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Quantum Group Structure in the Fock Space Resolutions of $\widehat{sl}(n)$ Representations

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Abstract. We describe a complex of Wakimoto-type Fock space modules for the affine Kac–Moody algebra $\widehat{sl}(n)$. The intertwining operators that build the complex are obtained from contour integrals of so-called screening operators. We show that a quantum group structure underlies the algebra of screening operators. This observation greatly facilitates the explicit determination of the intertwiners. We conjecture that the complex provides a resolution of an irreducible highest weight module in terms of Fock spaces.

1. Introduction

There are basically two procedures for constructing the correlation functions of a given conformal field theory on a general Riemann surface. The first consists of solving a set of differential equations arising from the symmetry structure of the theory. This method has proved to be useful in a number of cases, but progress seems to be limited due to the complicated nature of the differential equations involved. The second procedure, that originates in "the old string days," is purely algebraic in origin and involves the explicit computation of the correlation functions by "sewing" fundamental three point functions. The latter procedure, however, seems only feasible for free field theories.

It has been known for some time that many (even non-free) two dimensional conformal field theories admit a free field realization, albeit their Hilbert space is only a subspace of the total Fock space of these free fields. For the minimal models of the Virasoro algebra [BPZ] this so-called Feigin–Fuchs realization ("Coulomb gas") was used elegantly in [DF1, DF2] to compute the correlation functions on a sphere. Generalization to higher genus surfaces, by sewing, requires a procedure for projecting out irreducible representations from this Fock space. It was realized recently that this projection can be achieved by taking alternating sums over an

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infinite set of such Fock spaces [Fe]. The reason is that these Fock spaces, together with the group invariant mappings between them (the so-called intertwiners, which were computed in [TK1, FeFu]) form a complex whose cohomology is nonvanishing at one point only, where it is exactly the irreducible representation. In other words, there exists a resolution of the irreducible Virasoro modules in terms of Fock spaces. The resolution was used for computing torus correlation functions in [Fe] and subsequently applied to higher genus surfaces in [FLMS1, BaGo, FS2, FLMS2].

Another important class of conformal field theories are the WZNW-models [WZ, Wi1, No], whose symmetry algebra is an affine Kac–Moody algebra. To apply similar techniques for these WZNW-theories one first requires a free field realization. That such a free field realization might also exist for affine Kac–Moody algebras (for general values of the central charge!) can be anticipated from the Weyl–Kac character formula of an irreducible integrable highest weight module L_Λ (see, e.g. [Ka])

$$\mathrm{ch}_{L_\Lambda}(z,\tau)=\sum_{w\in\hat{W}}(-1)^{l(w)} \cdot \frac{e^{2\pi i\tau h(w*\Lambda)}e^{2\pi i(w*\Lambda,z)}}{\prod_{n\geq 1}((1-e^{2\pi in\tau})^{\ell}\prod_{\alpha\in\Delta_+}(1-e^{2\pi in\tau}e^{2\pi i(\alpha,z)})(1-e^{2\pi i(n-1)\tau}e^{-2\pi i(\alpha,z)}))}, \tag{1.1}$$

where $h(\Lambda)=(\Lambda,\Lambda+2\rho)/2(k+h^\vee)$ and $z\in\mathbf{h}$. (For an explanation of the various symbols we refer to the end of the introduction and Appendix 2).

The right-hand side of (1.1) may be recognized as the alternating sum of traces over Fock spaces $F_{w*\Lambda}$ of a set of bosonic $\beta\gamma$-fields, one for every positive root α of $\mathbf{g}$, and a set of rank $\mathbf{g}$ scalar fields ϕ^i, where $\mathbf{g}$ is the underlying finite dimensional Lie algebra, i.e.

$$\mathrm{ch}_{L_\Lambda}(z,\tau)=\sum_{w\in\hat{W}}(-1)^{l(w)}\,\mathrm{Tr}_{F_{w*\Lambda}}(e^{2\pi i\tau L_0}e^{2\pi i(z,H)}). \tag{1.2}$$

From the expression of $h(\Lambda)$, and the requirement that one obtains the correct central charge and the correct isospin for the $\beta\gamma$-system, one may already guess that

$$\begin{aligned} T(z)&=-\tfrac{1}{2}{:}\partial\phi(z)\cdot\partial\phi(z){:}-i\alpha_0\rho\cdot\partial^2\phi-\sum_{\alpha\in\Delta_+}{:}\beta^\alpha(z)\partial\gamma^\alpha(z){:},\\ H^i(z)&=\sum_{\alpha\in\Delta_+}\alpha^i{:}\beta^\alpha(z)\gamma^\alpha(z){:}+\frac{i}{\alpha_0}\partial\phi^i(z), \end{aligned} \tag{1.3}$$

where $\alpha_0^2=(k+h^\vee)^{-1}$.

This consideration would apply to an arbitrary affine Kac–Moody algebra $\hat{\mathbf{g}}$, in particular no distinction has to be made between simply- or non-simply laced. The problem is to show that the formulae (1.3) extend to a realization of the complete Kac–Moody algebra. For $\widehat{sl}(2)_k$ such a realization was discovered by Wakimoto [Wa] and reads

$$\begin{aligned} &e(z)=\beta(z),\quad h(z)=2{:}\gamma(z)\beta(z){:}+\sqrt{2(k+2)}\,i\partial\phi(z),\\ &f(z)=-{:}\gamma(z)\gamma(z)\beta(z){:}-\sqrt{2(k+2)}\,\gamma(z)i\partial\phi(z)-k\partial_\gamma(z). \end{aligned} \tag{1.4}$$

This realization bears a close resemblance to the following well-known realization of the finite dimensional Lie algebra $\widehat{sl}(2)$ in terms of differential operators on the space of polynomials of a single complex variable z

$$e = \frac{d}{dz}, \quad h = -2z\frac{d}{dz} + 2j, \quad f = -z^2\frac{d}{dz} + 2jz. \tag{1.5}$$

In fact (1.5) is exactly the zero mode piece of (1.4). It is also well-known that realizations of the type (1.5) arise from the (right) group action on a suitable flag manifold, and it is this observation that, in principle, makes the extension to general Lie algebras straightforward [FeFr1].

To apply this free field realization to the computation of correlation functions one needs to find a resolution of the irreducible module in terms of the Fock space modules. The intertwining operators in this complex will be built from so-called screening operators. The screening operators already arise in the study of the realizations of finite dimensional Lie algebras in terms of differential operators as in (1.5). There they have a natural geometric origin as generators of the (left) group action on the flag manifold: they satisfy the Lie algebra of the positive root generators of $\mathbf{g}$. In addition, the intertwining operators are in 1-1 correspondence with the singular vectors in a Verma module of $\mathbf{g}$.

The main results of this paper can be summarized as the following generalizations of these statements to the case of the affine Kac–Moody algebra $\hat{\mathbf{g}}$. The screening operators satisfy the identities of the positive root part of the quantum group $\mathscr{U}_q(\mathbf{g})$ (within suitably chosen contour integrals). Moreover, given a singular vector in a quantum group Verma module we can build an intertwining operator. We conjecture that the converse statement is also true.

The paper is organized as follows: In Sect. 2 we will discuss the Verma module, Fock space module and the corresponding resolutions for a simple finite-dimensional Lie algebra with particular emphasis on $sl(n)$. We do this mainly to establish notations. None of the results in this section are new, we merely present those issues, which, in our opinion are necessary to appreciate the discussion in the affine case.

Section 3 deals with Fock space modules for affine Lie algebras and contains the main results of this paper. The presentation closely follows the finite-dimensional analogue of Sect. 2. We will briefly discuss the affine counterpart of the BGG-resolution. Next, we explain how to obtain a Fock space module for an affine Kac–Moody algebra, and give explicit formulae for $\widehat{sl}(n)$. Screening operators are introduced and it is shown that they satisfy quantum group identities. The exact correspondence between intertwiners of Fock space modules and representations of quantum groups is revealed and used to make the complex explicit for $sl(3)$.

Section 4 contains a discussion of the results and comparison to related work. The results of this paper were announced in [BMP1].

Throughout the paper we will use the following notations (see e.g. [Ka]):
$\mathbf{g}$ a complex simple Lie algebra
$\mathbf{h}$ its Cartan subalgebra with dual $\mathbf{h}^*$

$\mathcal{U}(\cdot)$ the universal enveloping algebra functor
$\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ a Cartan (triangular) decomposition
$\mathbf{b}_\pm = \mathbf{n}_\pm \oplus \mathbf{h}$ the two Borel subalgebras
$G, H, N_\pm, B_\pm$ denote the corresponding groups
ℓ the rank of $\mathbf{g}$
$e_i, h_i, f_i, i = 1, \ldots, \ell$ a system of Cartan generators
$\Delta_\pm$ system of positive/negative roots
$M = \mathbf{Z} \cdot \Delta_+$ is the root lattice of $\mathbf{g}$
W the Weyl group of $\mathbf{g}$
r_α reflection in the root $\alpha \in \Delta_+$, r_i reflection in a simple root α_i
$(,)$ bilinear form on $\mathbf{h}$ or $\mathbf{h}^*$, sometimes also denoted by $\cdot$
$\langle , \rangle$ dual pairing between $\mathbf{h}$ and $\mathbf{h}^*$
ρ the element of $\mathbf{h}^*$ such that $\langle \rho, h_i \rangle = 1$, $\forall i$
$w * \lambda = w(\lambda + \rho) - \rho$ for $w \in W, \lambda \in \mathbf{h}^*$
$\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$
$h^\vee$ is the dual Coxeter number of $\mathbf{g}$
P, P_+ set of integral, and integral dominant weights, respectively.

In Sect. 3 we will distinguish between quantities of the affine Kac–Moody algebra $\hat{\mathbf{g}}$ and its underlying finite-dimensional Lie algebra $\mathbf{g}$ by putting hats on the former. Additional notations in Sect. 3 are (see also Appendix 2 for some additional notation concerning the affine Weyl group):

$\hat{P}^{(k)}, \hat{P}^{(k)}_+$ integral, and dominant integral weights of level k
$\hat{\Delta}_{\mathrm{re}}$ set of real roots.

2. Resolutions for Finite-Dimensional Lie Algebras

2.1 BGG *Resolution of a Verma Module.* In this section we will describe the Bernstein–Gel'fand–Gel'fand resolution of an irreducible highest weight module L_Λ in terms of Verma modules. Recall that a Verma module M_Λ with highest weight Λ is defined as the induced module $M_\Lambda = \mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{b}_+)} \mathbf{C}_\Lambda$, where $\mathbf{C}_\Lambda$ is the 1-dimensional $\mathbf{b}_+$-module, with character determined by $\Lambda \in \mathbf{h}^*$, i.e. $M_\Lambda = \mathcal{U}(\mathbf{g}) \cdot v_\Lambda$, where v_Λ is a (highest weight) vector such that $\mathbf{n}_+ \cdot v_\Lambda = 0$, $h \cdot v_\Lambda = \langle \Lambda, h \rangle v_\Lambda$ for $h \in \mathbf{h}$. In general, a Verma module is not irreducible. To describe the irreducible subspace in terms of a cohomology complex we need the concept of an intertwining operator.

Definition 2.1. *Let V and W be $\mathbf{g}$-modules. An intertwining operator $Q: V \to W$ is a homomorphism $V \to W$ commuting with the $\mathbf{g}$-action on V and W (and hence with the action of $\mathcal{U}(\mathbf{g})$). The set of such intertwining operators is denoted as* $\mathrm{Hom}_{\mathcal{U}(\mathbf{g})}(V, W)$.

It is clear that for $Q \in \mathrm{Hom}_{\mathcal{U}(\mathbf{g})}(V, W)$ both $\mathrm{Ker}\, Q$ and $\mathrm{Im}\, Q$ are invariant subspaces of V and W, respectively.

The set of all intertwining operators between Verma modules was determined in [Ve, BGG1]. Let us describe the result relevant for the complex of an integral dominant weight $\Lambda \in P_+$.

Recall that the Weyl group W of $\mathbf{g}$ is the (finite) group generated by the reflections r_i in the simple roots α_i of $\mathbf{g}$. Every element $w \in W$ can thus be written in the form $w = r_{i_1} \cdots r_{i_n}$, and the length $l(w)$ of w is defined as the minimal number of reflections r_i required. Denote $W^{(k)} = \{w \in W | l(w) = k\}$. A shifted action of W on $\Lambda \in \mathbf{h}^*$ is defined by $w * \Lambda = w(\Lambda + \rho) - \rho$.

