identified with the orbit of point $(x;\xi)$ in $S^*(S_n)$ under the geodesic flow.

So lower order terms are well defined in the Uribe calculus, and we get a complete asymptotic expansion!

Having established expansion (2.17) Uribe proceeds to derive some basic properties of Symbolic calculi, like the Product Rule, i.e. complete expansion of $\sigma_{AB}(z;k)$, for a pair of operators A, B.

Then he calculates some examples, among them

i) Operator $A = \sqrt{-\Delta}$, and "functions of A" $f(A)$. Here (2.17) takes the form

$\sigma_A = k$ (exact!), and $\sigma_{f(A)} \sim f_0 + f_1 k^{-1} + \cdots$; for $f(x) = f_0 + f_1 x^{-1} + \cdots$.

ii) The multiplication operator $V: f \to Vf$, and its average V^{av}. The coefficients of V^{av} are given by the Radon transform of V and certain polynomial differential expressions in Δ, applied to V. The first 3 of them are

$$\sigma_{V^{\mathrm{av}}} \sim \mathfrak{R}(V) - \tfrac{1}{4}\mathfrak{R}(\Delta V)k^{-1} + \tfrac{1}{32}\mathfrak{R}(\Delta^2 + 2(n+1)\Delta)[V]k^{-2} + \cdots. \tag{2.18}$$

Let us remark that (2.18) involves only the even part of V, as its odd part is annihilated by $\mathfrak{R}$.

The Uribe calculus enables us to combine the contribution of both (even and

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[10] Indeed, $\Delta[(s\cdot z)^k] = k(k-1)(z\cdot z)(s\cdot z)^{k-2}$, for the $\mathbb{R}^{n+1}$-Laplacian Δ

[11] They are also interesting from the standpoint of the representation theory. Indeed, each circle $C = C_z$ in S_n, fixes a 2-plane $\mathscr{V}_1 = \mathrm{Span}(C_z)$ in $\mathbb{R}^{n+1}$, and the whole space $\mathbb{R}^{n+1}$ can be decomposed into the orthogonal sum of 2-planes, $\mathscr{V}_1 \oplus \mathscr{V}_2 \oplus \cdots$. With any such partition one associates a maximal abelian (Cartan) subgroup T of $SO(n+1)$. Group T consists of all block-diagonal 2-D rotations $\{u = (u(\theta_1); u(\theta_2); \ldots)\}$, each $u(\theta_j)$ rotating $\mathscr{V}_j$ by an angle θ_j. Then function Y_{kz} gives the so called *highest weight vector* of an irreducible representation λ^k of $SO(n+1)$ in space $\mathscr{H}_k$, i.e. $\lambda^k_u[Y_{kz}] = e^{ik\theta_1} Y_{kz}$, for all $u \in T$

odd) parts of V to get an expansion to order -3 of the average operator W of (2.10),

$$\chi_W(z;k) = \Re(V) - k^{-1}\{\tfrac{1}{4}\Re(\Delta V)\} + k^{-2}\{\tfrac{1}{32}\Re(\Delta^2 + 2(n+1)\Delta)[V] + \sigma_{R^{\mathrm{av}}}\}, \quad (2.19)$$

where $\sigma_{R^{\mathrm{av}}}$ is given by (2.15).

The first 3 terms of (2.19) will provide the asymptotic coefficients $\{a_{km}, b_{km}, c_{km}\}$ of (2.13) for spectral shifts μ_{km} in the zonal case. But this will require yet another step.

Step 3°. Zonal Reduction. Zonal reduction means the restriction of axisymmetric ($SO(n)$-invariant) operators from $L^2(S_n)$ to eigensubspaces $\{\mathscr{L}_m\}$ (1.1) of the angular momentum algebra $\mathscr{M}$, or its invariant Laplacian Δ_{n-1}.

The S_n-Laplacian in spherical coordinates $\{(\phi;\theta_1;\ldots\theta_{n-1})\}$ has the form

$$\Delta = \frac{1}{\sin^{n-1}\phi}\partial_\phi \sin^{n-1}\phi\,\partial_\phi + \frac{1}{\sin^2\phi}\Delta_\theta,$$

where Δ_θ is the S_{n-1}-Laplacian.

We denote the $\mathscr{L}_m$-reduced Laplacian and Schrödinger operators by

$$L_m = -\Delta|\mathscr{L}_m; \quad \text{and} \quad H_m = H|\mathscr{L}_m,$$

and use the same convention for all other axisymmetric operators, like

$$A = \sqrt{-\Delta} \to A_m = A|\mathscr{L}_m; \quad B = \sqrt{H} - A \to B_m, \text{ etc.}$$

Each operator L_m is isomorphic to a $d(m)$-multiple ($d(m) = \dim \mathscr{H}_m(S_{n-1})$) of an *associated Gegenbauer/Legendre operator* G_m ([Sz], ch. 4),

$$G_m = G_\beta = -\left(\partial_\phi{}^2 + (n-1)\cot\phi\,\partial_\phi + \frac{\beta(1-\beta)}{\sin^2\phi}\right) - \alpha^2,$$

with

$$\alpha = \frac{n-1}{2}, \quad \beta = \alpha + m, \quad (2.20)$$

in the standard $L^2(\sin^{n-1}\phi\, d\phi)$-realization of the subspace $\mathscr{L}_m$, or

$$G_\beta = -\left(\partial^2 + \frac{\beta(1-\beta)}{\sin^2\phi}\right) - \alpha^2$$

in the $L^2[d\phi]$-realization.

The eigenfunctions of G_β in the form (2.20) are given in terms of the well known Gegenbauer (ultraspherical) polynomials

$$\psi_k = C_{k-m}^{(\alpha+m)}(\cos\phi), \quad \text{or} \quad \psi_k = \sin^{2/n-1}\phi\, C_{k-m}^{(\alpha+m)}(\cos\phi),$$

and the corresponding eigenvalues are $\{\lambda_k = (k+\beta)^2 - \alpha^2; k = 0; 1; \ldots\}$.

Remembering the exact values of α and β (2.20), we see that $\mathrm{Spec}(L_m) = \{\lambda_k = (k+m)(k+m+n-1)\}_{k=0}^{\infty}$, coincides, as expected, with the m^{th} tail $\{\lambda_k : k \geqq m\}$, of $\mathrm{spec}(\Delta_n)$, and has a uniform multiplicity $d_m = \dim \mathscr{H}_m$.

The reduced Schrödinger operator H_m then becomes a perturbation of G_m with potential $V = V(\cos\phi)$,

$$H_m \simeq (G_m + V) \oplus I_{d_m}.$$

The *zonal reduction procedure* of our previous work [Gur4-5] exploited the underlying $SO(n)$-symmetry of the Problem, both on the classical and on the Quantum (operator) level, and combined it with symbolic calculations that involved *principal symbols* (cf. [Ka]; [KM]). The key observation made in [Gur4] was to show that all the above (averaging) constructions, expansions and estimates of axisymmetric operators, can be transferred from S_n to $\mathscr{L}_m$-reduced operators on $[0;1]$.

