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Diff(S^1) and the Teichmüller Spaces

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Abstract. Precisely two of the homogeneous spaces that appear as coadjoint orbits of the group of string reparametrizations, $\widehat{\mathrm{Diff}(S^1)}$[1], carry in a natural way the structure of infinite dimensional, holomorphically homogeneous complex analytic Kähler manifolds. These are $N = \mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ and $M = \mathrm{Diff}(S^1)/\mathrm{Möb}(S^1)$. Note that N is a holomorphic disc fiber space over M. Now, M can be naturally considered as embedded in the classical universal Teichmüller space $T(1)$, simply by noting that a diffeomorphism of S^1 is a quasisymmetric homeomorphism. $T(1)$ is itself a homomorphically homogeneous complex Banach manifold. We prove in the first part of the paper that the inclusion of M in $T(1)$ is *complex analytic*.

In the latter portion of this paper it is shown that the *unique* homogeneous Kähler metric carried by $M = \mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ induces precisely *the Weil–Petersson metric* on the Teichmüller space. This is via our identification of M as a holomorphic submanifold of universal Teichmüller space. Now recall that every Teichmüller space $T(G)$ of finite or infinite dimension is contained canonically and holomorphically within $T(1)$. Our computations allow us also to prove that every $T(G)$, G any infinite Fuchsian group, projects out of M *transversely*. This last assertion is related to the "fractal" nature of G-invariant quasicircles, and to Mostow rigidity on the line.

Our results thus connect the loop space approach to bosonic string theory with the sum-over-moduli (Polyakov path integral) approach.

Introduction

Part I: The Complex Structures. The group $\mathrm{Diff}(S^1)$ and its universal central extension, the Virasoro group, occurs in string theory as the space of reparametrizations of a closed string. Two coadjoint orbit spaces of $\widehat{\mathrm{Diff}(S^1)}$, namely, $N = \mathrm{Diff}(S^1)/S^1$ and $M = \mathrm{Diff}(S^1)/SL(2,\mathbb{R})$, have occurred in the physics literature

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[1] The Virasoro group (see Witten [16])

as critically important because precisely these two carry the structure of infinite dimensional, holomorphically homogeneous, complex (Kähler) manifolds (Witten [16]). The complex structures on these spaces are obtained by placing a natural physics-motivated almost complex structure (arising from Kirillov–Kostant representation theory) on the appropriate spaces of real vector fields on S^1. This complex structure, arising from "conjugation" of Fourier series, appears, for example, in Pressley [18].

Now, considering diffeomorphisms of S^1 as quasisymmetric homeomorphisms one can naturally identify M as embedded in the classical universal Teichmüller space $T(1)$. $T(1)$ is a holomorphically homogeneous complex Banach domain from the famous Ahlfors–Bers theory of the Teichmüller spaces. Our first main result is that this inclusion of M into $T(1)$ is *complex analytic*. In fact, M is one leaf of a holomorphic foliation of $T(1)$. Also, since N is a holomorphic disc fiber space over M, it seems to us that this naturally and directly connects the string reparametrization complex manifolds with the complex analytic moduli of Riemann surfaces. It appears to have been an important question (see, for example, Bowick [5], Bowick and Rajeev [6]) to relate these reparametrization spaces with the spaces of moduli of Riemann surfaces because that would connect the loop space ("geometrical quantization") approach to string theory with the path integral ("sum over moduli") approach. Indeed, $T(1)$ contains canonically within itself as complex submanifolds all the Teichmüller spaces of arbitrary Riemann surfaces or Fuchsian groups. If, therefore, strings are reparametrized using the more general quasisymmetric homeomorphisms of the circle (rather than only by smooth diffeomorphisms), then the corresponding $SL(2,\mathbb{R})$ orbit space *is* the universal Teichmüller space of Riemann surfaces.

