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The Weak Coupling Limit as a Quantum Functional Central Limit

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Abstract. We show that, in the weak coupling limit, the laser model process converges weakly in the sense of the matrix elements to a quantum diffusion whose equation is explicitly obtained. We prove convergence, in the same sense, of the Heisenberg evolution of an observable of the system to the solution of a quantum Langevin equation. As a corollary of this result, via the quantum Feynman–Kac technique, one can recover previous results on the quantum master equation for reduced evolutions of open systems. When applied to some particular model (e.g. the free Boson gas) our results allow to interpret the Lamb shift as an Ito correction term and to express the pumping rates in terms of quantities related to the original Hamiltonian model.

1. Introduction

In the quantum theory of irreversible evolutions, the weak coupling limit was originally formulated as a device to extract the long time cumulative effect of a small perturbation of the global Hamiltonian of a composite system on the reduced evolution of a subsystem [9, 29]. As far as we know, the consideration of the weak coupling limit dates back to Friedrichs [18] in the context of the well-known Friedrichs model. However, in the physical literature the weak coupling limit is known as the van Hove limit, since van Hove [31] was the first author to consider the limit $\lambda \to 0$, $t \to \infty$, with $\lambda^2 t$ held constant, in the derivation of an irreversible evolution of semigroup type for the macroscopic observables of a large quantum system.

The original problem of van Hove has not been set into a fully rigorous form yet, although related rigorous results have been obtained by Martin and Emch [27] and Dell'Antonio [14]. On the other hand, theorems on the weak coupling limit for specific models of open quantum systems have been proved by Davies [9] and Pulé [28]. A general formulation in terms of the master equation approach

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was given in a series of papers by Davies [9–11]. More precisely: we consider a spatially confined quantum system (the "system" S), coupled to another (infinitely extended) quantum system (the "reservoir" or "heat bath' R), initially in a given reference state φ_R (which is usually a quasi-free state on the Weyl or the CAR algebra over some Hilbert space), through an interaction of the form λV, where V is a given self-adjoint operator. Denote by $\mathscr{A}_S$ and by $\mathscr{A}_R$ the W^*-algebras of observables of the system and of the reservoir respectively. Typically, $\mathscr{A}_S$ will be the algebra of all bounded linear operators on a separable Hilbert space $\mathscr{H}_S$, and $\mathscr{A}_R$ will be the weak closure of the GNS representation of the C^*-algebra of the reservoir determined by the reference state φ_R. Let

$$H_\lambda = H_S \otimes 1 + 1 \otimes H_R + \lambda V \tag{1.1}$$

be the total Hamiltonian of the composite system (in self-explanatory notations). For each x in $\mathscr{A}_S$, let $x^\lambda(t)$ be the element of $\mathscr{A}_S \otimes \mathscr{A}_R$ defined by

$$\begin{aligned} x^\lambda(t) &= \exp[iH_\lambda t/\lambda^2]\cdot\exp[-iH_0 t/\lambda^2](x\otimes 1)\exp[iH_0 t/\lambda^2]\cdot\exp[-iH_\lambda t/\lambda^2] \\ &= U_{t/\lambda^2}^{(\lambda)+}(x\otimes 1)U_{t/\lambda^2}^{(\lambda)}, \end{aligned}$$

where

$$U_{t/\lambda^2}^{(\lambda)} = \exp[iH_0 t/\lambda^2]\cdot\exp[-iH_\lambda t/\lambda^2], \tag{1.2}$$

i.e. we consider the Heisenberg evolute, in the interaction representation, of an observable of the system S in a time scale of order $1/\lambda^2$. Then [9, 28] in the limit as $\lambda \to 0$ and under suitable assumptions, there exists a semigroup T_t of weakly-*-continuous completely positive normal linear maps of $\mathscr{A}_S$ into itself (a *quantum dynamical semigroup* on $\mathscr{A}_S$ in the sense of Gorini Kossakowski and Sudarshan [23], Lindblad [26], a *quantum Markovian semigroup* in the sense of Accardi [1]) such that, for all x in $\mathscr{A}_S$ and for all normal states φ_S on $\mathscr{A}_S$ and $t \geqq 0$ one has

$$\lim_{\lambda\to 0} (\varphi_S \otimes \varphi_R)(x^\lambda(t)) = \varphi_S(T_t(x)).$$

We refer to the books of Davies [12, 13] for a presentation of the physical ideas and of the mathematical structures relevant for this phase of development of the problem. Under some assumptions on the interaction, which amount to the *rotating wave approximation*, familiar in the laser models, one sees (cf. [20]), considering the perturbation expansion of $U_{t/\lambda^2}^{(\lambda)}$, that the first order term does not depend on the field operators of the reservoir but on some time averages of them of the form

$$A_t^{(\lambda)} = \lambda \int_0^{t/\lambda^2} e^{-i\omega s} A(S_s^0 g)\, ds$$

(cf. Sects. 2 and 3 below for the notations). The normalization defining the "collective annihilation operator" $A_t^{(\lambda)}$ is strongly resemblant of the normalization of the classical invariance principles. This analogy suggests that, as already stated in Spohn [29], the weak coupling limit should be a manifestation of some kind of functional central limit effect. That is we expect that, in analogy with the quantum invariance principle proved in [2], the collective creation and annihilation processes

$A_t^{(\lambda)\pm}$ converge, in some sense to be specified, to some of the quantum analogues of the Wiener process, namely the quantum Brownian motions. A heuristic discussion of this approach to the weak coupling limit has been sketched in Frigerio [20], with some preliminary lemmas and some conjectures.

Moreover, if the quantum dynamical semigroup obtained in the weak coupling limit is norm continuous with infinitesimal generator G given by

$$G(x) = K^+x + xK + \sum_{j=1}^{n} L_j^+ x L_j; \quad x \in \mathscr{A}_S$$

with $L_j, K \in \mathscr{A}_S$ satisfying

$$K^+ + K + \sum_{j=1}^{n} L_j^+ L_j = 0$$

then we have, for all x in $\mathscr{A}_S$ and t in $\mathbf{R}_+$,

$$T_t(x) = E_0[U^+(t)(x \otimes 1_{\mathbf{R}})U(t)], \tag{1.3}$$

where $U(t)$ is the solution of the quantum stochastic differential equation, in the sense of Hudson and Parthasarathy [25],

$$dU(t) = \left\{ K\,dt + \sum_{j=1}^{n} [L_j dA_j^+(t) - L_j^+ dA_j(t)] \right\} U(t), \quad U(0) = 1, \tag{1.4}$$

and where $A_j(t), A_j^+(t)$ are mutually independent Fock quantum Brownian motions and E_0 is the vacuum conditional expectation. Then it is natural to conjecture that, under suitable assumptions and in a sense to be specified, one has, for all t in $\mathbf{R}_+$,

$$\lim_{\lambda \to 0} U_{t/\lambda^2}^{(\lambda)} = U(t) \tag{1.5}$$

and, for all x in $\mathscr{A}_S$,

$$\lim_{\lambda \to 0} x^\lambda(t) = U^+(t)(x \otimes 1_R)U(t). \tag{1.6}$$

The fact that the weak coupling limit should lead to a unitary process, satisfying a quantum stochastic differential equation was first noted by von Waldenfels [35] in connection with the Wigner–Weisskopf model. The explicit form of the stochastic equation, for the Wigner–Weisskopf model was obtained independently by Maasen [27a] in the Fock case. A thorough study of this equation, in the finite temperature case, is due to Applebaum and Frigerio [7b]. In all these cases the stochastic differential equation is not deduced as a (weak coupling) limit of Hamiltonian systems, but postulated ab initio.

In the present paper, using the notion of convergence for quantum processes introduced in [2], we give a precise statement and proof of the above conjecture (here we use the terminology "weakly convergent in the sense of the matrix elements" since, as remarked by a referee, the convergence considered in [2], when restricted to the Abelian case, gives a convergence weaker than the convergence in low). We shall only give here the proof of the first two statements above in the case when φ_R is the Fock state. The proof of (1.6) and the case of a thermal state at finite temperature is in [5]. The Fermion case introduces no additional difficulties (cf. [6]).

Among the motivations for the present work the following deserves to be mentioned. There are widespread misgivings concerning use of quantum Brownian motion as a (boson or fermion) reservoir in the description of open systems; in particular it is objected that:

(i) the one-particle energy is unbounded from below as well as from above;
(ii) the reference state satisfies the KMS condition not for the automorphism giving the time evolution of the reservoir, but for a much more trivial one, consisting of multiplying the creation operators by a phase factor $\exp[-i\omega_0 t]$.

Our results show how these features arise precisely in the weak coupling limit starting from a perfectly "legal" dynamics. A detailed discussion of the KMS condition is given in [5].

A preliminary version of the present paper has appeared in [7a]. Here we have greatly improved the uniform estimate, due to our improvement of Pule's inequality. Moreover we have changed two important notations with respect to [7a]:

1. We have particularized our Definition (2.3) of quantum Brownian motion (in the commutative case our previous definition reduced to the usual one only up to a "random time change").
2. The notion of weak convergence in the sense of matrix elements (cf. Definition (2.2)) was called in [7a] "convergence in low." However, without further qualifications of the random variables, also this definition might lead to incongruence, in the abelian case, with the standard terminology.

These changes were motivated by some constructive critiques of the referee of this paper, to whom we express our gratitude.

2. Statement of the Problem, Notations, Results

By a Hilbert space we mean a complex separable Hilbert space and by a pre-Hilbert space we mean a complex vector space endowed with a (possibly degenerate) sesquilinear form whose induced topology is separable. The $*$-algebra of continuous linear operators on a pre-Hilbert space $\mathscr{K}$ will be denoted $B(\mathscr{K})$.

If $\mathscr{K}$ is a Hilbert space, with scalar product denoted by $<\cdot,\cdot>$, we denote

$$L^2(R, dt; \mathscr{K}) \cong L^2(R, dt) \otimes \mathscr{K},$$

then Hilbert space of the square integrable $\mathscr{K}$-valued functions—the integral being meant in Bochner's sense. If $\mathscr{K} = \mathbf{C}$, we simply write $L^2(R)$.

Throughout this paper, H_1 will denote a fixed Hilbert space (the "second quantization" of H_1 in a suitable sense may be interpreted as the "reservoir state space"). Q will denote a self-adjoint operator defined on a dense subspace $D(Q)$ of H_1 and such that, on this domain,

$$Q \geqq 1 \tag{2.1}$$

$S_t^0: H_1 \to H_1$ will denote a strongly continuous 1-parameter unitary group on H_1 commuting with Q, in the sense that:

$$S_t^0 D(Q) \subseteq D(Q), \tag{2.2}$$

$$S_t^0 Q = Q S_t^0 \quad \text{on} \quad D(Q). \tag{2.3}$$

Our basic assumption on S_t^0 and Q will be the following:

There exists a non-zero subspace $K \subseteq D(Q)$ (in all the examples it will be a dense subspace) such that:

$$\int_{\mathbf{R}} |\langle f_1, S_t^0 f_2 \rangle| dt < +\infty; \quad \int_{\mathbf{R}} |\langle f_1, S_t^0 Q f_2 \rangle| dt < +\infty \quad \forall f_1, f_2 \in K. \tag{2.4}$$

This condition implies (cf. Lemma (3.2)) that, for any real number ω, the sesquilinear form

$$f_1, f_2 \in K \mapsto (f_1 | f_2)_Q := \int_{\mathbf{R}} e^{-i\omega t} \langle f_1, S_t^0 Q f_2 \rangle dt \tag{2.5}$$

defines a pre-scalar product on K. We shall denote K_Q the associated Hilbert space, i.e. the completion of the quotient of K by the zero $(\cdot|\cdot)_Q$-norm elements for the norm induced by the scalar product (2.5). In particular, for $Q = 1$, we simply write $\{K_1, (\cdot|\cdot)\}$.

Let $W(K)$ be the Weyl C^*-algebra over K and let φ_Q be the quasi-free state on $W(K)$ characterized by

$$\varphi_Q(W(f)) = e^{-1/2\langle f, Qf \rangle}; \quad f \in K. \tag{2.6}$$

We denote

$$\{\mathscr{H}_Q, \pi_Q, \Phi_Q\}$$

the GNS triple associated to $\{W(K), \varphi_Q\}$. We shall write

$$W_Q(f) = \pi_Q(W(f)); \quad f \in K. \tag{2.7}$$

Because of (2.3), there exists a unique φ_Q-preserving 1-parameter group of $*$-automorphisms u_t of $W(K)$ characterized by

$$u_t(W(f)) = W(S_t^0 f); \quad f \in K, \tag{2.8}$$

and we denote $U_t^Q: \mathscr{H}_Q \to \mathscr{H}_Q$ the associated unitary operator:

$$U_t^Q \cdot W_Q(f) \cdot \Phi_Q = W_Q(S_t^0 f) \cdot \Phi_Q; \quad f \in K. \tag{2.9}$$

The field, creation and annihilation operators of the representation (2.7) will be denoted

$$B_Q(f), \quad A_Q^+(f), \quad A_Q(f); \quad f \in K. \tag{2.10}$$

To simplify the notations in the following we shall often omit the index Q whenever we feel that this cannot create any confusion. Let $\mathscr{A}_R$ denote the weak closure of $W_Q(K)$ in $\mathscr{H}_Q$; let u_t^R denote the restriction to $\mathscr{A}_R$ of $\mathrm{Ad}U_t^R = U_{-t}^R \cdot (\cdot) \cdot U_t^R$, where U_t^R is the same as U_t^Q; and let φ_R be the restriction of the state $\langle \Phi_Q, (\cdot) \Phi_Q \rangle$ to $\mathscr{A}_R$. The W^*-dynamical system $\{\mathscr{A}_R, u_t^R, \varphi_R\}$ will be called the *reservoir*, or the *heat bath*. Now let $\mathscr{H}_0$ be another pre-Hilbert space (called the *system state space* or the *initial space*); let $U_t^S: \mathscr{H}_0 \to \mathscr{H}_0$ be a 1-parameter unitary group on $\mathscr{H}_0$ and denote

$$u_t^S = \mathrm{Ad}\, U_t^S = U_{-t}^S \cdot (\cdot) \cdot U_t^S : \mathscr{A}_S \to \mathscr{A}_S. \tag{2.11a}$$

We denote

$$U_t^0 = U_t^S \otimes U_t^R \in B(\mathscr{H}_0 \otimes \mathscr{H}_Q). \tag{2.11b}$$

The Heisenberg evolution, associated to U_t^0, i.e.

$$u_t^0 = \mathrm{Ad}\, U_t^0 = u_t^S \otimes u_t^R : \mathscr{A}_S \otimes \mathscr{A}_R \to \mathscr{A}_S \otimes \mathscr{A}_R \tag{2.11c}$$

will be called the *free evolution* of the composite system.

