

Asymptotic Mass Degeneracies in Conformal Field Theories

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Abstract. By applying a method of Hardy and Ramanujan to characters of rational conformal field theories, we find an asymptotic expansion for degeneracy of states in the limit of large mass which is *exact* for strings propagating in more than two uncompactified space-time dimensions. Moreover we explore how the rationality of the conformal theory is reflected in the degeneracy of states. We also consider the one loop partition function for strings, restricted to physical states, for arbitrary (irrational) conformal theories, and obtain an asymptotic expansion for it in the limit that the torus degenerates. This expansion depends only on the spectrum of (physical and unphysical) relevant operators in the theory. We see how rationality is consistent with the smoothness of mass degeneracies as a function of moduli.

Introduction

One of the characteristic features of strings is the prediction of the existence of infinitely many massive particles. Based on the fundamental principle of modular invariance (world-sheet reparametrization invariance) one deduces that the number of particles grows exponentially with mass. This growth is universal and independent of which background strings are propagating in. One of the simple consequences of this fundamental behaviour is the existence of a limiting temperature (the Hagedorn temperature) [1] beyond which the canonical partition function diverges.

For various reasons one might be interested in more detailed information about the large energy degeneracy of particle spectrum in string theory. Although the leading behaviour of degeneracies is independent of the background, as mentioned above, the subleading terms *do* depend on the background in which strings propagate. The subleading terms might be useful in computing certain thermodynamical quantities. In fact this point in connection with extending the thermodynamical computations in [2] was one of the main motivations for undertaking the present work. Another application of these subleading terms might be in probing high energy aspects of string theory. Finally, these expansions might

be relevant in distinguishing and classifying conformal theories. This is the stringy analogue of the famous mathematical problem “*can one hear the shape of a drum*” (i.e., by the knowledge of the eigenvalues of the Laplacian on a drum, what can be learned about the geometry of the drum).

A seemingly unrelated issue is the notion of rationality of a conformal theory. A conformal theory is rational if the correlation functions are finite sums of holomorphic times anti-holomorphic functions of the world-sheet coordinates. The main motivation for considering rational conformal theories in the physics literature, comes from the fact that they arise naturally in many examples, and can be usually solved exactly by solving certain differential equations. A simple adaptation of the method of Hardy and Ramanujan [3], which we employ, can be used for finding the subleading corrections for asymptotic expansion for mass degeneracies of rational conformal field theories. Similar methods can be adapted to the computation of the on-shell partition function at one-loop, as the torus degenerates ($\tau_2 \rightarrow \infty$). This can be done for both irrational and rational conformal theories.

The organization of this paper is as follows. Section 2 is intended as a short review of RCFT’s with emphasis on their modular invariance. The invariance under modular transformations in these theories leads to some basic relationships between the ratios of the matrix elements of the modular transformation $S: \tau \rightarrow -1/\tau$ and certain formal ratios involving the “dimensions” of the irreducible modules on which the chiral characters are defined [4]. This result will be reviewed and extended in this section. As a prelude to the more powerful analytic methods described in detail in Sect. 3 of this paper, we also discuss briefly in this section the well known application of modular invariance and the Cauchy integral theorem in determining the leading order behavior of the coefficients in the power series expansions of the chiral characters.

In Sect. 3 we will describe the method prescribed by Hardy and Ramanujan in deriving an asymptotic expansion for a large class of partition functions. We will consider in some detail their main example of the oscillator partition function

for a single free holomorphic boson $f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$. There we shall rederive

their asymptotic expression for this partition function using a somewhat less complicated but equivalent approach, and discuss in somewhat less involved terms their proof of the exactness of this asymptotic expansion. Next we will apply their analysis to obtain asymptotic expansions for some common fermionic partition functions which are, of course, not only of substantial physical interest but also of some mathematical interest. Some of these partitions have also been considered in [3]. Part of this section is somewhat technical and some of the more technical aspects are postponed to Appendix A.

In Sect. 4 we shall expound on the arguments given in Sect. 2 about the determination of the chiral characters in an arbitrary RCFT by means of modular invariance. As a simple application and extension of the method of Hardy and Ramanujan, here we will first derive the asymptotic expansion for the coefficients $a_n^{(i)}$ in the power series expressions for these characters $\chi_i(q)$ and then we will consider the question of the exactness of this expansion. It is argued that though

the expansion may not be exact, it is so that the error introduced by this “perturbative” method is in fact bounded. In other words we shall prove that the expansion determines the coefficients $a_n^{(i)}$ up to a constant which is independent of n . A simple example of the Ising model is used at this point to demonstrate these ideas. To illustrate these ideas even further, we also consider the example of bosonic strings compactified on a circle with rational squared radius.

It is a simple matter to extend our analysis of the conformal field theories to the spectrum of physical particles in string theory which is the subject of Sect. 5. One new novelty is that the partition function in general has a non-trivial modular weight $(2 - D)/2$, where D is the number of uncompactified dimensions. It is shown that the asymptotic expansion for the mass degeneracies is exact for $D > 2$; more precisely, for large enough mass m a finite number of terms, about $O(m^2)$ terms, can be summed to obtain the *exact* degeneracy. For $D \leq 2$ it can be shown that the series is exact up to some fixed power of n . A general principle which seems to underly this behavior is that the exactness of the asymptotic expansions depends only on the modular weight of the class of modular forms under investigation and not on any of the other details of their mixing under modular transformations. In particular the exactness of the asymptotic series is independent of the form of the modular transformation matrices S and T . Strings compactified on rational tori are an important class of string theories on rational backgrounds which we analyze as an example in Sect. 5.

In Sect. 6 we begin studying irrational conformal theories. In particular we study the physical partition function for such theories at one loop, in the limit as $\tau_2 \rightarrow \infty$. This limit probes the asymptotic degeneracy of states. This allows one to consider the behaviour of the asymptotic degeneracies as one continuously changes the moduli. In Sect. 7 we compare the two methods, and explain how rationality could coexist with continuity of moduli in the asymptotic expansions.

We present our conclusions in Sect. 8 and discuss some unresolved issues and possible future applications of the ideas introduced in this paper. Appendix A fills some technical gaps from the analysis of Sect. 3, Appendix B computes certain properties of some of the coefficients appearing in the asymptotic expansion, and finally Appendix C completes some of the analysis needed in Sect. 6.

2. Some Applications of Modular Invariance in RCFT's

RCFT's have been extensively studied in the last few years. They were first emphasized, in connection with classification of conformal theories by Friedan and Shenker. Rational Conformal Field Theories (RCFT's) are characterized by the defining property that the spectrum of the irreducible representations of the chiral algebra is finite dimensional. Hence the vector space of the conformal blocks for any correlation function in such theories has a finite dimension. Equivalently the Hilbert space in this kind of theory can be decomposed into a finite direct sum of tensor products of the space of irreducible representations of the left and right chiral algebras \mathcal{A} and $\bar{\mathcal{A}}$

$$\mathcal{H} = \bigoplus_{i, \bar{i}} \phi_i \otimes \bar{\phi}_{\bar{i}},$$

where ϕ_i and $\bar{\phi}_i$ for $i=0, 1, \dots, N-1$ denote the irreducible highest weight representations of \mathcal{A} and $\bar{\mathcal{A}}$. Of particular importance to us is that as a consequence of the above decomposition for the Hilbert space \mathcal{H} , the one-loop partition function for an arbitrary RCFT, being the trace over the Hilbert space, decomposes in a similar form:

$$Z(\tau, \bar{\tau}) = \sum_{i=0}^{N-1} \chi_i(\tau) \bar{\chi}_i(\bar{\tau}) \tag{2.1}$$

in which the functions χ_i and $\bar{\chi}_i$ are the chiral characters correspondingly of the left and right modules ϕ_i and $\bar{\phi}_i$ and are given as the traces over the corresponding sectors ϕ_i and $\bar{\phi}_i$ of the Hilbert space: $\chi_i = \text{tr}_{\phi_i}(q^{L_0 - c/24})$, with similar expression for $\bar{\chi}_i$ with quantities q, L_0, ϕ_i and c replaced by their barred expressions, where the parameter q is defined in terms of the one-loop modular parameter τ as $q = \exp(2\pi i\tau)$ (see Fig. 1).

Modular invariance is an important physical restriction imposed on the partition function $Z(\tau, \bar{\tau})$. It is the statement of the invariance of the partition function Z under an arbitrary modular transformation of the parameter τ :

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \tag{2.2}$$

This is the group $SL(2, \mathbb{Z})/Z_2$ of unimodular 2×2 matrices with integer coefficients, modulo a Z_2 action which changes the sign of all the integer coefficients a, b, c and d , hence leaving the transformation above of the parameter τ unchanged. The modular group is generated by two transformations $T: \tau \rightarrow \tau + 1$ and $S: \tau \rightarrow -1/\tau$

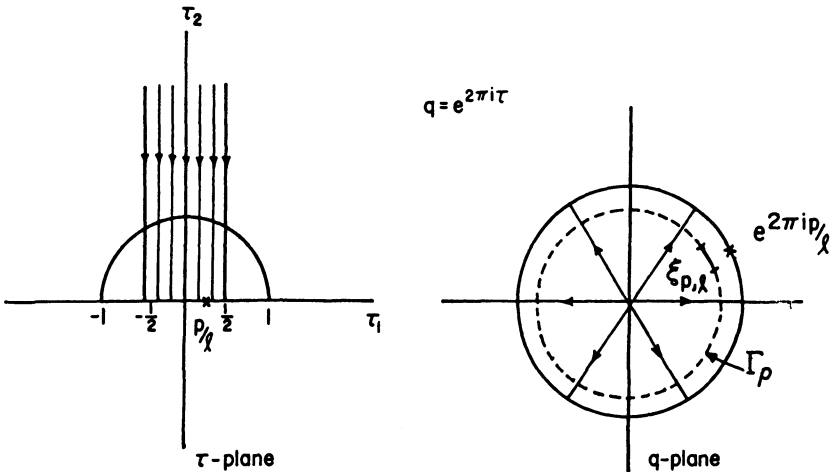


Fig. 1. The τ -plane is the Teichmüller space for a torus. The stripped region (excluding the part interior to the unit circle) is a fundamental domain for $PSL(2, \mathbb{Z})$. The transformation $q = e^{2\pi i\tau}$ maps the full stripped region in the τ -plane, to the interior of the unit disc in the q -plane. The contour Γ_ρ must be divided into arc segments $\xi_{p,l}$ near each rational singularity $e^{2\pi i p/l}$

of the modular parameter τ . The symmetry under the action of this group describes the physical requirement that the partition function should remain invariant under a change in the basis for the two vectors generating the lattice which defines the torus, over which the conformal theory at one loop order is constructed. The modular invariance of the theory (invariance under T) implies that the conformal dimensions of all the states in the spectrum (h, \bar{h}) must satisfy $h - \bar{h} \in \mathbf{Z}$ and furthermore, the left–right decomposition of the Hilbert space implies that the dimensions of all the states in a given module ϕ_i be the same, modulo addition of integers. As a consequence of this fact, the chiral characters can be given the following expansion in terms of the parameter q :

$$\chi_i(q) = q^{-c/24 + h_i} \left(a_0^{(i)} + \sum_{n=1}^{\infty} a_n^{(i)} q^n \right), \quad (2.3)$$

here h_i are the holomorphic conformal dimension of the highest weight vectors in the chiral modules ϕ_i . For the vacuum subsector $\phi_0 = 1$, the chiral character χ_i has the form (2.3) with $h_0 = 0$, and if the CFT is unitary then all other conformal dimensions are positive; $h_i > 0, i \neq 0$. Also the coefficients $a_n^{(i)}$ are non-negative integers for every value of $n \geq 0$. The value of $a_0^{(0)}$ depends on the degeneracy of the vacuum and is equal to 1 for the theories in which the vacuum is non-degenerate. In the course of our analysis we shall see that these simple facts are not without significance in determining the general expansions for the chiral characters. Precisely the same arguments of course apply to the right chiral sector, hence a similar expansion for the chiral characters $\bar{\chi}_i$ exist and is obtained from (2.3) by replacing all the quantities by their barred counterparts. For the present, therefore, it suffices to restrict our attention to the left chiral sector only. It follows from these arguments that the transformation matrix T in the space of chiral characters is diagonal with eigenvalues $\alpha_i \equiv e^{2\pi i(h_i - c/24)}$, referred to as the ‘‘Dehn twist’’ phases for this conformal theory. Furthermore the modular transformation matrix S describes how the chiral characters transform among one another under the transformation $\tau \rightarrow -1/\tau$. Therefore the action of the two generators of the modular group in the space of chiral characters is given as:

$$T: \chi_i \rightarrow \alpha_i \chi_i \quad \text{and} \quad S: \chi_i \rightarrow S_i^j \chi_j, \quad (2.4)$$

S is known to be symmetric: $S = S^T$ and unitary: $SS^* = 1$. Denoting by C the conjugation matrix which connects the chiral characters of the conjugate representations; $C: \chi_i \rightarrow \chi_{\bar{i}}$, there exist the following two relations among the transformation matrices S and T :

$$S^2 = C \quad \text{and} \quad (ST)^3 = C. \quad (2.5)$$

Indices are raised and lowered using the conjugation matrix C . Furthermore, if all the representations ϕ_i of the right chiral algebra are self-conjugate, then one has the usual $SL(2, \mathbf{Z})$ relationships: $S^2 = (ST)^3 = 1$, otherwise since C is idempotent; $C^2 = 1$, one has: $S^4 = (ST)^6 = 1$.

Transformation properties of the chiral characters χ_i under S and T are known to lead to severe constraints on the structure of the characters. For example, the most basic applications of the transformation properties described by (2.4) would

immediately give interesting relationships between the formal dimensions $D^{(i)}$ of the Hilbert subspaces ϕ_i and the matrix elements of the transformation matrices S and T . For instance a well known equality exists [4] between the ratios of the dimensions $D^{(i)}/D^{(0)}$ to the specific ratios of the matrix elements of the transformation matrix $S: D^{(i)}/D^{(0)} = S_{i0}/S_{00}$. Let us review this result. Using (2.3), $D^{(i)}$, being the sum of 1 over all the states in the representation ϕ_i , is formally expressed in terms of the coefficients $a_n^{(i)}$ as $D^{(i)} \equiv \sum_{n=0}^{\infty} a_n^{(i)} = \chi_i(q=1)$. Furthermore, under modular transformations $S: q = \exp(2\pi i\tau) \rightarrow q' = \exp(-2\pi i/\tau)$, the point $q = 1$ is mapped onto the point $q' = 0$ (allowing $\tau \rightarrow 0$ in upper half-plane), hence it is readily observed that (for unitary theories):

$$\frac{D^{(i)}}{D^{(0)}} = \lim_{q \rightarrow 1} \frac{\chi_i(q)}{\chi_0(q)} = \lim_{q' \rightarrow 0} \frac{S_{ij}\chi_j(q')}{S_{0j}\chi_j(q')} = \frac{S_{i0}}{S_{00}}, \tag{2.6}$$

where we have used the fact that in the limit $q' \rightarrow 0$ the dominant contribution arises from the vacuum sector for which $\chi_0(q') = q'^{-c/24} \left(1 + \sum_{n=1}^{\infty} a_n^{(0)} q'^n \right) \sim q'^{-c/24}$.

This can be recast in a slightly more canonical form (which does not treat the vacuum sector preferentially). To do this, consider the diagonal, modular invariant partition function $\sum_{i=0}^{N-1} \chi_i(q)\bar{\chi}_i(\bar{q})$ and for fixed i form the ratio of $\chi_i(q)\bar{\chi}_i(\bar{q})$ to this function. The modular invariance of this partition function implies that: $\sum_{i=0}^{N-1} \chi_i(q=1)\bar{\chi}_i(\bar{q}=1) = \sum_{i=0}^{N-1} \chi_i(q'=0)\bar{\chi}_i(\bar{q}'=0)$. This together with Eq. (2.4) gives:

$$\frac{\chi_i(q=1)\bar{\chi}_i(\bar{q}=1)}{\sum_{i=0}^{N-1} \chi_i(q=1)\bar{\chi}_i(\bar{q}=1)} = \frac{[S_{ij}\chi_j(q'=0)][S_{ik}^*\bar{\chi}_k(\bar{q}'=0)]}{\sum_{i=0}^{N-1} \chi_i(q'=0)\bar{\chi}_i(\bar{q}'=0)}. \tag{2.7}$$

The dominant contribution to the right-hand side of this equation comes from the vacuum as before. An application of (2.3) thus leads to the desired conclusion:

$$\frac{|D^{(i)}|^2}{\sum_{i=0}^{N-1} |D^{(i)}|^2} = |S_{i0}|^2. \tag{2.8}$$

One will further note that the sum over the index i of both sides of this equality is equal to unity, on the right-hand side this being due to the unitarity of the transformation matrix S .

An immediate generalization of the previous arguments is reached if in place of the point $q = 1$, one instead considers a general "rational" point $q = e^{2\pi ip/l}$ on the unit circle in the q -plane, where p and l are positive¹ co-prime integers and $p < l$. Using an appropriate modular transformation, which we presently denote

¹ Except for $l = 1$, in which case only $p = 0$ is to be considered

by $M^{p,l}$, a point $q = e^{2\pi ip/l}$ is mapped onto the point $q = 0^2$. It then follows that the arguments leading to the previous conclusions may be identically applied to any such point in place of the point $q = 1$. Making use of the notation:

$$D^{(i),p,l} \equiv \sum_{n=0}^{\infty} a_n^{(i)} e^{2\pi inp/l},$$

one readily finds the indicated extensions of (2.6):

$$\frac{D^{(i),p,l}}{D^{(0),p,l}} = (e^{-2\pi ih_i p/l}) \frac{[(M^{p,l})^{-1}]_{i0}}{[(M^{p,l})^{-1}]_{00}}, \tag{2.9}$$

an application of which, by means of identical reasoning as used to arrive at (2.8), also yields a generalization of the latter as follows:

$$\frac{|D^{(i),p,l}|^2}{\sum_{l=0}^{N-1} |D^{(i),p,l}|^2} = |[(M^{p,l})^{-1}]_{i0}|^2, \tag{2.10}$$

where the sum over index i of each side is again unity, on the right-hand side this being due to the unitarity of the modular transformation matrix $M^{p,l}$. It is clear that for $p=0$ and $l=1$ Eq. (2.9) reduces to (2.6). In general $D^{(i),p,l}$ gives us information about the relative degeneracy of levels modulo l . If for example we take $p=1$ and $l=2$ in (2.9), on the left-hand side we get the ratios of the difference between the dimensions of “even” and “odd” subspaces $D_{\text{even}}^{(i)} - D_{\text{odd}}^{(i)}$ of the i^{th} sector to that of the vacuum sector $D_{\text{even}}^{(0)} - D_{\text{odd}}^{(0)}$. To evaluate the right-hand side of (2.9) in this case, we note that the desired modular transformation in this case is given by: $\tau \rightarrow \tau/(1 - 2\tau)$ which maps the point $q = -1$ on the unit circle onto the point $q = 0$ (more precisely we define a parameter z by letting $\tau = p/l + iz/l$). Upon performing the modular transformation we will let $z \rightarrow 0^+$). The transformation $M^{p,l}$ affecting this modular transformation in the space of chiral characters can be clearly taken to have the decomposition $M^{1,2} = ST^2S$. Using this form and the definition of transformation matrices T and S as given by (2.4) one can evaluate the right-hand side of (2.9) and find the relation³:

$$\frac{D_{\text{even}}^{(i)} - D_{\text{odd}}^{(i)}}{D_{\text{even}}^{(0)} - D_{\text{odd}}^{(0)}} = e^{-\pi ic/24} (\alpha_i)^{-1/2} \frac{\sum_{l=0}^{N-1} \alpha_l^{-2} S_{il} S_{l0}}{\sum_{l=0}^{N-1} \alpha_l^{-2} (S_{l0})^2}. \tag{2.11}$$

That the right-hand side of this equation is a real number is a simple consequence of the relation (B.1) of Appendix B.