Our method of proof is to show that the almost complex structure obtained by the physicists (Bowick and Rajeev [6, 7]; Bowick and Lahiri [8]) on real vector fields on S^1 modulo the Möbius vector fields coincides with the almost complex structure of $T(1)$ at the origin. The holomorphic homogeneity of both M and $T(1)$ under the action of (right-) translation then implies that the complex structures are compatible everywhere.

Part II: The Kähler Structures. The infinite dimensional holomorphically homogeneous (Fréchet) complex manifold $M = \text{Diff}(S^1)/SL(2,\mathbb{R})$ is shown in Part I to be naturally embedded in universal Teichmüller space, $T(1)$, as one leaf of a holomorphic foliation of $T(1)$. That result showed that these two "universal moduli spaces" are intimately related. There is a *unique* (up to a scaling factor) homogeneous Kähler metric, g, on M; this metric and its curvature have been studied intensively by many physicists including Bowick, Rajeev, Lahiri, Zumino, Kirillov (see [5–8, 12, 13]). The chief result now is that this canonical metric g produces precisely the *Weil–Petersson* Kähler metric on the Teichmüller spaces.[2]

Let us be more precise. The metric g assigns a hermitian inner product on smooth real vector fields on the unit circle S^1. (S^1 is to be thought of as the

[2] The Kähler form of the metric g is is precisely the symplectic form that M carries by virtue of being a coadjoint orbit manifold. See Witten [16] and Kirillov [19]

boundary of the open unit disk Δ.) Now, tangent vectors to the Teichmüller space are represented by Beltrami coefficients on Δ (modulo the infinitesimally trivial ones). The vector field on S^1 corresponding to a Beltrami coefficient μ is $\dot{w}[\mu](z)(\partial/\partial z)$, $z \in S^1$. We are able to express the metric g as a pairing on these Beltrami coefficients; indeed, the formula is:

$$(*) \qquad g(\mu, \nu) = \iint_\Delta \times \iint_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4} d\xi d\eta \cdot dx dy.$$

This formula converges (as it must) whenever $\mu, \nu \in L^\infty(\Delta)$ represent *smooth* (C^2 is enough smoothness) vector fields on S^1. However, as it stands, (*) must diverge whenever μ, ν represent non-zero tangent vectors to any Teichmüller space $T(G)(\subset T(1))$, where G is any infinite Fuchsian group. This implies that each $T(G)$ sits in $T(1)$ *intersecting transversely* the leaves of the foliation of $T(1)$ by M and its Mod(G) translates.

On the other hand, *the formula* (*) *for the canonical metric g can still be used to recover the Weil–Petersson metric on any of the finite dimensional Teichmüller spaces* $T(G)$, (which is where the Weil–Petersson metric is defined classically). This is accomplished via a simple regulation of the improper integral (*)—as explained in Sect. II.3. Thus, the unique Kähler structure of Diff$(S^1)/SL(2,\mathbb{R})$ really does tie up with the Weil–Petersson Kähler metric on the Teichmüller spaces in a very convincing fashion.

The transversality we have mentioned above, of each $T(G)$ with M, relates intimately to various facts about the relationship of smooth diffeomorphisms to general quasisymmetric homeomorphisms (on S^1). Recall (see Sect. II.2) that the universal Teichmüller space $T(1)$ can be thought of as the space of all (Möbius-normalized) quasidisks. Now, the quasidisks corresponding to the analytic subset M are precisely the ones with C^∞ boundaries. Bowen [4] had proved that the boundary of any quasidisk corresponding to a (non-origin) point of $T(G)$, (where Δ/G is a compact Riemann surface), must be very non-smooth—indeed "fractal." These results (discussed in Sect. II.4) are thus compatible with, and shed new light on, what we have proved in this paper.

Part I: The Complex Structures

I.1. The Complex Structure of Diff$(S^1)/S^1$ *and* Diff$(S^1)/SL(2,\mathbb{R})$. Let N and M denote respectively these two orbit spaces. We will think of them as right coset spaces. Using $S^1 = \mathrm{Rot}(S^1)$ and $PSL(2,\mathbb{R}) = \mathrm{M\ddot{o}b}(S^1)$ to normalize a given C^∞ diffeomorphism (by following the given diffeomorphism by a normalizing one) we can identify N (and M) as those diffeomorphisms of S^1 that fix one (respectively, three) points of S^1.