We now introduce an interaction between the system and the reservoir of the form that is familiar in laser theory (cf. [32]), i.e.

$$\lambda V_g = -\frac{\lambda}{i}[D \otimes A^+(g) - D^+ \otimes A(g)], \tag{2.12}$$

where λ is a positive real number (the coupling constant), $g \in K$ and D is a bounded operator on H_0 satisfying the condition

$$u_t^S(D) = e^{-i\omega_0 t} D, \tag{2.13}$$

where ω_0 is a fixed positive real number (interpreted as the proper frequency of the laser). This is the type of interaction which arises in the *rotating wave approximation*. Our techniques are applicable to a wider class of interactions, but this will be shown elsewhere. Denoting

$$V_g(t) = u_t^0(V_g); \quad t \in \mathbf{R} \tag{2.14}$$

we see that, from (2.13) and the antilinearity of A, we have

$$u_t^0(V_g) = -\frac{1}{i}[D \otimes A^+(S_t g) - D^+ \otimes A(S_t g)],$$

where we have introduced the notation

$$S_t g = e^{-i\omega_0 t} S_t^0 g.$$

Clearly the conditions (2.2), (2.3), (2.4) are satisfied by S_t^0 if and only if they are satisfied by S_t. We will assume that the iterated series

$$1 + \sum_{n=1}^{\infty} (-i)^n \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V_g(t_1) V_g(t_2) \cdots V_g(t_n) \tag{2.15}$$

is uniformly convergent, for λ small enough and t bounded on the domain $H_0 \otimes \mathscr{E}_Q$, where $\mathscr{E}_Q$ is the linear space algebraically spanned by the coherent vectors in $\mathscr{H}_Q$ and the tensor product is algebraic. Moreover we assume that the series (2.15) defines a unitary operator $U_t^{(\lambda)}$ on $H_0 \otimes \mathscr{H}_Q$ which, on $H_0 \otimes \mathscr{E}_Q$ satisfies the Schrödinger equation in interaction representation:

$$\frac{\partial}{\partial t} U_t^{(\lambda)} = \frac{\lambda}{i} V_g(t) \cdot U_t^{(\lambda)}; \quad U_0^{(\lambda)} = 1. \tag{2.16}$$

This is an assumption on D which is always fulfilled if, e.g., D is a bounded operator. In the following, to avoid unnecessary technicalities, we shall always assume that D is bounded. For each $\lambda > 0$ the 1-parameter family $(U_t^{(\lambda)})$ is a left u_t^0-cocycle, i.e.

$$U_{s+t}^{(\lambda)} = u_t^0(U_t^{(\lambda)}) \cdot U_s^{(\lambda)}, \tag{2.17}$$

hence the 1-parameter family (V_t^λ), defined by

$$V_t^{(\lambda)} = U_{-t}^{(\lambda)} \cdot U_t^0; \quad t \in \mathbf{R} \tag{2.18}$$

is a strongly continuous unitary group whose formal generator coincides with

$$H_S \otimes 1 + 1 \otimes H_R + \lambda V_g, \tag{2.19}$$

where

$$U_t^R = e^{-itH_R}; \quad U_t^S = e^{-itH_S}. \tag{2.20}$$

(In the case of the Laplacian acting on $L^2(\mathbf{R})$, this is rigorously true on the domain $H_0 \otimes \mathscr{E}'$, where $\mathscr{E}'$ is the linear space generated by the coherent vectors corresponding to smooth test functions.) The Heisenberg dynamics, associated to $V_t^{(\lambda)}$, i.e.

$$u_t^{(\lambda)} = \mathrm{Ad}\, V_t^{(\lambda)*} = V_t^{(\lambda)} \cdot (\cdot) \cdot V_t^{(\lambda)+} = U_t^{(\lambda)+} \cdot u_t^0(\cdot) U_t^{(\lambda)} \tag{2.21}$$

is called the *interacting dynamics*.

Our goal is to study the time evolution, under the interacting dynamics, of some physically interesting quantity in the van Hove limit, i.e.

$$\lambda \to 0; \quad t \to \infty; \quad \lambda^2 t = O(1) = \text{of order } 1. \tag{2.22}$$

Since this limit extracts the long time cumulative behaviour of the interacting dynamics, we expect its effects to be best revealed on those observables and those states which depend on this long time cumulative behaviour. To make this remark precise, in Sect. 3 we introduce, as a continuous time analogue of the construction in [2], the *collective Weyl operators*

$$W\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du \right), \tag{2.23}$$

and the corresponding *collective coherent vectors*

$$\Phi_Q\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du \right) = W_Q\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du \right) \cdot \Phi_Q. \tag{2.24}$$

The family of all these vectors, with $f \in K$ and $-\infty < S < T < +\infty$, will be denoted $\mathscr{D}_Q(\lambda)$.

Now let us recall, from [2] the definitions of stochastic process and of convergence in law of stochastic processes.

Definition (2.1). A quantum stochastic process indexed by a set T over an Hilbert space H is a triple

$$X = \{H, \mathscr{D}, X(t)(t \in T)\},$$

where

i) H is a Hilbert space.
ii) T is a set.

iii) $\mathscr{D}$ is a total subset in H and $X(t)(t\in T)$ is a family of preclosed operators on H, called the random variables of the process, such that for any $t\in T$,

$$\mathscr{D}\subseteqq D(X(t)):=\text{domain of } X(t)$$

and the set $\{X(t)\}$ is self-adjoint in the sense that for each $t\in T$ there exists an uniquely determined element $t^+\in T$ such that the identity

$$X(t^+)=X^+(t):=X(t)^+$$

holds on $\mathscr{D}$.

Definition (2.2). Let $\mathscr{I}$ be an increasing net, partially ordered by a relation $\prec$. We say that a family

$$X_\alpha=\{H_\alpha,\mathscr{D}_\alpha,X_\alpha(t)(t\in T)\},\quad \alpha\in\mathscr{I}$$

of quantum stochastic processes converges to the quantum stochastic process

$$X=\{H,\mathscr{D},X(t)(t\in T)\}$$

weakly in the sense of the matrix elements, if the domains $\mathscr{D}_\alpha$ and $\mathscr{D}$ are invariant for the random variables of the respective processes and if for any $\alpha\in\mathscr{I}$ there exists a map

$$F_\alpha:\mathscr{D}\to\mathscr{D}_\alpha;\quad t_\alpha:T\to T$$

such that, for any fixed integer k, for all k-tuples $t_1,\dots,t_k\in T$ satisfying $t_\alpha(t_h)\to t'_h\in T$, $h=1,\dots,k$, and for all Ψ, $\Phi\in\mathscr{D}$, one has:

$$\lim_\alpha\langle F_\alpha(\Psi),X_\alpha(t_\alpha(t_1))\cdot\ldots\cdot X_\alpha(t_\alpha(t_k))F_\alpha(\Phi)\rangle=\langle\Psi,X(t'_1)\cdots X(t'_k)\Phi\rangle$$

Notice that, if the X_t are bounded, then we can take $\mathscr{D}_\alpha=H_\alpha$ and $\mathscr{D}=H$, so that the invariance of the domains, required in Definition (2.2) is automatically satisfied.

As shown in [2] (Theorem (9.2)) the notion of stochastic process given in Definition (2.1) is equivalent, in several important cases, to the ones given by [3], however it is better suited to deal with unbounded processes and nonfaithful states. In [2], it is also shown how to modify Definition (2.1) so that, in the commutative case, it includes all the classical stochastic processes. For our purposes, Definition (2.1) will be sufficient.

Definition (2.3). Let $\mathscr{K}$ be a Hilbert space, T an interval in $\mathbf{R}$, $Q\geqq 1$ be a self-adjoint operator on $\mathscr{K}$ and let

$$\{\mathscr{H}_Q,\pi_Q,\Phi_Q\}\tag{2.25}$$

denote the GNS representation of the CCR over $L^2(T,dt;\mathscr{K})$ with respect to the quasi-free state φ_Q on W $(L^2(T,dt;\mathscr{K}))$ characterized by

$$\varphi_Q(W(\xi))=e^{-1/2\langle\xi,1\otimes Q\xi\rangle};\quad \xi\in L^2(T,dt;\mathscr{K}).\tag{2.26}$$

Denote $\mathscr{D}$ the set of all vectors of the form $\pi(W(\xi))\Phi_Q=W_Q(\xi)\cdot\Phi_Q$ with

$\xi \in L^2(T, dt; \mathscr{K})$. The stochastic process

$$\{\mathscr{H}_Q, \mathscr{D}, W_Q(\chi_{(s,t]} \otimes f); (s,t] \subseteq T, f \in \mathscr{K}\} \tag{2.27}$$

is called the *Q-quantum Brownian motion* on $L^2(T, dt, \mathscr{K})$.

If $Q = 1$ we speak of the *Fock Brownian Motion* on $L^2(T, dt, \mathscr{K})$; if Q is the multiplication by a constant ($\beta \geqq 1$), then we speak of the *finite temperature quantum Brownian Motion*, in the terminology of [34] or of the *universal invariant quantum Brownian Motion* in the terminology of [24].

Sometimes, when no confusion can arise, we call quantum Brownian motion also the process

$$\{\mathscr{H}_Q, \mathscr{D}, A(\chi_{(s,t]} \otimes f), A^+(\chi_{(s,t]} \otimes f); s, t \in T, f \in K\}, \tag{2.28}$$

where $A(\cdot), A^+(\cdot)$ denote respectively the annihilation and creation fields in the representation (2.25). For the normalized coherent vectors we use the notation:

$$W_Q(\chi_{[s,t]} \otimes f) \cdot \Phi_Q = \Phi_Q(\chi_{[s,t]} \otimes f).$$

With these notatons we can state our main results:

Theorem (I). *Let H_1 be an Hilbert space and let $Q, (S_t^0)$, K satisfy the conditions (2.1), (2.2), (2.3), (2.4). Then, as $\lambda \to 0$ the stochastic process*

$$\left\{\mathscr{H}_Q, \mathscr{D}_Q(\lambda), W\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du\right), S, T \in \mathbf{R}, f \in K\right\} \tag{2.29}$$

with $\mathscr{H}_Q$ and Φ_Q defined after (2.6), converges weakly in the sense of the matrix elements to the Q-quantum Brownian Motion on $L^2(R, dt; K_1)$.

Theorem (II). *Let $Q = 1$, then for each $u, v \in H_0, f_1, f_2, g \in K_1$, $S_1, S_2, T_1, T_2 \in \mathbf{R}$ ($S_j \leqq T_j$) the limit*

$$\lim_{\lambda \to 0} \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1\, du\right), U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2\, du\right)\right\rangle \tag{2.30}$$

exists and is equal to

$$\langle u \otimes \Phi(\chi_{[S_1, T_1]} \otimes f_1), U_t v \otimes \Phi(\chi_{[S_2, T_2]} \otimes f_2)\rangle, \tag{2.31}$$

where the scalar product is meant in the space $H_0 \otimes \Gamma(L^2(\mathbf{R}, dt; K_1))$ and U_t is the solution of the quantum stochastic differential equation

$$dU_t = [D \otimes dA_g^+(t) - D^+ \otimes dA_g(t) - (g|g)_- D^+ D \otimes 1\, dt] \cdot U_t; \quad U_0 = 1 \tag{2.32}$$

in the sense of [25] and where

$$(g|g)_- = \int_{-\infty}^{0} \langle g, S_u g\rangle du. \tag{2.33}$$

Theorem (III). *In the notations and assumptions of Theorem (II), for any $X \in \mathscr{B}(H_0)$, the limit*

$$\lim_{\lambda \to 0} \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1\, du\right), U_{t/\lambda^2}^{(\lambda)} \cdot (X \otimes 1) \cdot U_{t/\lambda^2}^{(\lambda)*} \cdot v \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2\, du\right)\right\rangle$$

exists and is equal to

$$\langle u \otimes \Phi(\chi_{[S_1,T_1]} \otimes f_1), U_t(X \otimes 1)U_t^* \cdot v \otimes \Phi(\chi_{[S_2,T_2]} \otimes f_2)\rangle$$

where, $U(t)$ is the same as in Theorem (II).

The first two of the above theorems are proved in the present paper and the third one in [7].

3. Convergence of the Collective Process to the Noise Process

Lemma (3.1). *For any $g \in D(Q)$ and for any $-\infty < S \leqq T < \infty$, the integral*

$$\int_S^T S_t g dt \tag{3.1}$$

is well defined and belongs to $D(Q)$, moreover

$$Q \cdot \int_S^T S_t g dt = \int_S^T Q S_t g dt. \tag{3.2}$$

Proof. By the strong continuity of S_t, the function $t \mapsto S_t g$ is weakly measurable and with a separable range. Since $\|S_t g\| = \|g\|$, it follows that $t \mapsto S_t g$ is Bochner integrable. Moreover, for each $f \in D(Q)$ one has, using (2.2) and (2.3):

$$\left|\left\langle Qf, \int_S^T S_t g dt\right\rangle\right| \leqq \int_S^T |\langle Qf, S_t g\rangle| dt = \int_S^T |\langle S_{-t} Qf, g\rangle| dt \leqq (T-S)\|Qf\| \cdot \|g\|,$$

hence $\int_S^T S_t g dt \in D(Q)$ and (3.2) follows from the definition of Bochner integral.

Lemma (3.2). *For any pair $f, g \in D(Q)$ satisfying (2.4), and for any $S_1, T_1, S_2, T_2 \in \mathbf{R}$ ($S_j \leqq T_j$) one has*

$$\lim_{\lambda \to 0} \left\langle \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f du, Q \cdot \lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_v g dv \right\rangle = \langle \chi_{[S_1,T_1]}, \chi_{[S_2,T_2]}\rangle \int_{\mathbf{R}} \langle f, S_t Q g\rangle dt, \tag{3.3}$$

where the scalar product of the characteristic functions is meant in $L^2(\mathbf{R})$ and the limit is uniform for S_1, T_1, S_2, T_2 in a bounded set of $\mathbf{R}$.