² It should be pointed out that this transformation is not unique, because any transformation of the form $T^m M^{p,l}$ for any integer m , would do the same job. However as we shall see, all physical quantities remain independent of the choice of m

³ An application of this relation to the Ising model, for instance, yields that for the character $\chi_{1/2}$ the right-hand side is equal to $\sqrt{2} - 1$ and that for the character $\chi_{1/16}$ it is identically zero. The vanishing result in the latter case is directly connected to the fact that the rational point $q = -1$ is not a singular point of the character $\chi_{1/16}$

Perhaps the most interesting application of (2.4), and what is the general theme of the present paper, is made by the use of the Cauchy integral formula. The use of the Cauchy integral formula in the analytic theory of partitions was advocated many years ago by Hardy and Ramanujan, as we shall describe in some detail in the next section, who made use of this formula to arrive at some remarkable asymptotic expansions for several important partition functions. According to the Cauchy integral formula, since the functions $\tilde{\chi}_i(q) = q^{c/24-h_i}\chi_i(q) = \sum_{n=0}^{\infty} a_n^{(i)}q^n$ is a regular function everywhere in the interior of the unit disc in the complex q -plane, we have the following integral representation of the coefficient $a_n^{(i)}$ for fixed values of i and n :

$$a_n^{(i)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\tilde{\chi}_i(q)}{q^{n+1}} dq = \frac{1}{2\pi i} \int_{\Gamma} \frac{\chi_i(q)}{q^{n_i+1}} dq. \tag{2.12}$$

Here $n_i \equiv n - c/24 + h_i$ and the contour of integration Γ in this formula is any closed contour surrounding the origin $q = 0$ and lying entirely interior to the unit disc in the complex q -plane, and may be taken to be any circle with the center at the origin and radius $\rho < 1$. To see how this integral behaves we let the radius tend to 1: $\rho \rightarrow 1^-$ and, as it is commonly done in the physics literature, use the modular transformation property of the partition function $\chi_i(q)$ to render an approximate evaluation of the integral using the saddle-point approximation [5]. To be precise, we note that the point $q = 1$ is roughly a saddle-point of the integrand in Eq. (2.12) in the limit of large n . Furthermore making use of (2.3) and (2.4), Eq. (2.12) leads to:

$$a_n^{(i)} = \lim_{\rho \rightarrow 1} \frac{1}{2\pi i} \int_{\Gamma_{\rho}} \frac{\chi_i(q)}{q^{n_i+1}} dq = \frac{1}{2\pi i} \int_{\Gamma_{\rho}} \frac{S_{ij}\chi_j(q')}{q^{n_i+1}} dq \sim S_{i0} \int_{\text{s.p.}} \frac{q^{-c/24}}{q^{n_i+1}} dq,$$

where the last step is made noticing that in this limit $q' \approx 0$ and therefore the dominant contribution to the sum $S_{ij}\chi_j(q')$ comes from the vacuum sector, a situation encountered previously. The integral on the right is normally evaluated by defining a change of variable: $q = e^{-2\pi\omega}$, and hence we also have $q' = e^{-2\pi/\omega}$. Finally one makes a saddle-point evaluation of the resulting integral, which is readily seen to give the result:

$$a_n^{(i)} \sim S_{i0} \int \exp \left[\frac{2\pi(c/24)}{\omega} + 2\pi n_i \omega \right] d\omega \sim \text{constant} \times S_{i0} \frac{e^{4\pi\sqrt{(c/24)n_i}}}{(n_i)^{3/4}}. \tag{2.13}$$

This equation actually gives the correct leading order dependence of the coefficients $a_n^{(i)}$ on n . As for the constant multiplying the leading-order expression, a slightly more careful treatment shows that this constant is in fact equal to $1/\sqrt{2}(c/24)^{1/4}$. The importance of this coefficient in thermodynamical considerations was pointed out in [2]. Thus we see that the leading order behavior of the coefficient $a_n^{(i)}$ for all values of the index i in a general RCFT is determined solely by the knowledge of the central charge of the Virasoro algebra c , and (certain) matrix elements of the transformation matrix S . To put it yet more simply, the value of this coefficient

is fixed by modular invariance. It is also interesting to notice that this result implies that in fact the ratio of individual coefficients (i.e. for fixed n) of the i^{th} sector to the corresponding ones of the zeroth sector is given at the leading order in n by the same ratio as of the matrix elements of the transformation matrix S as in (2.6):

$$\frac{a_n^{(i)}}{a_n^{(0)}} = \frac{S_{i0}}{S_{00}} + \text{non-leading contributions.} \quad (2.14)$$

This relation will of course receive corrections from subleading terms. Can we determine the non-leading contribution to the integral expression (2.12) for the coefficients $a_n^{(i)}$? As we shall describe in detail in Sect. 4, our answer to this question is positive. Let us give a brief review of our strategy in dealing with this question. Fix the value of index i for now. Consider the function $\tilde{\chi}_i(q)$ above. This function is regular for every point in the interior of the unit disc in the complex q -plane and has essential singularities for some or all of the rational points on the boundary of this disc. As we let the radius ρ of the circular contour Γ_ρ approach value 1, the integral on the right-hand side of (2.12) will pick up a contribution from each of the singularities of the function χ_i on the boundary of the disc. The most “dominant” singularity is at the point $q = 1$ as one may intuitively expect. This leads to the contribution we have (in some sense) already computed using the saddle-point approximation. However to obtain a more exact expression for $a_n^{(i)}$ we must not only perform the integral in (2.13) exactly, but also include the contributions from all the other singularities. This is the strategy of HR which we will now turn to.

3. Analysis of Hardy and Ramanujan

Our intention in the present section is to give a self-contained account of a remarkable approach proposed by Hardy and Ramanujan [3] in the beginning of this century to study the asymptotic behavior of the class of partitions described by elliptic modular functions or by functions closely related to them. Their approach makes essential use of two important features of this kind of partition function. The first feature common to all elliptic modular functions is that every point interior to the unit disc in the complex plane is a regular point of these functions. At the same time, some or all the rational points lying on the boundary of this unit disc are essential singularities of such a function, with the function not defined for points outside the unit disc. The second of these features is that modular functions possess well-defined transformation properties under modular transformations, a feature crucial in studying the behavior of these functions near their essential singularities, a study which is at the heart of this approach as we shall see.

It is instructive to give a rough account of the use of these principles first, though we have already alluded to the basic ideas involved in this kind of application in the introduction of this paper as well as some in the preceding section. Let us presently concentrate on the main example considered by these authors, the characteristic function $f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$ associated with the

number of distinct unrestricted partitions $p(n)$ of a number n as the sum of positive integers. In physical terms, $f(q)$ of course describes the oscillator partition function for a single free boson on the torus world sheet with modular parameter τ related to q via the relation $q = \exp(2\pi i\tau)$. The partition function $f(q)$ is essentially given by the reciprocal of the Dedekind η function: $f(q) = q^{1/24}/\eta(q)$. Therefore its behavior under modular transformations readily follows from the notably simple linear properties of the elliptic modular function $\eta(q)$, being a function of modular weight of $+1/2$. To be specific, under an arbitrary modular transformation (2.2) this behavior is given by:

$$\eta(q) \rightarrow \zeta(c\tau + d)^{1/2} \eta(q) \tag{3.1}$$

in which ζ is some 24th-root of unity depending on the parameters a, b, c and d whose exact form will be given shortly. From this the transformation property of $f(q)$ is immediately deduced to be:

$$f(q') = \zeta^{-1}(c\tau + d)^{-1/2} q'^{1/24} q^{-1/24} f(q) \tag{3.2}$$

in which $q' = \exp\{2\pi i(a\tau + b)/(c\tau + d)\}$ is the image of the point q under this transformation.

As transparent from its definition, f is analytic at $q = 0$ and has a well-defined power series expansion in the neighborhood of this point. Thus $f(q) = \sum_{n=0}^{\infty} p(n)q^n$ with the integer coefficient $p(n)$ being clearly the number of distinct unrestricted partitions of number n and where $p(0) = 1$ by definition. The analyticity of the function f in the interior region of the unit disc, however, also makes possible the use of the Cauchy integral theorem in giving a contour integral representation of these coefficients. Thus fixing n we can write the coefficient $p(n)$ in the usual way as:

$$p(n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(q)}{q^{n+1}} dq, \tag{3.3}$$

where Γ is any arbitrary closed contour surrounding the origin $q = 0$ and lying entirely interior to the unit disc. Thus we may take the contour Γ to be any circle of radius $\rho < 1$.

As clear from its expression, the partition function $f(q)$ has an essential singularity for every point on the unit circle with a rational⁴ argument $q = e^{2\pi ip/l}$ in which one may take p and l such that $0 < p < l$ and as coprime integers⁵, without any loss in generality (see Fig. 1). The singularity arises because infinitely many terms in the denominator of f go to zero for these rational points. Inspection of the denominator indicates that, roughly speaking, the degree of the singularity of this function at each point substantially depends on the denominator l of its fractional argument; the smaller the value of the integer l , the more zero's we get

⁴ The function $f(q)$ is known to vanish for *almost* all the points on the unit circle with irrational arguments. Some brief comments on this behavior at irrational points may be found at the end of Appendix A

⁵ See footnote 1

in the denominator, and thus the more singular is the behavior of the partition function f at this point. We will shortly rigorize this observation.

The main idea of this method is to allow the radius of the contour of integration Γ_ρ to converge to $1(\rho \rightarrow 1^-)$ in an n -dependent way, and at the same time to take advantage of the modular transformation properties of the function f to determine the precise singular form that this function admits as we approach each one of its rational singularities on the unit circle. To put it slightly differently, the modular transformation properties of the partition function $f(q)$ allows us to find a “test function” which approximates the function f near each singularity with a substantial accuracy. Indeed, in the limit $q' \rightarrow 0$ the expression $f(q')$ rapidly approaches unity, a fact which is transparent from the defining expression for f . Consequently, if we consider the rational point $q = e^{2\pi i p/l}$ on the unit circle in the complex q -plane and a modular transformation $M^{p,l}$ which maps this point onto the origin $q' = 0$ of this plane, then an application of (3.2) yields an approximation of the function $f(q)$ by means of some specific exponentially singular test function in the vicinity of this rational point. A useful way of doing this is to define a new parameter z in this manner:

$$q = \exp [2\pi i(p/l + iz/l)], \quad q' = \exp [2\pi i(p'/l + i/lz)], \quad (3.4)$$

then, for p odd we set⁶:

$$a = -p', \quad b = \frac{1 + pp'}{l}, \quad c = -l, \quad d = p, \quad (3.5)$$

and that for l odd we set:

$$a = p', \quad b = -\frac{1 + pp'}{l}, \quad c = l, \quad d = -p, \quad (3.6)$$

in which the (non-unique) positive integer p' is chosen so that b is also an integer. If both p and l are odd, either of the above definitions will be satisfactory. Assuming further that we write $q = e^{2\pi i\tau}$ and $q' = e^{2\pi i\tau'}$, in view of (3.4) it is easily verified that: $\tau' = (a\tau + b)/(c\tau + d)$. The two choices of the set of parameters $a-d$ given above obviously lead to equivalent linear transformations. However they are needed in order to represent the phase ζ for an arbitrary set of parameters by means of a single formula given by [3, 6]:

$$\zeta = \left(\frac{c}{d}\right) \exp \left[\pi i \left\{ \frac{1}{4}(d-1) - \frac{1}{12}[d(c-b) + ac(d^2-1)] \right\} \right] \quad (3.7)$$

in which the symbol $\left(\frac{c}{d}\right)$ is the Legendre–Jacobi symbol taking only two values $+1$ or -1 . The complete set of rules defining this symbol may be found in reference [6] and will not be repeated here. The details of this point and particulars of the transformation theory of η -function, however, have been long known by mathematicians and may be found in [6], thus we choose not to repeat them here. Combining these definitions and assignments and by virtue of (3.2) one finds the

⁶ Because p and l are relatively prime, a pair of integers b and p' can always be found so that $bl - pp' = 1$

following exponential form of singularity as $q \rightarrow e^{2\pi ip/l}$ along any regular path:

$$f(q) \sim \omega_{p,l} z^{1/2} \exp\left(\frac{\pi}{12lz} - \frac{\pi z}{12l}\right) \tag{3.8}$$

in which $\omega_{p,l}$ is defined for p odd as:

$$\omega_{p,l} = (-i)^{1/2} \exp\left[\frac{\pi i}{12}(p/l - p'/l)\right] \zeta_{(p,-1)}$$

and for l odd as:

$$\omega_{p,l} = i^{1/2} \exp\left[\frac{\pi i}{12}(p/l - p'/l)\right] \zeta_{(-p,l)},$$

where the principle square roots are intended in all these equations. To this end it should be sufficient to give the final expressions for the phases. For l odd this phase is given by the formula [3]:

$$\omega_{p,l} = \left(\frac{-p}{l}\right) \exp\left[-\pi i \left\{\frac{1}{4}(l-1) + \frac{1}{12}\left(l - \frac{1}{l}\right)(2p - p' + p'p^2)\right\}\right], \tag{3.9}$$

and for p odd by the formula:

$$\omega_{p,l} = \left(\frac{-l}{p}\right) \exp\left[-\pi i \left\{\frac{1}{4}(2 - pl - p) + \frac{1}{12}\left(l - \frac{1}{l}\right)(2p - p' - p'p^2)\right\}\right], \tag{3.10}$$

where the symbol $\left(\frac{p}{q}\right)$ is the Legendre–Jacobi symbol mentioned earlier. Of particular importance is the fact that the expressions given here for $\omega_{p,l}$ are independent of the choice of the integer p' , the only criterion for which is that $(1 + pp')/l$ should be an integer, and therefore the substitution $p' \rightarrow p' + ml$ for any integer m should leave the phases invariant. Equivalently, this is to say that the phases and thus the expression of the partition function should depend only on integers p and l , which together fully specify the positions of the singularities of the partition function, and not on any other parameter as such. It is really to (3.8) that the analysis of Hardy and Ramanujan is owed.

The evaluation of the integral in (3.3) would now become possible if we could in some sense arrange to dissect the circle Γ_ρ into small arc segments $\xi_{p,l}$, one for each rational singularity $e^{2\pi ip/l}$ of the function $f(q)$, so that on points of any given arc segment the function f may be replaced by the test function, as given by (3.8) (see Fig. 1). Then we may express (3.3) at least formally as a sum over these arc segments:

$$p(n) = \sum_{p,l} \omega_{p,l} e^{-2\pi inp/l} \left(\frac{-i}{l}\right) \int_{\xi_{p,l}} z^{1/2} \exp\left[\frac{\pi}{12lz} + \frac{2\pi z}{l}(n - 1/24)\right] dz. \tag{3.11}$$

Rational points are unfortunately dense on the unit circle and thus for our prescription to be useful we have to adopt a somewhat more subtle approach to the matter of dissection of the circle mentioned here.

Now that we have disclosed the rough strategy, let us begin with a rigorous

treatment of the method. Here we will rather closely follow the discussion given in Ref. [7] which is a variant of the discussion given by Hardy and Ramanujan in their original work. We have chosen to do so because the analysis of [7], though more concise and less involved, nevertheless exhibits all the essential features of the original arguments by Hardy and Ramanujan. Let us begin with a note on the nature of the singularities of the partition function $f(q)$. To do this, we focus our attention instead on the partial or “restricted” expression $f_N(q) = \prod_{n=1}^N (1 - q^n)^{-1}$, in which the exponents involved in the product are bounded by some fixed integer N , and think of function $f(q)$ as the large- N limit of these restricted expressions. The function f_N here is clearly the generating function for the restricted partitions $p_N(n)$ of a number n as the sum of integers not exceeding N , and indeed for $N \geq n$ it reduces to the unrestricted partition $p(n)$. Moreover, for fixed N , the function f_N has finite order singularities, or poles, at those rational points on the unit circle for which the argument has the denominator not exceeding N ; $l \leq N$. The order of the pole at one such rational point is thus seen to be given by $[N/l]$, where the bracket denotes the integer part of the expression contained. It is clear that the order of singularity is larger in inverse proportion to l .

The set of rational points p/l , including 0 and 1, with $1 \leq l \leq N$ when considered as an ordered set, under the usual “ $<$ ” ordering of numbers, form what is commonly referred to as the Farey’s series F_N of order N . This set has many interesting arithmetic properties which do not particularly concern us here, but some of which may also be found in [8]. The relevance of F_N to the analysis of Hardy and Ramanujan is that it naturally provides the aforementioned dissection Σ_N of the full circle into arc segments $\xi_{p,l}$, each in effect centered around a rational point $e^{2\pi i p/l}$. Hence in the limit $N \rightarrow \infty$, and in the manner to which we alluded above, this leads to the desirable situation where all the rational singularities of the partition function f are taken into account when evaluating (3.3) or equivalently (3.11). Though many equivalent descriptions of the dissection Σ_N of the circle may be given, the original description was given by considering for a fixed rational number p/l in Farey’s series of order N , three consecutive rational numbers in this series $p'/l', p/l, p''/l''$. We then consider an arc segment $\xi_{p,l}$ defined by the angular range: $\phi'_{p,l} < \phi < \phi''_{p,l}$, where the endpoints $\phi'_{p,l}$ and $\phi''_{p,l}$ are defined as⁷:

$$\phi'_{p,l} = \frac{p}{l} - \frac{1}{l(l+l')}, \quad \phi''_{p,l} = \frac{p}{l} + \frac{1}{l(l+l'')} \tag{3.12}$$

A remarkable property of Farey’s series of arbitrary order is that for any two consecutive terms p'/l' and p/l in such a series one has the relation $pl' - p'l = 1$. Using this property it is simple to show [3] that (3.12) indeed leads to a dissection of the full circle. This we write symbolically as $\sum_{p,l} \xi_{p,l} = 1$. Moreover it has been shown that the length of each arc segment is larger than $1/2lN$ and smaller than

⁷ Note that the Farey’s series of order N begins at 0 and terminates at 1. Thus the intervals described here would clearly form a dissection of the circle should we choose the end points for $l = 1$ say in the form: $\phi'_{0,1} = 1 - \frac{1}{N+1}$ and $\phi''_{0,1} = \frac{1}{N+1}$

$1/lN$ which we write symbolically in the form: $1/2lN < \xi_{p,l} < 1/lN$. The last property turns out to be essential in evaluating (3.11) to the desirable approximation.

Using these ideas, we are now equipped to disclose the precise sense in which the summation in (3.11) has to be performed in order to lead to a rigorous estimation for the $p(n)$. Starting afresh from Eq. (3.3), and fixing the value of integer n , we consider a circular contour of integration Γ_ρ centered at the origin and having the radius $\rho = e^{-2\pi/n}$. Clearly $\rho \rightarrow 1^-$ as n approaches infinity. We shall have to postpone for the present the justification for the precise way we have chosen this contour to approach the unit circle; suffices it to say for the moment that this choice is the optimal choice one can make in order to obtain the most accurate estimation of $p(n)$. Letting $q = \rho e^{2\pi i\varphi}$, and by virtue of (3.3) we have that:

$$p(n) = \rho^{-n} \int_0^1 f(\rho e^{2\pi i\varphi}) e^{-2\pi i n\varphi} d\varphi. \tag{3.13}$$

Breaking the range of the integration by means of the dissection Σ_N we may further define a ‘‘local’’ angular variable θ in each arc segment $\xi_{p,l}$ associated with this dissection by means of the substitution: $\varphi = p/l + \theta$ and at the same time we may define a new parameter z via the relation: $z = l(1/n - i\theta)$. Thus we rewrite the last equation as:

$$p(n) = -i \sum_{p,l} e^{-2\pi i n p/l} \frac{1}{l} \int_{\xi_{p,l}} f(\exp [2\pi i(p/l + iz/l)]) \exp\left(\frac{2\pi n z}{l}\right) dz \tag{3.14}$$

in which by letting $-\theta'_{p,l}$ and $\theta''_{p,l}$ denote respectively the upper and lower bounds of the angular integration, the limits of z -integration are simply noted to be given respectively by $l(1/n - i\theta''_{p,l})$ and $l(1/n + i\theta'_{p,l})$. It follows from the definition of the arc segments $\xi_{p,l}$ that:

$$\theta'_{p,l} = \frac{1}{l(l+l')} \quad \text{and} \quad \theta''_{p,l} = \frac{1}{l(l+l'')}$$

with l' and l'' defined earlier.