For $w_1, w_2 \in W$ we write $w_1 \leftarrow w_2$ if $w_1 = r_\alpha w_2$ for some $\alpha \in \Delta_+$, and $l(w_1) = l(w_2) + 1$. A partial ordering (Bruhat ordering) on W is defined by: $w \prec w'$ if and only if there exists $w_1, \ldots, w_k \in W$ such that $w \leftarrow w_1 \leftarrow w_2 \leftarrow \cdots \leftarrow w_k \leftarrow w'$. We have

Proposition 2.2. [BGG1] *$\Lambda \in P_+, w, w' \in W$, then*

$$\mathrm{Hom}_{\mathscr{U}(\mathbf{g})}(M_{w*\Lambda}, M_{w'*\Lambda}) = \begin{cases} \mathbf{C} & \text{if } \; w \preccurlyeq w' \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $w \preccurlyeq w'$, every such intertwiner is a multiple of the canonical embedding $i_{w,w'}: M_{w*\Lambda} \to M_{w'*\Lambda}$.

A singular (or primitive) vector in a Verma module M_Λ is a vector v such that $\mathbf{n}_+ \cdot v = 0$. Every Verma submodule of M_Λ is generated by a singular vector. Consequently the above proposition shows that the singular vectors in a Verma module $M_\Lambda, \Lambda \in P_+$, are in 1-1 correspondence with elements of the Weyl group and moreover gives a complete description of the embedding pattern of the submodules generated by these singular vectors.

Using these embeddings one may give a description of the irreducible submodule L_Λ of M_Λ, the so-called Bernstein–Gel'fand–Gel'fand resolution, as follows

Theorem 2.3. [BGG2] *Let L_Λ be an irreducible finite dimensional $\mathbf{g}$-module with highest weight Λ, and M_Λ the Verma module with highest weight Λ. There exists a complex of $\mathbf{g}$-modules:*

$$0 \xleftarrow{d^{(0)}} M_\Lambda^{(0)} \xleftarrow{d^{(1)}} M_\Lambda^{(1)} \xleftarrow{d^{(2)}} \cdots \xleftarrow{d^{(s)}} M_\Lambda^{(s)} \longleftarrow 0,$$

where $s = \dim \mathbf{n}_+ = |\Delta_+|$, and

$$M_\Lambda^{(i)} = \bigoplus_{w \in W^{(i)}} M_{w*\Lambda}.$$

The cohomology of this complex is concentrated in the "zeroth" dimension

$$H^i(d) = \frac{\operatorname{Ker} d^{(i)}}{\operatorname{Im} d^{(i+1)}} = \begin{cases} L_\Lambda & \text{if } \; i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The key observation in this construction is that for $w_1, w_2 \in W$ such that $l(w_1) = l(w_2) + 2$ the number of elements $w \in W$ such that $w_1 \leftarrow w \leftarrow w_2$ is equal to either zero or two. In the last case the quadruple (w_1, w_3, w_4, w_2) is called a square. To each arrow $w \leftarrow w'$ one can now assign a sign $s(w, w') = \pm 1$, such that for every square

$$\begin{tikzcd}
& w_3 \arrow[dl] & \\
w_1 & & w_2 \arrow[ul] \arrow[dl] \\
& w_4 \arrow[ul] &
\end{tikzcd}$$

in the complex, the product of signs equals -1. This can be done consistently througout the complex. One now defines for $w_1 \in W^{(i)}, w_2 \in W^{(i-1)}$,

$$d^{(i)}_{w_1, w_2} = \begin{cases} s(w_1, w_2) \iota_{w_1, w_2} & \text{if} \quad w_1 \leftarrow w_2 \\ 0 & \text{otherwise,} \end{cases}$$

and $d^{(i)} = \oplus d^{(i)}_{w_1, w_2}$. The signs ensure that $d^{(i)} d^{(i+1)} = 0$.

Remark. One usually summarizes both statements of Theorem 2.3 by saying that the sequence

$$0 \leftarrow L_\Lambda \leftarrow M_\Lambda^{(0)} \leftarrow \cdots \leftarrow M_\Lambda^{(s)} \leftarrow 0$$

is exact. We have chosen the "unconventional" formulation above to be able to treat the affine case in complete analogy.

2.2 Fock Space Realization of $sl(n)$. In this section we will describe a realization of a simple finite-dimensional Lie algebra in terms of linear differential operators on a certain space of holomorphic functions. The type of realization is known under a wide variety of names, depending on the context, such as the Bargmann realization, coherent state realization [Pe], multiplier realization [Ko2], etc. For many purposes it is sufficient to restrict the function space to polynomials, in which case one can interpret the module as the Fock space of a (finite) set of harmonic oscillators. This will be a convenient description when generalizing these kind of modules to affine Kac–Moody algebras. So, henceforth, we will refer to these modules as "Fock space modules." Our main purpose in the rest of this chapter will be to describe a resolution of an irreducible highest weight module in terms of these Fock space modules, along the lines of the BGG-resolution described in Sect. 2.1.

Let Y be a Schubert cell of maximal dimension in the flag manifold $B_- \backslash G$ (where $g \sim bg$ for $b \in B_-$). We will denote by z both a point in Y and its coordinates. Denote by R_Λ the space of holomorphic sections of a line bundle over Y determined by a character $\chi_\Lambda : B_- \to \mathbf{C}^*$. We will henceforth implicitly identify these sections with functions in $C^\infty(G)$ satisfying the relation $f(bg) = \chi_\Lambda(b) f(g)$, $\forall b \in B_-$, $\forall g \in G$. The group G acts as a transformation group on Y by right multiplication. This induces a representation σ_Λ of $\mathbf{g}$ on R_Λ in terms of linear differential operators. Explicitly,

$$\sigma_\Lambda(x) = \xi(x) + h_\Lambda(x), \quad x \in \mathbf{g}, \tag{2.1}$$

where $\xi : \mathbf{g} \to \operatorname{diff} Y$ and $h_\Lambda : \mathbf{g} \to C^\infty(Y)$. They are obtained as follows [Ko2]. Let

$f \in C^\infty(Y)$, then

$$(\xi(x)f)(z) = \frac{d}{dt} f(ze^{tx})\bigg|_{t=0},$$

and

$$(h_\Lambda(x)f)(z) = \langle \Lambda_*, (\mathrm{Ad}\, z)x \rangle f(z),$$

where $z \in Y$, and $\Lambda_* \in \mathbf{g}'$ (the dual of $\mathbf{g}$) is defined by

$$\langle \Lambda_*, y \rangle = \begin{cases} 0 & y \in \mathbf{n}_+ \\ \frac{d}{dt} \chi_\Lambda(e^{ty})\Big|_{t=0} & y \in \mathbf{b}_- \end{cases}.$$

One may identify $Y \sim N_+$, thus we have as many coordinates z_α as the number of positive roots $\alpha \in \Delta_+$. As remarked before, we may also restrict the realization of $\mathbf{g}$ to the "Fock space" $F_\Lambda = \mathrm{Pol}(z_\alpha)$.

For reasons that will become clear in Sect. 3 it is convenient to encode Λ in terms of another set of operators p^i and q^i with commutation relations

$$[q^i, p^j] = i\delta^{ij}.$$

To this end we identify F_Λ with $\mathrm{Pol}(z_\alpha) \otimes \mathbf{C}_\Lambda$, where $\mathbf{C}_\Lambda$ is the one-dimensional space obtained from the "vacuum vector" $|\Lambda\rangle$, satisfying

$$p^i |\Lambda\rangle = \Lambda^i |\Lambda\rangle,$$

where Λ^i denotes the components of Λ with respect to some orthonormal basis in $\mathbf{h}^*$. The realization on $\mathrm{Pol}(z_\alpha) \otimes \mathbf{C}_\Lambda$ is given by replacing $\Lambda^i \to p^i$ in (2.1). Due to the identity

$$[p^i, e^{i\Lambda \cdot q}] = \Lambda^i e^{i\Lambda \cdot q},$$

we can identify $|\Lambda\rangle = e^{i\Lambda \cdot q}|0\rangle$. So the translation operator $e^{i\Lambda \cdot q}$ "connects" all Fock spaces with different Λ, hence we will refer to this realization as the "universal representation." By slight abuse of notation, we will denote the image of $x \in \mathbf{g}$ in this universal realization simply by x.

Let us now make the aforementioned realization somewhat more explicit in the case of $sl(n)$. The following theorem gives the expressions for the simple root generators in the Chevalley basis, which we recall is defined by the following commutators

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, \\ [h_i, e_j] &= a_{ij} e_j, \\ [h_i, f_j] &= -a_{ij} f_j, \end{aligned} \tag{2.2}$$

and Chevalley–Serre relations

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0, \tag{2.3}$$

where a_{ij} is the Cartan matrix of $\mathbf{g}$.

Proposition 2.4. *The following expressions define a realization, with highest weight Λ, of $sl(n)$ on the space of polynomials in z_{ij}, $1 \leqq i < j \leqq n$,*

$$\begin{aligned}
\sigma_\Lambda(e_i) &= \frac{\partial}{\partial z_{ii+1}} + \sum_{j \leqq i-1} z_{ji} \frac{\partial}{\partial z_{ji+1}}, \\
\sigma_\Lambda(f_i) &= (\Lambda, \alpha_i) z_{ii+1} + \sum_{j \leqq i-1} z_{ji+1} \frac{\partial}{\partial z_{ji}} - \sum_{j \geqq i+2} z_{ij} \frac{\partial}{\partial z_{i+1j}} \\
&\quad - z_{ii+1} \left(\sum_{j \geqq i+1} z_{ij} \frac{\partial}{\partial z_{ij}} - \sum_{j \geqq i+2} z_{i+1j} \frac{\partial}{\partial z_{i+1j}} \right), \\
\sigma_\Lambda(h_i) &= (\Lambda, \alpha_i) - 2 z_{ii+1} \frac{\partial}{\partial z_{ii+1}} + \sum_{j \leqq i-1} \left(z_{ji} \frac{\partial}{\partial z_{ji}} - z_{ji+1} \frac{\partial}{\partial z_{ji+1}} \right) \\
&\quad + \sum_{j \geqq i+2} \left(z_{i+1j} \frac{\partial}{\partial z_{i+1j}} - z_{ij} \frac{\partial}{\partial z_{ij}} \right).
\end{aligned} \tag{2.4}$$

Proof. For $G = SL(n, \mathbf{C})$ we have the following Gauss decomposition (see, e.g. [BaRa] Exercise 11.6.1)

$$g = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ (\zeta)_{pq} & & 1 \end{pmatrix} \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix} \begin{pmatrix} 1 & & (z)_{pq} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (\zeta)(\delta)(z) \in N_- H N_+,$$

with

$$\begin{aligned}
\delta_p &= \frac{\Delta_p}{\Delta_{p-1}}, & & 1 \leqq p \leqq n, \\
z_{pq} &= \frac{1}{\Delta_p} \begin{bmatrix} 1 & \cdots & p-1 & p \\ 1 & \cdots & p-1 & q \end{bmatrix}, & & 1 \leqq p < q \leqq n, \\
\zeta_{pq} &= \frac{1}{\Delta_q} \begin{bmatrix} 1 & \cdots & q-1 & p \\ 1 & \cdots & q-1 & q \end{bmatrix}, & & 1 \leqq q < p \leqq n,
\end{aligned}$$

where

$$\begin{bmatrix} p_1 & \cdots & p_m \\ q_1 & \cdots & q_m \end{bmatrix} = \begin{vmatrix} g_{p_1 q_1} & \cdots & g_{p_1 q_m} \\ \vdots & & \vdots \\ g_{p_m q_1} & \cdots & g_{p_m q_m} \end{vmatrix},$$

and

$$\Delta_p = \begin{bmatrix} 1 & \cdots & p \\ 1 & \cdots & p \end{bmatrix}, \quad \Delta_0 = 1.$$

Thus, for instance, for $g = e^{te_i}$ we have

$$zg = \begin{pmatrix} 1 & \cdots & \cdots & z_{1i+1} + tz_{1i} & \cdots & z_{1n} \\ & 1 & \cdots & \vdots & & \\ & & \ddots & & & \vdots \\ & 0 & & z_{ii+1} + t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

so

$$\xi(e_i)f(z) = \frac{d}{dt} f(ze^{te_i})\bigg|_{t=0} = \left(\frac{\partial}{\partial z_{ii+1}} + \sum_{j \leqq i-1} z_{ji} \frac{\partial}{\partial z_{ji+1}}\right) f(z).$$

The other expressions are proved similarly. □

Let $x^a, a = 1, \ldots, \dim \mathbf{g}$ be an orthonormal basis of $\mathbf{g}$. Let $C_2 = \sum x^a x^a$ denote the (second order) Casimir element of $\mathbf{g}$. We have

Proposition 2.5. *Let $\sigma_\Lambda : \mathbf{g} \to \operatorname{End}(F_\Lambda)$ be as in (2.1), then*

$$\sigma_\Lambda(C_2) = (\Lambda, \Lambda + 2\rho) id,$$

or equivalently, in the universal representation

$$C_2 = (p, p + 2\rho).$$

2.3 Screening and Intertwining Operators on Fock Spaces. In the previous section we have described the so-called Fock space realization of a finite-dimensional simple Lie algebra. It is clear that this representation is not irreducible, and moreover, not even completely reducible. We would like to characterize the subspace of F_Λ corresponding to the irreducible module L_Λ of highest weight Λ. Normally one would try to characterize L_Λ as a subspace of a module selected by certain eigenvalues of the Casimir operators of $\mathbf{g}$. This procedure does not specify L_Λ in this case (comp., Proposition 2.5). To make progress we need the slightly more general concept of an intertwining operator. In this section we will explain how to obtain all intertwining operators between Fock spaces, and show that they completely characterize the irreducible space L_Λ.