Proof. From Lemma (3.1) it follows that

$$\begin{aligned}
&\left\langle \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du, Q \cdot \lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_v f_2 dv \right\rangle \\
&= \lambda^2 \int_{S_1/\lambda^2}^{T_1/\lambda 2} du_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} du_2 \langle S_{u_1} f_1, S_{u_2} Q f_2\rangle \\
&= \lambda^2 \int_{S_1/\lambda^2}^{T_1/\lambda^2} du_1 \int_{S_2/\lambda^2 - u_1}^{T_2/\lambda^2 - u_1} du \langle f_1, S_u Q f_2\rangle \\
&= \int_{S_1}^{T_1} du_1 \int_{(S_2-u_1)/\lambda^2}^{(T_2-u_1)/\lambda^2} du \langle f_1, S_u Q f_2\rangle.
\end{aligned} \tag{3.4}$$

Now notice that for each $u_1 \in (S_1, T_1) \cap (S_2, T_2) = (S_1 \vee S_2, T_1 \wedge T_2)$, one has $S_2 - u_1 < 0$ and $T_2 - u_1 > 0$, hence

$$\lim_{\lambda \to 0} \int_{(S_2-u_1)/\lambda^2}^{(T_2-u_1)/\lambda^2} du \langle f_1, S_u Q f_2 \rangle = \int_{\mathbf{R}} \langle f, S_t Q g \rangle dt. \tag{3.5}$$

On the other hand, because of (2.4) for each $u_1 \in [S_1, T_1]$, the limit on the left-hand side of (3.5) is non-zero only if $S_2 - u_1 \leqq 0$ and $T_2 - u_1 \geqq 0$, that is if $u_1 \in [S_2, T_2]$. Therefore, by dominated convergence, we obtain:

$$\begin{aligned}\lim_{\lambda \to \infty} \int_{S_1}^{T_1} du_1 \int_{(S_2-u_1)/\lambda^2}^{(T_2-u_1)/\lambda^2} du \langle f_1, S_u Q f_2 \rangle &= \int_{S_1}^{T_1} \chi_{[S_2,T_2]} du_1 \lim_{\lambda \to 0} \int_{(S_2-u_1)/\lambda^2}^{(T_2-u_1)/\lambda^2} du \langle f_1, S_u Q f_2 \rangle \\ &= \langle \chi_{[S_1,T_1]}, \chi_{[S_2,T_2]} \rangle \cdot \int_{\mathbf{R}} \langle f, S_t Q g \rangle dt. \end{aligned} \tag{3.6}$$

To prove the uniformity of the convergence it will be sufficient to consider separately the two cases: (i) $[S_1, T_1] = [S_2, T_2]$; (ii) $[S_1, T_1] \cap [S_2, T_2] = \emptyset$. In case (i) we have:

$$\begin{aligned} &\left| \lambda^2 \int_{S_1/\lambda^2}^{T_1/\lambda^2} du_1 \int_{S_1/\lambda^2}^{T_1/\lambda^2} du_2 \langle S_{u_1} f_1, S_{u_2} Q f_2 \rangle - \langle \chi_{[S_1,T_1]}, \chi_{[S_1,T_1]} \rangle \cdot \int_{\mathbf{R}} \langle f, S_t Q g \rangle dt \right| \\ &\leqq \int_{S_1}^{T_1} du_1 \left| \int_{(S_1-u_1)/\lambda^2}^{(T_1-u_1)/\lambda 2} du \langle f_1, S_u Q f_2 \rangle - \int_{\mathbf{R}} \langle f, S_t Q g \rangle dt \right| \\ &\leqq \int_{S_1}^{T_1} du_1 \left(\int_{(T_1-u_1)/\lambda^2}^{\infty} du |\langle f_1, S_u Q f_2 \rangle| + \int_{-\infty}^{(S_1-u_1)/\lambda^2} du |\langle f_1, S_u Q f_2 \rangle| \right) \end{aligned}$$

whence the uniform convergence in case (i) follows. In case (ii) one has

$$\left| \lambda^2 \int_{S_1/\lambda^2}^{T_1/\lambda^2} du_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} du_2 \langle S_{u_1} f_1, S_{u_2} Q f_2 \rangle \right| \leqq \int_{S_1}^{T_1} du_1 \int_{(S_2-u_1)/\lambda^2}^{(T_2-u_1)/\lambda^2} du |\langle f_1, S_u Q f_2 \rangle|. \tag{3.7}$$

Assuming, without loss of generality, that $0 \leqq S_1 \leqq T_1 \leqq S_2 \leqq T_2$ and choosing $\varepsilon > 0$, arbitrarily small, the right-hand side of (3.7) is majorized by:

$$\varepsilon \cdot |(f_1 | Q f_2)| + (T_1 - S_1) \cdot \int_{(S_2-T_1+\varepsilon)/\lambda^2}^{(T_2-S_1+\varepsilon)/\lambda^2} |\langle f_1, S_u Q f_2 \rangle| du \tag{3.8}$$

which again implies the uniform convergence.

Remark. In the following we shall use the notation

$$(f | g)_Q := \int_{\mathbf{R}} \langle f, S_t Q g \rangle dt.$$

From (3.3) it is clear that the sesquilinear form $(\cdot | \cdot)_Q$ is of positive type. In particular, it defines a scalar product on K, as anticipated in Sect. 2.

Corollary (3.3). *On the space $L^2(\mathbf{R}) \otimes K_Q \cong L^2(\mathbf{R}, dt; K_Q)$, the operator $1 \otimes Q \geqq 1$ on the domain given by the linear combinations of vectors of the form $\psi \otimes f$, where ψ is a step function in $L^2(\mathbf{R})$ and $f \in D(Q)$.*

Proof. That $1 \otimes Q \geqq 1$ on the domain specified above, follows easily from (3.3) and the fact that $Q \geqq 1$.

The following theorem includes the poof of Theorem (I) of Sect. 2.

Theorem (3.4). *As $\lambda \to 0$, the quantum stochastic process*

$$\left\{\mathscr{H}, \Phi\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du\right), W\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u g\, du\right)\right\} \tag{3.9}$$

($S < T \in \mathbf{R}$, f, $g \in K$) converges weakly in the sense of the matrix elements, to the Q-quantum Brownian Motion on $L^2(\mathbf{R}, dt; K_Q)$ in the sense of Definition (2.3). Moreover, denoting

$$\{\mathscr{H}_Q, \pi_Q, \Psi_Q\}$$

the cyclic quasi-free representation of the CCR *over $L^2(\mathbf{R}, dt; K_Q)$ characterized by:*

$$\langle \Psi_Q, W_Q(\chi \otimes f)\Psi_Q \rangle = e^{-1/2\|\chi\|^2 \cdot \langle f, Qf \rangle}; \quad \chi \in L^2(\mathbf{R}), \quad f \in K_1 \tag{3.10}$$

one has that for each $f_1, \dots, f_n \in K$, $S_1, T_1, \dots, S_n, T_n, x_1, \dots, x_n \in \mathbf{R}$, the limit

$$\begin{aligned}&\lim_{\lambda \to 0} \left\langle \Phi_Q, W\left(x_1 \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1\, du\right) \cdots W\left(x_n \lambda \int_{S_n/\lambda^2}^{T_n/\lambda^2} S_u f_n\, du\right) \Phi_Q \right\rangle \\ &\quad = \langle \Psi_Q, W_Q(x_1 \chi_{[S_1, T_1]} \otimes f_1) \cdots W_Q(x_n \chi_{[S_n, T_n]} \otimes f_n) \Psi_Q \rangle \end{aligned} \tag{3.11}$$

exists uniformly for $x_1, \dots, x_n, S_1, \dots, S_n, T_1, \dots, T_n$ in a bounded set of $\mathbf{R}$.

Proof. By the CCR and (2.6) it follows that

$$\begin{aligned}&\left\langle \Phi_Q, W\left(x_1 \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1\, du\right) \cdots W\left(x_n \lambda \int_{S_n}^{T_n/\lambda^2} S_u f_n\, du\right) \Phi_Q \right\rangle \\ &\quad = \exp\left(-i \operatorname{Im} \sum_{1 \leqq j < k \leqq n} x_j x_k \lambda^2 \int_{S_j/\lambda^2}^{T_j/\lambda^2} \int_{S_k/\lambda^2}^{T_k/\lambda^2} \left\langle S_{u_1} f_j, S_{u_2} f_k \right\rangle du_1\, du_2\right) \cdot \\ &\quad \cdot \exp\left(-\frac{1}{2} \sum_{j,k=1}^{n} \lambda^2 x_j x_k \int_{S_j/\lambda^2}^{T_j/\lambda^2} \int_{S_k/\lambda^2}^{T_k/\lambda^2} \langle S_{u_1} f_j, Q S_{u_2} f_k \rangle\, du_1\, du_2\right)\end{aligned} \tag{3.12}$$

and by Lemma (3.2), as $\lambda \to 0$, this tends to

$$\begin{aligned}&\exp\left(-i \operatorname{Im} \sum_{1 \leqq j < k \leqq n} x_j x_k \langle \chi_{[S_j T_j]}, \chi_{[S_k T_k]} \rangle \cdot (f_j | f_k)\right) \\ &\quad \cdot \exp\left(-\frac{1}{2} \sum_{j,k=1}^{n} x_j x_k \langle \chi_{[S_j T_j]}, \chi_{[S_k T_k]} \rangle \cdot (f_j | f_k)_Q\right) \\ &\quad = \langle \Psi_Q, W_Q(x_1 \chi_{[S_1, T_1]} \otimes f_1) \cdots W_Q(x_n \chi_{[S_n, T_n]} \otimes f_n) \Psi_Q \rangle\end{aligned} \tag{3.13}$$

uniformly for $x_1, \dots, x_n, S_1, \dots, S_n, T_1, \dots, T_n$ in a bounded set of $\mathbf{R}$.

Corollary (3.5). *In the notation of Theorem (3.4) and (2.10), for each $n \in N$ and for* *each* $f_1, f_2, g_1 \cdots g_n \in K$, the expression:

$$\left\langle W\left(\lambda \int_{a_1/\lambda^2}^{b_1/\lambda^2} S_u f_1 du\right)\cdot \Phi_Q, B\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u g_1 du\right)\cdots B\left(\lambda \int_{S_n/\lambda^2}^{T_n/\lambda^2} S_u g_n du\right)\right.$$
$$\left. \cdot W\left(\lambda \int_{a_2/\lambda^2}^{b_2/\lambda^2} S_u f_2 du\right)\cdot \Phi_Q\right\rangle \tag{3.14}$$

converges as $\lambda \to 0$ to

$$\langle W_Q(\chi_{[a_1,b_1]}\otimes f_1)\cdot \Psi_Q, B(\chi_{[S_1,T_1]}\otimes g_1)\cdots B(\chi_{[S_n,T_n]}\otimes g_n)\cdot W_Q(\chi_{[a_2,b_2]}\otimes f_2)\cdot \Psi_Q\rangle \tag{3.15}$$

uniformly for $a_1, b_1, a_2, b_2, S_1, T_1, \ldots, S_n, T_n$ in a bounded subset of $\mathbf{R}$.

Proof. We know from [4] (Lemma (3.2)) that the expression (3.14) is equal to

$$\left\langle W\left(\lambda \int_{a_1/\lambda^2}^{b_1/\lambda^2} S_u f_1 du\right)\Phi_Q, W\left(\lambda \int_{a_2/\lambda^2}^{b_2/\lambda^2} S_u f_2 du\right)\Phi_Q\right\rangle\cdot P_n(s_1^{(\lambda)}, \ldots, s_n^{(\lambda)}, t_{1,2}^{(\lambda)}, \ldots, t_{n-1,n}^{(\lambda)}), \tag{3.16}$$

where P_n is a polynomial in the variables:

$$s_j^{(\lambda)} = i\,\mathrm{Re}\left[\lambda^2 \int_{a_2/\lambda^2}^{b_2/\lambda^2} ds \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt \langle S_s f_2, QS_t g_j\rangle - \lambda^2 \int_{a_1/\lambda^2}^{b_1/\lambda^2} ds \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt \langle S_s f_1, QS_t g_j\rangle\right]$$
$$+ i\,\mathrm{Im}\left[\lambda^2 \int_{a_2/\lambda^2}^{b_2/\lambda^2} ds \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt \langle S_s f_2, S_t g_j\rangle + \lambda^2 \int_{a_1/\lambda^2}^{b_1/\lambda^2} ds \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt \langle S_s f_1, S_t g_j\rangle\right]; \tag{3.17}$$

$$t_{h,j}^{(\lambda)} = \mathrm{Re}\,\lambda^2 \int_{S_h/\lambda^2}^{T_h/\lambda^2} ds \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt \langle S_s g_h, QS_t g_j\rangle + i\,\mathrm{Im}\,\lambda^2 \int_{S_h/\lambda^2}^{T_h/\lambda^2} ds \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt \langle S_s g_h, S_t g_j\rangle. \tag{3.18}$$

The polynomial P_n is of degree n if the variables $s_i^{(\lambda)}$ are considered to be of degree 1 and the variables $t_{ij}^{(\lambda)}$ of degree 2 and universal in the class of quasi-free representations. By Lemma (3.2)

$$\lim_{\lambda\to 0} P_n(\{s_i^{(\lambda)}\}, \{t_{ij}^{(\lambda)}\}) = P_n(\{s_i\}, \{t_{ij}\}). \tag{3.19}$$

Therefore, using the result of Theorem (3.3) to control the scalar product in (3.16) and Lemma (3.2) to control the limit of the variables (3.17), (3.18), we obtain, using again Lemma (3.2) of [4], that the limit of (3.16) for $\lambda \to 0$ is equal to (3.15). In the rest of this paper we shall always consider the case $Q = 1$ and we shall simply write Φ for Φ_Q.