Our next step is to make use of the approximate form of the function $f(q)$ in the vicinity of its rational singular points as described by (3.8). In fact the full expression may be given for an arbitrary point $q = \exp [2\pi i(p/l + iz/l)]$ in view of (3.2):

$$f(q) = \omega_{p,l} z^{1/2} \exp\left(\frac{\pi}{12lz} - \frac{\pi z}{12l}\right) + \sum_{m=1}^{\infty} p(m) \omega_{p,l} e^{2\pi i m p/l} z^{1/2} \exp\left[\frac{2\pi}{lz}(1/24 - m) - \frac{\pi z}{12l}\right]. \tag{3.15}$$

The appropriate limit for us to be considering is the limit $z \rightarrow 0^+$ in which limit clearly $q \rightarrow e^{2\pi i p/l}$ and also $q' \rightarrow 0$. It should be clear that the term involving summation in the last equation is vanishingly small in this limit, thus serving only as a tiny correction to the singular first term in a small neighborhood of each rational point on the unit circle. Inserting this expression for $f(q)$ in Eq. (3.3), we

arrive at the final form of the integral:

$$p(n) = -i \sum_{p,l} \omega_{p,l} e^{-2\pi i n p/l} \frac{1}{l} \int_{\xi_{p,l}} z^{1/2} \exp\left(\frac{\pi}{12lz} + \frac{2\pi \hat{n} z}{l}\right) dz + R_n, \tag{3.16}$$

where $\hat{n} \equiv n - 1/24$ and the range of l in the sum is $1 \leq l \leq N$ and p ranges through numbers smaller than and relatively prime to l . This result is clearly the same as the formal result (3.11) except that here we have managed to attach precise meanings to the otherwise formal sums and integrals within that equation. We have still to make a proper choice for the integer N in this sum as it has been so far left arbitrary and must be thus chosen to lead to the most accurate estimation of $p(n)$. An examination of this point can be shown to lead to the determination of N as $N = \text{constant} \times \sqrt{n}$, but we have to postpone this analysis for the present time. Since the number of elements in the Farey series F_N is $O(N^2)$ we end up with $O(n)$ terms in (3.16). The remainder term R_n has the same structure as the first term and is given by:

$$R_n = -i \sum_{p,l} \omega_{p,l} e^{-2\pi i n p/l} \frac{1}{l} \sum_{m=1}^{\infty} p(m) e^{2\pi i m p/l} \int_{\xi_{p,l}} z^{1/2} \exp\left[\frac{2\pi}{lz}(1/24 - m) + \frac{2\pi \hat{n} z}{l}\right] dz. \tag{3.17}$$

Even though the complete details of the computation of the integral expressions above are delegated to Appendix A, the basic line of reasoning and the results may be succinctly described here as follows: As in Fig. 2, the first term in (3.16) is computed by adjoining several additional line segments; ξ_s for $s=0, 1, \dots, 6$ to the original path segment ξ along which this complex integration is to be performed. By virtue of the Cauchy theorem and the analyticity of the integrand everywhere in the punctured z -plane with a cut along the negative real axis, the integration

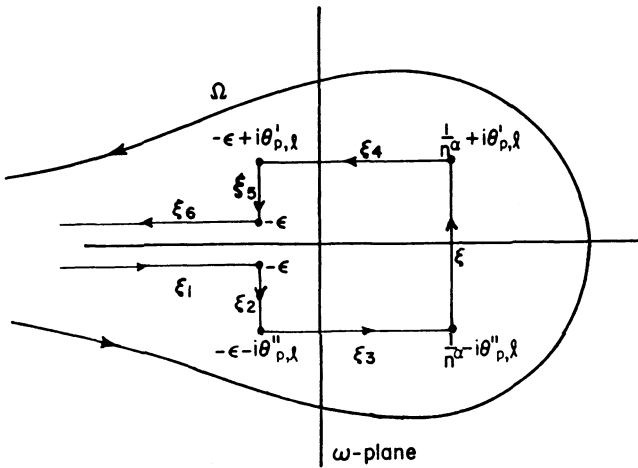


Fig. 2. The line integral over $\xi_{p,l}$ can be deformed, by using analyticity, to $-\xi_3 - \xi_2 - \xi_1 + \Omega - \xi_6 - \xi_5 - \xi_4$. ω is related to q by $q = e^{-2\pi\omega} e^{2\pi i p/l}$

along the combined path in the counter-clockwise direction equals the integration along the path Ω also shown in the figure. The integration over the contour Ω is more commonly denoted as $\int_{-\infty}^{(0+)} dt$, and in the present case the integral can be computed exactly with the aid of a loop-integral formula for the Bessel functions of the first kind J_ν :

$$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp(t - z^2/4t) dt. \tag{3.18}$$

Moreover, study of the integrals along the path segments $\xi_s, s = 0, 1, \dots, 6$ reveals that these integrals are all at most of order $O(n^{-1/4})$, a negligible correction in comparison with otherwise growing exponential behavior of the main contribution to this integral as emerging from Ω -integration. In addition, examination of the integral R_n in (3.17) demonstrates that the contribution to $p(n)$ which comes from this term is also of the same order of smallness: $R_n = O(n^{-1/4})$. The details of all these assertions may be found in Appendix A, however. Thus the final answer is found to be (see Appendix A):

$$p(n) = \sum_{l \leq O(\sqrt{n})} \sum_{(p,l)=1} (\omega_{p,l} e^{-2\pi i n p/l}) \frac{\sqrt{l}}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{(\pi/l)\sqrt{(2/3)(n-1/24)}}}{\sqrt{n-1/24}} \right) + O(n^{-1/4}). \tag{3.19}$$

The leading contribution to this sum comes from the leading order singularity $q = 1$ which corresponds to $l = 1$ and $p = 0$, and is readily found to be of an exponential form [3]:

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{\pi\sqrt{(2/3)(n-1/24)}}}{\sqrt{n-1/24}} \right) + \dots \tag{3.20}$$

The largest non-leading contribution to the partition $p(n)$ comes from the second most significant singularity $q = -1$ which corresponds to $l = 2$ and $p = 1$. It is found to have the form:

$$\frac{(-1)^n}{2\pi} \frac{d}{dn} \left(\frac{e^{(\pi/2)\sqrt{(2/3)(n-1/24)}}}{\sqrt{n-1/24}} \right). \tag{3.21}$$

Looking at one more term, the two singularities of the next order of importance $q = e^{2\pi i/3}$ and $q = e^{4\pi i/3}$ which correspond to $l = 3$ with two respective p -values $p = 1$ and $p = 2$, together lead to the following contribution:

$$\frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\frac{2n\pi}{3} - \frac{\pi}{18}\right) \frac{d}{dn} \left(\frac{e^{(\pi/3)\sqrt{(2/3)(n-1/24)}}}{\sqrt{n-1/24}} \right). \tag{3.22}$$

All the other terms in the asymptotic series (3.19) may be computed without any additional difficulties and they are found to be typically of the same structure as say the last expression. Particularly noteworthy is the form we have found for the error in this ‘‘perturbative’’ series for the partition function $p(n)$. Its form clearly indicates that for large values of n the series is not only asymptotic but it is also

exact, as follows from the integrality of $p(n)$ and vanishing of this error term in this limit. This is the surprising aspect of the work of HR, namely by adding a finite number of terms, about $O(n)$ terms, we can obtain an *exact* result for $p(n)$ by taking the closest integer to the sum. We have shown in Appendix A that as a general rule the modular weight of a given modular partition function (being $-1/2$ for $f(q)$) determines the structure of the error term one finds for the corresponding asymptotic series. Specifically for modular forms of weight d we find that the error term is of the order $O(n^{d/2})$ for $d \leq 0$ and $O(n^d)$ for $d > 0$. We shall encounter several examples of this fact in subsequent sections of this paper. Another item of great importance to the remainder of this section and to the rest this paper is that the method we have described in this section may not only be applied to a variety of modular forms with simple linear transformation rules such as (3.1) or (3.2) but it is equally applicable to any family of modular functions which transform linearly among one another under modular transformations. A major part of the present paper is devoted to the analysis of this kind of situation as will be discussed in more detail later.

Let us consider some simple generalizations and applications. An immediate application of this method, as discussed by Hardy and Ramanujan [3], is to the family of functions whose general form in terms of $f(q)$ is given by:

$$\frac{\{f(\pm q^{a_1})\}^{\alpha_1} \dots \{f(\pm q^{a_s})\}^{\alpha_s}}{\{f(\pm q^{b_1})\}^{\beta_1} \dots \{f(\pm q^{b_h})\}^{\beta_h}}, \tag{3.23}$$

where a_i, α_i, b_j and β_j are any arbitrary set of positive integers and also taking notice of the fact that $f(-q)$ is expressible in terms of $f(q)$ as: $f(-q) = (\{f(q^2)\}^3 / f(q)f(q^4))$. Even though most partitions of this type are of combinatorial and number theoretic value, among them there are some partition functions of fundamental physical importance. For instance consider the partition function:

$$F(q) = \frac{\{f(q)\}^2}{f(q^2)} = \frac{\prod_{n=1}^{\infty} (1 + q^n)}{\prod_{n=1}^{\infty} (1 - q^n)} \tag{3.24}$$

which is (essentially) the partition function for the free theory formed by a single holomorphic boson and a single holomorphic fermion with AP spin-structure on the torus. Note that $F(q)$ can be described in terms of the character $\chi_{1/16}$ of the irreducible representation of the Virasoro algebra with $h = 1/16$ and $c = 1/2$, i.e., $F(q) = (1/\eta)\chi_{1/16}$. It also possesses a description in terms of the basic Jacobian theta-function $\mathcal{G}_2(q)$ as $F(q) = (1/\sqrt{2\eta})\sqrt{\mathcal{G}_2/\eta}$. We let a_n denote the coefficient of q^n in the series expansion for this partition function; thus we see that:

$$F(q) = \sum_{n=0}^{\infty} a_n q^n = 1 + 2q + 4q^2 + 8q^3 + \dots \tag{3.25}$$

Using (3.8) in (3.24) it is not difficult to show that for $q \rightarrow e^{2\pi i p/l}$ the behavior

of F tends exponentially to an infinity which for l odd is of the form:

$$F(q) \sim \frac{1}{\sqrt{2}} \frac{\omega_{p,l}^2}{\omega_{[2p],l}} \sqrt{z} \exp\left(\frac{\pi}{8lz}\right) \tag{3.26}$$

in which the bracketed expression $[2p]$ is such that $[2p] < l$ and is equivalent to $2p$ modulo additions of l . For the rational points with l even the behavior is however not singular near these rational points:

$$F(q) \sim \frac{\omega_{p,l}^2}{\omega_{[p],l/2}} \sqrt{z} \tag{3.27}$$

in which again $[p] < l/2$ and is equivalent to p modulo additions of $l/2$. Thus it should be clear that there shall be no contributions from the rational points for which the denominator l is even to the asymptotic expansion for the coefficients a_n of this partition function. Hence we will reach the final form of this expansion with no additional difficulty:

$$a_n = \sum_{l \leq O(\sqrt{n})} \sum_{(p,l)=1} (\varrho_{p,l} e^{2\pi i n p/l}) \frac{\sqrt{l}}{4\pi} \frac{d}{dn} \left(\frac{e^{(\pi/l)\sqrt{n}}}{\sqrt{n}} \right) + O(n^{-1/4}) \tag{3.28}$$

in which $\varrho_{p,l}$ denotes the combination of phases which we found in (3.26). The first couple of terms of this expansion are then readily shown to be [3]:

$$a_n = \frac{1}{4\pi} \frac{d}{dn} \left(\frac{e^{\pi\sqrt{n}}}{\sqrt{n}} \right) + \frac{\sqrt{3}}{2\pi} \cos\left(\frac{2n\pi}{3} - \frac{\pi}{6}\right) \frac{d}{dn} \left(\frac{e^{(\pi/3)\sqrt{n}}}{\sqrt{n}} \right) + \dots \tag{3.29}$$

Other fermionic partition functions which occur commonly in superstring theory, and often in company with $F(q)$, are those associated with other spin structures for the chiral fermion on the torus. Let us name the partition function associated with the AA spin-structure G and that associated with the PA spin-structure H . These have the familiar forms:

$$G(q) = q^{-1/16} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})}{\prod_{n=1}^{\infty} (1 - q^n)} \quad \text{and} \quad H(q) = q^{-1/16} \frac{\prod_{n=1}^{\infty} (1 - q^{n-1/2})}{\prod_{n=1}^{\infty} (1 - q^n)}. \tag{3.30}$$

These partition functions also have simple descriptions in terms of the Jacobi theta-functions $\vartheta_3(q)$ and $\vartheta_4(q)$ respectively as: $G(q) = (1/\eta)\sqrt{\vartheta_3/\eta}$ and $H(q) = (1/\eta)\sqrt{\vartheta_4/\eta}$. We would naturally be interested in carrying out similar analyses for these partitions. Separately, however, neither of these functions satisfies the analytic criterion of the method of Hardy and Ramanujan. Nevertheless, as it is quite well known, appropriate linear combinations of these functions (which correspond to GSO projected subsectors) do possess the required structure, these being related to the two remaining irreducible characters $\chi_0(q)$ and $\chi_{1/2}(q)$ of the $c = 1/2$ Virasoro algebra in the same manner that $F(q)$ is related to the character $\chi_{1/16}(q)$.

Thus:

$$G(q) = \frac{1}{\eta}(\chi_0 + \chi_{1/2}) \quad \text{and} \quad H(q) = \frac{1}{\eta}(\chi_0 - \chi_{1/2}) \tag{3.31}$$

for which the characters have the required analytic expansions to which our method visibly applies:

$$\frac{\chi_0}{\eta}(q) = q^{-1/16} \sum_{n=0}^{\infty} b_n q^n = q^{-1/16}(1 + q + 3q^2 + 5q^3 + \dots), \tag{3.32}$$

and similarly

$$\frac{\chi_{1/2}}{\eta}(q) = q^{7/16} \sum_{n=0}^{\infty} c_n q^n = q^{7/16}(1 + 2q + 4q^2 + 7q^3 + \dots). \tag{3.33}$$

Even though perhaps no simple expression in terms of $f(q)$, such as (3.24), exists for these characters, it is nevertheless possible to use the transformation properties of $f(q)$ to establish, in a manner similar to what we did for the partition function $F(q)$, the asymptotic series for these partitions. This may be done for instance by studying the asymptotic series of the partition function:

$$R(q) = \frac{\{f(q^2)\}^2}{f(-q)} = \frac{f(q)f(q^4)}{f(q^2)} = \frac{\prod_{n=1}^{\infty} (1 + q^{2n-1})}{\prod_{n=1}^{\infty} (1 - q^{2n})}. \tag{3.34}$$

It is quite simple to show that by virtue of the definitions of the characters $(1/\eta)\chi_0$ and $(1/\eta)\chi_{1/2}$, their respective coefficients b_n and c_n are given in terms of the coefficient d_n of R ; $R(q) = \sum_{n=0}^{\infty} d_n q^n$, by the simple relations: $b_n = d_{2n}$ and $c_n = d_{2n+1}$.

Our problem is therefore reduced to finding the asymptotic expansion for d_n , a problem whose solution is very similar to that for the function $F(q)$, i.e. using the description in terms $f(q)$ to detect the exponential singularities of this function. Indeed for l odd we find:

$$R(q) \sim \frac{\omega_{[2p],l}^2}{\omega_{[2p+1],2l}} \sqrt{2z} \exp\left(\frac{\pi}{16lz} - \frac{\pi z}{4l}\right), \tag{3.35}$$

similarly for even l 's of the form $l = 4m$ we find:

$$R(q) \sim \frac{\omega_{[p],l/2}^2}{\omega_{[p+l/2],l}} \sqrt{z} \exp\left(\frac{\pi}{4lz} - \frac{\pi z}{4l}\right), \tag{3.36}$$

but for even l 's of the form $l = 4m + 2$ we find a non-singular form:

$$R(q) \sim \frac{\omega_{[p],l/2}^2}{\omega_{[p/2+l/4],l/2}} \sqrt{2z} \exp\left(\frac{-\pi z}{4l}\right). \tag{3.37}$$

Allowing $\varepsilon_{p,l}$ to denote the phase expression in the relation (3.35) and $\delta_{p,l}$ the phase

expression in (3.36), the asymptotic series for d_n takes the final form:

$$d_n = \sum_{l \in 2\mathbb{Z}+1} \sum_p (\epsilon_{p,l} e^{-2\pi i n p/l}) \frac{\sqrt{l}}{2\pi} \frac{d}{dn} \left(\frac{e^{(\pi/l)\sqrt{(1/2)(n-1/8)}}}{\sqrt{n-1/8}} \right) + \sum_{l \in 4\mathbb{Z}} \sum_p (\delta_{p,l} e^{-2\pi i n p/l}) \cdot \frac{\sqrt{l}}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{(\pi/l)\sqrt{(1/2)(n-1/8)}}}{\sqrt{n-1/8}} \right) + O(n^{-1/4}). \tag{3.38}$$

Using this and the relation of d_n to the coefficients of our interest b_n and c_n mentioned above, we readily find the desired asymptotic series for the latter. Looking at the first few terms of each series we find:

$$b_n = d_{2n} = \frac{1}{4\sqrt{2\pi}} \frac{d}{dn} \left(\frac{e^{\pi\sqrt{n-1/16}}}{\sqrt{n-1/16}} \right) + \frac{1}{2\pi} \cos\left(n\pi + \frac{\pi}{8}\right) \frac{d}{dn} \left(\frac{e^{(\pi/2)\sqrt{n-1/16}}}{\sqrt{n-1/16}} \right) + \frac{\sqrt{3}}{2\sqrt{2\pi}} \cos\left(\frac{2n\pi}{3} - \frac{\pi}{6}\right) \frac{d}{dn} \left(\frac{e^{(\pi/3)\sqrt{n-1/16}}}{\sqrt{n-1/16}} \right) + \dots \tag{3.39}$$

and also

$$c_n = d_{2n+1} = \frac{1}{4\sqrt{2\pi}} \frac{d}{dn} \left(\frac{e^{\pi\sqrt{n+7/16}}}{\sqrt{n+7/16}} \right) + \frac{1}{2\pi} \cos\left(n\pi - \frac{3\pi}{8}\right) \frac{d}{dn} \left(\frac{e^{(\pi/2)\sqrt{n+7/16}}}{\sqrt{n+7/16}} \right) + \frac{\sqrt{3}}{2\sqrt{2\pi}} \cos\left(\frac{2n\pi}{3} + \frac{\pi}{2}\right) \frac{d}{dn} \left(\frac{e^{(\pi/3)\sqrt{n+7/16}}}{\sqrt{n+7/16}} \right) + \dots \tag{3.40}$$

So far we have discussed few examples from a rather special class of partition functions of the form (3.23); those with the modular weight $-1/2$. There should be no inherent difficulties in generalizing the prescription of Hardy and Ramanujan to modular functions of arbitrary weight. Thus consider, as an example, the partition functions:

$$P(q) = \frac{f(q)}{f(q^2)} = \prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1} \tag{3.41}$$

which is another member of the family of partitions functions (3.23). From an arithmetic standpoint, $P(q)$ is the generating function for the partitions of a number n into odd parts, or equivalently into unequal parts. It also satisfies the relation: $P(q) = q^{-1/24} \chi_{1/16}$ and it is essentially the partition function for a single chiral fermion with PA boundary conditions on the torus, thus being of physical interest to us. It is apparent from (3.41) that $P(q)$ has vanishing modular weight. In fact by virtue of (3.8), and in analogy with (3.26)–(3.27), the exponential singular behavior of this partition function in a small neighborhood of the rational point $q = e^{2\pi i p/l}$ is observed for l odd to be:

$$P(q) \sim \frac{1}{\sqrt{2}} \frac{\omega_{p,l}}{\omega_{[2p],l}} \exp\left(\frac{\pi}{24lz} + \frac{\pi z}{12l}\right). \tag{3.42}$$

However, for l even we find a non-singular behavior:

$$P(q) \sim \frac{\omega_{p,l}}{\omega_{[p],l/2}} \exp\left(\frac{-\pi}{12lz} + \frac{\pi z}{12l}\right) \quad (3.43)$$

from which it follows that there are, as we may have expected from the second equality in (3.41), no contributions from the rational points on the unit circle whose denominator is even. A rather simple computation is then required to reach the final form of the asymptotic expansion for the coefficient $s(n)$ of q^n in the power series expansion of this partition function: $P(q) = \sum_{n=0}^{\infty} s(n)q^n$. General arguments of the kind required to obtain this result will be discussed in Sect. 5 and we need not go into the details of these arguments here. The final result is thus found to be:

$$s(n) = \sum_{l \leq O(\sqrt{n})} \sum_{(p,l)=1} (\sigma_{p,l} e^{-2\pi i n p/l}) \frac{1}{\sqrt{2}} \frac{d}{dn} J_0\left(\frac{i\pi}{l} \sqrt{\frac{1}{3}\left(n + \frac{1}{24}\right)}\right) + O(1) \quad (3.44)$$

in which $\sigma_{p,q}$ denotes the combination of phases present in (3.42) and J_0 is the Bessel function of the first kind with an imaginary argument. One notices that the error introduced by this expansion is here found to be a constant independent of n and hence does not seem to indicate the exactness of the series looked at previously. We will have more to say on this later on in Sect. 5 and Appendix A. The first two terms may thus be found to be as follows [3]:

$$s(n) = \frac{1}{\sqrt{2}} \frac{d}{dn} J_0\left(i\pi \sqrt{\frac{1}{3}\left(n + \frac{1}{24}\right)}\right) + \sqrt{2} \cos\left(\frac{2n\pi}{3} - \frac{\pi}{9}\right) \frac{d}{dn} J_0\left(\frac{i\pi}{3} \sqrt{\frac{1}{3}\left(n + \frac{1}{24}\right)}\right) + \dots \quad (3.45)$$

A similar analysis can be performed for the other two even spin structures, which we leave to the reader.