The reduced (1-D) ψDO's had principal symbols defined on the $(x;\xi)$-phase plane of the type:

$$\sigma_{A_m} = a = \sqrt{(1-x^2)\xi^2 + \frac{m^2}{(1-x^2)}}; \quad \sigma_B \sim \frac{V}{2a} + \cdots, \text{ etc.}$$

Consequently, all symbolic manipulations, and the ensuing formulae for eigenvalues and spectral shifts, were reduced to a one-variable "singular-degenerate" symbolic calculus on $[0;1]$.

The reduction procedure of the present work is somewhat different and leads more directly to our goal, formulae for the spectral shifts. As we shall see in the present scheme the $\mathscr{L}_m$-reduction is paramount to "quantizing" (in the sense of "discretizing" in the "joint eigenbasis") a commuting pair of operators: $A = \sqrt{-\Delta_n}$ and $P = \sqrt{-\Delta_{n-1}}$.

Precisely, let us introduce the algebra $\mathscr{A}$ of axisymmetric (zonal) operators $\{B\}$ on $L^2(S_n)$, i.e. the commutator of the angular momentum $\mathscr{M} \simeq so(n)$. Restricted on S_{n-1} all such B are given by certain "functions of Δ_{n-1} (or P)", $B = f(P)$.

So the intersection of two algebras $\mathscr{A} \cap \mathfrak{U}$ represents the joint commutator of A and P. Furthermore, any B in $\mathscr{A} \cap \mathfrak{U}$ is expanded into a power series (2.16) with coefficients $(B_j\}$ given by certain functions of P

$$B = \sum_{j \geqq 0} A^{-j} f_j(P). \tag{2.21}$$

From (2.21) one immediately reads off the eigenvalues of B,

$$\mu_{km}(B) = \sum_{j \geqq 0} k^{-j} f_j(m).$$

Our goal is to link such expansions to symbolic expansions of the preceding step 2°.

The angular momentum algebra $\mathscr{M} \simeq so(n)$ is generated by infinitesimal rotations about the x_1-axis. Symbols of all infinitesimal rotations $\eta \in \mathscr{G} = so(n+1)$, considered as 1st order differential operators on S_n(vector fields) $D_\eta = i\partial_\eta$, were computed in [Url], in terms of the *momentum map* $\Phi: T^*(s_n) \to \mathscr{G}^*$—dual space to the Lie algebra[12] $\mathscr{G}$. For any $\eta \in \mathscr{G}$ symbol of operator D_η is linear in k and equal to

$$\chi(z;k) = k\langle \eta; \Phi(z) \rangle, \tag{2.22}$$

[12] The momentum map arises whenever a Lie group G acts transitively by smooth transformations on a manifold X. Indeed, the differential ψ of the map $\psi: g \to x^g$, from G onto X, takes the Lie algebra $\mathscr{G}$ of G onto the tangent space T_x at $\{x\}$, hence the dual map $\Phi = (\partial\psi)^*$ embeds the cotangent space T^* into $\mathfrak{g}^*$.

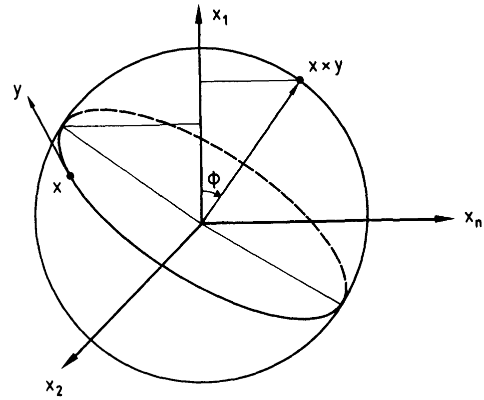

Fig. 2. Angle $\phi = \phi(x; y)$ between vector $x \times y$ and the x_1-axis

where $\langle ; \rangle$ denotes the natural pairing between the Lie algebra element η and the point $z \in S^*(S_n)$ lifted to the dual $\mathfrak{g}^*$ by the momentum map Φ. Let us remark that expansion (2.22) is exact, no l.o.t.!

To illustrate (2.22) we take the 2-sphere and the generator of rotations about x_1-axis, $D = i\partial_\theta$. Let $z = x + iy$, where $x \in S_2$, $y \in T_x^*$.

The peculiar feature of dim 3 is that the Lie algebra $so(3)$ can be identified with $\mathbb{R}^3$ itself, so that the Lie bracket turns into the cross product, $[x; y] \to x \wedge y$. The same holds for the dual space of $so(3)$, $\mathfrak{g}^* \simeq \mathfrak{g} \simeq \mathbb{R}^3$ (via the Killing form or the pairing $\langle x; y \rangle = \mathrm{tr}(xy)$!). It is easy to show that the momentum map on S_2 is also given by the cross product,

$$\Phi_x(y) = x \wedge y.$$

Consequently, symbol of D_η (2.22) becomes the triple product

$$\chi_\eta(z; k) = k\eta \cdot x \wedge y = k \det(\eta; x; y).$$

Specifying the latter for the field $D_\eta = i\partial_\theta$ (i.e. vector $\eta = (1; 0; 0)$), we find

$$\chi_\theta(z; k) = k \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = k(x \wedge y)_1 \text{—the 1}^{\text{st}} \text{ component of the cross product.}$$

It remains to rewrite the formula in polar coordinates on S_n and to observe that $(x \wedge y)_1 = \cos\phi$, in terms of the angle $\phi = \phi(x, y)$ between the orthogonal disk (polar axis) of the great circle C_{xy} and the x_1-axis (see Fig. 2). But $\cos\phi = \sqrt{1 - x_1{}^2 - y_1{}^2}$, in terms x_1-coordinates of vectors x and y, so the resulting expression for the symbol of $i\partial_\theta$ on S_2 takes the form

$$\chi_\theta(z; k) = k\cos\phi(x, y) = k\sqrt{1 - x_1{}^2 - y_1{}^2}.$$

On the n-sphere the role of ∂_θ is played by the operator $P = \sqrt{-\Delta_{n-1}}$. A similar argument applies to show

$$\chi_P(z; k) = k\cos\phi(x, y) = k\sqrt{1 - x_1{}^2 - y_1{}^2} = k\sqrt{1 - \rho^2},$$

where $\rho = \rho(C)$ is the "highest point" parameter (1.4) of the great circle $C = C_{xy}$ (Fig. 1).

Once the symbol of P is found we immediately get the symbol of any associated "function of P,"

$$\chi_{f(P)} = f(k\sqrt{1-\rho^2}). \tag{2.23}$$

Next we want to reduce (restrict) zonal operators on S_n to eigensubspaces $\{\mathscr{L}_m\}$ of operator P. Obviously, $f(P)|\mathscr{L}_m = f(m)I$, in other words the m^{th} reduction of such operators amounts to evaluating (restricting!) function f to the set of integers $\{m\}$.