4. Estimate of the Negligible Terms: The Fock Case

The next step in our program is to estimate the asymptotic behaviour, as $\lambda \to 0$, of expressions of the form

$$\left\langle u\otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_{u_1} f_1 du_1\right), U_{t/\lambda^2}^{(\lambda)}\cdot v\otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_{u_2} f_2 du_2\right)\right\rangle \tag{4.1}$$

with $u, v \in \mathscr{H}_0$, $S_1, T_1, S_2, T_2 \in \mathbf{R}$, $S_j \leqq T_j$, $f_1, f_2 \in K$, i.e. of matrix elements of the time-rescaled intersection cocycle $U^{(\lambda)}_{t/\lambda^2}$ with respect to pairs of collective coherent vectors times some vectors u, v in the system space. Using the iteration series (2.15), this leads to estimate terms of the form:

$$\lambda^n \cdot \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\cdot \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_{u_1} f_1 du_1 \right), V_g(t_1) \cdots V_g(t_n) v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_{u_2} f_2 du_2 \right) \right\rangle \tag{4.2}$$

with $t \geqq t_1 \geqq t_2 \geqq \cdots \geqq t_n$ and

$$V_g(t) = i(D \otimes A^+(S_t g) - D^+ \otimes A(S_t g)). \tag{4.3}$$

With the notations

$$D_0 = -D^+; \quad D_1 = D \tag{4.4}$$

$$A^0 = A; \quad A^1 = A^+ \tag{4.5}$$

one obtains:

$$V_g(t_1) \cdot V_g(t_2) \cdots V_g(t_n) = \sum_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n} i^n D_{\varepsilon_1} \cdots D_{\varepsilon_n} \otimes A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_n}(S_{t_n} g), \tag{4.6}$$

and this leads to the problem of estimating matrix elements of products of the form

$$A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_n}(S_{t_n} g) \tag{4.7}$$

with respect to pairs of collective coherent vectors. To this goal, we introduce now some notations which shall be used throughout the paper in the following.

For given $n \in \mathbf{N}$ and $\varepsilon \in \{0,1\}^n$, let $k = k(\varepsilon)$ denote the number of ones in the n-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, i.e. the number of creation operators in (4.7), and let $(j_1, \ldots, j_k) \subseteqq (1, \ldots, n)$ be the ordered set of the indices of time in (4.7), corresponding to the creation operators.

Lemma (4.1). *Any product of the form* (4.7) *can be written as a sum of two terms:*

$$A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_n}(S_{t_n} g) = I^{\varepsilon}_g + II^{\varepsilon}_g \tag{4.8}$$

with

$$I^{\varepsilon}_g = \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leqq r_1 < \cdots < r_m \leqq k \\ \{0, j_1, \ldots, j_k\} \cap \{j_{r_1} - 1, \ldots, j_{r_m} - 1\} = \emptyset}} \prod_{\alpha=1}^{m} \langle S_{t_{j_{r_\alpha} - 1}} g, S_{t_{j_{r_\alpha}}} g \rangle$$

$$\cdot \prod_{j \in \{j_1, \ldots, j_k\} - \{j_{r_1}, \ldots, j_{r_m}\}} A^+(S_{t_j} g) \cdot \prod_{j \in \{1, \ldots, n\} - [\{j_1, \ldots, j_k\} \cup \{j_{r_1} - 1, \ldots, j_{r_m} - 1\}]} A(S_{t_j} g) \tag{4.9a}$$

$$II^{\varepsilon}_g = \sum_{m=0}^{k \wedge (n-k)} \sideset{}{'}\sum_{(q_1, p_1, \ldots q_m, p_m)} \prod_{\alpha=1}^{m} \langle S_{t_{p_\alpha}} g, S_{t_{q_\alpha}} g \rangle$$

$$\cdot \prod_{j \in \{j_1, \ldots, j_k\} - \{q_1, \ldots, q_m\}} A^+(S_{t_j} g) \cdot \prod_{j \in \{1, \ldots, n\} - [\{j_1, \ldots, j_k\} \cup \{p_1, \ldots, p_m\}]} A(S_{t_j} g) \tag{4.9b}$$

where, by definition, $\prod_{\alpha=1}^{0} = 1$ *and, where the symbol* $\sideset{}{'}\sum_{(q_1, p_1, \ldots, q_m, p_m)}$ *denotes summation* *over all the 2m-tuples* $(q_1, p_1, \ldots, q_m, p_m)$ *such that for all* $\alpha, \beta = 1, \ldots, m$

$$p_\alpha \neq p_\beta, q_\beta; \quad q_\alpha \neq q_\beta \ (\alpha \neq \beta); \quad p_\alpha < q_\alpha \tag{4.10}$$

and for some α

$$q_\alpha - p_\alpha \geqq 2. \tag{4.11}$$

Notice that possibly by renumbering the pairs (p_α, q_α), one can always assume that

$$q_1 < q_2 < \cdots < q_m. \tag{4.12}$$

Remark that $\{q_1, \ldots, q_m\}$ respects, as a set, with $\{j_{r_1}, \ldots, j_{r_m}\}$ and $\{p_1, \ldots, p_m\}$ with $\{j_{r_1} - 1, \ldots, j_{r_m} - 1\}$. They differ only in the order. However, from (4.9b) it is clear that the indices p_α, q_β enter only in the product of scalar terms, so that the order is not relevant.

Proof. In the above notations one has:

$$\begin{aligned} A^{\varepsilon_1}(g_1) \cdots A^{\varepsilon_n}(g_n) &= \cdots A(g_{j_{r_1}-1}) \cdot A^+(g_{j_{r_1}}) \cdots A(g_{j_{r_m}-1}) \cdot A^+(g_{j_{r_m}}) \cdots \\ &= \cdots (A^+(g_{j_{r_1}}) \cdot A(g_{j_{r_1}-1}) + \langle g_{j_{r_1}-1}, g_{j_{r_1}} \rangle) \cdots \\ &\quad \cdot (A^+(g_{j_{r_m}}) \cdot A(g_{j_{r_m}-1}) + \langle g_{j_{r_m}-1}, g_{j_{r_m}} \rangle) \cdots, \end{aligned} \tag{4.13}$$

where the dots stand for products of creators or of annihilators not containing terms of the form $A(g_{j_{r_j}-1}) \cdot A^+(g_{j_{r_j}})$. Expanding the products in the right-hand side of (4.13), we find an expression of the form

$$\sum_{F \subseteqq \{1, \ldots, m_\varepsilon\}} \left(\prod_{\alpha \in F} \langle g_{j_{r_\alpha}-1}, g_{j_{r_\alpha}} \rangle \right) \left(\prod_{\alpha \in \{1, \ldots, m_\varepsilon\} - F} (\cdots A(g_{j_{r_\alpha}-1}) \cdot A^+(g_{j_{r_\alpha}}) \cdots) \right), \tag{4.14}$$

where the sum runs over all the subsets F of $\{1, \ldots, m_\varepsilon\}$ and the product of operators is meant of increasing order from left to right. The products of creators and annihilators appearing in the sum (4.14) have the following property: either they are in Wick ordered form, or they are not Wick ordered, but in this case they contain a term of the form $A(g_p)A^+(g_q)$, such that $q - p \geqq 2$. For this reason in bringing to normal order the products in (4.8), only two kinds of terms will appear

(i) The sum over all the terms in (4.14) which are already in normally ordered form.
(ii) The sum collecting all the terms which contain at least one commutator of the form

$$[A(g_p), A^+(g_q)] = \langle g_p, g_q \rangle \quad \text{with} \quad q - p \geqq 2. \tag{4.15}$$

The terms of type (i) are those we denoted by I_g^ε and the terms of type (ii) are those we denoted by II_g^ε. To complete the proof of the identity (4.9), we note that since the indices $j_{r_1}, \ldots, j_{r_m}$ label pairs of annihilation—creation operators, the number of these pairs is less than or equal to the total number of creators or annihilators, i.e.

$$m_\varepsilon \leqq k \wedge (n - k) \leqq n/2;$$

moreover, due to the meaning of the indices r_α, it follows that for all indices m, in both sums (4.49a), (4.9b) such that $m > m_\varepsilon$, one has necessarily

$$\{j_1,\dots,j_k\}\cap\{j_{r_1}-1,\dots,j_{r_m}-1\}\neq\emptyset,$$

hence in the first sum of (4.9a) the terms with $m > m_\varepsilon$ give zero contribution.

Finally, also in the second sum the index m is $\leqq k\wedge(n-k)$ since the appearance of a scalar product implies that one creation and one annihilation operator have been eliminated.

Now, we begin to estimate the terms of type II.

Lemma (4.2). *Denote*

$$\Delta_{m,n}^{(\lambda)}=\lambda^n\cdot\int_0^{t/\lambda^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\cdot\prod_{j=1}^m|\langle S_{t_{p_j}}g,S_{t_{q_j}}g\rangle|$$
$$\cdot\prod_{k\in\{1,\dots,n\}-\{p_1,q_1,\dots,p_m,q_m\}}\lambda\cdot\int_{S_k/\lambda^2}^{T_k/\lambda^2}|\langle S_{u_k}f_k,S_{t_k}g\rangle|du_k \tag{4.16}$$

with $n,k\in\mathbf{N}$, $m=0,\dots,n/2$, $S_1,\dots,S_k$, $T_1,\dots,T_k,t,\lambda\in\mathbf{R}$, $f_1,\dots,f_k,g\in K$, and for any choice of $p_1,\dots,p_m,q_1,\dots,q_m\in\{1,\dots,n\}$ such that the conditions (4.10), (4.11), (4.12) are fulfilled, then

$$\Delta_{m,n}^{(\lambda)}\leqq\frac{t^{n-m}c_1^mc_2^{n-m}}{(n-m)!} \tag{4.17}$$

with

$$c_1=\int_{\mathbf{R}}|\langle g,S_ug\rangle|du, \tag{4.18}$$

$$c_2=\max_{h=1,\dots,k}\int_{\mathbf{R}}|\langle f_h,S_ug\rangle|du \tag{4.19}$$

uniformly in $\lambda\in(0,+\infty)$. Moreover

$$\lim_{\lambda\to0}\Delta_{m,n}^{(\lambda)}=0. \tag{4.20}$$

Proof. With the change of variables $v_k=u_k-t_k$, the quantity $\Delta_{m,n}^{(\lambda)}$ becomes

$$\lambda^{2n-2m}\int_0^{t/\lambda^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\prod_{j=1}^m|\langle S_{t_{p_j}}g,S_{t_{q_j}}g\rangle|$$
$$\cdot\prod_{k\in\{1,\dots,n\}-\{p_1,q_1,\dots,p_m,q_m\}}\int_{S_k/\lambda^2-t_k}^{T_k/\lambda^2-t_k}|\langle f_k,S_{v_k}g\rangle|dv_k, \tag{4.21}$$

hence, with the further change of variable $s_k=\lambda^2t_k$ $(k=1,\dots,n)$, one finds:

$$\Delta_{m,n}^{(\lambda)}=\frac{1}{\lambda^{2m}}\cdot\int_0^t ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{n-1}}ds_n\prod_{j=1}^m|\langle g,S_{(s_{q_j}-s_{p_j})/\lambda^2}g\rangle|$$
$$\cdot\prod_{k\in\{1,\dots,n\}-\{p_1,q_1,\dots,p_m,q_m\}}\int_{(S_k-s_k)/\lambda^2}^{(T_k-s_k)/\lambda^2}|\langle f_k,S_{v_k}g\rangle|dv_k$$
$$\leqq c_2^{n-2m}\cdot\frac{1}{\lambda^{2m}}\cdot\int_0^t dt_1\int_0^{s_1}ds_2\cdots\int_0^{s_{n-1}}ds_n\prod_{j=1}^m|\langle g,S_{(s_{q_j}-s_{p_j})/\lambda^2}g\rangle|. \tag{4.22}$$

Now we do the change of variables

$$s_{q_j} - s_{p_j} = t_{q_j}; \quad j = 1, \dots, m, \tag{4.23}$$

$$s_\alpha = t_\alpha, \quad \alpha \neq q_j, \quad j = 2, \dots, m. \tag{4.24}$$

The right-hand side of (4.22) then becomes:

$$c_2^{n-2m} \cdot \frac{1}{\lambda^{2m}} \cdot \int_0^t dt_1 \cdots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_{-t_{p_1}}^{t'_{q_1-1}-t_{p_1}} dt_{q_1} \int_0^{t_{q_1}+t_{p_1}} dt_{q_1+1} \cdots \int_0^{t_{q_m-2}} dt_{q_m-1} \int_{-t_{p_m}}^{t'_{q_m-1}-t_{p_m}} dt_{q_m}$$

$$\cdot \int_0^{t_{q_m}+t_{p_m}} dt_{q_m+1} \int_0^{t_{q_m}+1} dt_{q_m+2} \cdots \int_0^{t_{n-1}} dt_n \prod_{j=1}^m |\langle g, S_{t_{q_j}/\lambda^2} g\rangle|, \tag{4.25}$$

where

$$t'_{q_j-1} = \begin{cases} t_{q_j-1}, & \text{if} \quad q_j - 1 \neq q_{j-1} \\ t_{q_{j-1}} + t_{p_{j-1}}, & \text{if} \quad q_j - 1 = q_{j-1} \end{cases}. \tag{4.26}$$

The further change of variable

$$t_{q_j}/\lambda^2 = R_{q_j} \tag{4.27}$$

brings the expression (4.25) to the form:

$$c_2^{n-2m} \cdot \int_0^t dt_1 \cdots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_{-t_{p_1}/\lambda^2}^{(t'_{q_1-1}-t_{p_1})/\lambda^2} dR_{q_1} \int_0^{\lambda^2 R_{q_1}+t_{p_1}} dt_{q_1+1} \cdots \int_0^{t_{q_m-2}} dt_{q_m-1} \int_{-t_{p_m}/\lambda^2}^{(t'_{q_m-1}-t_{p_m})/\lambda^2}$$

$$\cdot dR_{q_m} \int_0^{\lambda^2 R_{q_m}+t_{p_m}} dt_{q_m+1} \int_0^{t_{q_m}+1} dt_{q_m+2} \cdots \int_0^{t_{n-1}} dt_n \prod_{j=1}^m |\langle g, S_{R_{q_j}} q\rangle|. \tag{4.28}$$