4. Extension to Rational Conformal Field Theories

We argued in a previous section that the prescription of Hardy and Ramanujan finds application in a variety of cases of physical interest and we saw some examples of their application in that section. One naturally expects to be able to apply the same prescription to find asymptotic expansions for the chiral characters of an arbitrary RCFT.

It should be clear from our discussion of the method of HR that the basic ingredient in finding our series is to study how the characters transform under modular transformations which takes a rational point on the boundary of the unit disc in q -plane, to $q = 0$. We have already seen a simple example of this as described by phases in (3.10)–(3.11). However, for RCFT's, since the modular transformations are more conveniently expressed in terms of S and T , we would like to express a given modular transformation in terms of products of them. So we will first discuss some basic facts about the modular transformations.

An arbitrary modular transformation $\tau' = (a\tau + b)/(c\tau + d)$ can always be

expressed in the following from using continued fractions (see for instance [9, 10]):

$$\tau' = a_* + (-1)^{N+1} \frac{1}{a_N + \frac{1}{a_0 - 1/\tau}} \tag{4.1}$$

here a_j for $j = 0, 1, \dots, N$ are non-zero integers and a_* is an integer which may also be zero. Following our conventions in the previous section, we would like to consider modular transformations with $a = -p'$, $b = ((1 + pp')/l)$, $c = -l$ and $d = p$, where p , l and p' are all non-negative integers. The constants a_j , $j = 0, 1, \dots, N$ in the above expression also occur in the continued fraction representation of the ratios l/p and p'/l as given by⁸:

$$l/p = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_N}}}, \quad p'/l = a_* + \frac{1}{a_N + \frac{1}{\ddots + \frac{1}{a_0}}} \tag{4.2}$$

It follows from (2.1) that a general modular transformation M has a decomposition in terms of the generators S and T of the modular group as in the form:

$$M = T^{a_*} S T^{(-1)^N a_N} S \dots T^{-a_1} S T^{a_0} S \tag{4.3}$$

in which the integers a_j are again positive with the exception of a_* which may also be zero. We have already noted in the previous section that there exists a kind of ‘‘gauge’’ arbitrariness in the choice of p' and that the quantities of interest to us are all independent of the choice of this gauge. Specifically our expansions are collectively invariant under: $p' \rightarrow p' + ml$. Notably this substitution affects the form of the modular transformation M in (3.3) in the manner:

$$p' \rightarrow p' + ml \Leftrightarrow M \rightarrow T^m M, \tag{4.4}$$

as is apparent from the second of Eqs. (4.2). The proofs of invariance under the choice of p' in each of the forthcoming contexts will thus make use of this fact as we shall see.

Let us now return to the main point of this section, the determination of the asymptotic series for the coefficients $a_n^{(j)}$ of the chiral characters $\chi_i(q)$ in Eq. (2.3). Following the prescription of the previous section, as already pointed out in Sect. 2 of this paper, we have to consider the contribution from each rational singularity of the character in question on the boundary of the unit disc. Again we first fix our attention on one specific such point: $q = e^{2\pi i p/l}$.

Consider the modular transformation $M^{p,l}$ mapping this point to the origin $q = 0$ of the complex q -plane. We will denote the unitary matrix affecting this

⁸ As a rule the division should be continued, making use of the replacement $r = r - \frac{1}{l}$ of the final element in the fraction, until the forms of the two continued fractions for p/l and p'/l are in agreement

transformation in the vector space of chiral characters by $M^{p,l}$. Hence we have that:

$$M^{p,l}: \chi_i(q) \rightarrow (M^{p,l})_i^j \chi_j(q). \quad (4.5)$$

Following precisely the same steps as those used for $p(n)$ this formally implies:

$$a_n^{(i)} = \frac{1}{2\pi i} \sum_{p,l} \int_{\xi_{p,l}} \frac{(M^{p,l})_i^j \chi_j(q')}{q^{n_i+1}} dq \quad (4.6)$$

in which we have defined $n_i \equiv n - c/24 + h_j$, and where the arcs $\xi_{p,q}$ forming the dissection of the unit circle are same as in the previous section. The next step is to break the integrand in two parts, a most dominant part, singular in case of characters for which the exponent $-c/24 + h_j < 0$, and a non-singular part⁹ which rapidly approaches zero as $q' \rightarrow 0$: $\chi_j(q') = a_0^{(j)} q'^{-c/24+h_j} + \sum_{m=1}^{\infty} a_m^{(j)} q'^{m-c/24+h_j}$. The counterpart of (3.8) for the character $\chi_i(q)$ is now only a simple exercise to derive. It is found that the behavior of this character in the vicinity of the rational $q = e^{2\pi i n p/l}$ has the form:

$$\chi_i(q) \sim a_0^{(j)} [(M^{p,l})^{-1}]_i^j e^{-2\pi i(c/24-h_j)p'/l} \exp\left[\frac{2\pi}{lz}(c/24-h_j)\right]. \quad (4.7)$$

Finally denoting the integral over the second part of the expansion above for χ_i by $R_n^{(i)}$, and using the same notations and arguments as in Sect. 3, this expression can be easily shown to be equal to:

$$a_n^{(i)} = -i \sum_{p,l} \mathcal{G}_{ij}^{p,l} e^{-2\pi i n p/l} \int_{\xi_{p,l}} \exp\left[\frac{2\pi(c/24-h_j)}{l^2\omega} + 2\pi\omega n_i\right] d\omega + R_n^{(i)}, \quad (4.8)$$

where now $\mathcal{G}_{ij}^{p,l}$ denotes the following expression:

$$\mathcal{G}_{ij}^{p,l} = a_0^{(j)} [(M^{p,l})^{-1}]_i^j \exp(2\pi i[(c/24-h_j)p/l - (c/24-h_j)p'/l]) \quad (4.9)$$

with no summation over j intended in this last expression and where $R_n^{(i)}$ has exactly the same form as the first term with $(c/24-h_j)$ in the argument of the exponential (in the integrand) replaced by $(c/24-h_j-m)$ and with an additional summation over j . We leave the details of the computations of these integrals to Appendix A and simply give the final result here:

$$a_n^{(i)} = \sum_{j=0}^{N-1} \sum_{l,p} (\mathcal{G}_{ij}^{p,l} e^{-2\pi i n p/l}) \frac{d}{dn} J_0\left(\frac{4\pi i}{l} \sqrt{(c/24-h_j)n_i}\right) + \text{constant}, \quad (4.10)$$

described in terms of the derivative of the Bessel function of the first kind J_0 with an imaginary argument. Though all values of j are being formally included in this sum, those values of j for which $c/24 - h_j \leq 0$ are clearly observed not to lead to exponentially growing terms, and are thus generally not of great interest to us.

⁹ For conformal theories with $c > 24$ some of the subleading terms might also be singular and have to be treated similarly. This will not be the case for string theory (as is clear in the light cone gauge). Also the methods discussed here obviously generalize to non-unitary conformal theories as long as we include all the singular terms

Thus in these and subsequent formulas the value of j is implicitly restricted to those for which $c/24 - h_j > 0$. For large values of n , it is simple to find the leading behavior of $a_n^{(i)}$, as a function of n , using the known asymptotic form of the Bessel function J_0 with imaginary argument: $J_0(iz) \sim e^z/\sqrt{2\pi z}$ and recover the previous result (2.13) with the value of the constant being $(1/\sqrt{2})(c/24)^{1/4}$, as mentioned in the paragraph following that equation. Note that the quantities $D^{(i),p,l}$ defined in Sect. 2 are proportional to $\mathcal{G}_{i,0}^{p,l}$ which correspond to leading singularities in (4.10). Using this we can recover (2.9).

The (constant) correction in the above expression cannot be determined using our analysis and requires a fundamentally different approach for its determination. This point has been illustrated in detail in Appendix A where we have shown the reason for the emergence of this error term in this case and its absence from say the simple bosonic oscillator partition function $f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$ discussed in Sect. 3. This is connected to the chiral characters having zero weight under modular transformations, as opposed to the bosonic case in which the modular weight of the function $f(q)$ is $-1/2$.

As for the proof of invariance of the expansion (4.10) under different choices of parameter p' alluded to earlier, it directly follows from (4.4) and (4.9):

$$\mathcal{G}_{ij}^{p,l} \rightarrow \alpha_j^{-m} e^{-2\pi i m(c/24 - h_j)} \mathcal{G}_{ij}^{p,l} = \mathcal{G}_{ij}^{p,l} \tag{4.11}$$

in which the last equality has resulted from the definition of α_j phases defined earlier.

The reality of the expression (4.10) for the coefficients $a_n^{(i)}$ in general RCFT follows from the symmetry and unitarity of the transformation matrices S and T as well as relation (2.5) which hold between them, together with the expansion (4.3) for the elements of the modular group in terms of these generators. The details of the proof as well as some properties of the coefficient matrix $\mathcal{G}_{ij}^{p,l}$ are however provided in Appendix B.

4.1. A Simple Example: Critical Ising Model on the Torus. An explicit example of the formalism developed in this section is furnished by the critical Ising model on the torus for which $c = \bar{c} = 1/2$. The unique modular invariant partition function in this theory is the diagonal invariant formed from three chiral characters

$$\chi_0 = \frac{1}{2} \begin{pmatrix} A \square + P \square \\ A \quad A \end{pmatrix}, \chi_{1/2} = \frac{1}{2} \begin{pmatrix} A \square - P \square \\ A \quad A \end{pmatrix} \text{ and } \chi_{1/16} = \frac{1}{\sqrt{2}} \begin{pmatrix} A \square \\ P \end{pmatrix},$$

corresponding to the three unitary irreducible representations of the $c = 1/2$ Virasoro algebra; $h = 0, 1/2, 1/16$, together with their conjugates: $Z_{\text{Ising}} = \chi_0 \bar{\chi}_0 + \chi_{1/2} \bar{\chi}_{1/2} + \chi_{1/16} \bar{\chi}_{1/16}$. The descriptions in terms of the spin structures, or equivalent descriptions in terms of the basic Jacobian theta functions, then leads to the well known q -expansions for the characters. The vacuum sector corresponding to $h = 0$ thus has the character expansion:

$$\chi_0(q) = \text{tr}_{[h=0]}(q^{L_0 - 1/48}) = q^{-1/48}(1 + q^2 + q^3 + 2q^4 + \dots), \tag{4.12}$$

and the character expansions for the other two sectors $h = 1/2$ and $h = 1/16$ are respectively given by:

$$\chi_{1/2}(q) = \text{tr}_{[h=1/2]}(q^{L_0 - 1/48}) = q^{23/48}(1 + q + q^2 + q^3 + \dots) \tag{4.13}$$

and also:

$$\chi_{1/16}(q) = \text{tr}_{[h=1/16]}(q^{L_0 - 1/48}) = q^{1/24}(1 + q + q^2 + 2q^3 + \dots). \quad (4.14)$$

It should be noted that vacuum is the only sector in this model which possesses a character expansion with negative leading exponential. Inspection of the asymptotic formula (4.8) in this case thus reveals that the only significant contributions to the expansions indicated by this formula must come only from this sector. Let us proceed to determine the first few terms of the asymptotic expansions for the coefficient $a_n^{(i)}$, for each of the three characters $i = 0, 1, 2$ in this theory. The well-known transformations of the chiral determinants $X \square_Y$ with $X, Y = P, A$ under $\tau \rightarrow -1/\tau: X \square_Y \rightarrow Y \square_X$ immediately determines the form of transformation matrix S in the basis $\{\chi_0, \chi_{1/2}, \chi_{1/16}\}$ to be:

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \quad (4.15)$$

Also from the conformal dimensions of the three subsectors, and the central charge, or from the direct inspection of (4.12)–(4.14), the form of the transformation matrix T in the same basis is readily determined to be:

$$T = \begin{pmatrix} e^{(-i\pi/24)} & 0 & 0 \\ 0 & -e^{(-i\pi/24)} & 0 \\ 0 & 0 & e^{(i\pi/12)} \end{pmatrix}. \quad (4.16)$$

A straightforward application of (4.8) is thus seen to give the following first few terms of the asymptotic expansion for the degeneracies of the three chiral characters in this theory, as the reader may readily check. For the zeroth sector, $h = 0$, the result is:

$$\begin{aligned} a_n^{(0)} &= \frac{1}{2} \frac{d}{dn} J_0 \left(i\pi \sqrt{\frac{1}{3} \left(n - \frac{1}{48} \right)} \right) + \cos \left(n\pi + \frac{\pi}{8} \right) \frac{d}{dn} J_0 \left(\frac{i\pi}{2} \sqrt{\frac{1}{3} \left(n - \frac{1}{48} \right)} \right) \\ &+ \cos \left(\frac{2n\pi}{3} + \frac{2\pi}{9} \right) \frac{d}{dn} J_0 \left(\frac{i\pi}{3} \sqrt{\frac{1}{3} \left(n - \frac{1}{48} \right)} \right) + \dots \end{aligned} \quad (4.17)$$

For $h = 1/2$ sector:

$$\begin{aligned} a_n^{(1)} &= \frac{1}{2} \frac{d}{dn} J_0 \left(i\pi \sqrt{\frac{1}{3} \left(n + \frac{23}{48} \right)} \right) + \cos \left(n\pi - \frac{3\pi}{8} \right) \frac{d}{dn} J_0 \left(\frac{i\pi}{2} \sqrt{\frac{1}{3} \left(n + \frac{23}{48} \right)} \right) \\ &+ \cos \left(\frac{2n\pi}{3} - \frac{4\pi}{9} \right) \frac{d}{dn} J_0 \left(\frac{i\pi}{3} \sqrt{\frac{1}{3} \left(n + \frac{23}{48} \right)} \right) + \dots, \end{aligned} \quad (4.18)$$

and finally for the $h = 1/16$ sector one finds:

$$a_n^{(2)} = \frac{1}{\sqrt{2}} \frac{d}{dn} J_0 \left(i\pi \sqrt{\frac{1}{3} \left(n + \frac{1}{24} \right)} \right) + \sqrt{2} \cos \left(\frac{2n\pi}{3} - \frac{\pi}{9} \right) \frac{d}{dn} J_0 \left(\frac{i\pi}{2} \sqrt{\frac{1}{3} \left(n + \frac{1}{24} \right)} \right) + \dots \tag{4.19}$$

We may note that the last expansion is identical with the expansion obtained earlier for the partition function $\sum_{n=1}^{\infty} (1 + q^n)$. This is of course expected as this function represents, as mentioned earlier, the oscillator partition function for a single holomorphic fermion on the torus with periodic (P) boundary conditions in spatial direction and (A) boundary conditions in the time direction, the precise characterization for the defining spin-structure of the chiral character $\chi_{1/16}(q)$.

4.2. Another Example: Strings on a Circle. A well-known class of rational conformal theories are obtained through the study of the strings propagating on backgrounds at special rational moduli. The simplest such example is provided by strings propagating on a circle of rational dimension. The rationality in this case, as we shall describe in detail shortly, refers to the square radius of the compactification circle being a rational number. For the reasons of convenience and without any loss of generality for rational circles, the radius R of the circle may be taken to have the form: $R = \sqrt{r/2s}$, where r and s are positive co-prime integers. The partition function of this theory has the well-known form:

$$Z_{\text{circle}} = \frac{1}{\eta\bar{\eta}} \sum_{n,m \in \mathbb{Z}} q^{(1/2)(n/R - mR)^2} \bar{q}^{(1/2)(n/2R + mR)^2} \tag{4.20}$$

which entails the trace over oscillator states in the Fock space, as described by the oscillator partition function $1/\eta\bar{\eta}$, and over the momentum and winding states associated with the compact dimension, as described by the partition function

$\sum_{p_L, p_R} q^{p_L^2/2} \bar{q}^{p_R^2/2}$ in which $p_L \equiv \frac{1}{2}p - w$ and $p_R \equiv \frac{1}{2}p + w$, and where p and w , denoting respectively the momentum and winding of the states in the compact direction, are quantized according to usual rules as: $p = n/2R$ and $w = mR$.

Let us first proceed to bring the partition function (4.20) into the standard form (2.1) for RCFT's. Defining $N \equiv 2rs$ we first bring Z_{circle} to the convenient form below:

$$Z_{\text{circle}} = \frac{1}{\eta\bar{\eta}} \sum_{n,m \in \mathbb{Z}} q^{(1/2N)(ns - mr)^2} \bar{q}^{(1/2N)(ns + mr)^2}. \tag{4.21}$$

Next we denote the integer quantity inside the parentheses in the q -exponent by $l: ns - mr \equiv l$. Fix the value of integer l and suppose that two integers n_1 and m_1 satisfy the latter equation with this value of l , hence $n_1s - m_1r = l$. Since r and s are designated to be relatively prime, there exist integers u and v , which are by

no means unique¹⁰, for which one has $us - vr = 1$. It is not difficult to show that the set of all pairs of integers (n, m) satisfying the equation $ns - mr = l$ is identically equal to the set $\{(ul + tr, vl + ts); t \in \mathbf{Z}\}$. It follows from this, together with a further definition $\lambda \equiv us + vr$, that the set of values of the expression $ns + mr$ is identically given by $\{\lambda l + tN; t \in \mathbf{Z}\}$. Putting all this together, we arrive at nearly the final form for the partition function:

$$Z_{\text{circle}} = \frac{1}{\eta\bar{\eta}} \sum_{l, t \in \mathbf{Z}} q^{(1/2N)l^2} \bar{q}^{(1/2N)(\lambda l + tN)^2}. \quad (4.22)$$

The final step may now be managed by noticing that the argument of the \bar{q} -exponential is invariant under shifting l by multiples of N . Using this fact, some simple manipulations and renaming of the variables finally lead us to the desirable form for the partition function of this theory:

$$Z_{\text{circle}} = \frac{1}{\eta\bar{\eta}} \sum_{k=0}^{N-1} \left(\sum_{n \in \mathbf{Z}} q^{(N/2)(n+k/N)^2} \right) \left(\sum_{m \in \mathbf{Z}} \bar{q}^{(N/2)(m+\lambda k/N)^2} \right) \quad (4.23)$$

which has the standard form (2.1) with left chiral characters χ_l now having the specific form:

$$\chi_k(q) = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} q^{(N/2)(n+k/N)^2} \quad (4.24)$$

and, defining $\bar{k} \equiv \lambda k$, with right chiral characters having exactly the same form but with q and k replaced by \bar{q} and \bar{k} .