Recall that in the previous section the Fock space representation was obtained from the right action of G on the coset space $B_- \backslash G$. Though the left action of G on $B_- \backslash G$ is not an isometry it turns out that the left action of N_+ on $B_- \backslash G$ still contains interesting information.

Define for every $x \in \mathbf{n}_+$ a vector field $\rho(x)$ on Y by

$$(\rho(x)f)(z) = \frac{d}{dt} f(e^{-tx}z)\bigg|_{t=0}, \tag{2.5}$$

where $f \in C^\infty(Y)$. We have the following

Lemma 2.6. [Ko2] *$\rho:\mathbf{n}_+ \to \operatorname{diff} Y$ defines a representation of $\mathbf{n}_+$. Moreover, ρ extends to an isomorphism between $\mathscr{U}(\mathbf{n}_+)$ and the set of all differential operators on Y which are invariant under the action of $\mathbf{n}_+$.*

It is clear, however, that $\rho(x)$ does not commute with the whole algebra $\mathbf{g}$ in general. In fact one easily shows that $\forall x \in \mathbf{h}$, $\forall y \in \mathscr{U}(\mathbf{n}_+)$ we have

$$[\sigma_\Lambda(x), \rho(y)] = \rho([x, y]),$$

for instance

$$[\sigma_\Lambda(h_i), \rho(e_j)] = a_{ij}\rho(e_j). \tag{2.6}$$

At this point one may investigate the subset of $\mathscr{U}(\mathbf{n}_+)$ that gives rise to the so-called quasi-invariant differential operators on F_Λ [Ko2]. We find it more convenient to think of these in terms of intertwining operators (see, e.g. [KV]). To this end we can rewrite (2.6) as

$$\sigma_{\Lambda-\alpha_j}(h_i)\rho(e_j) = \rho(e_j)\sigma_\Lambda(h_i).$$

In other words, we may think of $\rho(x)$ as a map $F_\Lambda \to F_{\Lambda - \deg x}$.

Once more it is convenient to return to the universal representation. Let us define

$$s_i = \rho(e_i)e^{-i\alpha_i \cdot q}.$$

With a little hindsight we will call these the "screening operator" of $\mathbf{g}$. We have

Lemma 2.7.

$$[e_i, s_j] = 0, \quad [h_i, s_j] = 0,$$
$$[f_i, s_j] = \delta_{ij} e^{-i\alpha_i \cdot q}\langle p, h_i\rangle.$$

Recall that we are using the same notations for $x \in \mathbf{g}$ and its image in the universal representation, and that $\langle , \rangle$ denotes the dual pairing between $\mathbf{h}$ and $\mathbf{h}^*$, e.g. within angle brackets h_i always denote the abstract element in $\mathbf{h}$.

The following lemma provides us with an important class of intertwining operators:

Lemma 2.8. *Let $\Lambda \in P$ such that $m = \langle \Lambda, h_i\rangle \in \mathbf{Z}_+$, then $[x, (s_i)^{m+1}] = 0$, $\forall x \in \mathbf{g}$,* i.e. $(\rho(e_i))^{m+1} \in \operatorname{Hom}_{\mathscr{U}(\mathbf{g})}(F_\Lambda, F_{r_i * \Lambda})$.

Proof. We have to prove that $[f_i, (s_i)^{m+1}] = 0$ on F_Λ. Using Lemma 2.7 we find for $v \in F_\Lambda$,

$$\begin{aligned}[f_i, (s_i)^{m+1}]\cdot v &= \sum_{0 \le j \le m} e^{-i\alpha_i \cdot q}(s_i)^{m-j}\langle p, h_i\rangle (s_i)^j \cdot v \\ &= \sum_{0 \le j \le m} \langle \Lambda - j\alpha_i, h_i\rangle e^{-i\alpha_i \cdot q}(s_i)^m \cdot v = 0,\end{aligned}$$

because $\sum_{0 \le j \le m} (m - 2j) = 0$. □

As we will show later the set of all $(s_i)^{m_i+1}, m_i = \langle \Lambda, h_i\rangle$, does not exhaust the set of all intertwiners. For the purpose of characterizing the irreducible subspace

L_Λ of highest weight Λ within F_Λ they are however sufficient, as the following theorem shows:

Proposition 2.9. [Ze] *Let* $\Lambda \in P_+$. *Define* $m_i = \langle \Lambda, h_i \rangle \in \mathbf{Z}_+$, $i = 1, \ldots, \ell$. *Let* $V = \{v \in F_\Lambda | (\rho(e_i))^{m_i+1} \cdot v = 0,\ \forall i = 1, \ldots, \ell\}$, *then* $V \cong L_\Lambda$.

Proof. It is clear that V is an invariant subspace of F_Λ. Using a convenient choice of coordinates it is easy to show that V is finite-dimensional (see, e.g. [Ze]), hence by Weyl's theorem V is completely reducible. However, as one easily verifies, v_Λ is the only highest weight vector of V, which implies $V \cong L_\Lambda$. □

Remark. Though this will not be explored further in this paper, there is an obvious relation with the Borel–Weil theorem (see, e.g. [Bo, Ko1]), which states that L_Λ is isomorphic with those sections which are holomorphic over the entire flag manifold $B_- \backslash G$. Recall the Bruhat decomposition $B_- \backslash G = \bigcup_{w \in W} C_w$ $(Y = C_{id})$ in terms of Schubert cells C_w, labelled by elements of the Weyl group W. One can prove that $\rho(e_i)^{m+1} \cdot v = 0$ if and only if v can be extended holomorphically over $C_{id} \cup C_{r_i}$.

Notice that, at this point, the structure of the Fock space obtained so far is strikingly similar to that of the Verma module. In fact, in the finite dimensional case, this similarity can be pushed further.

Theorem 2.10. [Ko2] *For* $\Lambda \in P$ *let* M_Λ *be the Verma module with highest weight* Λ, *and* v_Λ *its highest weight vector. We define a map* $\gamma: (M_\Lambda)(\lambda) \to \mathrm{Hom}(F_\Lambda, F_{\Lambda - \lambda})$ *by*

$$\gamma(f_{i_1} \cdots f_{i_n} \cdot v_\Lambda) = \rho(e_{i_1} \cdots e_{i_n}).$$

Here $M_\Lambda = \bigoplus_{\lambda \geqq 0} (M_\Lambda)(\lambda)$, *where* $(M_\Lambda)(\lambda) = \{v \in M_\Lambda | h \cdot v = \langle \Lambda - \lambda, h \rangle v\}$ *denotes the weight space decomposition of* M_Λ. *(Notice that though the expression for the vector* $f_{i_1} \cdots f_{i_n} \cdot v_\Lambda$ *is not unique, the map* γ *is properly defined because* f_i *and* e_i *generate isomorphic algebras.)* γ *is an isomorphism between the set of all singular vectors in* $(M_\Lambda)(\lambda)$ *and* $\mathrm{Hom}_{\mathscr{U}(\mathfrak{g})}(F_\Lambda, F_{\Lambda - \lambda})$.

Proof. We have

$$\begin{aligned}[e_i, f_{i_1} \cdots f_{i_n}] \cdot v_\Lambda &= \sum_{j;\, i_j = i} f_{i_1} \cdots f_{i_{j-1}} h_i f_{i_{j+1}} \cdots f_{i_n} \cdot v_\Lambda \\ &= \sum_{j;\, i_j = i} (\langle \Lambda, h_i \rangle - a_j) f_{i_1} \cdots \hat{f}_{i_j} \cdots f_{i_n} \cdot v_\Lambda, \end{aligned} \tag{2.7}$$

where $a_j = a_{i i_{j+1}} + \cdots + a_{i i_n}$, and ^ denotes omission.

Similarly, on F_Λ, we have

$$[f_i, s_{i_1} \cdots s_{i_n}] = \sum_{j;\, i_j = i} e^{-i\alpha_i \cdot q} (\langle \Lambda, h_i \rangle - a_j) s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}. \tag{2.8}$$

Now observe that the right-hand side of (2.7) vanishes if and only if the right-hand side of (2.8) vanishes. Using the fact that every element in $\mathrm{Hom}_{\mathscr{U}(\mathfrak{g})}(F_\Lambda, F_{\Lambda - \lambda})$ can be written in the form $\rho(e_{i_1} \cdots e_{i_n})$ (see Lemma 2.6), the theorem is proved. □

In the next section we will show how this set of intertwining operators allows us to formulate a resolution of the Fock space module completely analogous to the BGG-resolution described in Sect. 2.1.

Finally, for $sl(n)$ the explicit form of the screening operators can easily be determined from their definition (2.5). In the notations of Proposition 2.4 we have:

$$\rho(e_i) = -\left(\frac{\partial}{\partial z_{i\,i+1}} + \sum_{j \geqq i+2} z_{i+1\,j} \frac{\partial}{\partial z_{ij}}\right). \tag{2.9}$$

Given these expressions, it is straightforward to verify the commutation relations of Lemma 2.7 explicitly.

2.4 Fock Space Resolution. In the previous section we have seen that there exists a 1-1 correspondence between the singular vectors in $(M_\Lambda)(\lambda)$ and the invariant homomorphisms (intertwiners) $F_\Lambda \to F_{\Lambda-\lambda}$. From the results of Sect. 2.1 we know that for $\Lambda \in P_+$ the singular vectors v_w are in 1-1 correspondence with the elements w in the Weyl group, and occur for the weights $\Lambda - \lambda = w * \Lambda$. Moreover, $v_{w'} \in M_{w*\Lambda}$ if and only if $w' \preccurlyeq w$, i.e. for $\Lambda \in P_+$,

$$\mathrm{Hom}_{\mathscr{U}(\mathfrak{g})}(F_{w*\Lambda}, F_{w'*\Lambda}) = \begin{cases} \mathbb{C} & w' \preccurlyeq w \\ 0 & \text{otherwise} \end{cases}.$$

Let us denote such a homomorphism by $Q_{w,w'}$. Completely analogous to Theorem 2.3 we can build a complex by combining the various intertwiners $Q_{w,w'}$.

Theorem 2.11. [Ke] *There exists a complex of Fock modules*

$$0 \longrightarrow F_\Lambda^{(0)} \xrightarrow{d^{(0)}} F_\Lambda^{(1)} \xrightarrow{d^{(1)}} \cdots \xrightarrow{d^{(s-1)}} F_\Lambda^{(s)} \longrightarrow 0,$$

where $s = |\Delta_+|$ *and*

$$F_\Lambda^{(i)} = \bigoplus_{w \in W^{(i)}} F_{w*\Lambda}.$$

As usual we define the cohomology of the complex by $H^i(d) = \mathrm{Ker}\, d^{(i)}/\mathrm{Im}\, d^{(i-1)}$. *We have*

$$H^i(d) = \begin{cases} L_\Lambda & i = 0 \\ 0 & \textit{otherwise} \end{cases}.$$

Clearly, for $i = 0$ this is the content of Proposition 2.9. The statement for $i \neq 0$ follows essentially from the fact that every $Q_{w,w'}$ is onto.