The crucial remark is that $t'_{q_j-1} - t_{p_j} \leqq 0$. In fact, if $t'_{q_j-1} = t_{q_j-1}$, i.e. $q_j - 1 > q_{j-1}$ then this is clear, while if $t'_{q_j-1} = t_{q_{j-1}} + t_{p_{j-1}}$, i.e. $q_j - 1 = q_{j-1}$ then, $p_j \leqq q_{j-1} - 1$ and

$$t'_{q_j-1} - t_{p_j} = t_{q_{j-1}} + t_{p_{j-1}} - t_{p_j} \leqq t_{q_{j-1}-1} - t_{p_j} \leqq 0. \tag{4.29}$$

Since $R_{q_j} \leqq (t'_{q_j-1} - t_{p_j})/\lambda^2 \leqq 0$ it follows that $0 \leqq \lambda^2 R_{q_j} + t_{p_j} \leqq t'_{q_j-1}$. Hence the expression (4.23) is majorized by:

$$c_2^{n-2m} \cdot \int_0^t dt_1 \cdots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_{-t_{p_1}/\lambda^2}^{(t'_{q_1-1}-t_{p_1})/\lambda^2} dR_{q_1} \int_0^{t_{q_1-1}} dt_{q_1+1} \cdots \int_0^{t_{q_m-2}} dt_{q_m-1} \int_{-t_{p_m}/\lambda^2}^{(t'_{q_m-1}-t_{p_m})/\lambda^2} dR_{q_m}$$

$$\cdot \int_0^{t_{q_m-1}} dt_{q_m+1} \int_0^{t_{q_m}+1} dt_{q_m+2} \cdots \int_0^{t_{n-1}} dt_n \prod_{j=1}^m |\langle g, S_{R_{q_j}} g\rangle|$$

$$\leqq c_2^{n-2m} \cdot c_1^m \cdot \int_0^t dt_1 \cdots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_0^{t_{q_1-1}} dt_{q_1+1} \cdots$$

$$\cdot \int_0^{t_{q_m-2}} dt_{q_m-1} \int_0^{t_{q_m-1}} dt_{q_m+1} \int_0^{t_{q_m}+1} dt_{q_m+2} \cdots \int_0^{t_{n-1}} dt_n = c_2^{n-2m} \cdot c_1^m \cdot \frac{t^{n-m}}{(n-m)!}, \tag{4.30}$$

and this proves (4.17). Finally, denote

$$j := \min\{\alpha; p_\alpha < q_\alpha - 1\}$$

if $q_j - 1 > q_{j-1}$, then $t'_{q_j-1} - t_{p_j} = t_{q_j-1} - t_{p_j} < 0$ almost everywhere; if $q_j - 1 = q_{j-1}$, then by the definition of j one has $p_{j-1} = q_{j-1} - 1$, so $p_j < q_{j-1} - 1$ and $t'_{q_j-1} - t_{p_j} \leqq t_{q_{j-1}-1} - t_{p_j} < 0$ almost everywhere. Moreover since $t \mapsto \langle g, S_t g\rangle$ is bounded, the expression

$$\prod_{j=1}^{m} \int_{-t_{p_j}/\lambda^2}^{(t'_{q_j-1} - t_{p_j})/\lambda^2} |\langle g, S_{R_{q_j}} g\rangle| dR_{q_j} \tag{4.31}$$

tends to zero, as $\lambda \to 0$, almost everywhere in the variables t_{p_j}, t_{q_j-1}, hence by dominated convergence the left-hand side of (4.22) tends to zero as $\lambda \to 0$ and this implies (4.20).

5. Uniform Estimates: The Fock Case

Throughout this section, we shall use the notations introduced at the beginning of Sect. 4 and in Lemmas (4.1) and (4.2). In particular, expanding the product $V_g(t_1) \cdots V_g(t_n)$ using the notations (4.3), (4.4), (4.5), we obtain

$$\sum_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n} i^n D_{\varepsilon_1} \cdots D_{\varepsilon_n} \cdot A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_n}(S_{t_n} g)$$
$$= \sum_{k=0}^{n} \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} i^n D_{\varepsilon_1} \cdots D_{\varepsilon_n} \cdot A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_n}(S_{t_n} g), \tag{5.1}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is uniquely determined by $(j_1, \ldots, j_k)$ and the sum over $(j_1, \ldots, j_k) \subseteq (1, \ldots, n)$ is extended to all the ordered subsets of $\{1, \ldots, n\}$ of cardinality k (remember that the indices $(j_1, \ldots, j_k)$ label the creation operators). Now, for each $\varepsilon \in \{0,1\}^n$, let $(j_{r_1}, \ldots, j_{r_m}) \subseteqq (j_1, \ldots, j_k) \subseteqq (1, \ldots, n)$ be as in (4.9a). Since the correspondence between the ε and the $(j_1, \ldots, j_k)$ is one-to-one, we can use the notation

$$D_{\varepsilon_1} \cdots D_{\varepsilon_n} = D_{(j_1, \ldots, j_k)}, \tag{5.2}$$

where $(j_1, \ldots, j_k)$ corresponds to $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ in the way indicated above.

Theorem (5.1). *For each $n \in N$, $u, v \in H$, $f_1, f_2, g \in K$ and $T_1, T_2, S_1, S_2 \in \mathbf{R}$ $(S_j \leqq T_j)$, the limit, for $\lambda \to 0$, of the quantity*

$$\left\langle u \otimes W\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right) \cdot \Phi, \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V_g(t_1) V_g(t_2) \cdots V_g(t_n) \right.$$
$$\left. \cdot v \otimes W\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du\right) \cdot \Phi \right\rangle \tag{5.3}$$

exists and is equal to

$$\sum_{k=0}^{n} \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leqq r_1 < \cdots < r_m \leqq k \\ \{0, j_1, \ldots, j_k\} \cap \{j_{r_1}-1, \ldots, j_{r_m}-1\} = \varnothing}} i^n \langle u, D_{(j_1, \ldots, j_k)} v \rangle$$

$$\cdot \int\limits_{0 \leqq t_n \leqq \cdots \leqq t_{j_{r_m}+1} \leqq \hat{t}_{j_{r_m}} \leqq t_{j_{r_m}-1} \leqq \cdots \leqq t_{j_{r_1}+1} \leqq \hat{t}_{j_{r_1}} \leqq t_{j_{r_1}-1} \leqq \cdots \leqq t_1 \leqq t} \cdots \int$$

$$\cdot dt_1 \cdots dt_{j_{r_1}-1} d\hat{t}_{j_{r_1}} dt_{j_{r_1}+1} \cdots dt_{j_{r_m}-1} d\hat{t}_{j_{r_m}} dt_{j_{r_m}+1} \cdots dt_n$$

$$\cdot \prod_{j \in \{j_1, \ldots, j_k\} - \{j_{r_1}, \ldots, j_{r_m}\}} \chi_{[S_1, T_1]}(t_j) \cdot (f_1 | g)^{k-m}$$

$$\cdot \prod_{j \in \{1, \ldots, n\} - [\{j_1, \ldots, j_k\} \cup \{j_{r_1}-1, \ldots, j_{r_m}-1\}]} \chi_{[S_2, T_2]}(t_j) \cdot (g | f_2)^{n-k-m} \cdot (g|g)_-^m$$

$$\cdot \langle W(\chi_{[S_1, T_1]} \otimes f_1) \cdot \Psi, W(\chi_{[S_2, T_2]} \otimes f_2) \cdot \Psi \rangle \tag{5.4}$$

where, by definition

$$(g|h)_- = \int_{-\infty}^{0} \langle g, S_u h \rangle du, \tag{5.5}$$

the symbol $\hat{t}_j$ means that the variable t_j is absent and Ψ is the vacuum vector of $\Gamma(L^2(\mathbf{R}, dt; K_1))$.

Proof. Expanding the product $V_g(t_1) \cdots V_g(t_n)$ and using (5.1), (5.2), the scalar product (5.3) becomes

$$\sum_{k=0}^{n} \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} i^n \langle u, D_{(j_1, \ldots, j_k)} v \rangle \cdot \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\cdot \left\langle W\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du \right) \cdot \Phi, A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_n}(S_{t_n} g) W\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du \right) \cdot \Phi \right\rangle. \tag{5.6}$$

Now, according to Lemma (4.1), the expression (5.6) can be split into two pieces

$$I_g(n, \lambda) + II_g(n, \lambda) \tag{5.7}$$

with

$$II_g(n, \lambda) = \sum_{k=0}^{n} \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} i^n \langle u, D_{(j_1, \ldots, j_k)} v \rangle \sum_{m=0}^{k \wedge (n-k)} \sum_{(q_1, p_1, \ldots, q_m, p_m)}{}'$$

$$\lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \prod_{\alpha=1}^{m} \langle S_{t_{p_\alpha}} g, S_{t_{q_\alpha}} g \rangle$$

$$\cdot \prod_{j \in \{j_1 \ldots, j_k\} - \{q_1, \ldots, q_m\}} \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} \langle S_{u_j} f_1, S_{t_j} g \rangle du_j$$

$$\cdot \prod_{j \in \{1, \ldots, n\} - [\{j_1 \ldots, j_k\} \cup \{p_1, \ldots, p_m\}]} \lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} \langle S_{t_j} g, S_{u_j} f_2 \rangle du_j$$

$$\cdot \left\langle W\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du \right) \Phi, W\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du \right) \Phi \right\rangle \tag{5.8a}$$

and

$$I_g(n,\lambda)=\sum_{k=0}^{n}\sum_{1\leqq j_1<\cdots<j_k\leqq n}\langle u, D_{(j_1,\ldots,j_k)}v\rangle$$

$$\cdot\left\langle W\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1\,du\right)\Phi, W\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2\,du\right)\Phi\right\rangle$$

$$\cdot\lambda^n\int_0^{t/\lambda^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\sum_{m=0}^{k\wedge(n-k)}\sum_{\substack{1\leqq r_1<\cdots<r_m\leqq k\\ \{0,j_1\ldots,j_k\}\cap\{j_{r_1}-1,\ldots,j_{r_m}-1\}=\varnothing}}$$

$$\cdot\prod_{\alpha=1}^{m}\langle S_{t_{j_{r_\alpha}-1}}g, S_{t_{j_{r_\alpha}}}g\rangle\prod_{j\in\{j_1\ldots,j_k\}-\{j_{r_1},\ldots,j_{r_m}\}}\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}\langle S_{u_j}f_1, S_{t_j}g\rangle du_j.$$

$$\cdot\prod_{j\in\{1,\ldots,n\}-[\{j_1\ldots,j_k\}\cup\{j_{r_1}-1,\ldots,j_{r_m}-1\}]}\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2}\langle S_{t_j}g, S_{u_j}f_2\rangle du_j. \tag{5.8b}$$

Using the notation (4.16), we obtain, for this piece, the estimate:

$$|II_g(n,\lambda)|\leqq\sum_{k=0}^{n}\sum_{1\leqq j_1<\cdots<j_k\leqq n}\sum_{m=0}^{k\wedge(n-k)}\sum_{(q_1,p_1,\ldots,q_m,p_m)}^{\prime}|\langle u, D_{(j_1,\ldots,j_k)}v\rangle|$$

$$\cdot\left|\left\langle W\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1\,du\right)\Phi, W\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2\,du\right)\Phi\right\rangle\right|\Delta_{n,m}^{(\lambda)}, \tag{5.9}$$

and the right-hand side of (5.9) tends to zero, as $\lambda\to 0$, by (4.20). Hence the limit of the expression (5.6) (if it exists) is equal to

$$\lim_{\lambda\to 0} I_g(n,\lambda).$$

And since, by Theorem (3.4), and in the notation (2.24), the scalar product of the collective coherent vectors converges to

$$\langle W(\chi_{[S_1T_1]}\otimes f_1)\Psi, W(\chi_{[S_2T_2]}\otimes f_2)\Psi\rangle,$$

the problem is reduced to proving that, for each $k=0,\ldots,m$ and $1\leqq j_1<\cdots<j_k\leqq n$, the limit of the quantity

$$\lambda^n\int_0^{t/\lambda^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\sum_{m=0}^{k\wedge(n-k)}\sum_{\substack{1\leqq r_1<\cdots<r_m\leqq k\\ \{0,j_1\ldots,j_k\}\cap\{j_{r_1}-1,\ldots,j_{r_m}-1\}=\varnothing}}$$

$$\cdot\prod_{\alpha=1}^{m}\langle S_{t_{j_{r_\alpha}-1}}g, S_{t_{j_{r_\alpha}}}g\rangle\prod_{j\in\{j_1\ldots,j_k\}-\{j_{r_1},\ldots,j_{r_m}\}}\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}\langle S_{u_j}f_1, S_{t_j}g\rangle du_j$$

$$\cdot\prod_{j\in\{1,\ldots,n\}-[\{j_1\ldots,j_k\}\cup\{j_{r_1}-1,\ldots,j_{r_m}-1\}]}\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2}\langle S_{t_j}g, S_{u_j}f_2\rangle du_j \tag{5.10}$$

as $\lambda\to 0$ exists and has the expression that one deduces from (5.4), (5.5). To this goal notice that, with the change of variables $u_j-t_j=v_j$, this expression becomes

$$\sum_{m=0}^{k\wedge(n-k)} \sum_{\substack{1\leqq r_1<\cdots<r_m\leqq k\\ \{0,j_1\ldots,j_k\}\cap\{j_{r_1}-1,\ldots,j_{r_m}-1\}=\varnothing}} \lambda^{2n-2m}\int_0^{t/\lambda^2} dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n$$

$$\cdot\prod_{\alpha=1}^{m}\langle S_{t_{j_{r_\alpha}-1}}g, S_{t_{j_{r_\alpha}-1}}g\rangle \prod_{j\in\{j_1\ldots,j_k\}-\{j_{r_1},\ldots,j_{r_m}\}}\int_{S_1/\lambda^2-t_j}^{T_1/\lambda^2-t_j}\langle S_{v_j}f_1,g\rangle dv_j.$$

$$\cdot\prod_{j\in\{1,\ldots,n\}-[\{j_1\ldots,j_k\}\cup\{j_{r_1}-1,\ldots,j_{r_m}-1\}]}\int_{S_2/\lambda^2-t_j}^{T_2/\lambda^2-t_j}\langle g,S_{v_j}f_2\rangle dv_j; \tag{5.11}$$

with the further change of variables $\lambda^2 t_j = s_j$, we obtain

$$\sum_{m=0}^{k\wedge(n-k)} \sum_{\substack{1\leqq r_1<\cdots<r_m\leqq k\\ \{0,j_1\ldots,j_k\}\cap\{j_{r_1}-1,\ldots,j_{r_m}-1\}=\varnothing}} \lambda^{-2m}\int_0^{t} ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{n-1}} ds_n\prod_{\alpha=1}^{m}\langle g,S_{(s_{j_{r_\alpha}}-s_{j_{r_\alpha}-1})/\lambda^2}g\rangle. \tag{5.12}$$

$$\prod_{j\in\{j_1\ldots,j_k\}-\{j_{r_1},\ldots,j_{r_m}\}}\int_{(S_1-s_j)/\lambda^2}^{(T_1-s_j)/\lambda^2}\langle S_{v_j}f_1,g\rangle dv_j \prod_{j\in\{1,\ldots,n\}-[\{j_1\ldots,j_k\}\cup\{j_{r_1}-1,\ldots,j_{r_m}-1\}]}$$

$$\cdot\int_{(S_2-s_j)/\lambda^2}^{(T_2-s_j)/\lambda^2}\langle g,S_{v_j}f_2\rangle dv_j. \tag{5.13}$$