As it is transparent from (4.23), the partition function Z_{circle} is in general an off-diagonal invariant in the space of characters χ_k , as the integer λ defined above in most cases is different from unity. In fact it is apparent from its definition that $\lambda = 1$ if either r or s is equal to unity. Some comments about the quantity λ and the partition function are in order at this point, however. Firstly, one should note that since, as mentioned in the previous paragraph, the integers u and v are not unique, then λ itself is not uniquely defined. In fact it should be clear from the definition of λ that this quantity is only defined modulo additions of integral multiples¹¹ of N . But the characters $\bar{\chi}_{\bar{k}}(\bar{q})$ remain well-defined as their definition only depends on the equivalence class $[\lambda]$ modulo N . Secondly, it follows from the definition of integers λ , u and v that

$$\lambda^2 \equiv 1 \pmod{2N}, \quad (4.25)$$

and in particular that λ and N are relatively prime integers. In fact this can be reversed, i.e., given an even integer N and an integer λ defined modulo N , satisfying (4.25), one can find r and s . Therefore we can make a change of variables from (r, s) to (N, λ) .

The matrix elements of the transformation matrices S and T in the space of characters are simple to obtain by making use of the Poisson resummation formula

¹⁰ It should be noted that any pair of integers $(u + mr, v + ms)$ for $m \in \mathbf{Z}$ may be used instead of the given pair (u, v) satisfying their defining relation: $us - vr = 1$

¹¹ See previous footnote

and the inspection of (4.24) and are given by:

$$(S_m^k) = \frac{1}{\sqrt{N}}(e^{2\pi i(k.m)/N}) \quad \text{and} \quad (T_m^k) = (e^{2\pi i((k^2/2N) - (1/24))} \delta_{km}). \tag{4.26}$$

Through an application of (4.10) this information may be used to arrive at the asymptotic series for the character coefficients $a_n^{(k)}$ for all the N chiral characters $k = 0, 1, \dots, N - 1$ of this theory. A rather lengthy computation gives the final result as we presently describe. Firstly the contribution from the leading value $l = 1$ ($p = 0$) in the summation (4.8) is given by:

$$a_n^{(k)} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \cos\left(\frac{2\pi \tilde{k} \cdot \tilde{m}}{N}\right) \frac{d}{dn} J_0(4\pi i \sqrt{\left(\frac{1}{24} - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}), \tag{4.27}$$

where $n_j = n - 1/24 + j^2/2N$, as defined previously and where, for the reason of simplifying the form of this expression, we have defined the hatted symbols as: $\tilde{j} = j$ if $j \leq N/2$ and $\tilde{j} = N - j$ if $j \geq N/2$. The dimensions h_j of the characters are thus given by $h_j = \tilde{j}^2/2N$. The contribution to the sum coming from $l = 2$ ($p = 1$) term in the summation can also be shown to be of the form:

$$\frac{1}{\sqrt{N}} C_1 \sum_{m=0}^{N-1} \cos\left(n\pi + \frac{\pi \tilde{k} \cdot \tilde{m}}{N}\right) \frac{d}{dn} J_0\left(2\pi i \sqrt{\left(\frac{1}{24} - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right), \tag{4.28}$$

and looking at one more value of l , we find this contribution from the two terms $p = 1, 2$ of the sum associated with the value $l = 3$:

$$\frac{1}{\sqrt{N}} C_2 \sum_{m=0}^{N-1} \cos\left(\frac{2n\pi}{3} + \frac{2\pi k \cdot \tilde{m}}{3N} + \frac{\pi}{36}\right) \frac{d}{dn} J_0\left(\frac{4\pi i}{3} \sqrt{\left(\frac{1}{24} - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right). \tag{4.29}$$

In the last two equations C_1 and C_2 denote constants which crucially depend on the equivalence classes of integers $k + m$ and $N/2$ in additions modulo 2 and 3 respectively. For instance it may be verified that C_1 is of the form:

$$C_1 = \sqrt{2} \quad \text{if} \quad \frac{N}{2} \in 2\mathbf{Z}, k + m \in 2\mathbf{Z} \quad \text{or} \quad \frac{N}{2} \in 2\mathbf{Z} + 1, \quad k + m \in 2\mathbf{Z} + 1,$$

$$C_1 = 0 \quad \text{if} \quad \frac{N}{2} \in 2\mathbf{Z}, k + m \in 2\mathbf{Z} + 1 \quad \text{or} \quad \frac{N}{2} \in 2\mathbf{Z} + 1, \quad k + m \in 2\mathbf{Z}.$$

Similar arithmetic structure is observed for the constant C_2 . In fact the same is true for all other terms of the asymptotic series. In particular, when finding the terms coming from the singularity at the rational point $e^{2\pi i p/l}$, the equivalence class of $N/2$ modulo l will be relevant in defining the coefficients of the asymptotic sum.

5. Extension to String Theory

It is a relatively simple task now to extend the analysis of Sect. 2 and 3 to the case of strings spatially compactified on rational backgrounds. In applications to

string theory, we have to decide how many uncompactified directions D we wish to have. The obvious phenomenological interest is for $D = 4$. However, for strings in the early universe, one might be interested in the case where all spatial directions are compact [2] and so one is dealing with $D = 1$. The main novelty for the partition function of particles in string theory, compared to that of a conformal theory, is that the modular weight of the partition function is not generally zero, and it depends on the number of uncompactified directions. The modular weight is $-(D-2)/2$. This is because, in the light cone gauge we have $D-2$ bosonic oscillators in addition to the other degrees of freedom coming from the internal conformal theory, each contributing a factor of $-1/2$.

If we assume the internal conformal theory is rational, with chiral characters χ_i , the string partition function will have the minor modification (for bosonic strings) of:

$$\chi_i \rightarrow \eta^{(-D+2)} \chi_i,$$

where η is Dedekind's function. Thus the following modification to transformation rules (2.4) for the characters is implied by this change

$$T: \chi_i \rightarrow \hat{\alpha}_i \chi_i \quad \text{and} \quad S: \chi_i \rightarrow \tau^{(-D+2)/2} \hat{S}_i^j \chi_j, \quad (5.1)$$

where we have defined $\hat{\alpha}_i = e^{2\pi i(-D+2)/24} \alpha_i$ and $\hat{S}_i^j = i^{(D-2)/2} S_i^j$, where α_i and S_i^j correspond to the characters χ_i . Equivalently, the transformation rule under the action of a general modular transformation $M: \tau \rightarrow (a\tau + b)/(c\tau + d)$ is given by:

$$M: \chi_i(q(\tau)) \rightarrow (c\tau + d)^{(-D+2)/2} \hat{M}_i^j \chi_j(q(\tau)), \quad (5.2)$$

where the transformation matrix \hat{M} is now described in terms of the hatted quantities. The goal of this section is to obtain similar asymptotic expansions for the coefficients $a_n^{(i)}$ as we did in the case of RCFT. In this case the n will have the physical interpretation of mass: $m^2 = n - 1 + h_i$. Our basic approach in this problem is precisely the same as that which we have used throughout this paper. Thus an immediate generalization of (4.6) leads to:

$$a_n^{(i)} = \frac{1}{2\pi i} \sum_{p,l} \int_{\xi_{p,l}} (-p + l\tau)^{(D-2)/2} \frac{(\hat{M}^{p,l})_i^j \chi_j(q')}{q^{n_i+1}} dq. \quad (5.3)$$

It is easy to convert the integral on the right-hand side of this equation to a form similar to (4.8):

$$a_n^{(i)} = -i \sum_{p,l} \tilde{\mathfrak{G}}_{ij}^{p,l} e^{-2\pi i n p/l} \int_{\xi_{p,l}} (l\omega)^{(D-2)/2} \exp \left[\frac{2\pi(1-h_j)}{l^2 \omega} + 2\pi \omega n_i \right] d\omega + R_n^{(i)}, \quad (5.4)$$

where $\tilde{\mathfrak{G}}_{ij}^{p,l}$ is given by: $\tilde{\mathfrak{G}}_{ij}^{p,l} = (i)^{(D-2)/2} \hat{\mathfrak{G}}_{ij}^{p,l}$, with $\hat{\mathfrak{G}}_{ij}^{p,l}$ given by the same expression as (4.9) with \hat{M} in place of M , and where the remainder term $R_n^{(i)}$ has exactly the same form as the first term with $(1-h_j)$ in the argument of the exponential now replaced by $(1-h_j-m)$ and with a further summation over j . The details of the computations of these integrals are similar to the previously encountered ones and are left to Appendix A. Here it suffices simply to give the final result. Hence for

the general case of D uncompactified dimensions we find:

$$a_n^{(i)} = \sum_{l,p} \tilde{g}_{ij}^{p,l} e^{-2\pi i n p/l} \frac{2\pi}{l} i^{D/2} \left(\frac{1-h_j}{n_i}\right)^{D/4} J_{-D/2} \left(\frac{4\pi i}{l} \sqrt{(1-h_j)n_i}\right) + O(n^{-(D-2)/4}) \tag{5.5}$$

(this form of the error is only valid for $D > 1$) where $J_{-D/2}$ is the Bessel function of the first kind of order $-D/2$. For $D = 1$, the case where all spatial dimensions are compactified, we obtain the following simple result:

$$a_n^{(i)} = \sum_{j=0}^{N-1} \sum_{l,p} (\tilde{g}_{ij}^{p,l} e^{-2\pi i n p/l}) \frac{e^{(4\pi i/l)\sqrt{(1-h_j)n_i}}}{\sqrt{2ln_i}} + O(n^{1/2}). \tag{5.6}$$

The case $D = 2$ corresponds to the vanishing modular weight which we thoroughly studied in Sect. 4 with the results recorded in Eq. (4.10).

The error in these sums are always $O(n^{d/2})$ for $d \leq 0$ and $O(n^d)$ for $d > 0$, where d is the modular weight of the characters, as discussed in Appendix A. It is interesting to note that except for $D = 1, 2$, the correction terms are suppressed for large n , and therefore, the asymptotic result can be used to obtain the exact result (again by using the integrality of the degeneracies). For $D = 2$, the error is an n -independent constant, just as was the case for rational conformal theories. For $D = 1$ the error could be as large as $n^{1/2}$. Though, it should be pointed out that we are not sure if the error is actually this large or not in this case. It might be, for reasons unknown to us, that the sum is a better approximation to the exact result as it is for $D > 2$. At any rate even for $D = 1$ the error is really small in comparison to all the exponentially rising terms in the sum ($\sim e^{\beta\sqrt{n}}$).

5.1. Example: Bosonic Strings Compactified on $(\Gamma_8)^3 \times (\bar{\Gamma}_8)^3 \times S^1$. An explicit example of the ideas in the previous section is furnished by the bosonic strings with 24 of the spatial dimensions compactified on three copies of the E_8 root lattice Γ_8 and the remaining spatial dimension compactified on a circle S^1 whose square radius is rational: $R^2 = r/2s$ in the conventions we adopted earlier. The choice of the lattice Γ_8 made here is entirely based on the particularly simple form and behaviour of its related partition function under modular transformations, thus allowing, without introducing any complications, the study of the dependence on the rational parameters r and s . The partition function is given by

$$Z = \sum_{k=0}^{N-1} \psi_k(q) \bar{\psi}_{\bar{k}}(\bar{q}),$$

where

$$\psi_k(q) \equiv \left[\frac{1}{2}(\mathcal{G}_2^8 + \mathcal{G}_3^8 + \mathcal{G}_4^8)\right]^3 \eta^{-24}(q) (\eta\chi_k)(q) \tag{5.7}$$

with $\chi_k(q)$ given as in (4.24) and with obvious modifications $\bar{\psi}_{\bar{k}}(\bar{q})$ is thus defined. It is by now a simple exercise to write down the asymptotic expansions for the coefficients $a_n^{(k)}$ of q^n in the series expansions $\psi_k(q) = q^{-1+\hat{k}^2/2N} \sum_{n=0}^{\infty} a_n^{(k)} q^n$ for the

characters of this theory which evidently possess weight $+1/2$ under arbitrary modular transformations. Due to the simple modular transformations of the (E_8) lattice partition function, one finds that the singular behaviour of this in the neighborhood of the rational point $q = e^{2\pi i p/l}$ is of the simple form:

$$\frac{1}{2}(\mathcal{G}_2^8 + \mathcal{G}_3^8 + \mathcal{G}_4^8) \sim \frac{1}{z^4} \quad (5.8)$$

in which phases have entirely disappeared and the coefficient is exactly unity.

Now we may take any of the two equivalent approaches to complete the problem. Either we combine the leading structures of the constituent partition functions near each rational point, noting that each constituent has been studied at some preceding point, or we can use the general algorithms laid out in this section. We shall do the latter thus obtaining the formula:

$$\psi_k(q) \sim (i)^{-1/2} [(\hat{M}^{p,l})^{-1}]^m e^{-2\pi i(1-h_m)p'/l} \frac{1}{\sqrt{z}} \exp\left[\frac{2\pi}{lz}(1-h_m)\right], \quad (5.9)$$

and then resulting series:

$$a_n^{(k)} = \sum_{m=0}^{N-1} \sum_{p,l} \tilde{\mathcal{G}}_{km}^{p,l} e^{-2\pi i n p/l} \frac{\exp\left(\frac{4\pi}{l} \sqrt{\left(1 - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right)}{\sqrt{2l n_{\tilde{k}}}} + O(n^{1/2}), \quad (5.10)$$

where $\tilde{\mathcal{G}}_{km}^{p,l}$ is once again defined by $\tilde{\mathcal{G}}_{km}^{p,l} = (i)^{-1/2} \hat{\mathcal{G}}_{km}^{p,l}$. The leading terms of this expansion are readily shown to be as follows:

$$a_n^{(k)} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \cos\left(\frac{2\pi \tilde{k} \cdot \tilde{m}}{N}\right) \frac{\exp\left(4\pi \sqrt{\left(1 - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right)}{\sqrt{2n_{\tilde{k}}}}, \quad (5.11)$$

and the first non-leading correction is obtained to be:

$$\frac{1}{\sqrt{N}} C_1 \sum_{m=0}^{N-1} \cos\left(n\pi + \frac{\pi \tilde{k} \cdot \tilde{m}}{N}\right) \frac{\exp\left(2\pi \sqrt{\left(1 - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right)}{\sqrt{4n_{\tilde{k}}}}, \quad (5.12)$$

where the constant C_1 in this expression is the same as defined in the paragraph following Eq. (4.29). The similarity between the form of this result and the results (4.28)–(4.29) found earlier in case of circular compactification is of course by no means accidental. This is in fact the result of our judicious choice for the lattice on which the 24 remaining spatial dimensions have been compactified. The interested reader is invited to compute the next few terms of this series, but it will not be really necessary for our purposes to do this here.

5.2. Superstrings Compactified on $(\Gamma_8) \times (\bar{\Gamma}_8) \times S^1$. For the sake of completeness it is appropriate to consider the fermionic counterpart of the preceding example. Thus let us briefly look at the type II superstrings, eight of whose left and right

spatial coordinates are compactified on Γ_8 and $\bar{\Gamma}_8$ respectively and the remaining spatial dimension on a circle S^1 of rational squared radius as before. The partition function of this theory is thus similarly given by $Z = \sum_k \psi_k(q)\bar{\psi}_k(\bar{q})$, where

$$\psi_k(q) = 16[\frac{1}{2}(\mathcal{G}_2^8 + \mathcal{G}_3^8 + \mathcal{G}_4^8)]F^8(q)(\eta\chi_k)(q) \tag{5.13}$$

with obvious definition for $\bar{\psi}_k(\bar{q})$ and with the superstring partition function $F(q)$ fully discussed in Sect. 3. Writing the power series expansion: $\psi_k(q) = q^{-1/2 + \tilde{k}^2/2N} \sum_{n=0}^{\infty} a_n^{(k)} q^n$ for the characters, it is a simple matter then to arrive at the exponential behavior in the neighborhood of the rational singular points:

$$\psi_k(q) \sim 16(i)^{-1/2} [(\hat{M}^{p,q})^{-1}]_k^m e^{-2\pi i(1/2 - h_m)p'/l} \frac{1}{\sqrt{z}} \exp\left[\frac{2\pi}{lz}(1/2 - h_m)\right] \tag{5.14}$$

and the associated asymptotic series for the coefficients:

$$a_n^{(k)} = 16 \sum_{m=0}^{N-1} \sum_{p,l} \tilde{g}_{km}^{p,l} e^{-2\pi i n p/l} \frac{\exp\left(\frac{4\pi}{l} \sqrt{\left(\frac{1}{2} - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right)}{\sqrt{2ln_{\tilde{k}}}} + O(n^{1/2}) \tag{5.15}$$

which is, as expected, entirely similar to its bosonic counterpart. Thus one finds the first couple of terms in this expansion to be, first:

$$a_n^{(k)} = \frac{16}{\sqrt{N}} \sum_{m=0}^{N-1} \cos\left(\frac{2\pi\tilde{k}\cdot\tilde{m}}{N}\right) \frac{\exp\left(4\pi \sqrt{\left(\frac{1}{2} - \frac{\tilde{m}^2}{2N}\right) n_{\tilde{k}}}\right)}{\sqrt{2n_{\tilde{k}}}}, \tag{5.16}$$

and next given by:

$$\frac{1}{\sqrt{N}} C_2 \sum_{m=0}^{N-1} \cos\left(\frac{2n\pi}{3} + \frac{2\pi\tilde{k}\cdot\tilde{m}}{3N} + \frac{5\pi}{12}\right) \frac{\exp\left(\frac{4\pi}{3} \sqrt{\left(\frac{1}{2} - \frac{\tilde{k}^2}{2N}\right) n_{\tilde{k}}}\right)}{\sqrt{6n_{\tilde{k}}}}, \tag{5.17}$$

where the n -independent coefficient C_2 is, except for an extra factor of 16, exactly as appeared in Sect. 3.1 and possesses characteristics similar to C_1 which we briefly described there.

6. Extension to Irrational Theories

So far our analysis is applicable only to RCFT's. It is obviously important to generalize this to arbitrary conformal theories. In this section we will concentrate on arbitrary conformal theories which could be used for constructing bosonic string vacua. In irrational theories, the notion of chiral characters are not very useful, as there are infinitely many characters, and hence our previous analysis based on the method of HR breaks down. In such cases the more natural thing to study is the full partition function $Z(\tau, \bar{\tau})$ which is a modular invariant form.

Even though the masses again appear in a discrete spectrum, it is no longer true, in these cases, that they appear at given integral values (shifted by a finite number of fixed values). Therefore, the question to be asking in this case should be somewhat different from the type we asked before: Let $d(m)$ be the number of *physical* states at mass m (i.e. number of states in the Hilbert space with $L_0 = \bar{L}_0 = m^2 + c/24$). Then $d(m)$ (or the density of physical states) as a function of m is not a very smooth function, and since we have no rationality to organize the states in subsectors to ask questions which depend smoothly on m , we have to look for another function to probe the large mass degeneracies of strings. In view of this we consider the function $Z(\beta)$ defined as

$$Z(\beta) = \sum_m d(m) e^{-\beta m^2}. \quad (6.1)$$

Note that this is a well defined function for all positive values of β , because $d(m) \sim \exp(\alpha m)$ from our previous analysis. For small β this function probes the large mass degeneracies of the conformal theory. Moreover, Z defined above is the physical (i.e., restricted to physical Hilbert space) partition function for the conformal theory in question. However, even for the case of rational theories, as long as we are interested in the degeneracies of physical states, the methods used in the previous sections have a shortcoming in determining the density function $d(m)$: In RCFT's to obtain the degeneracy of particles, we will have to put both the left- and the right-movers and require that the left energy be equal to right energy. So the number of particles coming from the i th character at mass squared $m^2 = -(c/24) + h_i + n = -(c/24) + \bar{h}_i + \bar{n}$ is given by

$$d(n, i) = a_n^{(i)} a_{\bar{n}}^{(i)},$$

where $\bar{n} \equiv n + h_i - \bar{h}_i$. For large enough n we have shown in the previous sections that we can determine $a_n^{(i)}$ up to power law corrections. But the error in $d(n, i)$ will unfortunately be exponential in n because the error for the degeneracy of the left-movers will multiply the exponentially rising right-mover degeneracy. For strings on rational backgrounds with the number of uncompactified space-time dimensions bigger than two, this multiplication of errors could be circumvented by choosing n large enough and taking the closest integer to the asymptotic estimation of the left-moving (and similarly right-moving) degeneracy, which gives rise to zero error as discussed in the previous sections and Appendix A. However, it would be very difficult to give an analytic expression for $d(n, i)$ even for such cases, because taking the integral part is not easy to accomplish analytically. In particular if we try to evaluate $Z(\beta)$ for such a theory, we will see that as $\beta \rightarrow 0$ we are unable to compute it with a good accuracy (error is of the form $\sim \exp(c/\beta)$). We are thus faced to compute $Z(\beta)$ not only for the irrational theories, but also for the rational ones.