Let us make this more explicit for $sl(3)$. Apart from the intertwiners provided by Lemma 2.8 there exist intertwiners corresponding to the Weyl group element $w = r_1 r_2 r_1$. They ensure that each square in the Fock space resolution, as shown in Fig. 1, is commutative. In practice their derivation is a simple matter of rearranging screening operators using classical identities (which the reader may find as the $q \to 1$ limit of the identities in Lemma A.1);

$$Q_{r_1, r_1 r_2} = \sum_{0 \leqq j \leqq l_2} b(l_2, l_3; j)(s_2)^{l_2 - j}(s_3)^j (s_1)^{l_2 - j},$$

$$Q_{r_2, r_2 r_1} = \sum_{0 \leqq j \leqq l_1} b(l_1, l_3; j)(s_1)^{l_1 - j}(-s_3)^j (s_2)^{l_1 - j},$$

$$\begin{tikzcd}
 & F_{r_1 * \Lambda} \arrow[r] \arrow[dr] & F_{r_1 r_2 * \Lambda} \arrow[dr] & \\
F_\Lambda \arrow[ur] \arrow[dr] & & & F_{r_1 r_2 r_1 * \Lambda} \\
 & F_{r_2 * \Lambda} \arrow[r] \arrow[ur] & F_{r_2 r_1 * \Lambda} \arrow[ur] &
\end{tikzcd}$$

Fig. 1. Fock space resolution for $sl(3)$

where $l_i = (\Lambda + \rho, \alpha_i)$ and

$$b(m, n; j) = \frac{m!n!}{j!(m-j)!(n-j)!}.$$

3. Resolutions for Affine Kac–Moody Algebras

3.1 BGG *Resolution for an Affine Kac–Moody Algebra.* For completeness we describe the BGG-resolution for an affine Kac–Moody algebra $\hat{\mathfrak{g}}$. Though the proof is more delicate, the outcome is remarkably similar to that in the finite-dimensional case, so we will make this exposition very brief.

The Weyl group is now infinite, but the notion of length and Bruhat ordering goes through as in the finite-dimensional case. Again, we have for an integral dominant weight $\Lambda \in \hat{P}_+$ that $\mathrm{Hom}_{\mathscr{U}(\hat{\mathfrak{g}})}(M_{w*\Lambda}, M_{w'*\Lambda})$ is nonvanishing if and only if $w \preccurlyeq w'$, and in that case the intertwiner is a multiple of the canonical embedding. We have the following resolution of an irreducible integrable highest weight module L_Λ in terms of Verma modules

Theorem 3.1. [GL, RCW] *Let $\Lambda \in \hat{P}_+$. There exists a complex of Verma modules*

$$0 \xleftarrow{d^{(0)}} M_\Lambda^{(0)} \xleftarrow{d^{(1)}} M_\Lambda^{(1)} \xleftarrow{d^{(2)}} M_\Lambda^{(2)} \xleftarrow{d^{(3)}} \cdots,$$

where

$$M_\Lambda^{(i)} = \bigoplus_{w \in \hat{W}^{(i)}} M_{w*\Lambda}.$$

The cohomology of this complex is given by

$$H^i(d) = \begin{cases} L_\Lambda & \text{if } \; i = 0 \\ 0 & \text{otherwise} \end{cases},$$

where L_Λ is the irreducible module with highest weight Λ.

The main difference with the finite-dimensional stituation is that now the resolution is infinite in one direction ("one-sided resolution"), due to the fact that the affine Weyl group is infinite. The resolution of L_Λ in terms of Fock spaces that we are going to discuss differs in two important aspects. First, the resolution is infinite in both directions ("two-sided resolution") and furthermore each term in the resolution will be an infinite sum over Fock space modules.

3.2 Fock Space Realization of $\widehat{sl}(n)_k$. In this section we present a Fock space realization of the affine Kac–Moody algebra $\widehat{sl}(n)$ that generalizes Wakimoto's realization of $\widehat{sl}(2)$ which we discussed in the introduction. Observe that we may interpret the coordinates z_{ij} and derivatives $\partial/\partial z_{ij}$ of Sect. 2.2 as the zero modes of a set of first order bosonic $\beta\gamma$-fields of conformal dimension 1 and 0, respectively;

$$\beta^{ij}(z) = \sum_{n\in\mathbf{Z}} \beta_n^{ij} z^{-n-1}, \quad \gamma^{ij}(z) = \sum_{n\in\mathbf{Z}} \gamma_n^{ij} z^{-n},$$

$$\gamma^I(z)\beta^J(w) = \delta_{IJ}\frac{1}{z-w} + \cdots, \quad (I \sim (ij)),$$

i.e. $\gamma_0^{ij} \sim -z^{ij}, \beta_0^{ij} \sim (\partial/\partial z_{ij})$. The pairs $\{p^i, q^i\}$ can be considered as the zero modes of a set of $n-1$ scalar fields $\phi^i(z)$,

$$\phi^i(z) = q^i - ip^i \log z + i \sum_{i\neq 0} \frac{1}{n} a_n^i z^{-n},$$

$$\phi^i(z)\phi^j(w) = -\delta_{ij}\log(z-w) + \cdots.$$

The modes $\beta_n, \ldots$ satisfy the commutation relations of free oscillators

$$[\gamma_m^I, \beta_n^J] = \delta^{IJ}\delta_{m+n,0}, \quad [a_m^i, a_n^j] = m\delta^{ij}\delta_{m+n,0}, \quad [p^i, q^j] = -i\delta^{ij}.$$

(We borrow techniques from 2-dimensional conformal field theory, where the commutation relations are encoded in so-called operator product expansions (see, e.g. [FMS, Gi] for an explanation of these techniques). The $+\cdots$ stands for terms which are regular in the limit $z \to w$.)

Let us denote the Lie algebra of oscillators by $\mathbf{a}$. The algebra $\mathbf{a}$ admits a Cartan decomposition $\mathbf{a} = \mathbf{a}_- \oplus \mathbf{a}_0 \oplus \mathbf{a}_+$, where $\mathbf{a}_-$ is spanned by oscillators β_n, a_n for $n < 0$ and $\gamma_n, n \leqq 0, \mathbf{a}_0$ is spanned by the p^i and finally $\mathbf{a}_+$ by $\beta_n, n \geqq 0$ and $\gamma_n, a_n, n > 0$.

In principle one might try to obtain a realization by interpreting the components β_n, γ_n, a_n partly as coordinates and partly as derivatives on some suitably chosen infinite dimensional flag manifold. We will follow a more pedestrian approach in that we straightforwardly "affinize" the realization (2.4) such that the zero mode part of the acquired realization agrees with (2.4), and the currents have conformal dimension 1. This obviously leads to some arbitrariness in terms of the form $(\gamma\cdots\gamma\partial\gamma)$ which have vanishing zero mode. This arbitrariness is fixed by requiring the correct central charge term in the operator product expansion of the currents.

Let us define for $\Lambda \in \mathbf{h}^*$ the Fock space module $F_\Lambda = \mathscr{U}(\mathbf{a})|\Lambda\rangle$, where $|\Lambda\rangle$ is a vector satisfying $\mathbf{a}_+|\Lambda\rangle = 0$, $p^i|\Lambda\rangle = \alpha_0\Lambda^i|\Lambda\rangle$. We obtain

Proposition 3.2. *On F_Λ we have an $\widehat{sl}(n)$ realization with highest weight $\hat{\Lambda} = (\Lambda, k)$, by*

$$\begin{aligned}
e_i(z) &= \beta^{ii+1} - \sum_{j\leqq i-1} \gamma^{ji}\beta^{ji+1},\\
f_i(z) &= -v\gamma^{ii+1}(\alpha_i\cdot i\partial\phi) - (k+i-1)\partial\gamma^{ii+1} - \sum_{j\leqq i-1}\gamma^{ji+1}\beta^{ji}\\
&\quad + \sum_{j\geqq i+2}\gamma^{ij}\beta^{i+1j} - \gamma^{ii+1}\left(\sum_{j\geqq i+1}\gamma^{ij}\beta^{ij} - \sum_{j\geqq i+2}\gamma^{i+1j}\beta^{i+1j}\right), \qquad (3.1)\\
h_i(z) &= v(\alpha_i\cdot i\partial\phi) + 2{:}\gamma^{ii+1}\beta^{ii+1}{:} - \sum_{j\leqq i-1} {:}(\gamma^{ji}\beta^{ji} - \gamma^{ji+1}\beta^{ji+1}){:}\\
&\quad + \sum_{j\geqq i+2} {:}(\gamma^{ij}\beta^{ij} - \gamma^{i+1j}\beta^{i+1j}){:},
\end{aligned}$$

where $v^2 = k + n$, and $:\cdots:$ denotes normal ordering. (We have suppressed the z-dependence on the right-hand side of the equations!) As usual, we have identified the modes of $x(z)$ with $x \otimes t^n$ in $L\mathbf{g} = \mathbf{g} \otimes \mathbf{C}[t, t^{-1}]$.

It is straightforward to check that these currents satisfy the correct operator product expansions corresponding to (2.2). Proving the analogue of (2.3) can explicitly be done in the lower rank cases, but the general proof seems, as in the finite-dimensional case, only feasible by geometric means [FeFr1].

As already remarked, the realization for $\widehat{sl}(2)$ was discovered by Wakimoto [Wa]. For general $\widehat{sl}(n)$ the realization was first discovered by Feigin and Frenkel [FeFr1], and has since then been rediscovered several times (see, [Za2, GMMOS, BeOo] for $\widehat{sl}(3)$, and [BMP2, ItKa] for $\widehat{sl}(n)$, and [GMMOS] for an interesting derivation of these realizations directly from the path-integral formulation of the WZNW-model).

To be able to use the above free field realization to compute conformal blocks of the associated WZNW-conformal field theory, we need to verify that the Sugawara stress energy tensor equals the stress energy tensor of the free fields β, γ and ϕ, as announced in the introduction. This is the content of the next proposition.

Proposition 3.3. *Let x^a denote an orthonormal basis of $sl(n)$. We have*

$$\begin{aligned} T_{\mathrm{Sug}}(z) &= \frac{1}{2(k+n)} \sum_a :x^a(z) x^a(z): \\ &= -\tfrac{1}{2} \cdot \partial\phi(z) \cdot \partial\phi(z) : - \alpha_0 \rho \cdot i \partial^2 \phi(z) - \sum_{1 \leqq i < j \leqq n} :\beta^{ij}(z) \partial\gamma^{ij}(z):, \end{aligned}$$

where $\alpha_0^{-2} = k + h^\vee = k + n$. The modes of $T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}$ generate a Virasoro algebra of central charge

$$c = \ell - 12\alpha_0^2 |\rho|^2 + 2|\Delta_+| = \frac{k(n^2 - 1)}{k + n}.$$

We omit the proof, which is based on a comparison with the finite dimensional result (Proposition 2.5) for its zero mode piece, and an explicit determination of the terms that are consistent with the requirements that $T(z)$ is a $\mathbf{g}$ singlet, and that every current $x^a(z)$ has conformal dimension 1.

3.3 Screening and Intertwining Operators. By analogy with the finite-dimensional case (Eq. (2.9)) we define for $\widehat{sl}(n)$ the following screening operators:

$$s_i(z) = \bar{e}_i : e^{-i\alpha_0(\alpha_i, \phi)} : (z), \tag{3.2}$$

where

$$\bar{e}_i(z) = -\left(\beta^{i\,i+1} - \sum_{j \geqq i+2} \gamma^{i+1\,j} \beta^{ij} \right). \tag{3.3}$$

These operators are primary fields of conformal dimension 1 with respect to the stress energy tensor of Proposition 3.3. The only nontrivial operator product expansion with the group currents is given by

$$f_i(z)s_j(w) = \delta_{ij}\partial_w\left(\frac{\bar{s}_i(w)}{z-w}\right), \tag{3.4}$$

where

$$\bar{s}_i(z) = -v^2 :e^{-i\alpha_0(\alpha_i,\phi)}:(z). \tag{3.5}$$

Contrary to the finite-dimensional case these screening operators do not generate the algebra $\hat{\mathfrak{n}}_+$, due to the nonlocality of $s_i(z)$ with respect to $s_j(w)$ for $a_{ij} \neq 0$. We will see later that this property is replaced by an equally powerful property.

We will now show how to build intertwining operators on Fock modules as appropriate contour integrals over products of screening operators.

Define the set $\mathscr{I}$ as the vector space generated by all operators of the form

$$[\![s_{i_1}\cdots s_{i_n}]\!] = \int_\Gamma dz_1\cdots dz_n s_{i_1}(z_1)\cdots s_{i_n}(z_n), \quad i_j \in \{1,\ldots,\ell\}, \tag{3.6}$$

where the contour Γ is taken as in Fig. 2, i.e. all contours taken counterclockwise from 1 to 1 and nested according to $|z_1| > \cdots > |z_n|$ for $z_i \neq 1$. The integral is defined by analytic continuation from a parameter region $0 < z_n < \cdots < z_1$ on the real axis, where the integrand is taken to be real.