The Lie algebra of the Fréchet Lie group Diff(S^1) is the algebra of C^∞ smooth real vector fields on S^1 (see Goodman [10]). The complexification of this Lie algebra is the Virasoro algebra generated by the $L_n = e^{in\theta}(\partial/\partial\theta) = iz^{n+1}(\partial/\partial z)$, $n \in \mathbb{Z}$. (Here $z = e^{i\theta}$.) A tangent vector to N at its origin is a linear combination:

$$\vartheta = \sum_{m \neq 0} \vartheta_m L_m, \quad \bar{\vartheta}_m = \vartheta_{-m}, \tag{1}$$

where $\vartheta = u(\theta)(\partial/\partial\theta)$ is the corresponding smooth real vector field on the circle and the ϑ_m are the Fourier coefficients of $u(\theta)$. (The ϑ_k decay faster than any negative power of k since $u(\theta)$ is C^∞. See Katznelson [11], p. 24.) For M, at its origin, a tangent vector will be of the form

$$\vartheta = \sum_{m \neq -1,0,1} \vartheta_m L_m, \quad \bar{\vartheta}_m = \vartheta_{-m}. \tag{2}$$

Here one loses the coefficients $\vartheta_{-1}, \vartheta_0, \vartheta_1$ because an infinitesimal Möbius transformation of Δ (the unit disc) allows one to normalize precisely these coefficients. One may also check that the Lie algebra generated by L_{-1}, L_0, L_1 is precisely (the complexification of) $sl(2, \mathbb{R})$, as would be expected.

The natural *almost complex structure* $\tilde{J}$ *at the origin* of these two spaces is then defined (in each case) by (motivated from representation theory; see Auslander–Kostant [17]):

$$\tilde{J}\vartheta = \sum_m -i \operatorname{sgn}(m) \vartheta_m L_m. \tag{3}$$

See Pressley [18], Bowick and Rajeev [7], and Bowick and Lahiri [8]. The formula (3) is, of course, the classic formula known in the theory of Fourier series as "conjugation." See, for example, Katznelson [11] Chapter III. One now follows [7, 8] to define the almost complex structure everywhere on these (right-) coset spaces by right-translation invariance. As shown in [7] using the rather obvious involutivity of the (1,0) vector fields this $\tilde{J}$ is seen to be integrable and the right translations by elements of $\mathrm{Diff}(S^1)$ act as biholomorphic automorphisms on N and M. (It is possible to get fairly explicit holomorphic coordinates on N and M as explained by Bruno Zumino in his July 1988 lectures at the ICTP. See Zumino [20].)

Remark. We are purposely using right translations and right-invariant objects in order to finally coincide with the usual version of the theory of Teichmüller spaces. It is of course possible, as indicated in the last remark of the next section, to modify the definition of the Teichmüller spaces so that the left-invariant theory of the M and N works compatibly.

I.2. The Universal Teichmüller Space $T(1)$. Let $\mathrm{Homeo}_{qs}(S^1)$ denote the group of quasisymmetric homeomorphisms of the unit circle. These are the ones which allow some quasiconformal extension into the unit disc Δ. By a well-known characterization due to Ahlfors (see [1] or [14]) these are the homeomorphisms that alter cross ratios of "symmetrically placed" points on S^1 by a bounded ratio. Now, Bers' universal Teichmüller space is

$$T(1) = \mathrm{Homeo}_{qs}(S^1)/SL(2, \mathbb{R}). \tag{4}$$

Again, $SL(2, \mathbb{R}) = \text{Möb}(S^1)$ can be thought of as normalizing a homeomorphism by following it by a Möbius transformation so that the composition fixes $+1, -1$ and $-i$ (say) on S^1.