We first note that $Z(\beta)$, being the physical partition function, has an integral representation as:

$$Z(\beta) = \int_0^1 Z(\tau, \bar{\tau}) d\tau_1 = \int_0^1 Z(q, \bar{q}) d\phi, \quad (6.2)$$

where in the first integral in the τ -plane, the value of τ_2 is fixed at $\beta/4\pi$, and in the second integral in the q - plane, the integration is performed over the contour Γ_β of radius $e^{-\beta/2}$. This contour is parametrized by the angular variable ϕ defined in terms of q by the relation $q = \exp(-\beta/2 + 2\pi i\phi)$. We are interested in evaluating (6.2) in the limit as $\beta \rightarrow 0^+$. The evaluation of $Z(\beta)$ now proceeds by applying to the last expression some of the techniques we developed in Sect. 3 in dealing with similar integrals in the complex q -plane. We thus introduce a dissection $\xi_{p,l}$ of the circular contour Γ_β in precisely the same manner as before and use the modular transformation properties of the integrand to rewrite $Z(\beta)$ in a form more apt for computation. Let us take the transformation rule for $Z(q, \bar{q})$ under arbitrary modular transformation (2.2) to be of the general form:

$$Z(q, \bar{q}) \rightarrow |(c\tau + d)|^{2d} Z(q, \bar{q}). \tag{6.3}$$

As before, this equation allows us to isolate the behavior of this function near its rational singularities on the unit circle. There are some rather important differences, however. Firstly, there are (essentially) no phases in the transformation properties to worry about, making our results somewhat simpler than before. Second and more importantly, we no longer have analytic (anti-analytic) expressions in the present case, thus the use of Cauchy integral theorem is not possible here. Notwithstanding these differences, our basic strategy remains unchanged. Specifically, in the limit $\beta \rightarrow 0^+$ ($|q| \rightarrow 1^-$), as we approach any given rational point $q = e^{2\pi i p/l}$ on the unit circle, the partition function behaves as:

$$Z(q, \bar{q}) \sim \sum_{(h, \bar{h})} e^{2\pi i(h - \bar{h})p'/l} (z\bar{z})^{-d} \exp \left[\frac{2\pi}{lz} (c/24 - h) + \frac{2\pi}{l\bar{z}} (c/24 - \bar{h}) \right] + \text{non - singular terms.} \tag{6.4}$$

Here $q = \exp [2\pi i(p/l + iz/l)]$ as before, and the summation is over all pair (h, \bar{h}) of the left and right conformal dimensions with the restriction that:

$$h + \bar{h} < \frac{c}{12}. \tag{6.5}$$

In the terminology of the Renormalization group flows, the states satisfying this condition correspond to relevant operators, i.e., operators which if used to perturb the action will lead to flows *away* from the conformal theory. The condition (6.5) ensures that the contribution of any such state in the vicinity of each singularity of the function $Z(q, \bar{q})$ will be of an exponentially singular form. We may further distinguish those states for which the relations $c/24 - h > 0$ and $c/24 - \bar{h} > 0$ simultaneously hold which in the case of interest in string theory it implies that $h = \bar{h}$. These states have zero two-dimensional spin (the spin being $h - \bar{h}$). Henceforth, we refer to these contributions to $Z(\beta)$ as “tachyonic” contributions because their contribution to m^2 is negative. The contributions coming from the remaining states for which only one of the latter conditions holds will also be referred to as “unphysical-tachyonic” contributions, because such states carry non-zero two dimensional spin (± 1 for the case of interest in string theory) and are projected out of the physical Hilbert space (because $h \neq \bar{h}$). It is ironic that we

are finding unphysical-tachyonic states relevant for capturing the spectrum of physical states at large masses. Moreover, since we are dealing with non-degenerate conformal theories (though not necessarily rational ones) in this analysis, there are only a finite number of tachyonic and unphysical-tachyonic pairs entering the summation. For instance, for the case studied earlier of closed bosonic strings spatially compactified on three copies of E_8 root lattice and a circle of radius R , having any positive real value, the only pairs satisfying this restriction ($h + \bar{h} < 2$) are those for which the momentum and winding quantum numbers n and m (assuming $R > 1$) are given by $n = 0, 1, \dots, [2\sqrt{2}R]$ and $m = 0$ or $n = 0$ and $m = \pm 1$ (for when $R < \sqrt{2}$), or finally by $n = \pm m = \pm 1$. The bracket here indicates the largest integral portion. The first and second cases correspond to the tachyonic pairs of conformal dimensions (h, \bar{h}) for which $h = \bar{h}$ and the latter to the unphysical-tachyonic pairs (h, \bar{h}) for which $h - \bar{h} = \pm 1$. The tachyonic pairs always exist, but the unphysical-tachyonic pairs can be shown to exist only for those radii for which R falls in the range $((1 - (\sqrt{3}/2))^{1/2}, (1 + (\sqrt{3}/2))^{1/2})$.

We may substitute for z in (6.4) its expression in terms of β and the local angular variable $\theta: z = l(\beta/4\pi - i\theta)$. Furthermore it is convenient to work with a new variable ρ in place of β , defined in terms of the latter by $\rho = \beta/4\pi$. The expression $Z(\beta)$ may then be readily rewritten as:

$$Z(\beta) = \sum_{\substack{(h, \bar{h}) \\ p, l}} e^{2\pi i(h - \bar{h})p/l} l^{-2d} \int_{\xi_{p, l}} (\rho^2 + \theta^2)^{-d} \exp \left[2\pi \frac{(c/12 - h - \bar{h})\rho - i(h - \bar{h})\theta}{l^2(\rho^2 + \theta^2)} \right] d\theta$$

+ correction terms, (6.6)

in which the sum is over all tachyonic and unphysical-tachyonic pairs (h, \bar{h}) . Here the corrections emerge because we have not included the non-singular non-tachyonic contributions to the integrand. It is a simple exercise to show, as has been done in Appendix C, that for the general case of modular weight d the corrections are of order $O(\beta^{-d})$. Our most important task, therefore, is to compute the integral expression in (6.6) with the largest possible accuracy, and it will be most desirable to be able to carry out this task with a degree of accuracy comparable to the correction terms we have just discussed, i.e. with power law corrections in parameter β . It is most remarkable that this in fact turns out to be possible for any value of the modular weight d . The details of this assertion are somewhat lengthy and have been delegated to Appendix C. There we will show that the tachyonic contributions can be computed in a closed form for arbitrary d involving gamma functions and hypergeometric functions. For the contribution of unphysical tachyonic states we have not been able to obtain a closed form expression for arbitrary d (though we give a closed form of it in Appendix C for the case $d = 1$). Below we shall describe some of the main features of this computation. Firstly, it is shown in Appendix C that the optimum choice for the range of the second summation in (6.6) corresponds to the Farey's series of order $1/\sqrt{\rho}$. Moreover, having included this many terms included in the sum, the final error of our forthcoming estimation for $Z(\beta)$ will be of order $O(\rho^{-d})$. Secondly, with the above choice of the Farey's series, the range of angles θ (in absolute value) belonging to

a given arc segment $\xi_{p,l}$ is, as we recall, bounded by $\sqrt{\rho/2l} < \theta < \sqrt{\rho}/l$. Hence, if we introduce a new variable $u = \theta/\rho$, for any fixed l , in the limit of vanishing ρ the range of the latter variable covers the entire set of real numbers. This suggests that we should evaluate the integrals with this new range of integration and, at the same time, estimate the amount of error thereby introduced for non-zero values of ρ . For $d \geq 1$ no problems arise. For $d < 1$, however, the extended integral is ill-defined. However, this could be resolved by analytic continuation. For example, in the case of the contribution of physical tachyons this integral has explicit expression in terms of hypergeometric functions, having simple poles (see Appendix C) for negative half-integral values of d and having otherwise well-defined analytic extensions for negative integral values of this parameter. We have seen that in string theory both half-integral and integral values of modular weight with $d < 1$ may arise. For negative integer values of d , the resulting expression is then precisely the desired result, up to the required power law error. For negative half-integer values of d , however, it is easy to show that the resulting expression after the truncation of the poles is in fact the desired result. The error introduced in the process is then readily proved to be of the stated power law form.

Thirdly, in the limit as $\beta \rightarrow 0$, the partition function $Z(\beta)$ has an exponentially singular dependence on β of the general form $Z(\beta) \sim e^{c/\beta}$. To be more specific, it is adequate to examine the leading behavior of the integral term in (6.6) in this limit. Let us denote this integral by $I_{p,l}^{(h,\bar{h})}$. Then this integral has a universal leading behavior for all values of modular weight d :

$$I_{p,l}^{(h,\bar{h})} = \frac{1}{2} e^{2\pi i(h-\bar{h})p'/l} [(c/24 - h)(c/24 - \bar{h})]^{-1/4} l^{-2d+1} (\beta/4\pi)^{-2d+3/2} \cdot \left[\exp \left\{ \frac{4\pi^2}{l^2\beta} \left(\frac{c}{12} - h - \bar{h} \right) + \frac{8\pi^2}{l^2\beta} \sqrt{(c/24 - h)(c/24 - \bar{h})} \right\} + c.c \right] + \text{non-leading} \tag{6.7}$$

which we have re-expressed in terms of β . It is therefore observed that the tachyonic contributions are purely of exponential growth form. In contrast, the unphysical-tachyonic contributions involve an oscillatory part which changes very rapidly in the intended limit of small β .

Lastly, the phases which enter Eq. (6.6) for $Z(\beta)$ can be collected, thus giving rise to a very simple and natural expression. To be precise, since in the second summation p can be replaced by p' and since no other p' (or p) dependence occurs throughout, except in the phases, we can perform this sum independent of other summations. Noting that $h - \bar{h} \in \mathbb{Z}$, this sum is precisely given in terms of the well-known Ramanujan function [8]: $C_n(m) = \sum_{(h,n)=1} \exp(2\pi i m h/n)$. Hence we have:

$$\sum_{(p',l)=1} \exp [2\pi i(h - \bar{h})p'/l] = C_l(h - \bar{h}). \tag{6.8}$$

The tachyonic case $h = \bar{h}$ is particularly simple. The Ramanujan function in this case simply degenerates to the sum $\sum_{(p',l)=1} 1$. This is the well-known Euler's function

$\phi(l)$, counting the total number of positive integers no larger than l and relatively prime to it. An exact expression for $I_{p,l}^{(h,h)}$ can also be given in terms of the confluent hypergeometric functions for all d values. This result may be found in Appendix C.

7. Rationality Versus Irrationality

Consider a conformal theory which can be deformed, by changing some of its defining parameters, to obtain some new conformal theories. These parameters are known as the moduli of the conformal theory. For example, in toroidal compactification choices for radii of the torus will be part of the moduli for string propagation on that space. The moduli space of a conformal theory has a natural metric which comes directly from the inner product structure of the Hilbert space [11]. For example, if we consider circle compactification, the moduli is just the choice of the radius of the circle R , and the metric in this case is simply $(dR/R)^2$. This metric is useful in describing low energy phenomena, for example for discussion of adiabatic variations of R with respect to uncompactified coordinates. Moduli fields correspond to massless scalars as viewed from the uncompactified dimensions, whose dynamics is described by a Lagrangian which to lowest adiabatic approximation is obtained from the metric. For instance the Lagrangian for the radius field R of a circle compactification is $(\nabla R/R)^2$. This description of the moduli space suggests that two different circle compactifications, with radii R_1 and R_2 , are easily deformable to one another. The closer the values R_1 and R_2 are to one another as real numbers, the easier is such a deformation (the lower is the action for such a process). However, this description is applicable only in the adiabatic approximation; i.e., the low energy domain. How can one think about moduli in the high energy domain? Could it be that the smoothness of physical quantities with respect to moduli is violated in this limit? We have been looking at one such physical quantity in this paper and that is the asymptotic degeneracy of states. So let us explore the smoothness of that with respect to moduli.

It should be noted that a priori it is not clear if the asymptotic density of states should vary smoothly with moduli. As we change the moduli of a conformal theory, the masses of particles change continuously. For example, in circle compactification, the winding mode energy $w = mR$ changes continuously with R . However, these changes take place in a non-uniform way with respect to energy. In other words, if we wish that the masses of string states change by no more than ϵ , no matter how small we vary R , this cannot be accomplished (for example the winding number m can be as large as we wish, and can magnify the variation drastically). Therefore it is conceivable that the asymptotic behaviour of density of states might not depend smoothly on R .

For the sake of argument we will concentrate on circle compactifications (together with three copies of E_8 lattice) discussed before, though we believe our qualitative conclusions hold quite generally. Undoubtedly the most striking feature of the series representations for the asymptotic degeneracy of states found in our circle example in Sect. 4 is the manner in which the dependence on the radius $R = \sqrt{r/2s}$ enters into the results. Any given term in any one such series is manifestly

an explicit function of one or both of the variables N and λ in place of the variable R . Notably the variables N and λ , by virtue of their definitions, have an entirely arithmetic dependence on r and s or, equivalently, on the ratio r/s viewed as a rational number. It is easy to argue, as we have done in Sect. 4, that the pairs (r, s) and (N, λ) are in correspondence, and that this correspondence is one-to-one if we should restrict the variable λ by $\lambda < N$ and λ satisfying (4.25). Geometrically, however, two arbitrarily close rational radii r_1/s_1 and r_2/s_2 (in the sense of real numbers) evidently do not give rise to close values of N or λ in any ordinary sense. Now if we consider the chiral characters, $\chi_i(q)$ for instance, we have already observed that their dependence (in view of the asymptotic series for $a_n^{(i)}$) on the compactification radius is strictly through N , and thus no explicit expression in terms of R exists for these characters. Speaking in more general terms, the chiral characters in a RCFT only reflect the rational structure of the moduli and not (say) their structure as a real number. Accordingly, it would be meaningless, for instance, to inquire as to how much the left- or right-mover characters independently vary as we make continuous changes in the moduli in such a conformal theory. It would therefore seem quite miraculous that physical quantities, such as the physical partition function $Z(\beta)$ of the previous section, should have an explicit dependence on R , thus exhibiting no sensitivity to the rationality of the moduli and reflecting only their essence as real numbers. The answer to this enigma lies, as one may expect, with modular invariance. It forces left- and right-moving character degeneracies to coalesce precisely in such a manner so as to regenerate the R dependence.

A simple illustration of this point is provided by the physical partition function $Z(\beta)$, studied in the previous section, specialized to the present case of circle compactification. In view of (6.1) we may write an approximate (smoothed out) form of this function as follows:

$$Z(\beta) = \sum_n \sum_{i=0}^{N-1} d(n^i, i) e^{-\beta n}, \quad (7.1)$$

where $n^i = n + [h_i] - 1$, with bracket denoting integer portion, and $d(m, i)$ is defined following Eq. (6.1). We have taken the slight liberty of ignoring the fractional portions of h_i in the exponent. By the same token we may replace $[h_i]$ by h_i . The final effect of all this is that the quantity n_i appearing in the asymptotic series (5.10) for the coefficients $a_n^{(i)}$ is now simply replaced by n . The smoothing thus simplifies the analysis but is seen to be of no fundamental consequence to its conclusions. Despite the fact mentioned earlier that this approach entails exponential errors, we would expect, nevertheless, to be able to obtain correct leading order behavior of $Z(\beta)$ from this approach. In fact this very observation would be adequate to our purpose.

Replacing for the coefficients $a_n^{(i)}$ and $a_n^{(j)}$ their series expression (5.10), there will be summations over two sets of pairs (p, l) and (\tilde{p}, \tilde{l}) to perform, that is aside from the summation over character labels i . The sum over different pairs can be seen to lead to rapidly oscillatory expressions (which may include exponential growth as well) in the limit $\beta \rightarrow 0$ and do not seem to be particularly relevant to our leading order investigation. There are other sources of oscillatory terms which

we shall shortly discuss. Our main concern for the present is, however, with those terms in the product for which $p = \tilde{p}$ and $l = \tilde{l}$. Here we can use the unitarity relation (B.4) to obtain:

$$Z(\beta) = \sum_{(h_i, \bar{h}_i)} \sum_{p, l; n} \frac{1}{2ln} \exp \left[\frac{4\pi}{l} \sqrt{n} \{ (1 - h_i)^{1/2} + (1 - \bar{h}_i)^{1/2} \} \right] e^{-\beta n} + \dots, \quad (7.2)$$

in which the first sum is performed over all pairs of conformal dimensions for which we have exponential growth. The remaining pairs would invariably lead to oscillatory terms which we are setting aside for the present discussion. Other expressions of this type also emerge when a left-moving (exponentially growing) term multiplies a general right-moving error term and vice versa. We have already pointed out in Sect. 6 that it is precisely the emergence of these terms that makes the accuracy of this approach in finding the physical state densities inadequate.

Having made these remarks, let us show that the pairs (h_i, \bar{h}_i) which are included in the first sum are precisely the tachyonic pairs of the previous section with equal left and right dimensions. We can show this starting directly from our earlier definitions of $h_i = (\tilde{r}^2/2N)$ and $\bar{h}_i = (\tilde{\lambda}i^2)/2N$, with the tilde denoting a number between zero and $N/2$ in equivalence modulo N . Since $h_i - \bar{h}_i \in \mathbb{Z}$, and since we require $1 - h_i > 0$ and $1 - \bar{h}_i > 0$ should both hold, it is clear that these two quantities must be equal. Thus we must have $\tilde{r} = \pm \tilde{\lambda}i$, or equivalently that:

$$(\lambda + 1)i = kN \quad \text{or} \quad (\lambda - 1)i = kN$$

for some integer k . However, it follows from the definition of λ that $\lambda + 1 = 2us$ and also $\lambda - 1 = 2vr$, with u and v defined in Sect. 4.2. Moreover, as a consequence of their definition, u and v are correspondingly prime with respect to r and s . It readily follows from this and the above equations that the only solutions are $i = mr$ or $i = ns$, for arbitrary integers m and n subject to the restriction that corresponding h_i and \bar{h}_i should be both less than 1. These solutions correspond respectively to conformal dimensions $h = \frac{1}{2}m^2R^2$ and $h = n^2/8R^2$. Together with the restrictions that these be less than 1, we arrive at the conditions we identified from an entirely different point of view of Sect. 6. Therefore the leading order dependence in RCFT's can also be recast as a function of R , with no particular dependence on the rationality of R^2 .