Let $v \in F_\Lambda$. Using the Campbell–Baker–Hausdorff formula we can write more explicitly

$$\begin{aligned}[\![s_{i_1}\cdots s_{i_n}]\!]\cdot v = \int_\Gamma dz_1\cdots dz_n \prod_{1\leqq k<l\leqq n}(z_k - z_l)^{\alpha_0^2 a_{i_k i_l}} \prod_{1\leqq k\leqq n} z_k^{-\alpha_0^2(\Lambda,\alpha_{i_k})} \\ \cdot :e^{-i\alpha_0\alpha_{i_1}\cdot\tilde{\phi}(z_1)}\cdots e^{-i\alpha_0\alpha_{i_n}\cdot\tilde{\phi}(z_n)}:\bar{e}_{i_1}(z_1)\cdots\bar{e}_{i_n}(z_n)\cdot v,\end{aligned} \tag{3.7}$$

where $\tilde{\phi}(z) = \phi(z)|_{p=0}$ and $\bar{e}_i(z)$ is defined in (3.3). Expressions for $[\![s_{i_1}\cdots s_{i_n}]\!]$ involving normal ordered products of $\beta\gamma$-fields are easily written down using Wick's theorem. However, as they do not play a role for the monodromy properties of the integrand, we will not give them explicitly.

To analyze properties of $[\![s_{i_1}\cdots s_{i_n}]\!]$ it is convenient to rewrite them into "elementary integrals" with a specific ordering of the variables on the unit circle; i.e. we define the operators

$$\begin{aligned}I_{i_1\cdots i_n} = \int_{0<\arg z_1<\cdots<\arg z_n<2\pi} dz_1\cdots dz_n \prod_{1\leqq k<l\leqq n}(z_k - z_l)^{\alpha_0^2 a_{i_k i_l}} \\ \cdot \prod_{1\leqq k\leqq n} z_k^{-\alpha_0^2(\Lambda,\alpha_{i_k})} :e^{-i\alpha_0\alpha_{i_1}\cdot\tilde{\phi}(z_1)}\cdots e^{-i\alpha_0\alpha_{i_n}\cdot\tilde{\phi}(z_n)}:\bar{e}_{i_1}(z_1)\cdots\bar{e}_{i_n}(z_n).\end{aligned} \tag{3.8}$$

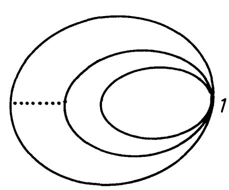

Fig. 2. Integration contour Γ

So we have for example

$$\llbracket (s_1)^m \rrbracket = \prod_{k=1}^{m} \left(\frac{1-q^{2k}}{1-q^2} \right) \times I_{1\cdots 1},$$

$$\llbracket s_1 s_2 \rrbracket = I_{12} + q^{-1} I_{21}, \quad \text{if} \quad a_{12} = -1, \tag{3.9}$$

where

$$q = \exp(\pi i \alpha_0^2) = \exp\left(\frac{\pi i}{k+n} \right). \tag{3.10}$$

Conventionally one might take for the contour $\boldsymbol{\Gamma}$ an element of the homology group $H_*(\mathscr{M}, \mathscr{S})$ of the manifold $\mathscr{M} = \mathbf{C}^{*n} \setminus \bigcup_{i \neq j} \{z_i = z_j\}$ with coefficients in a local system $\mathscr{S}$ defined by the multivalued integrand in (3.7) [DM, TK, FeFu]. The reader should then note that our choice of $\boldsymbol{\Gamma}$ in (3.6) is *not* always in $H_*(\mathscr{M}, \mathscr{S})$. However, the intertwiners we build involve a linear combination of operators $\llbracket s_{i_1} \cdots s_{i_n} \rrbracket$, with permutations of indices, such that the resulting contour *is* in $H_*(\mathscr{M}, \mathscr{S})$.

Proposition 3.4. *Within the contour-integrals* $\boldsymbol{\Gamma}$, i.e. *in the "words"* $\llbracket s_{i_1} \cdots s_{i_n} \rrbracket$, *the* s_i*'s satisfy the defining identities of the quantum group* $\mathscr{U}_q(\mathbf{n}_+)$, *where* $q = \exp(\pi i/(k+n)) = \exp(\pi i \alpha_0^2)$,

$$s_i s_j - s_j s_i = 0, \quad \text{if} \quad a_{ij} = 0,$$

$$s_i s_i s_j - (q + q^{-1}) s_i s_j s_i + s_j s_i s_i = 0, \quad \text{if} \quad a_{ij} = -1. \tag{3.11}$$

Proof. For $a_{ij} = 0$ the statement is trivial, because then the operators $s_i(z)$ and $s_j(w)$ commute. To prove the statement for $a_{ij} = -1$ we write $\llbracket s_{i_1} \cdots s_{i_n} \rrbracket$ in terms of elementary integrals (3.8), and consider the various terms with different orderings. Suppose for notational simplicity that the three operators occur in the first three entries of $\llbracket \cdots \rrbracket$, i.e. we want to show

$$\llbracket (s_i s_i s_j - (q + q^{-1}) s_i s_j s_i + s_j s_i s_i) s_{i_4} \cdots s_{i_n} \rrbracket = 0. \tag{3.12}$$

Consider the overall coefficient of the term $I_{iij\ldots}$. Suppose this term occurs with coefficient $(1+q^2)\mathscr{A}(q)$ in $\llbracket s_i s_i s_j \cdots \rrbracket$, where $\mathscr{A} \in \mathbf{Z}[q, q^{-1}]$ is some polynomial (the factor $(1+q^2)$ which is taken out, comes from interchanging $i_1 \leftrightarrow i_2$). Then it will occur with coefficients $q^{-1}(1+q^2)\mathscr{A}(q)$ and $q^{-2}(1+q^2)\mathscr{A}(q)$ in $\llbracket s_i s_j s_i \cdots \rrbracket$ and $\llbracket s_j s_i s_i \cdots \rrbracket$, respectively. So clearly the total coefficient for (3.12) vanishes. The coefficient of $I_{jii\ldots}$ works similarly. For the term $I_{iji\ldots}$ the respective coefficients are $q^{-1}(1+q^2)\mathscr{B}(q), 2\mathscr{B}(q)$ and $q^{-1}(1+q^2)\mathscr{B}(q)$, so that again the total coefficient in (3.12) vanishes. This completes all different cases, hence the proposition is proved. $\square$

Proposition 3.5. *We have the following commutator*

$$[f_i(z), \llbracket s_{i_1} \cdots s_{i_n} \rrbracket] = \frac{\bar{s}_i(1)}{z-1} q^{(\Lambda, \alpha_i) + \xi} \sum_{j;\, i_j = i} (q^{-(\Lambda, \alpha_i) + a_j} - q^{(\Lambda, \alpha_i) - a_j}) \llbracket s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n} \rrbracket, \tag{3.13}$$

where

$$a_j = a_{i i_{j+1}} + \cdots + a_{i i_n} \tag{3.14}$$

and

$$\xi = a_{ii_1} + \cdots + a_{ii_{j-1}} + a_{ii_{j+1}} + \cdots + a_{ii_n}.$$

Proof. By (3.4) we have

$$[f_i(z), [\![s_{i_1} \cdots s_{i_n}]\!]] = \sum_{j; i_j = i} \int_\Gamma dz_1 \cdots dz_j \cdots dz_n s_{i_1}(z_1) \cdots \left(\frac{\bar{s}_i(z_j)}{z - z_j} \right) \Bigg|_{\arg z_j = 0}^{\arg z_j = 2\pi} \cdots s_{i_n}(z_n)$$

$$= \sum_{j; i_j = i} \int_\Gamma dz_1 \cdots dz_j \cdots dz_n s_{i_1}(z_1) \cdots (q^{-2(\Lambda,\alpha_i) + 2a_j} - 1) \frac{\bar{s}_i(1)}{z - 1} \cdots s_{i_n}(z_n).$$

The evaluation of the integral at $\arg z_j = 2\pi$ acquires a phase factor from pulling z_j around $0, z_{j+1}, \ldots, z_n$. The term $\bar{s}_i(1)$ can be written in front of the expression at the expense of an additional phase factor q^{b_j}, where $b_j = a_{ii_1} + \cdots + a_{ii_{j-1}}$, from pulling z_j across $z_1, \ldots, z_{j-1}$. Now notice that $\xi = a_j + b_j$ is independent of j, so the proposition is proved. □

Let us review briefly the definition of the quantum group $\mathscr{U}_q(\mathbf{g})$ [Ji, Dr1, Dr2]. Suppose $\mathbf{g}$ is a finite-dimensional Lie algebra with Cartan matrix a_{ij} of rank ℓ. Fix positive integers d_i such that $d_i a_{ij} = d_j a_{ji}$. Fix a complex number q such that $q^{2d_i} \neq 1$ $(1 \leq i \leq \ell)$. Introduce q-numbers, q-factorials and q-binomials as in Appendix 1. Then, $\mathscr{U}_q(\mathbf{g})$ is the associative $\mathbf{C}$-algebra with generators $e_i, f_i, k_i^{\pm 1}$, $(1 \leq i \leq \ell)$ $(k_i \sim q^{h_i})$, and relations (we use the conventions of [Lu2])

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$
$$k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j, \qquad k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q^{d_i} - q^{-d_i}},$$
$$\sum_{\kappa=0}^{1-a_{ij}} (-1)^\kappa \begin{bmatrix} 1 - a_{ij} \\ \kappa \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-\kappa} e_j e_i^\kappa = 0,$$
$$\sum_{\kappa=0}^{1-a_{ij}} (-1)^\kappa \begin{bmatrix} 1 - a_{ij} \\ \kappa \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-\kappa} f_j f_i^\kappa = 0. \tag{3.15}$$

This algebra is endowed with a co-multiplication, co-unit and antipode which makes it into a Hopf algebra. We refrain from giving their definitions as we will not need them here.

We now describe the definition of the "quantum Verma module" M_Λ^q [Lu1]. We define $M_\Lambda^q = \mathscr{U}_q(\mathbf{g}) \cdot v_\Lambda$, where v_Λ is a (highest weight) vector satisfying,

$$e_i \cdot v_\Lambda = 0, \quad k_i \cdot v_\Lambda = q^{\langle \Lambda, h_i \rangle} v_\Lambda, \quad i = 1, \ldots, \ell.$$

The space M_Λ^q has an (overcomplete) basis consisting of monomials $f_{i_1} \cdots f_{i_n} \cdot v_\Lambda$, and is a $\mathscr{U}_q(\mathbf{g})$ module under the action

$$e_i(f_{i_1} \cdots f_{i_n} \cdot v_\Lambda) = \sum_{j; i_j = i} \frac{q^{\langle \Lambda, h_i \rangle - d_i a_j} - q^{-\langle \Lambda, h_i \rangle + d_i a_j}}{q^{d_i} - q^{-d_i}} f_{i_1} \cdots \widehat{f_{i_j}} \cdots f_{i_n} \cdot v_\Lambda,$$
$$f_i(f_{i_1} \cdots f_{i_n} \cdot v_\Lambda) = f_i f_{i_1} \cdots f_{i_n} \cdot v_\Lambda, \tag{3.16}$$
$$k_i(f_{i_1} \cdots f_{i_n} \cdot v_\Lambda) = q^{\langle \Lambda, h_i \rangle} q^{-d_i(a_{ii_1} + \cdots + a_{ii_n})} f_{i_1} \cdots f_{i_n} \cdot v_\Lambda,$$

where a_j is defined in (3.14). This module is integrable for $\Lambda \in P_+$ and reduces to the conventional Verma module for $\mathbf{g}$ in the limit $q \to 1$, i.e. is a deformation of M_Λ [Lu1]. The module has a weight space decomposition

$$M^q_\Lambda = \bigoplus_{\lambda \geqq 0} (M^q_\Lambda)(\lambda),$$

where

$$(M^q_\Lambda)(\lambda) = \{v \in M^q_\Lambda \mid k_i \cdot v = q^{\langle \Lambda - \lambda, h_i \rangle} v\}.$$

Using the aforementioned description of the quantum Verma module we find the following immediate consequence of Proposition 3.5.

Theorem 3.6. *There exists a map from the set of simgular* (i.e. *primitive*) *vectors in* $(M^q_\Lambda)(\lambda)$ *to elements in* $\mathrm{Hom}_{\mathscr{U}(\hat{\mathbf{g}})}(F_\Lambda, F_{\Lambda-\lambda}) \cap \mathscr{I}$. *The map is given by*

$$f_{i_1} \cdots f_{i_n} \cdot v_\Lambda \to [\![s_{i_1} \cdots s_{i_n}]\!].$$

Proof. Combining Proposition 3.5 with the aforementioned description of the quantum Verma module M^q_Λ we see that the action of $f_i(z)$ on $\mathscr{I}$ (by commutation) is, up to an overall nonzero operator, the same as the action of e_i on M^q_Λ. This proves the theorem, exactly as in the finite-dimensional situation of Theorem 2.10. □

The converse would follow from Proposition 3.5, if it could be proven that the operators $[\![s_{i_1} \cdots \hat{s}_{1_j} \cdots s_{i_n}]\!]$ do not conspire to cancel in such a way that the right-hand side of (3.13) vanishes "accidently." We are not able to evaluate these contour integrals in general. Further we do not have an argument which selects precisely which of the fundamental integrals of (3.8) are independent. However, by analogy with the finite dimensional case it is reasonable to expect that the converse to Theorem 3.6 is true.