Naturally, symbols of scalar (reduced) operators are expected to be scalars (constant functions). Therefore equating the right-hand side of (2.23) with the "would-be" symbol of the reduced operator, $\chi_{f(P)|\mathscr{L}_m} = f(m)$, we get

$$f(k\sqrt{1-\rho^2}) = f(m), \quad \text{for all } f, \quad \text{or} \quad m = k\sqrt{1-\rho^2}.$$

The latter explains the meaning of m^{th} reduction of zonal operators on the level of symbols, it corresponds to *quantizing the values of the parameter* $\rho = \rho(C_z)$ *in the Uribe (principal) symbol to a discrete set of values,*

$$\rho = \sqrt{1-\left(\frac{m}{k}\right)^2}; \quad k = 0; 1; \ldots.$$

Hence for any zonal operator $B = f(A; P)$ with Uribe symbol $\chi_B(z; k)$, the m^{th} reduced operator $B_m = B|\mathscr{L}_m$ has symbol

$$\chi_{B_m} = \chi_B\left(\sqrt{1-\left(\frac{m}{k}\right)^2}; k\right). \tag{2.24}$$

As we mentioned earlier reduced operators B_m are scalar multiples of the (1-variable) Gegenbauer/Legendre type ψdo's on $[-1; 1]$, $B_m = f(L_m)$. Symbols (2.24) acquire then particularly simple meaning: integer k represents the eigenvalue parameter of the reduced Laplacian $A_m = \sqrt{L_m}$, so all operators $\{B_m\}$ are simultaneously diagonalized in the eigenbasis of L_m, and "symbol B_m" means "diagonal B_m."

In particular, (2.24) yields eigenvalues of zonal operators B_m in terms of the Uribe symbol of B,

$$\mu_k(B_m) = \chi_B(\sqrt{1-(m/k)^2}; k). \tag{2.25}$$

To complete the proof of Theorem 1 we apply (2.25) to the average operator W restricted on the subspace $\mathscr{L}_m$. As we have shown operator W commutes with A (modulo l.o.t.), and for zonal potential V it belongs to the zonal algebra $\mathscr{A}$.

Thus spectral shifts (eigenvalues) $\{\mu_{km}\}$ of W_m, are given in terms of its Uribe symbol (2.19),

$$\mu_{km} = \chi_W(\sqrt{1-(m/k)^2}; k) = a_{km} + k^{-1}b_{km} + k^{-2}c_{km} + \cdots$$

with coefficients,

$$\begin{aligned} a_{km} &= \tilde{V}_{\text{ev}}(\sqrt{1-(m/k)^2}); \\ b_{km} &= -\tfrac{1}{4}(\Delta V_{\text{ev}})^{\sim}(\sqrt{1-(m/k)^2}); \\ c_{km} &= \{\tfrac{1}{32}\mathfrak{R}(\Delta^2 + 2(n+1)\Delta)V_{\text{ev}} + \sigma_{R^{\text{av}}}\}(\sqrt{1-(m/k)^2}). \end{aligned} \tag{2.26}$$

It remains to find the principal symbol of the operator R^{av} (2.15) for zonal potentials V. For the sake of presentation we carried out the calculation in the Appendix. Here we shall only state the final result,

$$\sigma_{R^{\text{av}}} = \tfrac{1}{2}(\mathfrak{R}(V_{\text{ev}}{}^2 + V_{\text{od}}{}^2) - (\mathfrak{R}V_{\text{ev}})^2) - \frac{1-\rho^2}{2\rho^2}\mathfrak{R}\left(x\frac{d}{dx}(V_{\text{ev}}{}^2 + V_{\text{od}}{}^2)\right), \tag{2.27}$$

where $\mathfrak{R}$ means the reduced Radon transform on $[0;1]$.

Formulae (2.26) along with the explicitly computed symbol $\sigma_{R^{\text{av}}}$ (2.27) produce the asymptotic coefficients $\{a;b;c\}$ of spectral shifts (2.13). The first two of them are equal to

$$a_{km} = a\left(\frac{m}{k}\right); \quad \text{with} \quad a = \mathfrak{R}(V_{\text{ev}}),$$

$$b_{km} = b\left(\frac{m}{k}\right); \quad \text{with} \quad b = -\tfrac{1}{4}\mathfrak{R}(\mathfrak{L}V_{\text{ev}}).$$

The Gegenbauer operator $\mathfrak{L} = (d/dx)(1-x^2)d/dx + (\alpha x/1-x^2)^2 - \alpha$, here arises from the Laplacian Δ_n in (2.18), reduced to the subspace of zonal functions.

Finally, the 3rd coefficient

$$c_{km} = c\left(\frac{m}{k}\right); \quad \text{where function } c = \{\tfrac{1}{32}\mathfrak{R}(\Delta^2 + 2(n+1)\Delta)V_{\text{ev}} + \sigma_{R^{\text{av}}}\},$$

with symbol $\sigma_{R^{\text{av}}}$ given by (2.27).

The completes the proof of Theorem 1.

3. Applications of Theorem 1: Inverse Problem, Band Invariants, Rigidity

In this section we shall apply Theorem 1 to various Problems in spectral theory of zonal Schrödinger operators. These will include

(i) *Inverse Problem*: determination of V from asymptotics of the joint spectrum of H and $\mathcal{M}$;

(ii) *Spectral Rigidity for zonal potentials*;

(iii) *Weinstein band-invariants of cluster distribution measures.*

We shall outline the results in special cases of even/odd V as well as in the general (even + odd) case.

1°. *Even Case.* In the even case asymptotics of Theorem 1 simplify to

$$\mu_{km} = \tilde{V}\left(\frac{m}{k}\right) - k^{-1}\mathfrak{L}\tilde{V}\left(\frac{m}{k}\right) + O(k^{-2}). \tag{3.1}$$

(i) *Inverse Problem.* From the leading term of (3.1) we immediately recover function $\tilde{V}$ on interval $[0;1]$,

$$\tilde{V}(x) = \lim_{m/k \to x} \mu_{km}.$$

Solution of the Inverse Problem for the joint spectrum then amounts to inverting the Radon transform, $\mathfrak{R}^{-1}:\tilde{V} \to V$. Let us observe that after the change of variables: $x \to \sqrt{x}$, $\rho \to \sqrt{\rho}$ (in the ρ-representation (1.3) of $\mathfrak{R}$), the Radon transform becomes a half line convolution kernel,

$$\mathfrak{R}: V(\sqrt{x}) \to \tilde{V}(\sqrt{\rho}) = \frac{1}{\pi}(x^{-1/2} V \circ \sqrt{x}) * x^{-1/2}. \tag{3.2}$$

The inversion is easily completed via the standard Laplace transform.