We complete our analysis by making a leading order estimation of the contributions to $Z(\beta)$ coming from such tachyonic pairs. Rather than doing this for a special case, it is instructive to look at the general case considered in Sect. 6 instead. We do this by replacing the sum over n , which we shall perform first, by an integral. Here we would generally require some positive cutoff at the lower tail of integration because of the fact that our series representations are correct only for large enough values of n . The exact value of this cutoff is, however, immaterial in the limit that β goes to zero. Thus making use of Eq. (5.6) (with $d = -(D - 2)/2$) we arrive at:

$$Z(\beta) \sim \sum_{h_i, p, l} l^{-2} (c/24 - h)^{-(d-1)} \int_{\Lambda}^{\infty} n^{d-1} J_{d-1}^2 \left(\frac{4\pi i}{l} \sqrt{(c/24 - h)n} \right) e^{-\beta n} dn. \quad (7.3)$$

Furthermore, to the leading order it suffices to replace the bessel function in the integrand by its asymptotic behavior (which up to some ν -dependent constant is given by $J_\nu(iz) \sim e^z/\sqrt{2\pi z}$). Changing the variable of integration to $x = \sqrt{n}$ and making a saddle point approximation of the integral, the end result is found to be:

$$Z(\beta) \sim \sum_h \sum_{p,l} \frac{1}{2\sqrt{1-h}} l^{-2d+1} (\beta/4\pi)^{-2d+3/2} \exp\left[\frac{16\pi^2}{l^2\beta}(1-h)\right]. \quad (7.4)$$

This result is (to this order) precisely the result found earlier given by Eq. (6.7). Thus we conclude that our two approaches are consistent, and rationality can live peacefully with irrationality.

8. Conclusions

In this section we present our conclusions and a few comments on the application and extensions of this work. We have seen that following the method of Hardy and Ramanujan and using modular invariance, it is possible to obtain an asymptotic expansion for degeneracy of high energy states in a rational conformal theory (or strings, with a rational conformal theory as its “internal” theory). This expansion depends only on the one-loop modular transformation matrix S , the dimensions h_i , ground state degeneracy $a_0^{(i)}$ of chiral characters, and the central charge c of the theory. For strings with more than two uncompactified space-time dimensions this expansion is *exact*, and the exact degeneracy of states for sufficiently large mass m can be obtained by taking a sum of $O(m^2)$ terms.

The asymptotic expansion we discussed has an arithmetic nature: Using the leading dependence of density of states on the radius of a circle compactification, we have shown that the arithmetic natural of the radius is relevant in this asymptotic expansion.

We also studied the asymptotic behavior of physical partition functions for arbitrary rational and irrational theories, and we found that it is completely fixed (up to a power law correction) by the tachyonic, and unphysical-tachyonic modes of the theory (in the bosonic string from the spectrum of states with $L_0 + \bar{L}_0 < 2$, i.e., the relevant operators). We also discussed how the rationality and continuity of spectrum with respect to moduli could coexist.

It therefore seems that the study of asymptotic degeneracy of states does not make a striking distinction between rational and irrational conformal theories. It would be interesting to look at some other physical questions and see if one can find a physical criteria distinguishing rational and irrational theories. In the absence of such a criteria the distinction between rational and irrational conformal theories should be viewed as a mathematically convenient, but physically inconsequential drawing line.

Appendix A

Our primary intention in this appendix is to present the relevant details of the computations as well as a rather thorough account of the errors for the asymptotic

series which we have encountered in this paper. We shall also briefly discuss the behaviour of characters at the irrational points on the circle of convergence. The most expeditious manner to proceed is by working with the general case of modular weight d , and then specializing to some of the cases which we have examined in various sections of the text. The analysis of errors will be also presented in a general setting, thus greatly simplifying our task by dealing with various cases all at once. In our analysis, we will follow the general method of [7] which is a simplification of the work of HR. The reference [7] deals only with the example of the η -function. Below we will generalize this to families of modular functions with arbitrary modular weights.

Consider an arbitrary character $\chi_i(q)$ of modular weight d . We start with the integral form of series expression for the coefficient $a_n^{(i)}$ of q^n in expansion of this character (see for instance (4.8)). Our analysis proceeds in two stages. In the first stage, we will make an accurate evaluation of the integral expressions due the singular tehyonic contribution to the partition function, so that we arrive at an explicit form for the asymptotic series. This, however, will entail introduction of small corrections, which we must also estimate at this stage. In the second stage to estimate the corrections $R_n^{(i)}$ due to ignoring non-singular contributions of massive modes, which we have seen invariably accompany the integral expressions in this method. This and the previous corrections together will constitute the total correction (which we like to see minimized) to our asymptotic series for $a_n^{(i)}$.

To carry out the first stage, we seek to evaluate the integral:

$$I_{p,l}^{(i)} = \int_{\xi_{p,l}} \omega^{-d} \exp \left[\frac{2\pi(c/24 - h_j)}{l^2 \omega} + 2\pi\omega n_i \right] d\omega \quad (\text{A.1})$$

(we have dropped unnecessary indices) for which the range of integration $\xi_{p,l}$ has been described in the paragraph following Eq. (3.14). The integrand is observed to be an analytic function everywhere on the punctured complex ω -plane with the origin $\omega = 0$ removed, with the possible exception (for half integral d) of a cut along the negative real axis. The Cauchy integral theorem may now be used to compute this integral if we complete the path of integration $\xi_{p,l}$ by adjoining to it various line segments ξ_s ; $s = 1, 2, \dots, 6$ as shown in Fig. 2. Let $I_s^{(i)}$ denote (henceforth, the subscripts p and l are implicitly understood) the integral along the path segment ξ_s . Then an application of Cauchy's theorem immediately implies that the integration over the combined path of all the line segments considered thus far is equal with the integral $I_\Omega^{(i)}$ over the contour Ω , also shown in Fig. 2.

Therefore we may write: $I^{(i)} = I_\Omega^{(i)} - \sum_{s=1}^6 I_s^{(i)}$. In its common mathematical notation

$I_\Omega^{(i)}$ is denoted by $\int_{-\infty}^{(0^+)}$ and, as we will presently see, it may be exactly computed:

$$I_\Omega^{(i)} = \int_{-\infty}^{(0^+)} \omega^{-d} \exp \left[\frac{2\pi(c/24 - h_j)}{l^2 \omega} + 2\pi n_i \omega \right] d\omega. \quad (\text{A.2})$$

The simplest way to evaluate this integral is by taking advantage of the following

well-known loop integral formula for Bessel functions of the first kind [12]:

$$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \int_{-\infty}^{(0^+)} t^{-\nu-1} \exp(t - z^2/4t) dt \tag{A.3}$$

by means of which we find:

$$I_\Omega^{(i)} = -2\pi i^{-d} l^{d-1} \left(\frac{n_i}{c/24 - h_j} \right)^{(d-1)/2} J_{d-1} \left(\frac{4\pi i}{l} \sqrt{(c/24 - h_j)n_i} \right). \tag{A.4}$$

Various special cases of interest in this paper are now simple to derive by using some basic identities satisfied by the Bessel functions. In the case $d = -1/2$, for instance, we use the identity $J_{-3/2}(z) = \sqrt{2z/\pi} (d/dz)(\cos z/z)$ to reach the desired result. The result of the case $d = 0$, corresponding to RCFT's, is obtained by using $J_{-1}(z) = (d/dz)J_0(z)$, and in the context of strings, the case $d = 1/2$ of spatially compactified bosonic strings, for instance, makes use of the identity $J_{-1/2}(z) = \sqrt{2/\pi z} \cos z$.

It is possible to make an exact evaluation of the integrals along the path segments ξ_1 and ξ_6 . We examine $I_1^{(i)}$ for the general case of modular weight d and $I_6^{(i)}$ may be subsequently checked to be given by exactly the same result. This is given by (setting ε to zero without any loss):

$$I_1^{(i)} = - \int_0^\infty \omega^{-d} \exp \left[\frac{-2\pi}{l^2 \omega} (c/24 - h_j) - 2\pi n_i \omega \right] d\omega, \tag{A.5}$$

which can be computed with aid the following definite integral identity for $\beta, \gamma > 0$:

$$\int_0^\infty x^{\nu-1} \exp \left(-\frac{\beta}{x} - \gamma x \right) dx = 2(\beta/\gamma)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}), \tag{A.6}$$

where K_ν is the Bessel function of the imaginary argument of order ν .

We are thus led to the result:

$$I_1^{(i)} = -2 \left(\frac{c/24 - h_j}{l^2 n_i} \right)^{(1-d)/2} K_{1-d} \left(\frac{4\pi}{l} \sqrt{(c/24 - h_j)n_i} \right). \tag{A.7}$$

Let us now focus on making estimations of the remaining quantities. We want to estimate the following expressions for $s = 1, 2, \dots, 6$:

$$W_{s,n}^{(i)} = -i \sum_{p,l,j} \tilde{\mathfrak{P}}_{ij}^{p,l} e^{-2\pi i n p/l} l^{-d} \int_{\xi_s} \omega^{-d} \exp \left[\frac{2\pi}{l^2 \omega} (c/24 - h_j) + 2\pi n_i \omega \right] d\omega. \tag{A.8}$$

To do this, we must first specify the range of the sum over the integer variable l . Furthermore, we must rigorously characterize the path segments ξ_s along which the latter integrals are performed, and also the precise manner by which the contour Γ_ρ converges to the unit circle. To attain an optimal choice, we allow all values of l such that $l \leq \text{constant} \times n^\alpha$ in the summation. At the same time we allow the radius ρ of the circular contour of integration, of which $\xi_{p,l}$ are sections, to have the rather general form: $\rho = e^{-2\pi/n^\beta}$, for some positive real numbers α and β as yet to be determined. The constant in the above inequality will not be determined by

this analysis, as it turns out to be of no consequence to the order of magnitude estimations which we are about to make. Henceforth the value of this constant is assumed for convenience to be unity.

Let us try to establish the bounds on the expressions $W_{s,n}^{(i)}$, for fixed but arbitrary allowed values of s and n . We can confine our attention to those values of j for which $c/24 - h_j \geq 0$. As argued earlier, only these lead to exponential singularities. As for remaining ones, the magnitude of their total contribution to the asymptotic series for $a_n^{(i)}$ is readily seen to be of the same order as the remainder term $R_n^{(i)}$ which we will shortly study, so there is no need for their independent estimation. Focusing on $W_{2,n}^{(i)}$ first, our main task is to place a bound on the integral $I_2^{(i)}$ which appears in this expression. Here we have to distinguish the case of $d \leq 0$ from $d > 0$. Let us focus on the case $d \leq 0$ for the time being. We will need a certain inequality for the matrix elements of the coefficient matrix \mathcal{G} which we now derive. For simplicity we shall assume $a_0^{(i)} = 1$ (though it is a trivial matter to generalize this). Then the set of coefficients $\hat{\mathcal{G}}_{ij}^{p,l}$, viewed collectively as a matrix in the indices i and j , forms a unitary matrix for any fixed p and l , the proof of which has been provided in Appendix B. This fact is easily seen to lead, for arbitrary p and l , to the relation:

$$\sum_{j=0}^{N-1} |\hat{\mathcal{G}}_{ij}^{p,l}| \leq \sqrt{N}. \quad (\text{A.9})$$

Now going back to our estimation, and using this inequality, with reference to Fig. 2 we have (here we have dropped h_j from the exponent without any change in our results):

$$|I_2^{(i)}| < \sqrt{N} \int_0^{\theta_{p,l}''} (l^2 \varepsilon^2 + l^2 u^2)^{-d/2} \exp \left[\frac{2\pi}{l^2} (c/24) \operatorname{Re} \left(\frac{1}{-\varepsilon - iu} \right) - 2\pi n_i \varepsilon \right] du, \quad (\text{A.10})$$

which upon using the property $1/2 \ln^\alpha < \theta_{p,l}'' < 1/\ln^\alpha$ and the inequality

$$\operatorname{Re} \left(\frac{1}{-\varepsilon - iu} \right) = \frac{-\varepsilon}{\varepsilon^2 + u^2} < 0 \quad (\text{A.11})$$

would immediately yield (by sending ε to zero¹²):

$$|I_2^{(i)}| < \sqrt{N} \frac{1}{\ln^\alpha} \left(\frac{1}{n^{2\alpha}} \right)^{-d/2}. \quad (\text{A.12})$$

Inserting this result into the expression $W_{2,n}^{(i)}$ then results in:

$$|W_{2,n}^{(i)}| < \sqrt{N} \frac{1}{n^\alpha} \left(\frac{1}{n^{2\alpha}} \right)^{-d/2} \sum_{p,l} l^{-1}. \quad (\text{A.13})$$

The precise meaning of the sums of the kind seen here has been discussed in the

¹² A more rigorous treatment of this point would involve setting $\varepsilon = n^{-h}$, for some positive real number h . The final result for the case $d \leq 0$, however, does not depend on the choice of h so long as $h \geq \beta$. Hence we may let $\varepsilon \rightarrow 0$ at this earlier stage without any problem

text and in general the result of their rough estimation, being nonetheless sufficient for our purposes is given by (for $r \geq -2$):

$$\sum_{p,l} l^r = \sum_{l \leq O(n^\alpha)} l^{r+1} = O(n^{\alpha(r+2)}). \tag{A.14}$$

The final result is thus:

$$|W_{2,n}^{(i)}| < \text{constant} \times n^{\alpha d} \tag{A.15}$$

for some constant not depending on either n or i but generally depending on N . A quick inspection of $W_{3,n}^{(i)}$ should be enough to convince us that the same result, with only a minor difference coming from replacing $\phi''_{p,l}$ with $\phi'_{p,l}$ which is entirely without significance to our present discussion, holds for this expression. Hence we need not repeat that here.

Next we turn our attention to $W_{3,n}^{(i)}$ in which the integration $I_3^{(i)}$ is performed along the path segment ξ_3 as in Fig. 2. Proceeding along the same lines as above, for $d \leq 0$ we obtain that:

$$|I_3^{(i)}| < \sqrt{N} e^{2\pi n_i/n^\beta} \int_{-\varepsilon}^{1/n^\beta} (l^2 v^2 + l^2 \phi''_{p,l})^{-d/2} \exp \left[\frac{2\pi}{l^2} (c/24) \operatorname{Re} \left(\frac{1}{v - i\phi''_{p,l}} \right) \right] dv. \tag{A.16}$$

Based on the inequality:

$$\frac{1}{l^2} \operatorname{Re} \left(\frac{1}{v - i\phi''_{p,l}} \right) = \frac{v}{l^2 v^2 + l^2 \phi''_{p,l}} < \frac{1/n^\beta}{l^2 v^2 + l^2 \phi''_{p,l}} < 4n^{2\alpha - \beta} \tag{A.17}$$

which is correct in the limit $\varepsilon \rightarrow 0$, this leads to the bound:

$$|W_{3,n}^{(i)}| < \sqrt{N} e^{2\pi n_i/n^\beta} n^{2\alpha - \beta} \left(\frac{1}{n^{2\beta}} + \frac{1}{n^{2\alpha}} \right)^{-d/2} \exp [8\pi n^{2\alpha - \beta} (c/24)]. \tag{A.18}$$

Once again it is evident that same result applies, without any noticeable change, to $W_{4,n}^{(i)}$.

Having completed the first stage (for $d \leq 0$), we begin the second stage by considering the remainder integral $R_n^{(i)}$ given by:

$$R_n^{(i)} = -i \sum_{p,l,j} \tilde{\mathfrak{P}}_{ij}^{p,l} e^{-2\pi i n p/l} l^{-d} \sum_{m=1}^{\infty} a_m^{(i)} e^{2\pi i m p/l} \int_{\xi_{p,l}} \omega^{-d} \cdot \exp \left[\frac{2\pi}{l^2 \omega} (c/24 - h_j - m) + 2\pi n_i \omega \right]. \tag{A.19}$$

To estimate this expression we note that, for a fixed value of l , for every value of the local angular variable θ (defined in terms of the variable ω via the relation $\omega = 1/n^\beta - i\theta$) which belongs to the arc segment $\xi_{p,l}$, we have $1/2 \ln^\alpha < \theta < 1/\ln^\alpha$, a fact already pointed out in Sect. 3. Hence making use of this and other facts discussed above we have, for any arbitrary point on $\xi_{p,l}$, that (for $d \leq 0$):

$$|l^{-d} \omega^{-d}| = (l^2/n^{2\beta} + l^2\theta^2)^{-d/2} < \left(\frac{1}{n^{2(\beta-\alpha)}} + \frac{1}{n^{2\alpha}} \right)^{-d/2}. \tag{A.20}$$

Furthermore, since:

$$\left| \exp \left[\frac{2\pi}{l^2 \omega} (c/24 - h_j - m) + 2\pi n_i \omega \right] \right| = e^{2\pi n_i / n^\beta} \exp \left[\frac{2\pi}{l^2} (c/24 - h_j - m) \operatorname{Re} \left(\frac{1}{\omega} \right) \right] \quad (\text{A.21})$$

and, because of the inequality:

$$\frac{1}{l^2} \operatorname{Re} \left(\frac{1}{\omega} \right) > \frac{n^\beta}{n^{2\alpha} + n^{2(\beta-\alpha)}} \quad (\text{A.22})$$

and the fact that we are dealing in unitary theories so that $c/24 - h_j - m \leq 0$ for $m \geq 1$ (and $c \leq 24$)¹³, we find:

$$|R_n^{(i)}| < \sqrt{N} e^{2\pi n_i / n^\beta} \left(\frac{1}{n^{2(\beta-\alpha)}} + \frac{1}{n^{2\alpha}} \right)^{-d/2} \sum_{m=1}^{\infty} a_m^{(i)} \exp \left[-2\pi \frac{n^\beta}{n^{2\alpha} + n^{2(\beta-\alpha)}} (m - c/24) \right] \quad (\text{A.23})$$

(we will see below that this sum is bounded with a suitable choice of α and β). In deriving (A.23) we have made use of two facts. First we have used (A.9). Secondly, the arc segments $\xi_{p,i}$ form, by virtue of their definition, a dissection of the full circle, hence we used the identity $\sum_{p,i} \int_{\xi_{p,i}} d\theta = 1$.

The final step in our analysis (it suffices to consider $d \leq 0$) is to determine the optimal choice of the parameters α and β in the above results. This may be done in a variety of ways, perhaps the quickest being by direct inspection of the bound (A.23) on $R_n^{(i)}$. Recall that our object is to minimize the total sum of the errors coming from $R_n^{(i)}$ as well as from all the auxiliary integrals we have been discussing. There are two conditions which must be met in order to minimize the correction coming from the $R_n^{(i)}$. First, in view of the fact that for large enough values of m , the coefficients $a_m^{(i)}$ behave exponentially in m with the approximate form $e^{\gamma\sqrt{m}}$ for some positive γ , we can deduce with no difficulty that the summation in (A.23) converges if and only if the ratio $n^\beta / (n^{2\alpha} + n^{2(\beta-\alpha)})$ behaves as an exponential of n with some non-negative exponent. However this manifestly requires that the relation $\beta = 2\alpha$ should hold. Note that this in particular implies that the sum appearing in (A.23) is bounded by some constant multiple of the value of the character $\chi_i(q)$ at $\tau = i/2$, which must be finite in any consistent conformal theory.

To find the suitable β we note that since by definition $n_i = n - c/24 + h_i$, from inspection of the exponential $e^{2\pi n_i / n^\beta}$ it follows that we must demand $\beta \geq 1$. This in turn implies that $\alpha \geq 1/2$ from the preceding relation. We therefore find:

$$|R_n^{(i)}| < \text{constant} \times n^{ad}. \quad (\text{A.24})$$

Furthermore it follows from (A.29) and (A.33) that for $s = 2, 3, 4, 5$ we have:

$$|W_{s,n}^{(i)}| < \text{constant} \times n^{ad} \quad (\text{A.25})$$

¹³ For a more general situation, we will have to retain all singular terms, as mentioned in the text, and the analysis of the error will be correct, except that the remainder R_n will involve large enough m , so that $c/24 - h_j - m \leq 0$

for some n and i independent constants generally depending on N (typically like \sqrt{N}) and s . One has further that for $s = 1, 6$ the expressions $W_{s,n}^{(i)}$, having the same form, possess a decaying exponential behavior large values of n , and are thus entirely negligible in the present context. For the case $d = 0$ this assertion follows from the well-known asymptotic behavior of the Bessel function of imaginary argument $K_1(z)$ in the limit of large z : $K_1(z) \sim \sqrt{\pi/2z} e^{-z}$.