The problems here are linked to the discussion below (3.10). We expect that $\mathrm{Hom}_{\mathscr{U}(\hat{\mathbf{g}})}(F_\Lambda, F_{\Lambda'}) \subset \mathscr{I}$, if we make an assumption that all intertwiners can be built from screening operators as in (3.6), by taking Γ a contour in $H_*(\mathscr{M}, \mathscr{S})$. But the dimension of this homology group can be larger than $\dim \mathscr{I}$ [FeFu]. Of course not all independent contours in $H_*(\mathscr{M}, \mathscr{S})$ give rise to a different operator of the form (3.6); for instance, permutations of variables corresponding to the same simple root do not change the operator (up to a phase). However, to clarify this issue we need to know which of the fundamental integrals are independent.

Let us make a final remark on the choice of contour Γ, as opposed to the "conventional" choice of contour Γ' [Fe] of Fig. 3, i.e. $z_2, \ldots, z_n$ are integrated over nested contours from z_1 to z_1 and finally z_1 is integrated over the unit circle.

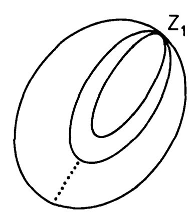

Fig. 3. Integration contour Γ'

The most important virtue of the contour Γ' is that those expressions evidently commute with the algebra, provided the last contour is "closed." For expressions containing only one type of screening operators $s_i(z)$ one easily shows that the two operators are proportional, i.e.

$$\int_{\Gamma} dz_1 \cdots dz_n s_i(z_1) \cdots s_i(z_n) = \frac{1}{n}\left(\frac{1-q^{2n}}{1-q^2}\right) \int_{\Gamma'} dz_1 \cdots dz_n s_i(z_1) \cdots s_i(z_n).$$

In general, however, this is not true for operator containing several types of screening operators. It proves to be useful to work with the contours Γ because it treats the variables $z_1, \ldots, z_n$ on equal footing, so that the quantum group relations (3.11) apply to all entries of $[\![s_{i_1} \cdots s_{i_n}]\!]$. A drawback is that the intertwining property is somewhat harder to prove. We believe that in the end the intertwining operators can all be rewritten in terms of contours Γ'. This we explicitly verified in some nontrivial examples, but we have not proved it in general.

3.4 The Fock Space Resolution. So far the discussion has been completely general. In particular, no restrictions have been put on the highest weight Λ, and its level k. To describe the complex of intertwiners we have to distinguish several cases however. For $k \notin \mathbf{Q}$, i.e. q is not a root of unity, the quantum Verma module M^q_Λ has the same singular vector structure as M_Λ [Ro], in which case the complex is exactly the same as in the finite-dimensional case, discussed in Sect. 2. For $k \in \mathbf{Q}$ we have $q^M = 1$ for some $M \in \mathbf{N}$. In this case the module M^q_Λ contains additional singular vectors. We will restrict the discussion below to highest weights $\Lambda \in \hat{P}^{(k)}_+$, $k \in \mathbf{Z}_+$, because they appear to be the most relevant in physical applications [GeWi]. The general case $k \in \mathbf{Q} \cap \{k > -n\}$ does not seem to be essentially more difficult.

For $k = -n$ however, the ϕ-field decouples, and we are left with a realization in terms of $\beta\gamma$-fields only. This "degenerated" realization can be used to verify the validity of the Kac–Kazhdan conjecture [KaKa] for the character formula of highest weight modules at the critical level (i.e. $k = -n$) [FeFr2].

From now on let $\Lambda \in \hat{P}^{(k)}_+$, $k \in \mathbf{Z}_+$. We recall that every $w \in \hat{W}$ can uniquely be written as $t_\alpha \bar{w}$ for some $\bar{w} \in W$ and some $\alpha \in M$. Here t_α is a translation operator (see, [Ka] and Appendix 2).

The following lemma provides us with a basic set of intertwiners

Lemma 3.7. *Let $w \in \hat{W}$. Write $w = t_\alpha \bar{w}$ for some $\alpha \in M$, $\bar{w} \in W$. Define for a given simple root $\alpha_i, i = 1, \ldots, \ell$*

$$\begin{cases} l = (\bar{w}(\Lambda+\rho), \alpha_i), & w' = t_\alpha r_i \bar{w} \quad \text{if} \quad (\bar{w}(\Lambda+\rho), \alpha_i) > 0 \\ l = (k+n) + (\bar{w}(\Lambda+\rho), \alpha_i), & w' = t_{\alpha-\alpha_i} r_i \bar{w} \quad \text{if} \quad (\bar{w}(\Lambda+\rho), \alpha_i) < 0 \end{cases}$$

(note that in both cases $0 < l < k+n$ since $\Lambda \in \hat{P}^{(k)}_+$). Then $(f_i)^l \cdot v_{w*\Lambda}$ is a singular vector in $M^q_{w*\Lambda}$, i.e. $Q_{w,w'} \equiv [\![(s_i)^l]\!] \in \mathrm{Hom}_{\mathscr{U}(\hat{\mathfrak{g}})}(F_{w*\Lambda}, F_{w'*\Lambda})$.

Proof. The proof of singularity of the vector is by straightforward calculation. We will present the proof in the case $(\bar{w}(\Lambda+\rho), \alpha_i) < 0$. The other case is similar, and in fact the singular vector in this case is just a q-deformation of the singular vector present for $q = 1$. By (3.16) we have

$$e_i (f_j)^l \cdot v_{w*\Lambda} = \delta_{ij} \frac{1}{q - q^{-1}} \sum_{0 \leqq m \leqq l-1} (q^{(w*\Lambda, \alpha_i) - 2m} - q^{-(w*\Lambda, \alpha_i) + 2m})(f_j)^{l-1} \cdot v_{w*\Lambda}.$$

Now $(w*\Lambda, \alpha_i) = (l-1) - (k+n)$, so the summation gives

$$\sum_{m=0}^{l-1} (q^{l-1-(k+n)-2m} - q^{-(l-1)+(k+n)+2m})$$
$$= -q^{l-1}\left(\frac{1-q^{-2l}}{1-q^{-2}}\right) + q^{-(l-1)}\left(\frac{1-q^{2l}}{1-q^{2}}\right) = 0. \quad \square$$

For a single simple root α_i the intertwining operators of Lemma 3.7 form a "grid" of $\widehat{sl}(2)$ complexes:

Lemma 3.8. *Let α_i be a simple root, and $w = t_\alpha \bar{w} \in \hat{W}$. Define*

$$\begin{cases} w' = t_\alpha r_i \bar{w}, & w'' = t_{\alpha-\alpha_i}\bar{w} \quad \text{if} \quad (\bar{w}(\Lambda+\rho), \alpha_i) > 0 \\ w' = t_{\alpha-\alpha_i} r_i \bar{w}, & w'' = t_{\alpha-\alpha_i}\bar{w} \quad \text{if} \quad (\bar{w}(\Lambda+\rho), \alpha_i) < 0 \end{cases}$$

then $Q_{w',w''}Q_{w,w'} = 0$.

Proof. We have $Q_{w',w''}Q_{w,w'} = [\![(s_i)^{k+n}]\!]$, which vanishes due to the phase factor in (3.9). $\square$

The two previous lemmas completely describe the situation for $\widehat{sl}(2)$. The resulting complex is depicted in Fig. 4 (see also [FeFr2, BF]). In general, the intertwiners corresponding to the simple root directions do not exhaust the set of all intertwiners. In fact, we expect intertwiners for every positive root direction. Unfortunately, not much is known about the structure (i.e. singular vectors, composition series) of a quantum Verma module. For $\widehat{sl}(3)$ however we can exhibit additional intertwiners which allow the formulation of a complex, by explicit "reshuffling" of the screening operators using the lemma's in the appendix.

So, let $\alpha_3 = \alpha_1 + \alpha_2$ be the third root of $\widehat{sl}(3)$, and define $l_i = (\Lambda + \rho, \alpha_i)$, $\bar{l}_i = (k+3) - l_i, i = 1,2,3$. We use the notation $Q_l^{(i)}$ for an intertwiner $Q_{w,w'} \in \mathrm{Hom}_{\mathscr{U}(\hat{\mathfrak{g}})}(F_{w*\Lambda}, F_{w'*\Lambda})$ which is such that $w*\Lambda - w'*\Lambda = l\alpha_i \bmod \mathbf{C}\delta, i = 1,2,3$. The reason for this notation is that since $q^{2(k+3)} = 1$ there is a "periodicity" property $M^q_{t_\alpha \bar{w}*\Lambda} \cong M^q_{\bar{w}*\Lambda}$, for all $\alpha \in M$, which implies that throughout the (infinite) complex only a finite number of different $Q_l^{(i)}$'s are needed. The operator $Q_l^{(i)}$ that acts on $F_{t_\alpha \bar{w}*\Lambda}$ is completely determined by $\bar{w} \in W$.

Theorem 3.9. *Let $w = t_\alpha \bar{w} \in \hat{W}$. Then we have, in addition to the intertwiners of Lemma 3.7, the following intertwiners $Q_{w,w'} \in \mathrm{Hom}_{\mathscr{U}(\hat{\mathfrak{g}})}(F_{w*\Lambda}, F_{w'*\Lambda})$, depending on $\bar{w} \in W$*

(i)

$$\bar{w} = r_1, \qquad Q^{(3)}_{l_2} = \sum_{0 \leqq j \leqq l_2} b_q(l_2, l_3; j)[\![(s_2)^{l_2-j}(s_3)^j(s_1)^{l_2-j}]\!];$$

(ii)

$$\bar{w} = r_2, \qquad Q^{(3)}_{l_1} = \sum_{0 \leqq j \leqq l_1} b_q(l_1, l_3; j)[\![(s_1)^{l_1-j}(\hat{s}_3)^j(s_2)^{l_1-j}]\!];$$

$$\cdots \longrightarrow F_{r_0 r_1 * \Lambda} \longrightarrow F_{r_0 * \Lambda} \longrightarrow F_\Lambda \longrightarrow F_{r_1 * \Lambda} \longrightarrow F_{r_1 r_0 * \Lambda} \longrightarrow \cdots$$

Fig. 4. Fock space resolution for $\widehat{sl}(2)$

(iii)

$$\bar{w} = r_1 r_2 r_1, \quad Q_{\bar{l}_3}^{(3)} = \sum_{0 \leqq j \leqq \bar{l}_3} b_q(\bar{l}_2, \bar{l}_3; j) [\![(s_1)^{\bar{l}_3 - j} (\hat{s}_3)^j (s_2)^{\bar{l}_3 - j}]\!];$$

where $s_3 = -s_2 s_1 + q^{-1} s_1 s_2, \hat{s}_3 = -s_1 s_2 + q^{-1} s_2 s_1$, *and*

$$b_q(m, n; j) = q^{j + (m-j)(n-j)} \frac{[m]_q! [n]_q!}{[j]_q! [m-j]_q! [n-j]_q!}.$$

This results in a diagram of Fock space modules $F_{w*\Lambda}$ *and mappings between them, part of which is depicted in Fig. 5. The diagram is invariant under* $\Lambda \to \Lambda + (k+3)\alpha$ *for* $\alpha \in M$, *and contains three types of hexagons (Fig. 6* (a), (b), *and* (c)) *in which the following relations are satisfied:*

(a)

$$Q_{l_2}^{(3)} Q_{l_1}^{(1)} = Q_{l_3}^{(1)} Q_{l_2}^{(2)}, \quad Q_{l_1}^{(2)} Q_{l_2}^{(3)} = Q_{l_2}^{(1)} Q_{l_3}^{(2)},$$
$$Q_{l_1}^{(3)} Q_{l_2}^{(2)} = Q_{l_3}^{(2)} Q_{l_1}^{(1)}, \quad Q_{l_2}^{(1)} Q_{l_1}^{(3)} = Q_{l_1}^{(2)} Q_{l_3}^{(1)};$$

(b)

$$Q_{\bar{l}_1}^{(3)} Q_{l_1}^{(2)} = Q_{\bar{l}_2}^{(2)} Q_{\bar{l}_3}^{(1)}, \qquad Q_{l_1}^{(1)} Q_{\bar{l}_3}^{(3)} = Q_{\bar{l}_3}^{(2)} Q_{\bar{l}_2}^{(1)},$$
$$Q_{l_1}^{(3)} Q_{\bar{l}_3}^{(1)} = (-1)^{l_1} Q_{\bar{l}_2}^{(1)} Q_{l_1}^{(2)}, \quad Q_{\bar{l}_3}^{(2)} Q_{l_1}^{(3)} = (-1)^{l_1} Q_{l_1}^{(1)} Q_{\bar{l}_2}^{(2)};$$