This result extends an earlier 2-D version of [Gur4] to S_n.

(ii) *Spectral Rigidity.* In this part we shall drop the assumption of the "bigraded joint spectrum of H and $\mathscr{M}$," and concentrate on spec(H) alone. In other words spectral shifts $\{\mu_{km}\}$ of the k^{th} cluster will now represent a collection of reals, labeled in an increasing order, rather than according to certain values of the angular momentum.

The Problem is stated as follows: givèn a collection of subsets $\Gamma_k \subset \mathbb{R}$, each of which consists of approximate values of an unknown function $f = \tilde{V}(x)$ on all k-partitions of $[0,1]$, $\Sigma_k = \{j/k : 0 \leqq j \leqq k\}$,

$$\Gamma_k = \left\{\mu_{kj} = f\left(\frac{j}{k}\right) + k^{-1}\mathfrak{L}f\left(\frac{j}{k}\right) + \cdots\right) \approx f(\Sigma_k), \quad k = 1;2;\ldots \tag{3.3}$$

we are asked to recover f, or rather to show that f is uniquely determined by such asymptotic data.

This Problem was solved in [Gur4] (Theorem 2), which showed that a generic Morse function f on $[0;1]$ (i.e. a function with finitely many critical points) is indeed locally uniquely determined from data (3.3).

Let us briefly sketch the argument.

First we show that the critical values of f along with their type (local min, max, inflection) are determined by subset $\{\Gamma_k\}$, in fact by singularities of the limiting distributon density $d\nu_f$,

$$d\nu_f = \lim_{k \to \infty} \#\left\{ j : \frac{j}{k} \in \Gamma_k \cap [y - \varepsilon; y + \varepsilon] \right\} / k = \text{"distribution function of } f(x)\text{."}$$

It remains then to recover the slope-function of $f(x)$ between any two subsequent critical values, i.e. along any monotone branch of f.

The crucial part of the argument, which also reveals of the "source" of rigidity of the n-sphere zonal Problem, lies in "almost arithmeticity" of subsets Γ_k, localized to a small vicinity of any noncritical value $\{y\}$. So the problem is then essentially reduced to the following elementary statement:

Rigidity of Arithmetic Sequences. If a subset $\Gamma \subset \mathbb{R}$ consists of a union of finitely many arithmetic sequences $\Gamma_i = \{a_i + jb_i : 1 \leqq j \leqq M\}$, with rationally independent differences $\{b_1;\dots b_N\}$, then all numbers $\{b_i\}$ can be uniquely recovered from Γ, i.e. the only arithmetic subsequences in Γ are those of some Γ_i!

Of course, our setting is compounded by the obvious nonarithmeticity of subsets $\Gamma_k \approx f(\Sigma_k) = \bigcup_j \Gamma_{ki}$ (index i labeling different branches of f), one reason being higher order corrections to $\{\mu_{km}\}$ (3.1), another the apparent global nonlinearity of $f(x)$! However, the above "pure arithmetic argument" can be refined by localizing Γ_k to sufficiently small intervals, $\Gamma_k \cap [y-\varepsilon; y+\varepsilon]$ where ε and k are allowed to change simultaneously so that $\varepsilon \to 0$, as $k \to \infty$. As a result one is able to recover exact values of slopes $\{b_j(y) = f'(x_i)\}$ at all preimages $\{x_j\}$ of a given generic value $\{y\}$, by a combination of the above "Arithmetic Rigidity" and a simple ergodic argument (see [Gur4] for details).

Once all slope-functions of different branches of f are recovered, there remains only a combinatorial (discrete) step in the reconstruction, hence the resulting discrete ambiguity! This step involves a suitable ordering of critical points and branches of f on [0; 1], whereon the exact distances between subsequent critical points will be *rigidly fixed* by the values of the appropriate slope-function (see Fig. 3).

(iii) *Band invariants.* The principal asymptotics (3.1) of spectral shifts yields the well known Weinstein 1st band invariant in a straightforward manner. Namely, a sequence of cluster distribution measures

$$d\nu_k = \frac{1}{d_n(k)} \sum_{m \leq k} d_{n-1}(m)\delta(x - \mu_{km}), \tag{3.4}$$

has a continuous limit as $k \to \infty$, $\beta_0 = \lim_{k\to\infty} d\nu_k =$ "distribution density of $\tilde{V}$ on $S^*(S_n)$."

The details will be omitted here, since later on we shall discuss the general case.

2°. *Odd Case.* The leading asymptotics of μ_{km} is now $O(k^{-2})$. Precisely,

$$\mu_{km} = k^{-2} c\left(\frac{m}{k}\right) + \cdots = k^{-2}\tfrac{1}{2}\left(\mathfrak{R} - \frac{r^2}{1-r^2}\mathfrak{R}\left(x\frac{d}{dx}\right)\right)[V^2] + \cdots.$$

Transform $\mathfrak{T} = \frac{1}{2}(\mathfrak{R} - (r^2/1-r^2)\mathfrak{R}(x(d/dx)))$ appeared first in [Gur5] in the 2-sphere case. As in the even case it allows explicit reconstruction of function V^2 from asymptotics on the joint spectrum. Namely,

(i) *Inverse Problem*: function $(\mathfrak{T}V^2)$ is found at all real points $r \in [0; 1]$, by

$$(\mathfrak{T}V^2)(r) = \lim_{m/k \to \Gamma} k^2 \mu_{km}.$$

To recover V^2 itself it remains to invert transform $\mathfrak{T}$. Using the above the ρ-parameter (1.4), and changing variables: $x \to \sqrt{x}$, $\rho \to \sqrt{\rho}$, transform $\mathfrak{T}$ is reduced to a psucdodifferential operation on [0; 1],

$$\mathfrak{T}: V^2(\sqrt{x}) \to \tfrac{1}{2}((1-x)|\partial|^{1/2} \operatorname{sgn} \partial - |\partial|^{-1/2} x^{-1/2})[V^2(\sqrt{x})], \tag{3.5}$$

applied to V^2.

Operation (3.5) is shown to be invertible[13] on suitable classes of functions on $[0; 1]$, which allows to recover V^2. Of course, V^2 does not yet determine V itself. But for generic V (e.g. functions with simple zeroes on $[0; 1]$), V is uniquely determined by the choice of its sign at any particular point $x_0 \in [0; 1]$. Thus from asymptotics of the joint spectrum we are able to recover $\pm V$.

Let us remark that the sign ambiguity $\pm V$ is to be expected, since two Schrödinger operators $-\Delta \pm V$ with odd V are obviously unitary equivalent.

(ii) *Spectral Rigidity* is established as in part 1° via Theorem 2 of [Gur4]. The unknown function here is $f = \mathfrak{T}V^2$, and we are given its approximate values on all k-partitions,

$$\Gamma_k = f(\Sigma_k) = \{k^2 \mu_{km} = (\mathfrak{T}V^2)(m/k) + \cdots; m \leqq k\}.$$

The rest of the argument proceeds as in the even case (see [Gur4]).