The complex analytic structure of $T(1)$ comes by thinking of it as equivalence classes of proper Beltrami coefficients on Δ. These Beltrami coefficients comprise the unit ball $L^\infty(\Delta)_1$ of the complex Banach space $L^\infty(\Delta)$. Given any $\mu \in L^\infty(\Delta)_1$

one solves the Beltrami equation

$$w_{\bar{z}} = \mu w_z \tag{5}$$

to get a quasi conformal self-homeomorphism $w = w_\mu$ of Δ. The boundary values of w_μ on S^1 (which always exist) is the quasisymmetric homeomorphism of S^1 representing the equivalence class $[\mu]$ in $T(1)$. Thus,

$$T(1) = L^\infty(\Delta)_1/\sim, \tag{6}$$

where $\sim$ is the equivalence relation saying $\mu \sim v$ if and only if w_μ and w_v (normalized as explained by post-composition with Möbius transformations) have identical boundary values on S^1.

Remark. The way to get w_μ given μ in Δ is explained in Sect. II.2. Much more about the Teichmüller spaces will be needed in Part II of this paper—see the section quoted above.

Bers proved that $T(1)$ inherits the structure of a complex Banach manifold from the complex structure of the unit ball $L^\infty(\Delta)_1$. Namely there is a unique induced complex structure on $T(1)$ such that the quotient projection $\Phi: L^\infty(\Delta)_1 \to T(1)$ becomes a *holomorphic submersion.* For complete proofs see Nag [14].

Notice that $T(1)$ is a group (though *not* a topological group). In fact, composition of quasisymmetric homeomorphisms corresponds to the following group law on Beltrami coefficients (see [14], p. 54–55 and p. 227–228)

$$\begin{aligned} \lambda \cdot \mu &= \text{Beltrami coefficient of } (w_\lambda \circ w_\mu) \\ &= \frac{\mu + (\lambda \circ w_\mu)\gamma_\mu}{1 + \bar{\mu}(\lambda \circ w_\mu)\gamma_\mu}, \quad \text{where} \quad \gamma_\mu = \frac{\overline{(w_\mu)_z}}{(w_\mu)_z}. \end{aligned} \tag{7}$$

Since formula (7) depends holomorphically on λ we see that right translations act as biholomorphic automorphisms on $T(1)$ (and on $L^\infty(\Delta)_1$).

Remark. If we redefine universal Teichmüller space by associating to $\mu \in L^\infty(\Delta)_1$ the boundary values of w_μ^{-1}, then the *left* translations act biholomorphically. The usual conventions in the physics literature regarding the orbit spaces of Diff(S^1) can then be retained. We prefer to stick to the classical conventions in Teichmüller space theory.

I.3. $M \hookrightarrow T(1)$ is a Holomorphic Inclusion. It is well-known that every diffeomorphism of S^1 extends to a diffeomorphism of the closed disk $\Delta \cup S^1$. So diffeomorphisms are certainly quasisymmetric. Consequently, $M = \text{Diff}(S^1)/\text{Möb}(S^1)$ sits canonically inside $T(1) = \text{Homeo}_{qs}(S^1)/\text{Möb}(S^1)$.

Theorem I.1. *The natural inclusion $M \hookrightarrow T(1)$ is holomorphic. M can be thought of as one leaf of a holomorphic foliation of $T(1)$ by injectively and holomorphically immersed leaves. (The leaves, which are the cosets of the subgroup M within $T(1)$, are not closed in $T(1)$.)*

Using the holomorphic homogeneity of both M and $T(1)$ under right translations, one only needs to check the identity of the almost complex structures at the origin. The first problem is therefore to get a description of the almost complex structure, J, of $T(1)$ at the origin (so as to be able to compare it with the $\tilde{J}$ of Sect. I.1).

Acknowledgement. The pretty description of J on $T(1)$ given in the Proposition below is essentially an idea of S. Kerckhoff. The idea was explained to the first author in oral communication by C. J. Earle at Cornell University (1987–88).