Now, putting

$$t_{j_{r_\alpha}}=(s_{j_{r_\alpha}}-s_{j_{r_\alpha}-1})/\lambda^2;\quad \alpha=1,\ldots,m, \tag{5.14}$$

$$t_j=s_j;\quad j\in\{1,\ldots,n\}-\{j_{r_1},\ldots,j_{r_m}\} \tag{5.15}$$

we obtain:

$$\sum_{m=0}^{k\wedge(n-k)} \sum_{\substack{1\leqq r_1<\cdots<r_m\leqq k\\ \{0,j_1\ldots,j_k\}\cap\{j_{r_1}-1,\ldots,j_{r_m}-1\}=\varnothing}} \int_0^{t} dt_1\int_0^{t_1} dt_2\cdots\int_{-t_{j_{r_1}-1}/\lambda^2}^{0} dt_{j_{r_1}}\langle g,S_{t_{j_{r_1}}}g\rangle$$

$$\cdot\int_0^{\lambda^2 t_{j_{r_1}}+t_{j_{r_1}-1}} dt_{j_{r_1}+1}\cdots\int_0^{t_{j_{r_m}-2}} dt_{j_{r_m}-1}\int_{-t_{j_{r_m}-1}/\lambda^2}^{0} dt_{j_{r_m}}\langle g,S_{t_{j_{r_m}}}g\rangle$$

$$\cdot\int_0^{\lambda^2 t_{j_{r_m}}+t_{j_{r_m}-1}} dt_{j_{r_m}+1}\cdots\int_0^{t_{n-1}} dt_n\cdot\prod_{j\in\{j_1\ldots,j_k\}-\{j_{r_1},\ldots,j_{r_m}\}}\int_{(S_1-t_j)/\lambda^2}^{(T_1-t_j)/\lambda^2}\langle S_{v_j}f_1,g\rangle dv_j$$

$$\cdot\prod_{j\in\{1,\ldots,n\}-[\{j_1\ldots,j_k\}\cup\{j_{r_1}-1,\ldots,j_{r_m}-1\}]}\int_{(S_2-t_j)/\lambda^2}^{(T_2-t_j)/\lambda^2}\langle g,S_{v_j}f_2\rangle dv_j. \tag{5.16}$$

Now, as $\lambda\to 0$,

$$\int_{-t_{j_{r_\alpha}-1}/\lambda^2}^{0} dt_{j_{r_\alpha}}\langle g,S_{t_{j_{r_\alpha}}}g\rangle\to(g|g)_- \tag{5.17}$$

$$\cdot\int_0^{\lambda^2 t_{j_{r_\alpha}}+t_{j_{r_\alpha}-1}} dt_{j_{r_\alpha}+1}\to\int_0^{t_{j_{r_\alpha}-1}} dt_{j_{r_\alpha}+1}$$

$$\cdot\int_{(S_\alpha-t_\alpha)/\lambda^2}^{(T_\alpha-t_\alpha)/\lambda^2}\langle S_{v_\alpha}f_\alpha,g\rangle dv_\alpha\to\chi_{[S_\alpha,T_\alpha]}(t_j)(f_\alpha|g);\quad \alpha=1,2$$

with $(g|g)_-$ given by (5.5). Since in all cases the convergence is dominated (due to $t < \infty$ and (2.4)), it follows that, as $\lambda \to 0$, the expression (5.14) converges to (5.4) and this ends the proof.

Lemma (5.2). *Let f_1, f_2, g, t, and $D_\pm$ be a fixed as in Theorem (5.1)* and let $I_g(n, 1)$, *be defined by (5.8a) respectively, then*

$$|I_g(n, \lambda)| \leqq \|u\| \cdot \|v\| c^n \frac{(t \vee 1)^n}{(n/2)!} \tag{5.18}$$

uniformly in $\lambda > 0$, where, c is a constant.

Proof. The terms of type $I_n(\lambda)$ have the form (5.8a) and therefore they are estimated using (5.16) which yields the majorization:

$$|I_g(n, \lambda)| \leqq \sum_{k=0}^{n} \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leqq r_1 < \cdots < r_m \leqq k \\ \{0, j_1, \ldots, j_k\} \cap \{j_{r_1} - 1, \ldots, j_{r_m} - 1\} = \varnothing}} \tag{5.19}$$

$$\left| \left\langle W\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du \right) \Phi, W\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du \right) \Phi \right\rangle \right| \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_{-t_{j_{r_1}}/\lambda^2}^{0} dt_{j_{r_1}}$$

$$\cdot |\langle g, S_{t_{j_{r_1}}} g \rangle| \int_0^{\lambda^2 t_{j_{r_1}} + t_{j_{r_1}-1}} dt_{j_{r_1}+1} \cdots \int_0^{t_{j_{r_m}-2}} dt_{j_{r_m}-1} \int_{-t_{j_{r_m}-1}/\lambda^2}^{0} dt_{j_{r_m}} |\langle g, S_{t_{j_{r_m}}} g \rangle|$$

$$\cdot \int_0^{\lambda^2 t_{j_{r_m}} + t_{j_{r_m}-1}} dt_{j_{r_m}+1} \cdots \int_0^{t_{n-1}} dt_n \prod_{j \in \{j_1, \ldots, j_k\} - \{j_{r_1}, \ldots, j_{r_m}\}} \int_{(S_1 - t_j)/\lambda^2}^{(T_1 - t_j)/\lambda^2} |\langle S_{v_j} f_1, g \rangle| dv_j$$

$$\cdot \prod_{j \in \{1, \ldots, n\} - [\{j_1, \ldots, j_k\} \cup \{j_{r_1} - 1, \ldots, j_{r_m} - 1\}]} \int_{(S_2 - t_j)/\lambda^2}^{(T_2 - t_j)/\lambda^2} |\langle g, S_{v_j} f_2 \rangle| dv_j.$$

Now, since $t_{j_{r_\alpha}} \in (-(1/\lambda^2) t_{j_{r_\alpha}-1}, 0)$, it follows that $\lambda^2 t_{j_{r_\alpha}} + t_{j_{r_\alpha}-1} \leqq t_{j_{r_\alpha}-1}$ and therefore, since $n - m \geqq n/2$, the expression (5.19) is dominated by

$$\|u\| \cdot \|v\| \cdot \|D\|^n n^2 2^n |(g|g)_-|^m \cdot |(f_1|g)|^{k-m} \cdot |(g|f_2)|^{n-k-m}$$

$$\cdot \int \cdots \int_{0 \leqq t_{n-1} \leqq \cdots \leqq \hat{t}_{j_{r_m}} \leqq \cdots \leqq \hat{t}_{j_{r_1}} \leqq \cdots \leqq t_1 \leqq t} dt_1 \cdots d\hat{t}_{j_{r_1}} \cdots d\hat{t}_{j_{r_m}} \cdots dt_n$$

$$\leqq \|u\| \cdot \|v\| c^n \max_{0 \leqq m \leqq n/2} \frac{(t \vee 1)^n}{(n-m)!} \leqq \|u\| \cdot \|v\| c^n \frac{(t \vee 1)^n}{(n/2)!}, \tag{5.20}$$

and this proves the lemma.

Lemma (5.3). *There exists a constant C, such that for each $n \in N$,*

$$|II_g(n, \lambda)| \leqq C^n \frac{(t \vee 1)^n}{([\frac{1}{3}n])!}. \tag{5.21}$$

Proof. From (5.9) we have that for each $n \in N$,

$$|II_g(n, \lambda)| \leqq \sum_{k=o}^{n} \sum_{1 \leqq j_1 < \cdots j_k \leqq n} \sum_{m=0}^{k \wedge (n-k)} \sum_{(p_1 \cdot q_1, \ldots, p_m, q_m)}' c_3^n \Delta_{n,m}^{(\lambda)}, \tag{5.22}$$

where c_3 is a constant satisfying:

$$\|D\|\cdot(1 \vee \|u\|\cdot\|v\|) \leqq c_3$$

and where $\sum'_{(p_1,q_1,\ldots,p_m,q_m)}$ has been defined by (4.10), (4.11), (4.12). From this definition, one easily verifies that the following identity holds:

$$\sum'_{(p_1,q_1,\ldots,p_m,q_m)} = \sum_{\substack{q_1<\cdots<q_m \\ \{q_h\}_{h=1}^m \subset \{j_h\}_{h=1}^k}} \sum_{\substack{\{p_h\}_{h=1}^m \subset \{1,\ldots,n\}-\{j_h\}_{h=1}^k \\ |\{p_h\}_{h=1}^m|=m}} \sum_{\sigma\in\mathcal{S}'_m} \tag{5.23}$$

where, denoting $\mathcal{S}_m$ the permutation group on $\{1,\ldots,m\}$ and

$$\mathcal{S}'_m = \{\sigma\in\mathcal{S}_m, p_{\sigma(h)} < q_h, h=1,\ldots,m\}.$$

Now, fix $k=0,1,\ldots,n$, $1\leqq j_1<\cdots<j_k\leqq n$, and let $m\leqq\frac{1}{3}n$, then, from (4.17) it follows that with c_1, c_2 given by (4.18), (4.19), one has:

$$\begin{aligned} |II_g(n,\lambda)| &\leqq n^2\cdot\left\{\max_{k=0,\ldots,n}\binom{n}{k}c_3^n\cdot \max_{m=0,\ldots,n/3}\left[\binom{k}{m}\binom{n-k}{m}m!t^{n-m}\frac{c_1^m c_2^{n-m}}{(n-m)!}\right]\right\} \\ &\leqq c_4^n(t\vee 1)^n 2^n\cdot n^2 4^n \max_{m\leqq n/3}\frac{m!}{(n-m)!} \leqq c_5^n\left(\left[\frac{n}{3}\right]\right)!\frac{(t\vee 1)^n}{([\frac{2}{3}n])!}. \end{aligned} \tag{5.24}$$

If $m\geqq\frac{1}{3}n$, then, for each fixed $q_1<\cdots<q_m$ and $p_1,\ldots,p_m$ as in (5.23), after the change of variables $\lambda^2 t_j = s_j$ in the expression (4.16) for $\Delta_{n,m}^{(\lambda)}$ we are led to estimate the quantity:

$$\lambda^{-2m}\sum_{\sigma\in\mathcal{S}'_m}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\prod_{h=1}^m|\langle g, S_{(t_{q_h}-t_{p_{\sigma(h)}})/\lambda^2}g\rangle|. \tag{5.25}$$

For this goal, notice that, for each $p\in\{1,\ldots,n\}-\{p_h,q_h\}_{h=1}^m$, the expression (5.25) is equal to:

$$\lambda^{-2m}\sum_{\sigma\in\mathcal{S}'_m}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{p-2}}dt_{p-1}\int_0^{t_{p-1}}dt_p\int_0^{t_p}dt_{p+1}\cdots\int_0^{t_{n-1}}dt_n \cdot\prod_{h=1}^m|\langle g,S_{(t_{q_h}-t_{p_{\sigma(h)}})/\lambda^2}g\rangle| \tag{5.25a}$$

where, the variable t_p does not appear in the interand. Since, for any such p, $t_p\leqq t_{p-1}\leqq t$, it follows that (5.25) is majorized by:

$$\lambda^{-2m}t\sum_{\sigma\in\mathcal{S}'_m}\int_0^t dt\int_0^{t_1}dt_2\cdots\int_0^{t_{p-2}}dt_{p-1}\int_0^{t_{p-1}}dt_{p+1}\cdots\int_0^{t_{n-1}}dt_n \cdot\prod_{h=1}^m|\langle g,S_{(t_{q_h}-t_{p_{\sigma(h)}})/\lambda^2}g\rangle|. \tag{5.26}$$

Repeating this estimate for each $p\in\{1,\ldots,n\}-\{p_h,q_h\}_{h=1}^m$, we obtain that the expression (5.25) is majorized by:

$$\lambda^{-2m}t^{n-2m}\sum_{\sigma\in\mathcal{S}'_m}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2m-1}}dt_{2m}\prod_{h=1}^m|\langle g,S_{(t_{q_h}-t_{p_{\sigma(h)}})/\lambda^2}g\rangle|. \tag{5.27}$$

Here, $1 \leqq q_1 < \cdots < q_m = 2m$, and $p_{\sigma(h)} < q_h$, for each $h = 1, \ldots, m$. Now, for each $\sigma \in \mathscr{S}'_m$, put

$$\varepsilon_\sigma(j) = \begin{cases} q_h, & \text{if } j = 2h, h = 1, \ldots, m \\ p_{\sigma(h)}, & \text{if } j = 2h-1, h = 1, \ldots, m \end{cases}. \tag{5.28}$$