There is, nevertheless, a further restriction which should a priori be placed on α . Consider a typical term in an arbitrary asymptotic series. It has more or less the form $e^{\gamma\sqrt{n}^l}/n^\sigma$ for some positive constants γ and σ . This form together with the constraint $l \leq \text{constant} \times n^\alpha$ strongly suggests that we should further demand $\alpha \leq 1/2$ so that at least some positive power of n remains in the exponent for all l . Detailed examination of this point further confirms its validity. The final conclusion of all this being that the asymptotic series involves corrections which are collectively of order $O(n^{d/2})$ for $d \leq 0$.

Before we go on let us briefly discuss the case $d > 0$. Here caution must be taken as to the precise manner by which ε approaches zero. In particular we may no longer let $\varepsilon \rightarrow 0$ in an arbitrary fashion, but must instead let $\varepsilon = n^{-h}$ and determine h together with α and β from minimizing the errors. It is a simple exercise to show that in this case one finds $h = \beta$, and although all other bounds remain more or less unchanged, the bound on $W_{2,n}^{(i)}$ now reads differently:

$$|W_{2,n}^{(i)}| < \text{constant} \times n^{hd}. \quad (\text{A.26})$$

The final result is then that $\alpha = 1/2$, $h = \beta = 1$ and that the total error in the asymptotic series in this case is $O(n^d)$, as opposed to $O(n^{d/2})$ we found for $d \leq 0$. Putting all this together, it becomes a trivial matter now to obtain Eq. (3.19), (4.10) and (5.6).

To this end, a few comments regarding the behavior of the modular functions in the neighborhood of the points $q = e^{2\pi i \xi}$, lying on the unit circle and having irrational values for their argument ξ , seems to be in order. Thus far we have strictly dealt with the behavior of the modular functions near the points with rational arguments. Their study, as we have seen, pivots crucially on the linear transformation theory of the functions in question. Our analysis has culminated in deriving an asymptotic series representation for the coefficients, including corrections which, notwithstanding some order of magnitude estimations, remain to a large part undetermined. In view of the methods of our computation it is quite reasonable to suppose that there exists, in a certain sense, an intimate relationship between these corrections and the behavior of a given modular function near the irrational points which lie on its circle of convergence. However, the problem of determining this behavior, even in most specific cases, turns out to be of the most non-trivial character and certainly far more difficult than the corresponding problem for the rational values, examined hitherto.

A rather remarkable study of the behavior of the Jacobi theta functions near irrational points has been carried out in [13]. It is not our intention to review all their findings here. Only some features of their conclusions would suffice for our rather qualitative purposes. Following their notation, let us consider the sum:

$s_n(x, \theta) = \sum_{v \leq n} e^{v^2 \pi i x} \cos 2\pi v \theta$, which is of course related with the elliptic modular function $\mathcal{G}_3(q, \theta)$ evaluated at the point $q = e^{2\pi i x}$ on the unit circle. Then one interesting result is that, for almost all values of x and uniformly in θ , one has: $s_n = O[n^{1/2}(\log n)^{1/2 + \delta}]$ for any $\delta > 0$ and that, in a certain sense, this is the best of such kinds of relations¹⁴. Moreover, if the function $g(q)$ is any one of the Jacobi theta functions, and allowing $q \rightarrow e^{2\pi i \xi}$ along a radius vector or, for that matter, along any regular path not touching the unit circle, it is true that: $g(q) = O[(1 - |q|^{1/2})^{-1/4}]$ for all of irrational values of the parameter ξ . This is again shown to be in a sense the best relation of its kind. These statements also hold true for the Dedekind η -function, for instance, having essentially the same kind of power series representation as the Jacobi theta functions. Thus the partition function $f(q) = q^{1/24}/\eta(q)$ is readily concluded to rapidly converge to zero as we approach irrational points on the boundary of the unit disc. Specifically, as also mentioned in [3], it is true in the limit $q \rightarrow e^{2\pi i \xi}$, for any irrational value of ξ that: $f(q) = O[(1 - |q|^{1/2})^{1/4}]$. The remarkable accuracy with which the asymptotic series (3.19) estimates the partition $p(n)$ is in fact owed to this particular behavior. Since our analysis has shown that the error introduced by the asymptotic series only depend on the modular weight of the partition function in question and not on its other characteristics, it is reasonable to assume, though we are not in a position to prove this point, that similar behavior occurs for all modular functions of negative modular weight. That the asymptotic series in these cases are exact is owed to the vanishing of the corresponding partition functions at the irrational points on the circle of convergence.

Having already discussed three examples of the modular functions whose modular weights are positive (and equal to $1/2$), it may also be concluded that the asymptotic series for partition functions of positive modular weight are not exact precisely because of their singular behavior near irrational points whose contributions, though quite small compared with those from rational points, lead in general to the unbounded error terms. The constancy of the error terms in the cases of modular weight zero can be addressed using these same arguments. For instance the ratio $\mathcal{G}_3(q)/\eta(q)$ is of vanishing modular weight and, from the ensued discussion, if singular at irrational points at all, this is at most of logarithmic form. However, with the aid of an elementary argument it is simple to illustrate that in general one should not expect modular functions of vanishing weight to have zeros at (almost) all irrational points on their circle of convergence, as in the case for negative modular weights. Consider the partition $Z = (1/\eta^{24}) \left(\sum_{v \in \Gamma_8} q^{1/2v^2} \right)^3$ of 24 bosons compactified on three copies of the root lattice Γ_8 . This is an invariant function under modular transformation and thus, in particular, of vanishing modular weight. Suppose then that we add a non-zero number λ to this partition function: $Z \rightarrow Z + \lambda$. It clearly remains to be an invariant modular function under

¹⁴ We refer the reader to Theorems 2.14, 2.141, 2.15, 2.22 and 2.31 of that paper for more details on these statements

this change. Thus if Z did vanish at some irrational point on the unit circle, the new modular function $Z + \lambda$ does not.

To summarize it would seem natural that the exactness of the asymptotic series expansions, obtained in this paper by studying the behavior of the partition functions in the vicinity of rational points, is closely linked with their behavior near the irrational points on the unit circle of their convergence.

Appendix B

Our goal in this appendix is to briefly derive some basic properties of the expressions $\mathcal{G}_{ij}^{p,l}$ whose exact form is provided in (4.9). These we shall in turn use to deduce some elementary facts regarding the coefficient $a_n^{(i)}$ represented by (4.10). First, under complex conjugation it may be shown that (assuming for simplicity of notation that $C = 1$):

$$\mathcal{G}_{ij}^{*l-p,l} = \mathcal{G}_{ij}^{p,l}. \tag{B.1}$$

The proof of this fact is primarily based on an elementary identity satisfied by the matrix elements of the modular transformation matrices $M^{p,l}$. In fact in view of Eqs. (4.1)–(4.3), it is a rather simple exercise to show that:

$$M^{*l-p,l} = T^{-1} M^{p,l} T. \tag{B.2}$$

This identity in conjunction with the fact that for the pair $(l - p, p)$ we must use $l - p'$ in place of p' which, by virtue of its definition, corresponds to the pair (p, l) . Of course we still have the arbitrariness of adding to this integer any multiple of l , this having absolutely no influence on our arguments. Simple algebraic manipulations of (B.2) are then readily observed to lead to the desired result. An important application of the relation (B.1) is found in proving that the asymptotic series expression (4.10) for the coefficients are indeed real, as expected in a conformal theory. Specifically, since replacing the variable p in the summation by $l - p$ leaves the summation in (B.1) invariant, we see, ignoring the inconsequential constant term, that:

$$a_n^{*(i)} = \sum_{j=0}^{N-1} \sum_{p,l} \mathcal{G}_{ij}^{*l-p,l} e^{2\pi i(l-p)/l} \frac{d}{dn} J_0 \left(\frac{4\pi i}{l} \sqrt{(c/24 - h_j)n_i} \right) = a_n^{(i)}. \tag{B.3}$$

Another useful property of the expressions $\mathcal{G}_{ij}^{p,l}$ is that when viewed as a matrix in i and j indices, it is essentially unitary. To be exact, we have that:

$$\sum_{i=0}^{N-1} \mathcal{G}_{ij}^{p,l} \mathcal{G}_{ik}^{*p,l} = (a_0^{(j)})^2 \delta_{jk}. \tag{B.4}$$

The proof of this formula follows quite simply from the unitarity of the transformation matrix $M^{p,l}$ via some elementary algebraic manipulations. Thus if $a_0^{(j)} = 1$ for all $j = 0, 1, \dots, N - 1$ then $\mathcal{G}_{ij}^{p,l}$ is precisely unitary:

$$(\mathcal{G}^{p,l})^{-1} = (\mathcal{G}^{p,l})^\dagger. \tag{B.5}$$

Some of the simple applications of this unitarity relation have already been encountered in the text and in Appendix A.

Finally it may be worth noting an amusing relationship which comes about as the result of the interchange $p \leftrightarrow p'$. Specifically this is the relation:

$$\mathcal{G}_{ji}^{*p',l} = \mathcal{G}_{ij}^{p,l}, \tag{B.6}$$

which by virtue of the unitarity relation established earlier, it may be written in the form of a matrix equality:

$$(\mathcal{G}^{p',l}) = (\mathcal{G}^{p,l})^{-1}. \tag{B.7}$$

The proof of this relation rests on a simple interdependence between the matrices $M^{p,l}$ and $M^{p',l}$:

$$M^{p',l} = (M^{p,l})^{-1}. \tag{B.8}$$

We will leave it to the reader to prove this equality using Eq. (4.1) and (4.2) of the text.

Appendix C

In this appendix we shall briefly examine the derivation of the partition function $Z(\beta)$, discussed in Sect. 6. It is imperative, as well as the primary object of the present discussion, that we should obtain a result which is correct up to power law expression in β .

We start by letting $I_{p,l}^{(h,\bar{h})}$ stand for the integral expression in (6.6) so that $Z(\beta) = \sum_{(h,\bar{h})} \sum_{p,l} I_{p,l}^{(h,\bar{h})}$. here we will consider the contributions coming from the tachyonic pairs with $h = \bar{h}$ for which we can derive a closed-form expression. To see this let $u = \theta/\rho$ (where $\rho = \beta/4\pi$), so that in this case we may write:

$$I_{p,l}^{(h,h)} = l^{-2d} \rho^{-2d+1} \int_{-\theta_{p,l}''/\rho}^{\theta_{p,l}'/\rho} (1+u^2)^{-d} \exp\left[\frac{4\pi(c/24-h)}{l^2\rho(1+u^2)}\right] du. \tag{C.1}$$

With the aid of one more change of the variable; $y = (1+u^2)^{-1}$, we reach the following form for this integral:

$$I_{p,l}^{(h,h)} = l^{-2d} \rho^{-2d+1} \int_{y'_{p,l}}^1 \frac{y^{d-3/2}}{\sqrt{1-y}} \exp\left[\frac{4\pi(c/24-h)}{l^2\rho} y\right] dy + (y'_{p,l} \leftrightarrow y''_{p,l}), \tag{C.2}$$

where we have set $y_{p,l} = [1 + (\theta_{p,l}/\rho)^2]^{-1}$. Shortly we will prove that the optimal choice for the number of terms in the series corresponds to Farey's series of order $1/\sqrt{\rho}$. Let us assume this for the moment. Then from the properties of the dissection $\xi_{p,l}$ it follows that $\sqrt{\rho}/2l < \theta_{p,l} < \sqrt{\rho}/l$. Hence for any fixed value of l the lower limits of integrations approach zero as we take the limit $\rho \rightarrow 0$. For $d < 1$ the integrals $I_{p,l}^{h,\bar{h}}$ are then clearly observed to be very sensitive to the values of their integration limits $y'_{p,l}$ and $y''_{p,l}$, and in particular on the manner we have chosen to dissect the contour Γ_β .

Inasmuch as we intend to evaluate these integrals with power law accuracies, it is notable that in fact this sensitivity to the tail ends of the integration domain is of the desired power law form $O(\rho^{-d})$ and not (say) of an exponential form $e^{c/l\rho}$.

This fact is based essentially on the observation that the argument of the exponential term in the integrand is nearly unity at the lower tail of the integration domain, thus not leading to the unwanted exponential form. Technically, this is observed to be equivalent to performing the integration over an extended domain $[0, 1]$ and viewing the integral as an analytic function in variable d . This function, or its analytic extension to be more precise, will have poles for a certain discrete set of values of d . The expression obtained by simple removing of these poles will be demonstrably equal to the value of the integral up to errors of order $O(\beta^{-d})$, which are quite acceptable in the light of our previous arguments. To be more exact, with the integration domain so modified the integral is expressible in terms of hypergeometric functions:

$$\int_0^1 \frac{y^{d-3/2}}{\sqrt{1-y}} \exp\left[\frac{4\pi(c/24-h)}{l^2\rho} y\right] dy = B\left(\frac{1}{2}, d-\frac{1}{2}\right) \Phi\left(d-\frac{1}{2}, d, \frac{4\pi(c/24-h)}{l^2\rho}\right), \quad (C.3)$$

where $\Phi(\alpha, \gamma, z)$ is the degenerate hypergeometric function and $B(x, y)$ is the beta function having the well-known expression $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ in terms of the Euler gamma functions.

There is a vast mathematical literature devoted to the study of hypergeometric functions, including their analytic properties in their various arguments, and it is certainly not our intention to discuss them here. What is relevant to us, for the present, is that $\Phi(\alpha, \gamma, z)$ is analytic in its second argument every where with the exception of the point $\gamma = 0, -1, -2, \dots$ at which it has simple poles, in much the same way as the gamma function. In fact there is a very useful formula which illuminates the behaviour of this function at its poles:

$$\lim_{\gamma \rightarrow -n} \frac{1}{\Gamma(\gamma)} \Phi(\alpha, \gamma, z) = z^{n+1} \binom{\alpha+n}{n+1} \Phi(\alpha+n+1, n+2, z) \quad (C.4)$$

with the second term on the right-hand side being the familiar binomial coefficient. This formula indicates that despite its appearance to be ill-defined, the extended integral has a well defined analytic extension for zero and negative integral values of modular weight d . Specifically, using this formula it is a simple matter to obtain the desired result, up to the forementioned error, for the case $d = -n$ as follows:

$$\begin{aligned} I_{p,l}^{(h)} &= \sqrt{\pi} l^{-2d} \rho^{-2d+1} \Gamma\left(d-\frac{1}{2}\right) \lim_{d \rightarrow -n} \frac{1}{\Gamma(d)} \Phi\left(d-\frac{1}{2}, d, \frac{4\pi(c/24-h)}{l^2\rho}\right) \\ &= \frac{\pi l^{2n} \rho^{2n+1}}{(n+1)!} \left[\frac{4\pi(c/24-h)}{l^2\rho}\right]^{n+1} \Phi\left(\frac{1}{2}, n+2, \frac{4\pi(1-h)}{l^2\rho}\right) + O(\rho^{-d}). \end{aligned} \quad (C.5)$$

Note that the right-hand side of this equation is well-defined.

The same may now be done for negative half-integral values of d . Here, however, we find that the integral has simple poles coming from $\Gamma(d - \frac{1}{2})$. To describe the structure of the integral near these poles we thus make use of a well-known relation:

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n+1) + O(\epsilon) \right] \quad (C.6)$$

in which $\psi(x)$ is the logarithmic derivative of the gamma function. It is then a simple matter, using this expression, to remove the poles existing at half-integral values of d . Thus taking the limit $\varepsilon \rightarrow 0^+$, and by some simple algebra, we reach the end result for $d = -n + \frac{1}{2}$:

$$I_{p,l}^{(h,h)} = \frac{\sqrt{\pi}(-1)^n}{n! \Gamma(-n + \frac{1}{2})} \left\{ \psi(n+1) \Phi\left(-n, -n + \frac{1}{2}, \frac{4\pi(c/24 - h)}{l^2 \rho}\right) + \Phi'\left(-n, -n + \frac{1}{2}, \frac{4\pi(c/24 - h)}{l^2 \rho}\right) \right\} + O(\rho^{-d}), \tag{C.7}$$

where prime indicates the differentiation with respect to n . Again note that, viewed as an analytic extension, the right-hand side of this equation is well-defined.

To see what of contributions we should expect from unphysical-tachyonic pairs, it suffices to consider the special case of $d = 1$, for which the computation is particularly simple. Starting from Eq. (6.5), we perform a convenient change of variable by setting $u = \tan(w/2)$, where u is as used in (C.1). It is then a simple matter, in this special case, to arrive at:

$$I_{p,l}^{(h,\bar{h})} = \frac{1}{2} e^{\pi(c/12 - h - \bar{h})/l^2 \rho} \int_{-\pi}^{\pi} \exp\left[\frac{\pi}{l^2 \rho} (c/12 - h - \bar{h}) \cos w + \frac{\pi i}{l^2 \rho} (h - \bar{h}) \sin w \right] dw, \tag{C.8}$$

which is easily integrated to give:

$$I_{p,l}^{(h,\bar{h})} = 2\pi e^{\pi(c/12 - h - \bar{h})/l^2 \rho} I_0\left(\frac{2\pi}{l^2 \rho} \sqrt{(c/24 - h)(c/24 - \bar{h})}\right), \tag{C.9}$$

where $I_0(z)$ is the bessel function of complex argument which asymptotically (for large $|z|$) behaves as $I_0(z) \sim e^z / \sqrt{2\pi z}$. It readily follows from this that the above expression is rapidly oscillating in the limit as ρ approaches zero. This in fact is observed to be a common feature of unphysical-tachyonic contributions for any modular weight d even though we have not been able to find a closed form expression for it for arbitrary d .

It is time to discuss the corrections to $Z(\beta)$ which are contributed by the non-tachyonic pairs. These contributions, denoted here by \mathfrak{R} , are easily seen to take the form:

$$\mathfrak{R}(\rho) = \sum_{(h,\bar{h})'} \sum_{p,l} \sum_{n,m \geq 1} a_n^{(h)} a_m^{(\bar{h})} \int_{\xi_{p,l}} (z\bar{z})^{-d} \exp\left[\frac{2\pi}{lz} (c/24 - h - n) + \frac{2\pi}{l\bar{z}} (c/24 - \bar{h} - m) \right], \tag{C.10}$$

where the prime in the first sum indicates that only non-tachyonic pairs of conformal dimensions are included in that sum. To characterize the second sum we assume that we are dealing with a Farey's series of order (up to some irrelevant constant) ρ^{-r} , for some positive real constant r . It then follows from the properties of the Farey's series that $\rho^r/2l < \theta < \rho^r/l$ for all the points belonging to $\xi_{p,l}$. Let us take the case $d \leq 0$. Then it is easy to see that for every such point (for small

enough ρ):

$$(z\bar{z})^{-d} \leq (\rho^{2+2r} + \rho^{2r})^{-d} < (2\rho^{2r})^{-d}.$$

Also since

$$\frac{1}{l} \operatorname{Re} \left(\frac{1}{z} \right) = \frac{\rho}{l^2(\rho^2 + \theta^2)} > \frac{\rho}{\rho^{2-2r} + \rho^{2r}},$$

we have:

$$\left| \exp \left[\frac{2\pi}{lz} (c/24 - h - n) \right] \right| < \exp \left[2\pi \frac{\rho}{\rho^{2-2r} + \rho^{2r}} (c/24 - h - n) \right].$$

Putting this altogether we obtain:

$$|\mathfrak{R}(\rho)| < (2\rho^{2r})^{-d} \sum_{(h,\bar{h})} \sum_{n,m \geq 1} a_n^{(h)} a_m^{(\bar{h})} \exp \left[2\pi \frac{\rho}{\rho^{2-2r} + \rho^{2r}} (c/12 - h - \bar{h} - n - m) \right]. \quad (\text{C.11})$$

It should be clear from the form of the exponent that in order for the remaining summations to converge in the limit as $\rho \rightarrow 0$ we must insist on having ρ -dependence in this limit of the form ρ^{-c} , for some $c > 0$. This is readily seen to be only possible for $r = 1/2$. Note that the sum appearing in this expression is bounded by the value of the partition function $Z(q, \bar{q})$ at the point $\tau = i/2$, which must be finite in any consistent theory. Hence we finally have:

$$|\mathfrak{R}(\rho)| < \text{constant} \times \rho^{-d}. \quad (\text{C.12})$$

The case $d > 0$ is very similar and it is a trivial exercise to check that it leads exactly to the same bound.

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