(c)

$$Q_{\bar{l}_3}^{(3)} Q_{l_2}^{(1)} = (-1)^{\bar{l}_3} Q_{\bar{l}_1}^{(1)} Q_{\bar{l}_3}^{(2)}, \quad Q_{l_2}^{(2)} Q_{\bar{l}_3}^{(3)} = (-1)^{\bar{l}_3} Q_{\bar{l}_1}^{(2)} Q_{\bar{l}_3}^{(1)},$$
$$Q_{l_2}^{(3)} Q_{\bar{l}_3}^{(2)} = (-1)^{l_2} Q_{\bar{l}_1}^{(2)} Q_{l_2}^{(1)}, \quad Q_{\bar{l}_3}^{(1)} Q_{l_2}^{(3)} = (-1)^{l_2} Q_{l_2}^{(2)} Q_{\bar{l}_1}^{(1)}.$$

Proof. The proof is a straightforward application of Lemma A.1 and Lemma A.2 in the appendix. □

As discussed in Appendix 2, there is a "modified length" $\tilde{l}$ defined on $\hat{W}$ so that all the intertwiners $Q_{w,w'}$ of Lemma 3.7 and Theorem 3.9 are such that $\tilde{l}(w') = \tilde{l}(w) + 1$. In particular, for $\widehat{sl}(3)$, we find that $w' = wr_i$ for $Q_{l_i}^{(\alpha)}, (i = 1, 2)$ and

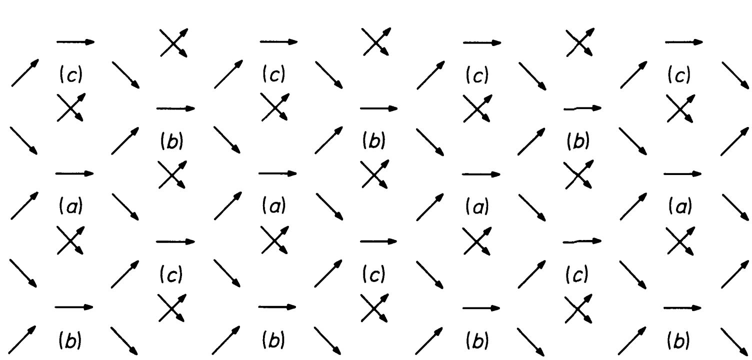

Fig. 5. Fock space resolution for $\widehat{sl}(3)$. The fundamental hexagons are those of Fig. 6 (a), (b) and (c)

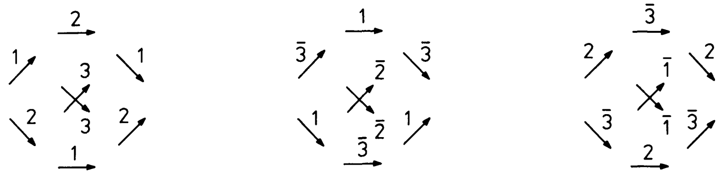

Fig. 6. Fundamental hexagons. The mappings stand for

$\nearrow Q^{(1)}, \qquad \searrow Q^{(2)}, \qquad \rightarrow Q^{(3)}$

and the labellings i and $\bar{i}$ give l_i and $\bar{l}_i$, respectively, e.g. $\xrightarrow{2}$ stands for $Q^{(3)}_{l_2}$

$w' = wr_0$ for $Q^{(\alpha)}_{\bar{l}_3}$. This also provides an easy procedure for reconstructing the Weyl group elements w for the Fock spaces $F_{w*\Lambda}$ in Fig. 5 in terms of simple reflections by walking along the edges of the hexagons starting from the middle Fock space F_Λ [FeFr2].

Let us emphasize once again that we have not proved that the intertwiners of Theorem 3.9 exhaust the set of all intertwiners. Two ingredients are lacking: First, we would have to show that all intertwiners are of the form (3.6), and secondly we lack a complete understanding of the quantum Verma module $\widehat{M^q_\Lambda}$. Nevertheless, we believe that the set we have found is complete (i.e. for $\widehat{sl}(3)$). Furthermore we should note that the relations (a), (b) and (c) in Theorem 3.9 ensure that the solutions $Q^{(3)}$ are well-defined and nonvanishing. Thus our explicit results for $\widehat{sl}(3)$ circumvent the remarks after Theorem 3.6. Moreover, they suffice to build the required complex, as the following theorem shows

Theorem 3.10. *Let* $\tilde{W}^{(i)} = \{w \in \hat{W} \,|\, \tilde{l}(w) = i\}$. *For every* $\Lambda \in \hat{P}_+$ *we have a complex of* $\widehat{sl}(3)$ *Fock space modules*

$$\cdots \xrightarrow{d^{(-3)}} F^{(-2)}_\Lambda \xrightarrow{d^{(-2)}} F^{(-1)}_\Lambda \xrightarrow{d^{(-1)}} F^{(0)}_\Lambda \xrightarrow{d^{(0)}} F^{(1)}_\Lambda \xrightarrow{d^{(1)}} \cdots,$$

where

$$F^{(i)}_\Lambda = \bigoplus_{w \in \tilde{W}^{(i)}} F_{w*\Lambda}.$$

Proof. We define $d^{(i)}: F^{(i)}_\Lambda \to F^{(i+1)}_\Lambda$ by its matrix elements $d^{(i)}_{w_1,w_2}$, $w_1 \in \tilde{W}^{(i)}, w_2 \in \tilde{W}^{(i+1)}$. Let $d^{(i)}_{w_1,w_2} = 0$ if there does not occur an intertwiner Q_{w_1,w_2} in Fig. 5. Otherwise, put $d^{(i)}_{w_1,w_2} = \hat{s}(w_1, w_2) Q_{w_1,w_2}$, where a possible choice of signs $\hat{s}(w_1, w_2) = \pm 1$ is given in Fig. 7. (Note that for $(-1)^{k+3} = 1$ the signs can be taken such that they differ only for different types of hexagons. For $(-1)^{k+3} = -1$ the "periodicity length" of signs is increased by a factor of two). The nilpotency property $d^{(i+1)}d^{(i)} = 0$ follows from the identities in Theorem 3.9. □

For $\widehat{sl}(n)$ the lemmas of the appendix again give additional intertwiners. Moreover, for general q, intertwiners will exist for every q-analogue of a singular vector present in the $q = 1$ case. These intertwiners give a diagram that is a (fundamental) $n!$-gon in $(n-1)$-dimensional space. We conjecture that the

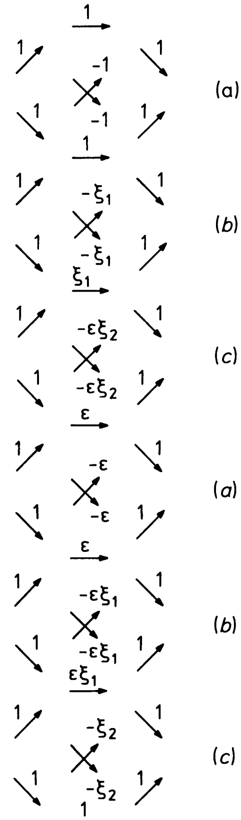

Fig. 7. Signs $\hat{s}(w_1, w_2)$ for a fundamental cell. We have put $\varepsilon = (-1)^{k+3}$, $\xi_i = (-1)^{l_i}$, $i = 1, 2$

additional intertwiners that exist for q a root of unity are such that the resulting diagram gives a tiling of $(n-1)$-dimensional space in terms of this fundamental $n!$-gon. In particular we conjecture that Theorem 3.10 holds similarly for $\widehat{sl}(n)$.

Finally, we conjecture that the complex described in Theorem 3.10 provides a (two-sided) resolution of the irreducible highest weight module L_Λ in terms of Fock space modules $F_{w*\Lambda}$, i.e.

Conjecture 3.11.

$$H^i(d) \cong \begin{cases} L_\Lambda & \textit{for} \quad i = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

For $\widehat{sl}(2)$ a proof has been given in [FeFr2, BF]. As we have indicated the structure simplifies drastically in this case. The two main simplifications are that the complex of Theorem 3.10 becomes "1-dimensional" since $|\tilde{W}^{(k)}| = 1$, $\forall k$, and secondly it is relatively easy to determine the complete "Fock space embedding pattern" by means of the Jantzen filtration [Ja].

As a final remark observe that since $(-1)^{l(w)} = (-1)^{\tilde{l}(w)}$ the validity of the

Conjecture 3.11 would provide one more proof of the Weyl–Kac character formula (1.1).

4. Discussion and Outlook

In this paper we have shown that a quantum group underlies the structure of the intertwining operators between various Fock space modules of $\widehat{sl}(n)$. For $\widehat{sl}(3)$ we explicitly computed the intertwining operators needed to build a complex of Fock spaces.

To make progress for general $\widehat{sl}(n)$ we obviously need a better understanding of the quantum group Verma module. That the results can be extended to other Lie algebras is obvious, though the actual computations might be tedious. Clearly, we would like a proof of the conjecture that the cohomology of the complex is concentrated in the "zeroth dimension," where it is exactly the irreducible module. Given its validity the (higher genus) conformal blocks for WZNW-models can be computed in the same spirit as for the minimal models [Fe, FLMS1, BaGo, FS2, FLMS2], using the screened vertex operators introduced in [BMP1].

It is well-known that two-dimensional rational conformal field theory seems to be a generalization of ordinary group theory, and it has been emphasized that its structure in fact resembles that of a quantum group (see, e.g. [MS, AGS, TK2, Sm, Wi2, MR]). The correspondence, however, was (to our knowledge) shown only indirectly by either the explicit computation of the braiding and fusion matrices [TK2, FS1, FFK] which are argued to correspond to the quantum group $6j$-symbols [MS, AGS], comparison of modular properties [AGS, Sm], or its relation to 3-dimensional topological field theories (braids/knot invariants) [Wi2].

The quantum group structure revealed in this paper is more fundamental in the sense that part of the quantum group relations are uncovered, and furthermore the relation to representation theory of the quantum group is pointed out. We believe that these observations will ultimately lead to a full understanding of, in particular, the monodromy properties of conformal blocks.

Though we restricted the discussion in this paper to affine Kac–Moody algebras, the generalization to the so-called $\mathscr{W}$-algebras [Za1, FL1, FL2, BBSS1, BBSS2] is straightforward. For instance, the $\mathscr{W}(\mathbf{g})$-algebra corresponding to a simply-laced Lie algebra $\mathbf{g}$ of rank ℓ possesses a realization in terms of ℓ scalar fields ϕ^i with background charge $\alpha_0 \rho$ and screening operators

$$s_i^{\pm} = \exp(i\alpha_{\pm}\alpha_i \cdot \phi), \quad i = 1, \dots, \ell,$$

where $\alpha_{\pm} = \frac{1}{2}(\alpha_0 \pm \sqrt{\alpha_0^2 + 4})$, and α_0 is related to the central charge of the Virasoro algebra by $c = \ell - 12\alpha_0^2 |\rho|^2$. The sets $\{s_i^+\}$ and $\{s_i^-\}$ will satisfy the identities of the quantum group $\mathscr{U}_q(\mathbf{n}_{\pm})$, with however different values of q, namely $q_{\pm} = \exp(\pi i \alpha_{\pm}^2)$. For the description of the complex we only need one set, say $\{s_i^+\}$. In fact, the complex for the $\mathscr{W}(\mathbf{g})$-algebra will have exactly the same structure as the corresponding one for the affine algebra $\hat{\mathbf{g}}$.

Though the sets $\{s_i^+\}$ and $\{s_i^-\}$ do *not* combine into a realization of $\mathscr{U}_q(\mathbf{g})$, because of the different q-values for $s_i^{\pm}$, it might be possible to broaden the definition of a quantum group. Obviously the enlarged structure will not have a classical limit as one cannot take the limit of both q_+ and q_- to 1 at the same time.

Again, one might anticipate that the occurrence of the quantum group $\mathscr{U}_q(\mathbf{n}_+)$ are reflected in the properties of the conformal blocks (see, e.g. [FZ2, Bi2, Bi3] for the computation of conformal blocks on the sphere and their braiding properties in the case of the $\mathscr{W}(sl(n))$-algebra).

Another application of the type of free field realization discussed in this paper is that, in principle, free field realizations can be obtained for arbitrary coset models G/H (for one approach see [GMM]). The procedure is to bosonize the $\beta\gamma$-system (see, [FMS]) and to "rotate" the scalar fields such that the H-piece can be factorized out. This program has been worked out in detail for the Fateev–Zamolodchikov parafermion algebra [FZ1] and their generalizations [Ge] in [Ne1, Ne2, Bi1, DQ, GMM, ItKa], and for the closely related $N=2$ superconformal algebra [DQ, It].