Then, ε_σ is a map from $\{1,\ldots,2m\}$ onto the set $\{q_1,\ldots,q_m, p_1,\ldots,p_m\}$ and $\varepsilon_\sigma(2) < \cdots < \varepsilon_\sigma(2m)$; $\varepsilon_\sigma(2h-1) < \varepsilon_\sigma(2h)$; $h = 1, \ldots, m$. Moreover, it is clear that if $\sigma \neq \sigma'$, then, $\varepsilon_\sigma \neq \varepsilon_{\sigma'}$. Identifying the set $\{q_1, \ldots, q_m, q_1, \ldots, p_m\}$ with $\{1, \ldots, 2m\}$, ε can be seen as a permutation on $\{1,\ldots,2m\}$ and the expression (5.27) can be written as:

$$t^{n-2m} \sum_{\substack{\varepsilon \in \mathscr{S}_{2m}, \varepsilon(2) < \cdots < \varepsilon(2m) \\ \varepsilon(2h-1) < \varepsilon(2h), h = 1, \ldots, m,}} \lambda^{-2m} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{2m-1}} dt_{2m} \prod_{h=1}^m |\langle g, S_{(t_{\varepsilon(2h)} - t_{\varepsilon(2h-1)})/\lambda^2} g\rangle|. \tag{5.29}$$

To estimate the expression (5.29), we adapt to our needs an argument due to Pulé ([28], Lemma (3)). Denote $\mathscr{P}^0_{2m}$ the set of all permutations σ of $\{1, \ldots, 2m\}$ satisfying

$$\sigma(2) < \sigma(4) < \cdots < \sigma(2m); \quad \sigma(2h-1) < \sigma(2h), \quad h = 1, \ldots, m$$

for $t > 0$ and natural integer k, let

$$S_t^{(k)} = \{x = (x_1, \ldots, x_k) \in \mathbf{R}^k : t \geqq x_1 \geqq \cdots \geqq x_k \geqq 0\}$$

finally, let $\mathscr{P}^0_{2m}$ act on $\mathbf{R}^{2m}$ by

$$\sigma(t_1, \ldots, t_{2m}) = (t_{\sigma(1)}, \ldots, t_{\sigma(2m)}).$$

With these notations, if $f : \mathbf{R}^m \to \mathbf{R}_+$ is a symmetric function, then

$$\begin{aligned} &\sum_{\sigma \in \mathscr{P}^0_{2m}} \lambda^{-2m} \int_{S_t^{(2m)}} f\left(\frac{\tau_{\sigma(2h)} - \tau_{\sigma(2h-1)}}{\lambda^2}\right) d\tau \\ &= \sum_{\sigma \in \mathscr{P}^0_{2m}} \lambda^{-2m} \int_{\sigma(S_t^{(2m)})} f\left(\frac{s_{2h} - s_{2h-1}}{\lambda^2}\right) ds \\ &= \lambda^{-2m} \int_{\cup_{\sigma \in \mathscr{P}^0_{2m}} \sigma(S_t^{(2m)})} f\left(\frac{s_{2h} - s_{2h-1}}{\lambda^2}\right) ds \end{aligned} \tag{5.30}$$

because the $\sigma(S_t^{(2m)})$ are disjoint for different σ. Now notice that, if $\sigma \in \mathscr{P}^0_{2m}$ and $\tau \in S_t^{(2m)}$, then

$$\frac{\tau_{\sigma(2h-1)} - \tau_{\sigma(2h)}}{\lambda^2} = \frac{s_{2h-1} - s_{2h}}{\lambda^2} =: x_h \in \mathbf{R}_+, \quad h = 1, \ldots, m \tag{5.31}$$

$$(\tau_{\sigma(2)}, \tau_{\sigma(4)}, \ldots, \tau_{\sigma(2m)}) = (s_2, s_4, \ldots, s_{2m}) =: (y_1, \ldots, y_m) \in S_t^{(m)}, \tag{5.32}$$

and therefore, under the change of variables (5.31), (5.32), the set $\bigcup_{\sigma \in \mathscr{P}^0_{2m}} \sigma(S_t^{(2m)})$ is transformed into a subset of $S_t^{(m)} \times \mathbf{R}^m_+$ so that the right-hand side of (5.30) is less than or equal to:

$$\int_{S_t^{(m)}} dy \int_{\mathbf{R}_+^m} f(x)dx = \frac{t^m}{m!} \int_{\mathbf{R}_+^m} f(x)dx.$$

Applying this argument to the function

$$f(x) = \prod_{j=1}^{m} |\langle g, S_{x_j} g \rangle|$$

we obtain that the expression (5.29) is majorized by:

$$\frac{t^{n-m}}{m!} c_6^n. \tag{5.33}$$

Putting together (5.29) and (5.33), we get eventually:

$$\begin{aligned} |II_g(n,\lambda)| &\leqq \sum_{k=0}^{n} \sum_{1 \leqq j_1 < \cdots j_k \leqq n} \left(\sum_{m=0}^{k \wedge (n-k) \wedge 1/3n} + \sum_{m=k \wedge (n-k) \wedge 1/3n}^{k \wedge (n-k)} \right) \sum_{(p_1,q_1,\ldots,p_m,q_m)}' \\ &\quad \cdot c_3^{n-2m} \lambda^{-2m} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} |\langle g, S_{(t_{q_h} - t_{p_h})/\lambda^2} g \rangle| \\ &\leqq c_5^n (t \vee 1)^n \frac{([\frac{1}{3}n])!}{([\frac{2}{3}n])!} + c_6^n (t \vee 1)^n \frac{1}{([\frac{1}{3}n])!} \\ &\leqq C^n \frac{1}{([\frac{1}{3}n])!}, \end{aligned} \tag{4.34}$$

where, C is an easily estimated constant.

We sum up our conclusions in the following:

Theorem (5.4). *For every $u, v \in H_0, S_1, T_1, S_2, T_2 \in \mathbf{R}$ $(S_j \leqq T_j), f_1, f_2 \in K$ and for every $T \in \mathbf{R}_+$ the limit*

$$\lim_{\lambda \to 0} \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du \right), U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du \right) \right\rangle \tag{5.35}$$

exists and is equal to

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{1 \leqq j_1 < \cdots < j_k \leqq n} \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leqq r_1 < \cdots < r_m \leqq k \\ \{0, j_1, \ldots, j_k\} \cap \{j_{r_1} - 1, \ldots, j_{r_m} - 1\} = \varnothing}} \langle u, D_{(j_1,\ldots,j_k)} v \rangle \\ &\cdot \int \cdots \int_{0 \leqq t_n \leqq \;\; \leqq t_{j_{r_m}+1} \leqq \hat{t}_{j_{r_m}} \leqq t_{j_{r_m}-1} \leqq \cdots \leqq t_{j_{r_1}+1} \leqq \hat{t}_{j_{r_1}} \leqq t_{j_{r_1}-1} \leqq t} \\ &\cdot dt_1 \cdots dt_{j_{r_1}-1} \widehat{dt}_{j_{r_1}} dt_{j_{r_1}+1} \cdots dt_{j_{r_m}-1} \widehat{dt}_{j_{r_m}} dt_{j_{r_m}+1} \cdots dt_n \\ &\cdot \prod_{\alpha \in \{j_1,\ldots,j_k\} - \{j_{r_1},\ldots,j_{r_m}\}} \chi_{[S_1,T_1]}(t_\alpha) \cdot (f_1|g)^{k-m} \prod_{\alpha \in \{1,\ldots,n\} - (\{j_1,\ldots,j_k\} \cup \{j_{r_1}-1,\ldots,j_{r_m}-1\})} \\ &\cdot \chi_{[S_2,T_2]}(t_\alpha) \cdot (g|f_2)^{n-k-m} \langle \Psi(\chi_{[S_1,T_1]} \otimes f_1), \Psi(\chi_{[S_1,T_1]} \otimes f_1) \rangle \cdot (g|g)_-^m \end{aligned} \tag{5.36}$$

where, $(g|h)_-$ is defined by (5.5).

Proof. Expanding $U^{(\lambda)}_{t/\lambda^2}$ with the iterative series one obtains a series which is absolutely and uniformly covergent in the pair $(\lambda, t)\in\mathbf{R}_+ \times [0, T]$ for any $T < +\infty$,

$$\left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), U^{(\lambda)}_{t/\lambda^2} v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du\right)\right\rangle$$

$$= \langle u,u\rangle\cdot\left\langle \Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), \Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du\right)\right\rangle$$

$$+\sum_{n=1}^{\infty}(-i)^n\lambda^n\cdot\int_0^{t/\lambda^2} dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n$$

$$\cdot\left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_{u_1} f_1 du_1\right), V_g(t_1)\cdots V_g(t_n)v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_{u_2} f_2 du_2\right)\right\rangle \tag{5.37}$$

expanding the product $V_g(t_1)\cdots V_g(t_n)$ as in (4.6) and using Lemma (4.1), the series (5.37) becomes

$$\sum_{n=0}^{\infty}(-i)^n I_g(n,\lambda)+\sum_{n=0}^{\infty}(-i)^n II_g(n,\lambda) \tag{5.38}$$

with $I_g(n,\lambda)$, $II_g(n,\lambda)$ defined respectively by (5.8b) and (5.8a). By Lemma (4.2) each term $II_g(n,\lambda)$ tends to zero as $\lambda\to 0$ and by Lemma (5.3), the series $\sum_{n=0}^{\infty}(-i)^n II_g(n,\lambda)$ is absolutely convergent, uniformly in λ and uniformly for t, S_1, S_2, T_1, T_2 in a bounded set. Hence

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty}(-i)^n II_g(n,\lambda)=0.$$

The estimate of Lemma (5.2) shows that the series (5.37) is absolutely and uniformly convergent for $\lambda, t, S_1, S_2, T_1, T_2$ as above. Therefore the statement immediately follows from Theorem (5.1).

6. The Stochastic Differential Equation in the Fock Case

Our goal in this section is to prove Theorem (II) of Sect. 2, that is: $Q=1$, then for each $u,v\in H_0, f_1, f_2, g\in K_1, S_1, S_2, T_1, T_2\in\mathbf{R}$ $(S_j\leqq T_j)$, the limit

$$\lim_{\lambda\to 0}\left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), U^{(\lambda)}_{t/\lambda^2} v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du\right)\right\rangle \tag{6.1}$$

exists and is equal to

$$\langle u\otimes\Psi(\chi_{[S_1,T_1]}\otimes f_1), U_t v\otimes\Psi(\chi_{[S_2,T_2]}\otimes f_2)\rangle, \tag{6.2}$$

where the scalar product is meant in the space $H_0\otimes\Gamma(L^2(\mathbf{R}, dt; K_1))$ and U_t is the solution of the quantum stochastic differential equation

$$dU_t=[D\otimes dA_g^+(t)-D^+\otimes dA_g(t)-(g|g)_- D^+D\otimes 1dt]\cdot U_t;\quad U_0=1 \tag{6.3}$$

in the sense of [39].

Notice that, by Theorem (5.4), the limit (6.1) exists.

We shall first prove that the limit (6.1) has the form

$$\langle u, G(t) \rangle, \tag{6.4}$$

where $t \mapsto G(t) \in H_0$ is a a.e.—weakly differentiable function. We then write the expression (6.2) in the form

$$\langle u, F(t) \rangle, \tag{6.5}$$

and we show that the functions $t \mapsto F(t)$, $G(t) \in H_0$ satisfy the same integral equation in H_0. The equality $F(t) = G(t)$ will then follow from the existence and uniqueness theorem for this integral equation in H_0.

Lemma (6.1). *There exists a* a.e.—*weakly differentiable map*

$$t \mapsto G(t) \in K$$

such that for all $u, v, f_1, f_2 \in K_0$ *and for all* S_1, T_1, S_2, T_2 *one has*

$$\begin{aligned}&\lim_{\lambda \to 0} \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 \, du\right), U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 \, du\right)\right\rangle \\ &\quad = \langle u, G(t) \rangle.\end{aligned} \tag{6.6}$$

Proof. The limit in the expression (6.1) exists, is sesquilinear in u, v and is dominated by $\|u\| \cdot \|v\|$. Hence there exists a contraction $V_t = V_t(f_1, f_2, S_1, S_2, T_1, T_2): H_0 \to H_0$ such that the limit of the left hand side of (6.6) is equal to

$$\langle u, V_t v \rangle.$$

Denoting $G(t) = V_t v$, one obtains (6.4). The weak differentiability of $t \to G(t)$ for $t \in \mathbf{R} \setminus \{S_1, T_1, S_2, T_2\}$ follows from Lemma (5.2), Lemma (5.3) and Theorem (5.4).

In order to obtain a differential equation for $G(t)$, first notice that, for fixed λ, one has:

$$\begin{aligned}&\frac{d}{dt} \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 \, du\right), U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 \, du\right)\right\rangle \\ &\quad = \left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 \, du\right), -\frac{1}{\lambda} \cdot [-D \otimes A(S_{t/\lambda^2} g)^+ + D^+ \otimes A(S_{t/\lambda^2} g)] \right. \\ &\quad \left. \cdot U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 \, du\right)\right\rangle.\end{aligned} \tag{6.7}$$

Now we introduce the notations:

$$\begin{aligned}I_\lambda = \frac{1}{\lambda} \cdot &\left\langle u \otimes \Phi\left(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 \, du\right), (D \otimes A(S_{t/\lambda^2} g)^+) \right. \\ &\left. \cdot U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 \, du\right)\right\rangle\end{aligned} \tag{6.8}$$

$$II_\lambda = -\frac{1}{\lambda}\cdot\left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), (D^+\otimes A(S_{t/\lambda^2}g))\right.$$
$$\left.\cdot U^{(\lambda)}_{t/\lambda^2} v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 du\right)\right\rangle, \tag{6.9}$$

and we study separately the limits of the quantities I_λ, II_λ as $\lambda\to 0$.

Lemma (6.2).

$$\lim_{\lambda\to 0} I_\lambda = \chi_{[S_1,T_1]}(t)(f_1|g)\langle D^+u, G(t)\rangle \quad \text{a.e.} \tag{6.10}$$

Proof. Using (6.8) we can define $G_\lambda(t)$ by

$$I_\lambda = \frac{1}{\lambda}\cdot\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} \langle S_u f_1, S_{t/\lambda^2} g\rangle du\cdot\langle D^+u, G_\lambda(t)\rangle \tag{6.11}$$

and, with the substitution $u - t/\lambda^2 = v$, the right-hand side of (6.11) becomes

$$\langle D^+u, G_\lambda(t)\rangle\cdot\int_{(S_1-t)/\lambda^2}^{(T_1-t)/\lambda^2} \langle S_v f_1 dv, g\rangle \tag{6.12}$$

which converges a.e., as $\lambda\to 0$, to

$$\langle D^+u, G(t)\rangle\chi_{[S_1,T_1]}(t)(f_1|g) = \langle u, DG(t)\rangle\chi_{[S_1,T_1]}(t)(f_1|g) \tag{6.13}$$

since D is a bounded operator.