Thus we get *local spectral rigidity* for generic zonal odd potentials V on S_n, which extends our previous S_2-result [Gur5].

(iii) *Band invariants.* The sequence of cluster distribution measures (3.4) is expanded now into the series,

$$d\nu_k \sim \delta_0 + k^{-2}\beta_2 + \cdots$$

since the leading asymptotics of $\{\mu_{km}\}$ in the odd case is $O(k^{-2})$. The band-invariant β_2 can be found from a modified sequence of measures,

$$d\tilde{\nu}_k = \frac{1}{d_n(k)} \sum_{m \leqq k} d_{n-1}(m)\delta(x - k^2\mu_{km}). \tag{3.6}$$

As above the modified sequence (3.6) is shown to converge to a continuous limit = "distribution function of $\mathfrak{T}(V^2)$", which confirms [Ur2].

3°. General Even + Odd Case. Now we shall use the complete expansion of Theorem 1,

$$\mu_{km} = a\left(\frac{m}{k}\right) + k^{-1}b\left(\frac{m}{k}\right) + k^{-2}c\left(\frac{m}{k}\right) + \cdots \tag{3.7}$$

with function-coefficients $\{a; b; c\}$ given by (1.9). Notice that V_{ev} contributes to all three terms of (3.7), whereas the contribution of V_{od} is limited to $\{c\}$. Let us also remark that the 1st and 2nd coefficients are interdependent, since both represent transforms of the same function V_{ev}: $a = \mathfrak{R}(V_{ev})$; $b = -\frac{1}{4}\mathfrak{R}\mathfrak{L}(V_{ev})$.

(i) *Inverse Problem.* From spectral data we recover the values of functions a; b; c at all real points of $[0; 1]$,

$$a(x) = \lim_{m/k \to \Gamma} \mu_{km}; \quad b(x) = \lim_{m/k \to \Gamma} k(\mu_{km} - a(m/k)); \quad c(x) = \lim_{m/k \to r} k^2(\mu_{km} - \cdots).$$

Having thus found $\{a; b; c\}$ we proceed in two steps: first invert the Radon

[13] In fact, the explicit form of two pseudodifferential operations (3.2) and (3.5) allows not only to invert the transforms $\mathfrak{R}$ and $\mathfrak{J}$, but also to study suitable classes of input/output data $\{V\}$ for the Inverse Problem, and stability of the inversion procedure: {asymptotics of μ_{km}} $\Rightarrow$ {transformed data: $\tilde{V}; \ldots$} $\Rightarrow$ {input data V}

transform to find $V_{ev} = \mathfrak{R}^{-1}(a)$. Then we subsitute V_{ev} into the 3rd coefficient

$$c(r) = \tfrac{1}{32}\mathfrak{R}\{\mathfrak{L}^2 + 2(n+1)\mathfrak{L}\}(V_{ev}) + \tfrac{1}{2}\{\mathfrak{L}(V_{ev}{}^2) - (\mathfrak{R}V_{ev})^2\}$$
$$+ \frac{r^2}{2(1-r^2)}\left\{\mathfrak{R}(V_{ev})\mathfrak{R}\left(x\frac{d}{dx}V_{ev}\right)\right\} + \mathfrak{T}(V_{od}{}^2),$$

write $\mathfrak{T}(V_{od}{}^2)$ in terms of V_{ev}, and finaly invert transform $\mathfrak{T}$ in the resulting expression,

$$V_{od}{}^2 = \mathfrak{T}^{-1}\{c - \cdots(V_{ev})\}.$$

As above this would yield $V_{od}{}^2$, hence $\pm V_{od}$, but two Schrödinger operators with potentials $V = (V_{ev} \pm V_{od})$ are obviously unitary equivalent.

Thus we get a unique (modulo $\pm V_{od}$) and explicit solution on the Inverse Problem for the joint spectrum in the general (even + odd) case on S_n.

2°. *Spectral rigidity.* The argument once again is based on a two step application of Theorem 2 of [Gur4]. Namely, from the leading asymptotics of $\{\mu_{km}\}$ we recover a generic Morse function $a(x)$. Then the Gegenbauer/Legendre operator $\mathfrak{L}$ yields $b(x) = -\frac{1}{4}\mathfrak{L}[a]$. Finally the reconstruction procedure of Theorem 2 of [Gur4] applies to a modified sequence

$$\tilde{\mu}_{km} = k^2\left\{\mu_{km} - a\left(\frac{m}{k}\right) - k^{-1}b\left(\frac{m}{k}\right)\right\} = c\left(\frac{m}{k}\right) + \cdots,$$

to recover function $c(x)$.

Of course, all three coefficients $\{a; b; c\}$ will be uniquely (rigidly) determined within local deformations, by the asymptotics of spectral data. As a consequence we get *local spectral rigidity* for generic zonal potentials on S_n.

3°. *Band invariants.* Asymptotic formula (3.7) for spectral shifts allows a straightforward derivation of the first 3 of Weinstein band-invariants for zonal potentials.

Let us recall the definition of band invariants. A sequence of cluster distribution measures $\{d\nu_k\}$ on $\mathbb{R}$, admits an asymptotic expansion in powers of k^{-1}

$$d\nu_k \sim \beta_0 + k^{-1}\beta_1 + k^{-2}\beta_2 + \cdots$$

with distributions β_0; β_1; β_2 on $\mathbb{R}$.

The 1st invariant β_0 is the well known distribution density of the Radon transform $\mathfrak{R}(V_{ev})$ [We], while higher order invariants are much harder to calculate (see [Ur1]).

Using asymptotics of spectral shifts (3.7) and spectral multiplicities we will be able to derive explicit formulae for $\{\beta_j\}$ in a straightforward manner. The multiplicites we have in mind are the dimension of the k^{th} cluster, $d_n(k) = \dim \mathscr{H}_k(D_n)$, and the multiplicity of the m^{th} spectral shift μ_{km}, $d_{n-1}(m) = \dim \mathscr{H}_m(S_{-1})$.