Appendix 1

In this appendix we collect some notations and lemmas which are used throughout the paper. Let $q\in\mathbf{C}$ be such that $q^2\neq 1$. We use the following definitions from q-number analysis:

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}},\quad [n]_q!=\prod_{k=1}^{n}[k]_q,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q=\frac{[m]_q!}{[n]_q![m-n]_q!},$$

known as the q-number, q-factorial and q-binomial, respectively. The following lemma proves to be useful for the explicit determination of the intertwiners

Lemma A.1. [Lu3] *Consider the associative algebra with two generators A, B and defining relations*

$$A^2B-(q+q^{-1})ABA+BA^2=0,$$
$$AB^2-(q+q^{-1})BAB+B^2A=0.$$

Define $C=-AB+q^{-1}BA$, then

(i)
$$AC=qCA,\quad qBC=CB;$$

(ii)
$$\frac{B^k}{[k]_q!}\frac{A^l}{[l]_q!}=\sum_{0\leqq j\leqq \min(k,l)} q^{j+(k-j)(l-j)}\frac{A^{l-j}}{[l-j]_q!}\frac{C^j}{[j]_q!}\frac{B^{k-j}}{[k-j]_q!};$$

(iii)
$$A^kB^{k+l}A^l=B^lA^{k+l}B^k;$$

(iv)
$$\frac{C^m}{[m]_q!}=\sum_{0\leqq j\leqq m}(-1)^{m-j}q^{-j}\frac{B^j}{[j]_q!}\frac{A^m}{[m]_q!}\frac{B^{m-j}}{[m-j]_q!}.$$

Proof. (i) is proved by straightforward calculation, (ii) and (iv) are proved by induction, and (iii) follows from (ii). □

In the limit $q \to 1$ these identities reduce to those of A_2-subalgebras of Lie algebras (see, e.g. [Ve]).

If in addition q is a root of unity, then there are additional relations.

Lemma A.2. *Let A, B and C be as in Lemma A.1. Let $q^M = -1$, and define $\bar{k} = M - k$ for $0 < k < M$, then for $0 < l < k < M$ we have*

(i)

$$A^{\bar{k}+l}B^l = (-1)^l \sum_{0 \leqq j \leqq l} q^{j+(l-j)(k-j)} \frac{[l]_q![k]_q!}{[j]_q![l-j]_q![k-j]_q!} A^{l-j}C^jB^{l-j}A^{\bar{k}};$$

(ii)

$$A^lB^{\bar{k}+l} = (-1)^l \sum_{0 \leqq j \leqq l} q^{j+(l-j)(k-j)} \frac{[l]_q![k]_q!}{[j]_q![l-j]_q![k-j]_q!} B^{\bar{k}}A^{l-j}C^jB^{l-j}.$$

Proof. First prove the identity for $l = 1$ by induction to $\bar{k}$, then prove the identity by induction to l. □

Appendix 2

In this appendix we summarize some facts about the Weyl group of an (untwisted) affine Kac–Moody algebra (see, e.g. [Ka] for more details) and introduce the concept of the modified length of a Weyl group element.

The affine Kac–Moody algebra $\hat{\mathbf{g}}$ can be obtained as the (unique) central extension of $L\mathbf{g} = \mathbf{g} \oplus \mathbf{C}[t, t^{-1}]$. We will introduce an additional element d (derivation) defined by $d(x \otimes t^n) = n(x \otimes t^n)$ for $x \in \mathbf{g}$. The Cartan subalgebra of $\hat{\mathbf{g}}$ is then given by $\hat{\mathbf{h}} = \mathbf{h} \oplus \mathbf{C}c \oplus \mathbf{C}d$ and its dual by $\hat{\mathbf{h}}^* = \mathbf{h}^* \oplus \mathbf{C}\Lambda_0 \oplus \mathbf{C}\delta$, where Λ_0 and δ are dual to c and d, respectively. The bilinear form (,) on $\hat{\mathbf{h}}^*$ is defined by the bilinear form on $\mathbf{h}^*$ and the additional relations $(\Lambda_0, \delta) = 1$, $(\Lambda_0, \Lambda_0) = (\delta, \delta) = 0$. We will denote an element $\hat{\lambda} \in \hat{\mathbf{h}}^*$ either by $\hat{\lambda}$ or by its components (λ, k, n) in $\mathbf{h}^* \oplus \mathbf{C}\Lambda_0 \oplus \mathbf{C}\delta$. For modules with a highest weight $\hat{\Lambda}$ we always choose $\hat{\Lambda}$ such that $n = 0$. We have a root space decomposition $\hat{\mathbf{g}} = \left(\bigoplus_{\hat{\alpha} \in \hat{\Delta}} \hat{\mathbf{g}}_{\hat{\alpha}} \right) \oplus \hat{\mathbf{h}}$, where $\hat{\Delta} = \{n\delta, n \in \mathbf{Z} \backslash \{0\}\} \cup \{n\delta + \alpha, n \in \mathbf{Z}, \alpha \in \Delta\}$ and $\hat{\mathbf{g}}_{n\delta} = \mathbf{h} \otimes t^n, \hat{\mathbf{g}}_{n\delta+\alpha} = \mathbf{g}_\alpha \otimes t^n$.

The system of positive roots is given by $\hat{\Delta}_+ = \{n\delta, n > 0\} \cup \{n\delta + \alpha, n \geqq 0, \alpha \in \Delta_+\} \cup \{n\delta - \alpha, n > 0, \alpha \in \Delta_+\}$. Every positive root can be written as positive integral combination of simple roots $\hat{\alpha}_0 = \delta - \theta, \hat{\alpha}_i = \alpha_i, i = 1, \ldots, \ell$, where θ is the highest root of $\mathbf{g}$ and $\alpha_i, i = 1, \ldots, \ell$ are the simple roots of $\mathbf{g}$.

The Weyl group $\hat{W}$ of $\hat{\mathbf{g}}$ is the group generated by the reflections $r_i, i = 0, \ldots, \ell$ in the simple roots $\hat{\alpha}_i$, i.e. $r_i\lambda = \lambda - (2(\lambda, \hat{\alpha}_i)/(\hat{\alpha}_i, \hat{\alpha}_i))\hat{\alpha}_i$ for $\lambda \in \hat{\mathbf{h}}^*$, and leaves the bilinear form (,) invariant.

Every Weyl group element $w \in \hat{W}$ can uniquely be written as $w = t_\alpha \bar{w}$ for some $\alpha \in M, \bar{w} \in W$, where the "translation operator" is defined as

$$t_\alpha \lambda = \lambda + (\lambda, \delta)\alpha - ((\lambda, \alpha) + \tfrac{1}{2}|\alpha|^2(\lambda, \delta))\delta, \quad \lambda \in \hat{\mathbf{h}},$$

i.e. in terms of the decomposition $\hat{\mathbf{h}}^* = \mathbf{h}^* \oplus \mathbf{C}\Lambda_0 \oplus \mathbf{C}\delta$,

$$t_\alpha(\lambda, k, n) = (\lambda + k\alpha, k, n - ((\lambda, \alpha) + \tfrac{1}{2}k|\alpha|^2)).$$

In particular we have $t_\theta = r_{\delta-\theta} r_\theta$ and $t_{w(\alpha)} = w t_\alpha w^{-1}$, $w \in W$. The length of an element $w \in \hat{W}$ is defined as the minimal number l that is required to write w in terms of reflections $w = r_{i_1} \cdots r_{i_l}, i_j \in \{0, \ldots, \ell\}$. One can show that $l(w) = |\Phi_w|$, where the (finite) set Φ_w is defined by $\Phi_w = \hat{\Delta}_+ \cap w(\hat{\Delta}_-)$. We have the following basic lemma which follows directly from [GL, Proposition 2.2].

Lemma A.3. *Let $w \in \hat{W}$ and $i \in \{0, \ldots, \ell\}$. Then*

$$\begin{aligned} w\alpha_i \notin -\Phi_w &\Rightarrow \Phi_{wr_i} = \Phi_w \cup \{w\alpha_i\}, \qquad l(wr_i) = l(w) + 1,\\ w\alpha_i \in \Phi_w &\Rightarrow \Phi_w = \Phi_{wr_i} \cup \{-w\alpha_i\}, \quad l(wr_i) = l(w) - 1. \end{aligned}$$

In particular it follows (by induction) that if $w = r_{i_1} \cdots r_{i_n}$ is a reduced expression for w then $\Phi_w = \{\alpha_{i_1}, r_{i_1}\alpha_{i_2}, \ldots, r_{i_1} \cdots r_{i_{n-1}}\alpha_{i_n}\}$.

To describe the *Fock space complex* we have to introduce the concept of the *modified length* $\tilde{l}(w)$ of a Weyl group element w. Let thereto $\hat{\Delta}_{+,re} = \hat{\Delta}'_+ \cup \hat{\Delta}''_+$ with

$$\begin{aligned} \hat{\Delta}'_+ &= \{\hat{\alpha} = n\delta + \alpha, \alpha \in \Delta_+, n \geqq 0\},\\ \hat{\Delta}''_+ &= \{\hat{\alpha} = n\delta - \alpha, \alpha \in \Delta_+, n > 0\}, \end{aligned}$$

and define for all $w \in \hat{W}$

$$\Phi'_w = \hat{\Delta}'_+ \cap w(\hat{\Delta}_-), \quad \Phi''_w = \hat{\Delta}''_+ \cap w(\hat{\Delta}_-), \quad \Phi_w = \Phi'_w \cup \Phi''_w.$$

Definition A.4. *For every $w \in \hat{W}$ we define the modified length $\tilde{l}(w)$ by*

$$\tilde{l}(w) = |\Phi'_w| - |\Phi''_w|,$$

and let $\tilde{W}^{(i)}$ denote the subset of $\hat{W}$ of elements of modified length i. (Note that, contrary to the subsets $\hat{W}^{(i)} \subset \hat{W}$, the subsets $\tilde{W}^{(i)} \subset \hat{W}$, the subsets $\tilde{W}^{(i)}$ consist of an infinite number of elements (except for $\hat{g} = \widehat{sl}(2)$).)

Lemma A.5. *Let $w = t_\beta \bar{w} \in \hat{W}$, and $\alpha \in \Delta_+$.*
(i) *If $\bar{w}^{-1}\alpha = \alpha_i$ for some $i = 1, \ldots, \ell$, then $t_\beta r_\alpha \bar{w} = wr_i$ and $\tilde{l}(wr_i) = \tilde{l}(w) + 1$.*
(ii) *If $\bar{w}^{-1}\alpha = -\theta$ (θ the highest root of $\mathbf{g}$), then $t_\beta t_{-\alpha} r_\alpha \bar{w} = wr_0$ and $\tilde{l}(wr_0) = \tilde{l}(w) + 1$.*

Proof.
(i) For all simple roots $\alpha_k, k = 0, \ldots, \ell$ we have $r_\alpha \bar{w}\alpha_k = \bar{w}\alpha_k - (\bar{w}\alpha_k, \alpha)\alpha = \bar{w}\alpha_k - a_{ki}\alpha = \bar{w}(\alpha_k - a_{ki}\alpha_i) = \bar{w}r_i\alpha_k$ so $r_\alpha \bar{w} = \bar{w}r_i$. Now, $w\alpha_i = t_\beta \bar{w}\alpha_i = t_\beta \alpha = \alpha - (\alpha, \beta)\delta$. For $(\alpha, \beta) > 0$ we have $w\alpha_i \in -\hat{\Delta}''_+$ so that $\Phi_w = \Phi_{wr_i} \cup \{-w\alpha_i\}$, for $(\alpha, \beta) \leqq 0$ we have $w\alpha_i \in \hat{\Delta}'_+$ so that $\Phi_{wr_i} = \Phi_w \cup \{w\alpha_i\}$. In both cases $\tilde{l}(wr_i) = \tilde{l}(w) + 1$.
(ii) We have $t_{-\alpha} r_\alpha \bar{w} = t_{-\alpha} \bar{w} r_\theta = \bar{w} t_{-\bar{w}^{-1}} r_\theta = \bar{w} t_\theta r_\theta = \bar{w} r_0$. The statement that $\tilde{l}(wr_0) = \tilde{l}(w) + 1$ is proved similarly as in (i). $\square$

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Note added. In the course of writing we received several papers by Feigin and Frenkel [FeFr3, FeFr4] in which Conjecture 3.11 is proved by geometrical methods. Though their method is elegant it has two important drawbacks compared to our approach. Firstly, it does not give explicit formulae for the intertwining operators which are needed if one ultimately wants to apply these techniques to compute (higher genus) correlation functions. Secondly, the method, as presented, only works for the integrable highest weight modules while treating rational k-values would be essentially the same in our approach. One may hope that the two approaches combine to give an even better understanding of the various issues involved.

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