Now we write the term II_λ as follows:

$$II_\lambda = \left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_2/\lambda^2} S_u f_1 du\right), \left(-\frac{1}{\lambda}\right)\cdot(D^+\otimes 1)\cdot U^{(\lambda)}_{t/\lambda^2}\cdot(1\otimes A(S_{t/\lambda^2}g))\right.$$
$$\left.\cdot v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 du\right)\right\rangle + \left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), \left(-\frac{1}{\lambda}\right)\cdot(D^+\otimes 1)\right.$$
$$\left.\cdot[(1\otimes A(S_{t/\lambda^2}g)), U^{(\lambda)}_{t/\lambda^2}]\cdot v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_1 du\right)\right\rangle = II_\lambda(a) + II_\lambda(b). \tag{6.14}$$

One easily sees, exactly as in the proof of Lemma (6.2), that

$$\lim_{\lambda\to 0} II_\lambda(a) = -\chi_{[S_2,T_2]}(t)(g|f_2)\langle u, D^+G(t)\rangle \quad \text{a.e.} \tag{6.15}$$

In order to evaluate the limit of $II_\lambda(b)$, we need the following remark:

Lemma (6.3). *Let $F\in L^1(\mathbf{R})$ and let for each $\lambda\in\mathbf{R}_+$, $G_\lambda:\mathbf{R}\to\mathbf{C}$ be a continuous function such that*

$$\sup_{(\lambda,t)\in\mathbf{R}_+\times\mathbf{R}} |G_\lambda(t)| \leqq C \tag{6.16}$$

for some constant $C < +\infty$ and

$$\lim_{\lambda\to 0} G_\lambda(t+\lambda^2 r) = G_0(t) \tag{6.17}$$

uniformly for r in each bounded subset of **R**. *Then*

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\int_0^t ds F\left(\frac{s-t}{\lambda^2}\right)G_\lambda(s)=G_0(t)\int_{-\infty}^0 F(s)ds. \tag{6.18}$$

Proof. The left-hand side of (6.18) is equal to:

$$\lim_{\lambda\to 0}\int_{-t/\lambda^2}^0 F(r)G_\lambda(\lambda^2 r+t)dr \tag{6.19}$$

and the statement follows by dominated convergence.

Lemma (6.4). *In the above notations, one has*:

$$\lim_{\lambda\to 0} II_\lambda(b)=-(g|g)_-\cdot\langle u, D^+DG(t)\rangle. \tag{6.20}$$

Proof. We consider the expression

$$II_\lambda(b)=\left(-\frac{1}{\lambda}\right)\cdot\left\langle Du\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right),[(1\otimes A(S_{t/\lambda^2}g)),U^{(\lambda)}_{t/\lambda^2}]\right.$$
$$\left.\cdot v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2}S_u f_1 du\right)\right\rangle, \tag{6.21}$$

and we split the proof in two steps: first we show that

$$\lim_{\lambda\to 0} II_\lambda(b)=-\lim_{\lambda\to 0}\sum_{n=1}^{\infty}\lambda^{n-1}(-i)^{n-1}\int_0^{t/\lambda^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\cdot$$
$$\cdot\langle S_{t/\lambda^2}g, S_{t_1}g\rangle\cdot\left\langle u\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right),(D^+D\otimes 1)\cdot V_g(t_2)\cdots V_g(t_n)\right.$$
$$\left.\cdot v\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right)\right\rangle, \tag{6.22}$$

and then, noticing that the right-hand side of (6.22) has the form

$$\frac{1}{\lambda^2}\cdot\int_0^t ds\langle S_{t/\lambda^2}g, S_{s/\lambda^2}g\rangle$$
$$\cdot\left\langle D^+Du\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right),U^{(\lambda)}_{s/\lambda^2}\cdot v\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right)\right\rangle, \tag{6.23}$$

and applying Lemma (6.3) with

$$G_\lambda(s)=\left\langle D^+Du\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right),U^{(\lambda)}_{s/\lambda^2}\cdot v\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2}S_u f_1 du\right)\right\rangle \tag{6.24}$$

$$G_0(s)=\langle D^+Du, G(t)\rangle, \tag{6.25}$$

$$F(s)=\langle g, S_s g\rangle, \tag{6.26}$$

we find that the limit (6.23) is equal to

$$-\langle D^+Du, G(t)\rangle\cdot \int_{-\infty}^{0} \langle g, S_s g\rangle ds, \tag{6.27}$$

which is the right-hand side of (6.20). To prove (6.22) we expand U_{t/λ^2} in series. Then, using the identity

$$[1\otimes A(S_t g), V_g(t_j)] = \langle S_t g, S_{t_j} g\rangle D\otimes 1$$

we obtain

$$II_\lambda(b) = -\sum_{n=1}^{\infty} \lambda^{n-1}(-i)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_2} dt_2 \cdots \int_0^{t_{n-1}} dt_n \cdot$$

$$\sum_{j=1}^{n} \langle S_{t/\lambda^2} g, S_{t_j} g\rangle\cdot\left\langle Du\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), V_g(t_1)\cdots V_g(t_{j-1})\right.$$

$$\left.\cdot(D\otimes 1)\cdot V_g(t_{j+1})\cdots V_g(t_n)\cdot v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du\right)\right\rangle. \tag{6.28}$$

As $\lambda\to 0$, the term with $j=1$ in the right-hand side of (6.28) is simply the right-hand side of (6.22). Therefore our thesis is equivalent to show that

$$\lim_{\lambda\to 0}\sum_{n=1}^{\infty} \lambda^{n-1}(-i)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \cdot$$

$$\sum_{j=2}^{n} \langle S_{t/\lambda^2} g, S_{t_j} g\rangle\cdot\left\langle Du\otimes\Phi\left(\lambda\int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du\right), V_g(t_1)\cdots V_g(t_{j-1})\right.$$

$$\left.\cdot(D\otimes 1)\cdot V_g(t_{j+1})\cdots V_g(t_n)\cdot v\otimes\Phi\left(\lambda\int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 du\right)\right\rangle = 0, \tag{6.29}$$

and the proof of this relation is exactly the same as the proof of the relation (4.20) in Lemma (4.2).

Summing up, we have shown that the limit (6.6) is a.e. differentiable and that

$$\langle u, G(t)\rangle = \lim_{\lambda\to 0}\langle u, G_\lambda(t)\rangle = \langle u, G(0)\rangle + \lim_{\lambda\to 0}\int_0^t \frac{d}{ds}\langle u, G_\lambda(s)\rangle ds$$

$$= \langle u, G(0)\rangle + \lim_{\lambda\to 0}\int_0^t (I_\lambda + II_\lambda) ds, \tag{6.30}$$

where I_λ and II_λ are bounded for $(\lambda, s)\in\mathbf{R}_+\times\mathbf{R}_+$. So, by (6.10), (6.15), (6.20) and dominated convergence, one obtains

$$\langle u, G(t)\rangle = \langle u, G(0)\rangle + \int_0^t (\chi_{[S_1,T_1]}(s)(f_1|g)\langle D^+u, G(s)\rangle$$

$$-\chi_{[S_2,T_2]}(s)(g|f_2)\langle u, D^+G(s)\rangle - (g|g)_-\cdot\langle u, D^+DG(s)\rangle) ds. \tag{6.31}$$

But, it is clear that, if U_t is the unique solution of (6.3) and we define $F(t)$ by (6.5), then the function $t\to\langle u, F(t)\rangle$ satisfies Eqs. (6.31) with F substituted everywhere for G and $F(0)=G(0)$. From this we conclude that, for each t,

$$\langle u, F(t)\rangle = \langle u, G(t)\rangle$$

and this proves the identity of (6.1) and (6.2).

7. Examples and Applications

It is instructive to calculate how the scalar product (2.5) looks like under some particular assumptions on the "one-particle free evolution" S_t^0 and on the covariance operator Q. We assume that this evolution has positive energy with absolutely continuous spectral measure, i.e.

$$S_t^0 = \int_0^\infty e^{it\omega} dE(\omega), \tag{7.1}$$

$$\langle f, dE(\omega) g\rangle = J_{f,g}(\omega) d\omega. \tag{7.2}$$

Furthermore we assume that Q has the form

$$Q = \int_0^\infty q(\omega) dE(\omega) \tag{7.3}$$

with $q:[0,+\infty)\to[1,+\infty)$ a continuous function. For example if, in the notations of Sect. 2, we choose $H_1 = L^2(\mathbf{R}^d)$ with $d \geqq 3$ and

$$S_t^0 = e^{-it\Delta}; \quad \Delta - \text{the Laplacian} \tag{7.4}$$

$$q(\omega) = \coth(\beta\omega/2), \tag{7.5}$$

then the sub-space K in (2.4) can be taken to consist of those functions f in $D(Q)$ such that f and Qf are $L^2(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$. Defining, as in Sect. 2, for some fixed $\omega_0 \in \mathbf{R}$,

$$S_t = e^{-i\omega_0 t} S_t^0 \tag{7.6}$$

we obtain

Lemma (7.1). *For all $f, g \in K$, the Radon–Nikodym derivative $J_{f,g}(\cdot)$ is a continuous function, vanishing at 0 and at $+\infty$. Moreover the expression*

$$(f|g)_Q := \int_{\mathbf{R}} \langle f, S_t Q g\rangle dt = 2\pi q(\omega_0) J_{f,g}(\omega_0) \tag{7.7}$$

defines a (usually degenerate) positive sesquilinear form on K.

Proof. For $f, g \in K$ the integral

$$(f|g)_Q(\omega_0) = \int_{\mathbf{R}} \langle f, S_t Q g\rangle dt = \int_{\mathbf{R}} e^{-it\omega_0} \langle f, S_t^0 Q g\rangle dt$$

is a continuous function of ω_0 vanishing at infinity by the Riemann–Lebesgue Lemma. Moreover

$$(f|g)_Q(\omega_0) = \int_{\mathbf{R}} dt \int_0^{+\infty} e^{it(\lambda-\omega_0)} q(\lambda) J_{f,g}(\lambda) d\lambda = 2\pi q(\omega_0) J_{f,g}(\omega_0). \tag{7.8}$$

Hence $J_{f,g}(\cdot)$ is a continuous function. Since it vanishes on the negative half line, by continuity it will vanish at $\omega_0 = 0$.

If S_t^0 and Q are as (7.4), (7.5), then $J_{f,g}$ can be computed explicitly and one finds

$$J_{f,g}(\omega_0) = \omega_0^{d-2/2} \int_{S^{(d-1)}} \hat{f}(\sqrt{\omega_0}, \sigma)^* \hat{g}(\sqrt{\omega_0}, \sigma)\, d\sigma_{d-1}, \tag{7.9}$$

where $S^{(d-1)} \subseteq \mathbf{R}^d$ is the unit sphere and $d\sigma_{d-1}$ the normalized measure on it and, where $\hat{f}$ is the normalized Fourier transform of f expressed in polar coordinates in momentum space. Denoting $L^2(S^{(d-1)})$ the space of square integrable complex valued functions on $S^{(d-1)}$ with the natural scalar product and considering the map

$$f \in L^2 \cap L^1(\mathbf{R}^d) \to \hat{f}\omega_0 = \hat{f}(\sqrt{\omega_0}, \cdot) \in L^2(S^{(d-1)})$$

from (7.8) and (7.9) we obtain

$$(f|g)_Q = (f|g)_Q(\omega_0) = 2\pi q(\omega_0)\omega^{d-2/2} \langle \hat{f}_{\omega_0}, \hat{g}_{\omega_0} \rangle_{L^2(S^{(d-1)})}.$$

Now we use this result to make more explicit the meaning of the scalar coefficient $(g|g)$ entering in the stochastic differential equation (2.32). Even though Theorem (II) is formulated only in the Fock case ($Q = 1$), we deal here with a general Q. In this case the stochastic differential equation, (2.32) becomes (cf. [5])

$$dU_t = \{D \otimes dA_g^+(t) - D^+ \otimes dA_g(t) - (g|g)_{Q_+}^- D^+ D \otimes 1\, dt + (g|g)_{Q_-}^- DD^+ \otimes dt\} U_t \tag{7.10}$$

with

$$(g|g)_{Q_\pm}^- = \int_{-\infty}^{0} \left\langle f, S_t\left(\frac{Q \pm 1}{2}\right) g \right\rangle dt. \tag{7.11}$$

In this case the Ito table for $dA_g^{\pm}(t)$ is

$$dA_g(t) \cdot dA_g^+(t) = 2\Re(g|g)_{Q_+}^- dt,$$
$$dA_g^+(t) \cdot dA_g(t) = 2\Re(g|g)_{Q_-}^- dt,$$

therefore, separating the real and the imaginary part in the scalar factors $(g|g)_{Q_\pm}^-$ amounts to separating the Ito correction term from a purely Hamiltonian term of the form

$$(\Im(g|g)_{Q_+}^- D^+ D \otimes 1 + \Im(g|g)_{Q_-}^- DD^+ \otimes 1) dt.$$

This is an operator generalization of the scalar Lamb shift. In order to see what the scalar terms (7.11) look like under the assumptions (7.4) and (7.5), we use the identity:

$$\int_{-\infty}^{0} e^{it\omega}\, dt = \pi\delta(\omega) - i\mathscr{P}\frac{1}{\omega},$$

where $\mathscr{P}\frac{1}{\omega}$ denotes the principal part distribution, to obtain

$$(g|g)_{Q\pm}^- = \int_{-\infty}^{0} dt \int_{\mathbf{R}} d\omega e^{it(\omega-\omega_0)}\left(\frac{q(\omega) \pm 1}{2}\right) J_{g,g}(\omega) = \frac{\pi}{2}(q(\omega_0) \pm 1) J_{g,g}(\omega_0)$$
$$- i\mathscr{P}\int_{\mathbf{R}} \frac{q(\omega) \pm 1}{2(\omega - \omega_0)} J_{g,g}(\omega)\, d\omega.$$

This gives the expression of the pumping rates and intensity of the energy shift in terms of the original Hamiltonian model.

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