Obviously, $d_n(k)$ and $d_{n-1}(m)$ are polynomial functions in variable k (respectively m),

$$d_n(k) = \binom{n+k}{n} - \binom{n+k-2}{n} = \frac{1}{n!}(pk^{n-1} + qk^{n-2} + sk^{n-3} + \cdots),$$

with coefficients depending on n. The first 3 of them are found to be

$$p = p_n = 2(n-1);$$
$$q = q_n = (n-1)(n^2-2n+2),$$
$$s = s_n = \tfrac{1}{2}(n-1)^2(n-2)(n^2-\tfrac{10}{3}n+4).$$

Consequently, the weights $w_k(m) = (d_{n-1}(m)/d_n(k))$, that appear in (3.7), can also be expanded in powers of k^{-1},

$$w_k(m) = k^{-1}\left\{w_0\left(\frac{m}{k}\right) + k^{-1}w_1\left(\frac{m}{k}\right) + k^{-2}w_2\left(\frac{m}{k}\right) + \cdots\right\}. \tag{3.8}$$

The 3 leading terms of (3.8) turn out to be polynomial functions with coefficients depending on the coefficients $\{p; q; s\}$ of $d_n(k)$, and coefficients $\{p' = p_{n-1}; q' = q_{n-1}; s' = s_{n-1}\}$ of $d_{n-1}(m)$. Precisely,

$$w_0(x) = \frac{np'}{p}x^{n-2};$$
$$w_1(x) = \frac{n}{p}\left(q'x^{n-3} - \frac{q}{p}x^{n-2}\right);$$
$$w_2(x) = \frac{n}{p}\left\{s'x^{n-4} - \frac{qq'}{p}x^{n-3} + p'\frac{q^2-ps}{p^2}x^{n-2}\right\}. \tag{3.9}$$

Let us observe that the leading weight w_0 coincides with the density of the induced measure $r^*(dS)$ defined on $[0;1]$ by the "polar-sphere" map $r: \mathcal{O} \to [0;1]$ (1.5), where dS is the natural $SO(n+1)$-invariant measure on the space $\mathcal{O} \simeq S^*(S_n)/\mathbb{T}$. This, of course, corresponds to "zonal reduction" of functions and their transforms from S_n (respectively from $S^*(S_n)$ and $\mathcal{O}$) to the interval $[0;1]$.

It remains to pair measures $d\nu_k = \sum w_{km}\delta(x - 4\mu_{km})$ to a test function f on $\mathbb{R}$, and to use asymptotic expansions (3.8) of weights $\{w_{km}\}$, and (3.7) of spectral shifts $\{\mu_{km}\}$,

$$\langle f; d\nu_k\rangle = \frac{1}{k}\sum_m f\circ(a + k^{-1}b + k^{-2}c)(w_0 + k^{-1}w_1 + k^{-2}w_2)\left(\frac{m}{k}\right).$$

The right-hand side can be treated as "partial Riemann sums" on $[0;1]$, and approximated by integrals. Namely, expanding f in the Taylor series, multiplying 2 series and collecting powers of k^{-1}, we end up with the following expansion,

$$\langle f; d\nu_k\rangle \sim \left(\int_0^1 (f\circ a)w_0\,dx\right) + k^{-1}\left(\int_0^1 \{w_0 b(f'\circ a) + w_1(f\circ a)\}dx\right)$$
$$+ k^{-2}\int_0^1 \frac{1}{2}w_0 b^2(f''\circ a) + (w_0 c + w_1 b)(f'\circ a) + w_2(f\circ a)\,dx + O(k^{-3}). \tag{3.10}$$

The first 3 terms of (3.10) provide explicit formulae for band-invariants $\{\beta_0; \beta_1; \beta_2\}$, in terms of 3 asymptotic coefficients $\{a; b; c\}$ of spectral shifts (1.9),

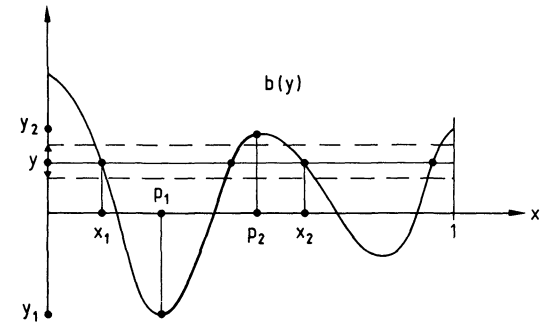

Fig. 3. A generic Morse function f on $[0; 1]$. Near any noncritical value $\{y\}$, subsets $\Gamma_k \cap [y-\varepsilon; y+\varepsilon]$ consist of asymptotically "almost arithmetic" sequences with differences $b_j = f'(x_j)$. After critical points $\{p_1; p_2\}$ are ordered, the branch-function $x = f_j^{-1}(y)$, is rigidly determined by the slope-function $b(y)$

and 3 weights $\{w_0; w_1; w_2\}$ (3.9). In particular, the well known invariant β_0 is the distribution function of the Radon transform $a = \tilde{V}_{ev}$, reduced on $[0; 1]$. Two higher invariants β_1 and β_2 represent certain distributions of order 1 and 2 on $\mathbb{R}$, which depend on the $\mathfrak{R}$- and $\mathfrak{T}$-transforms of V (V_{ev}, V_{od}), and on polynomial weights $\{w_0; w_1; w_2\}$.

Let us observe the following properties of distributions $\beta_1; \beta_2$:

(i) *singular support of* $\{\beta_j\}$ *coincides with critical values*[14] *of the Radon transform* $a = \tilde{V}_{ev}$.

Furthermore, one can explicitly characterize the type of singularity of β_j at each critical value.

(ii) *at a critical value* $y_0 = a(x_0)$, *of degree m* (i.e. $a(x) \approx a_0(x - x_0)^m$), *distribution* β_1, *localized to* $[y_0 - \varepsilon; y_0 + \varepsilon]$, *is equal to*

$$\beta_1 = C_0(y - y_0)^{-s} \frac{d}{dy} \chi + \text{"regular part"};$$

where $s = (m - 1/m)$, *and* χ *denotes the Heavy side function of the interval* $(\ldots; y_0]$ *or* $[y_0; \ldots)$. *Similarly,*

$$\beta_2 = (y - y_0)^{-s} \left\{ C_1 \frac{d^2}{dy^2} + C_2 \frac{d}{dy} \right\} (\chi) + \text{"regular part."}$$

The proof of both statements easily follows by truncating the $[0; 1]$-integration in (3.10) into the sum of integrals over different monotone branches of $a(x)$, which connect subsequent critical values (Fig. 3).

Appendix. Principal Symbol of the Reduced Average Operator R^{av}

Our goal here is to compute the principal symbol of the average operator $R = R^{av}$ given by (2.9) for zonal potentials V.

[14] The role of critical values of $\tilde{V}$, as certain accumulation levels of spectral shifts (so-called "clustering within bands") was observed earlier by A. Uribe [Ur2]

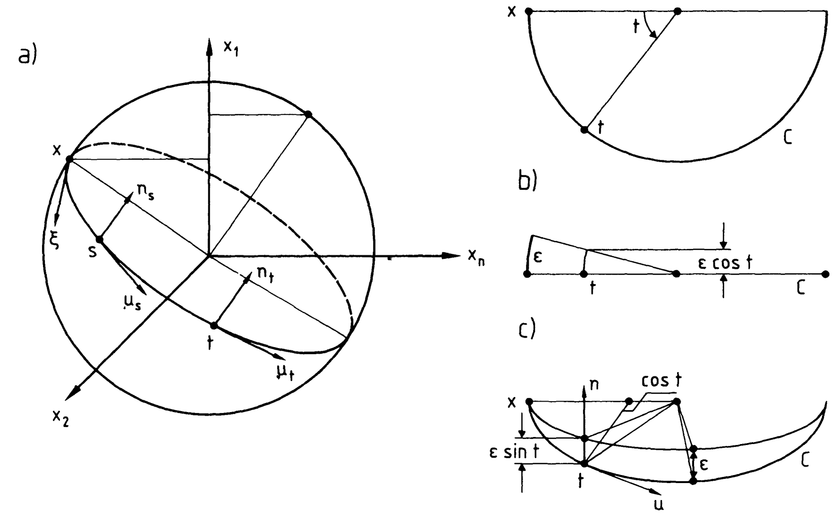

Fig. 4. The derivative formulae *(A2)* are illustrated here: (b) infinitesimal rotation by an angle ε in the direction perpendicular to C at $\{x\}$ results in infinitesimal shift by $\varepsilon \cos t$ at the point $\{t\}$, whence follows $\partial_x V(t) = \mathbf{u}\partial_{\mathbf{u}} V + \mathbf{n} \cos t \partial_{\mathbf{n}} V$; (c) uses polar coordinates $(r;\theta)$ in ξ-plane to write $\partial_\xi V(t) = \mathbf{u}\partial_r V(rt) + (1/r)\mathbf{n} \sin rt \partial_\theta V|_{r=1}$, whence follows the 2nd formula *(A2)*

We recall the general formula (2.15) of σ_R

$$\sigma_R = \tfrac{1}{4}\langle V^2\rangle^{\mathrm{av}} - (V^{\mathrm{av}})^2) + \tfrac{1}{4}\{\tau(V); V^{\mathrm{av}}\} - \frac{1}{8\pi}\int_0^{2\pi}\int_0^t \{V(t); V(s)\}\, ds\, dt. \tag{A1}$$

Here $V(t)$; $V(s)$ abbreviate symbol/function V composed with the geodesic flow $\exp t\Xi$ (or $\exp s\Xi$), in other words V evaluated along the great circle $C_{x\xi}$ at two points $\{t\}$, $\{s\}$, on the angular distance t and s from $\{x\}$.

The 2nd and the 3rd terms of σ_R involve the Poisson Bracket of symbols $V(t)$ and $V(s)$ on $S^*(S_n)$, so we shall first derive the formula for $\{V(t); V(s)\}$.

Let us remark that both symbols are homogeneous of degree zero in the ξ-variable. So their bracket $\{V(t); V(s)\}$ is homogeneous of degree -1, hence equal to $|\xi|^{-1}h(x;\xi')$, where ξ' denotes a unit vector in the direction ξ. In formula (A1) we dropped the homogeneous factor $|\xi|^{-1}$, so both Poisson brackets really mean functions on $S^*(S_n)$.

The calculation of $\{V(t); V(s)\}$ is illustrated by Fig. 4. Namely, we take a great circle $C = C_{x\xi}$, fix two points $\{s;t\}$ on C, and denote by $\mathbf{n} = \mathbf{n}_t(\mathbf{n}_s)$ the unit normals to C at $\{t;s\}$, and by $\mathbf{u} = \mathbf{u}_t(\mathbf{u}_s)$—unit tangent vectors at $\{t;s\}$.

Then we find the x- and ξ-derivatives of $V(t)$,

$$\begin{aligned}\partial_x V(t) &= \mathbf{u}(\partial_u V)(t) + \mathbf{n} \cos t(\partial_n V)(t);\\ \partial_\xi V(t) &= t\mathbf{u}(\partial_u V)(t) + \mathbf{n} \sin t(\partial_n V)(t),^{15}\end{aligned} \tag{A.2}$$

[15] Note the subtle difference of the meaning of **u**, **n** in the above formulae. Whereas *subscripts* **u**, **n** in the derivatives indicate unit tangent/normal at the point $\{t\}$ (respectively $\{s\}$), the *coefficients* **u**, **n** refer to the tangent/normal at the point $\{x\}$ (i.e. $t = 0$). Of course, the former are parallel transported initial vectors **u**, **n**, along the geodesics C

and similar relations for $V(s)$. Then the Poisson bracket of two functions becomes,

$$\{V(t); V(s)\} = (t-s)\partial_{\mathbf{u}}V(t)\partial_{\mathbf{u}}V(s) + \sin(t-s)\partial_{\mathbf{n}}V(t)\partial_{\mathbf{n}}V(s).$$

Writing the tangent and normal derivatives as $\partial_{\mathbf{u}}V = (d/dt)V$ and $\partial_{\mathbf{n}}V = V_{\mathbf{n}}$, we recast it in the form

$$\{V(t); V(s)\} = (t-s)\frac{d}{dt}V(t)\frac{d}{ds}V(s) + \sin(t-s)V_{\mathbf{n}}(t)V_{\mathbf{n}}(s). \tag{A3}$$

We need to compute the 2[nd] and 3[rd] terms of (A.1). By the definition of averages $\tilde{V}$ and $\tau(V)$ (2.14) it follows

$$\{\tilde{V}; \tau(V)\} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}(2\pi - s)\left((t-s)\frac{d}{dt}V(t)\frac{d}{ds}V(s) + \sin(t-s)V_{\mathbf{n}}(t)V_{\mathbf{n}}(s)\right)ds\,dt, \tag{A4}$$

while

$$\iint\{V(t); V(s)\}ds\,dt = \iint(t-s)\frac{d}{dt}V(t)\frac{d}{ds}V(s) + \sin(t-s)V_{\mathbf{n}}(t)V_{\mathbf{n}}(s). \tag{A5}$$

Both integrands (A4) and (A5) consist of two parts, one made of the tangent derivatives of $V(t)$, $V(s)$, and the other of normal. Integrating the "tangential part" of (A4) by parts, we get

$$\frac{1}{4\pi^2}\iint(t-s)\frac{d}{dt}V(t)V(s) = (V - \tilde{V})\tilde{V},$$

while the same term of (A5) yields

$$\iint(t-s)\frac{d}{dt}V(t)\frac{d}{ds}V(s) = 2\pi(V\tilde{V} - \tilde{V}^2).$$

Remembering the constant factors, accompanying two integrals ($\frac{1}{4}$ for (A4) and $-(1/8\pi)$ for (A5)), products $V\tilde{V}$ cancel out, so the total contribution of two tangential part becomes

$$\tfrac{1}{4}\{(V^2)^{\sim} - \tilde{V}^2\}. \tag{A6}$$

Notice, that so far we did not use the assumption of *zonal* V. However, this assumption will feature prominently in evaluating the "normal contributions" of both integrals.

To find normal derivatives of zonal V, we introduce angle ϕ between the great circle $C_{x\xi}$ and the x_1-axis (see Fig. 5), and write

$$(\partial_{\mathbf{n}}V)(t) = \sin\phi V'(\cos\phi\cos t), \text{ and similarly } (\partial_{\mathbf{n}}V)(s). \tag{A7}$$

Here V' means an ordinary derivatives (dV/dx_1), as a function on $[-1;1]$.

Introducing new variables $\rho = \rho(C) = \cos\phi$, and $\begin{cases} x = \cos\phi\cos t = \rho\cos t \\ y = \cos\phi\cos s = \rho\cos s \end{cases}$; (Fig. 5) we verify the following relations:

$$\sin t V'(\cos\phi\cos t) = -\frac{1}{\cos\phi}\frac{d}{dt}V(\ldots t); \text{ and similarly for } \sin s\, V'(\ldots s).$$

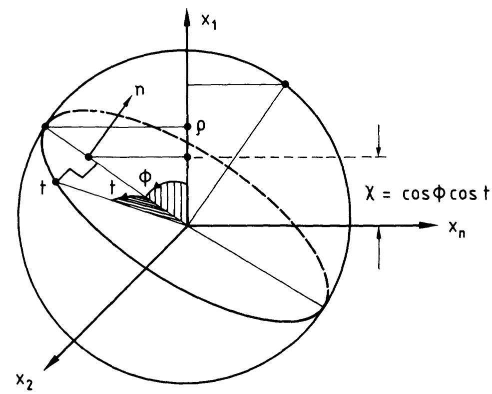

Fig. 5

The latter upon substitution in (A3) with normal derivatives given by (A7) yields the identity,

$$\sin(t-s)V'(\ldots t)V'(\ldots s)=\frac{1}{\cos^2\phi}(\cot s-\cot t)\frac{d}{dt}[V(\ldots t)]\frac{d}{ds}[V(\ldots s)]. \quad \text{(A8)}$$

Now we want to replace $\{t; s; \phi\}$ with new variables $\{\rho; x; y\}$. Let us observe the relation between the old and new derivatives,

$$\cot t\frac{d}{dt}[V(\ldots t)]=-x\frac{d}{dx}V(x). \quad \text{(A9)}$$

Incorporating (A8) and (A9) in the "normal component" of (A3) we get

$$\sin(t-s)V_n(t)V_n(s)=\frac{1-\rho^2}{\rho^2}\left(-\frac{d}{dt}V(\ldots t)y\frac{d}{dy}V(y)+x\frac{d}{dx}V(x)\frac{d}{ds}V(\ldots s)\right). \quad \text{(A10)}$$

The right-hand side of (A10) contains the mixture of the old and new variables, which will be convenient to maintain for the subsequent integration.

Indeed, when (A10) is inserted in the "normal part" of (A4) the resulting double integrals break into products of ordinary integrals, each accompanied by the factor $2(1-\rho^2)/\rho^2$,

$$\left(\int_{-\rho}^{\rho} x\frac{d}{dx}V\frac{dx}{\sqrt{\rho^2-x^2}}\right)\left(\int_0^{2\pi}(2\pi-s)\frac{d}{ds}V(\ldots s)\right)-\left(\int_0^{2\pi}\frac{d}{dt}V(\ldots t)\right)$$

$$\cdot\left(\int_{-\rho}^{\rho}(2\pi-s)\,y\frac{d}{dy}V\frac{dy}{\sqrt{\rho^2-y^2}}\right).$$

The first factor of the 2nd product clearly vanishes (due to periodicity of $V(\ldots t)$!), while the second factor of the 1st product, upon integration by parts, yields $-2\pi V(\rho)$. So the resulting "normal contribution" of $\{\tilde{V}; \tau(V)\}$ becomes

$$-8\pi V(\rho)\int_0^\rho x\frac{d}{dx}V\frac{dx}{\sqrt{\rho^2-x^2}} = -8\pi V(\rho)\mathfrak{R}\left(x\frac{d}{dx}V\right) \tag{A11}$$

in terms of the reduced Radon transform $\mathfrak{R}$(1.3).

A similar calculation with the "normal part" of (A5) yields

$$4\frac{1-\rho^2}{\rho^2}\left(\int_0^\rho x\frac{d}{dx}(V^2)_{\mathrm{ev}}\frac{dx}{\sqrt{\rho^2-x^2}} - V(\rho)\mathfrak{R}\left(x\frac{d}{dx}V\right)\right). \tag{A12}$$

We write $(V^2)_{\mathrm{ev}}$ in (A12), since integration, $\int_0^{2\pi}\cdots dt$, of functions f of the form $f(\rho\cos t)$ maintains only the even part of f.

Once again the coefficients of the product $V(\rho)\mathfrak{R}(\ldots V)$ in (A11) and (A12) perfectly match, so they cancel out, and the only surviving term of two normal contributions is

$$-\frac{1-\rho^2}{2\rho^2}\mathfrak{R}\left(x\frac{d}{dx}(V^2)_{\mathrm{ev}}\right).$$

Combining the latter with the 1st term of (A1) and (A6), we obtain

$$\sigma_R = \frac{1}{2}(\tilde{V}^2 - \tilde{V}^2) - \frac{1}{2}\left(\frac{1-\rho^2}{\rho^2}\right)\mathfrak{R}\left(x\frac{d}{dx}\right)(V^2)_{\mathrm{ev}}, \tag{A13}$$

Rewriting (A13) in terms of the even and odd parts of V we get the final result needed in the proof of Theorem 1,

$$\sigma_R = \frac{1}{2}(\tilde{V}_{\mathrm{ev}}{}^2 + \tilde{V}_{\mathrm{od}}{}^2 - \tilde{V}_{\mathrm{ev}}{}^2) - \frac{1}{2}\left(\frac{1-\rho^2}{\rho^2}\right)\mathfrak{R}\left(x\frac{d}{dx}(V_{\mathrm{ev}}{}^2 + V_{\mathrm{od}}{}^2)\right).$$

Remark. Though we conducted the above argument in the case of the 2-sphere it remains valid for all S_n. The necessary modifications include replacing normal derivatives in formulae (A2), (A3), etc. by the dot product of *normal components* of $\partial V(t)$, $\partial V(s)$.

Then for zonal functions V we proceed by reducing the problem to the S_2-case. Indeed, any great circle on S_n that does not pass through its North Pole, can be rotated by the axisymmetric group $SO(n)$ into a circle that lies in a "3-D slice of S_n" (a 2-sphere cut out by a subspace $\simeq \mathbb{R}^3$). In fact, a 3-D subspace can be chosen as a span of coordinates $\{x_1; x_i; x_j\}$, where C is not orthogonal to a $(i,j)^{\mathrm{th}}$ coordinate plane of $\mathbb{R}^{n+1}$.

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Communicated by B. Simon

Received September 8, 1989; in revised form January 11, 1990