A quasisymmetric real vector field on S^1, to be thought of as an arbitrary tangent vector at the origin of $T(1)$, is obtained from a one-parameter flow of quasisymmetric homeomorphism $w_{t\mu}$, for any $\mu \in L^\infty(\Delta)$. The vector field on S^1 is then $\vartheta = \dot{w}[\mu]\partial/\partial z$, where $w_{t\mu}$ has the perturbation expansion:

$$w_{t\mu}(z) = z + t\dot{w}[\mu](z) + o(t), \quad t \to 0. \tag{8}$$

The problem is to obtain $J\vartheta$ on S^1, where

$$J\vartheta = \dot{w}[i\mu](z)\frac{\partial}{\partial z}, \quad \text{given} \quad \vartheta = \dot{w}[\mu](z)\frac{\partial}{\partial z}. \tag{9}$$

(Recall that the complex structure of $T(1)$ is inherited from the complex structure of the space of μ's, as explained in Sect. I.2. So J corresponds to sending μ to $i\mu$.)

Proposition. *Using θ as coordinate on S^1, $z = e^{i\theta}$, we can write $\vartheta = u(\theta)(\partial/\partial\theta)$, where $\dot{w}[\mu](z) = izu(z)$, $z \in S^1$. Then, $J\vartheta = u^*(\theta)(\partial/\partial\theta)$, where $\dot{w}[i\mu](z) = izu^*(z)$, $z \in S^1$. The formula for u^* is:*

$$u^*(z) = \operatorname{Im}(D(z)) + (cz + \bar{c}\bar{z} + b) \quad \text{on } S^1 \tag{10}$$

for a certain $b \in \mathbb{R}$, $c \in \mathbb{C}$. Here $D(z)$ is a member of the disc algebra $A(\Delta)$ (namely, functions holomorphic in Δ and continuous on $\Delta \cup S^1$) such that $\operatorname{Re} D = u$ *on S^1.*

Remark. Notice that $u(z)$ is simply the *magnitude* of the vector field ϑ at the point $z \in S^1$. Note also that u^* is being shown to be essentially **the Hilbert transform** of u.

Proof. The first variation term $\dot{w}[\mu]$ can actually be explicitly written down (see [14], or Eq. (24) of Part II here) in the form

$$\dot{w}[\mu](z) = \iint_{\mathbb{C}} \tilde{\mu}(\zeta)R(z,\zeta)d\zeta \wedge d\bar{\zeta}, \tag{11}$$

where $\tilde{\mu}$ is the extension of μ to the whole plane by reflection across S^1, as explained in Part II, Sect. II.2. (Explicitly, $\tilde{\mu}(1/\bar{w}) = \overline{\mu(w)}(w^2/\bar{w}^2)$ for w in Δ.) Here $R(z,\zeta)$ is a certain rational function. The main feature of $\dot{w}[\mu]$ (from which actually formula (11) can be derived) is that

$$\bar{\partial}\dot{w}[\mu] = \mu \text{ a.e. on } \Delta, \quad (\text{here } \bar{\partial} = \partial/\partial\bar{z}). \tag{12}$$

See Nag [14] pp. 39–40 and p. 171 for a proof of this critical property. We also note for later use that formula (11) implies, since μ is L^∞, that $u(z) = (\dot{w}[\mu](z))/iz$ (on S^1) satisfies a Hölder condition ($|u(z_1) - u(z_2)| \leqq c|z_1 - z_2|^\lambda, 0 < \lambda < 1$)—in fact with λ arbitrarily close to 1.

Construct the function

$$F(z) = \dot{w}[i\mu](z) - i\dot{w}[\mu](z), \quad \text{on} \quad \Delta \cup S^1. \tag{13}$$

By formula (12) we see that $\bar{\partial}F = 0$ on Δ, so F is in the disc algebra $A(\Delta)$. Therefore the critical fact is:

$$izu^*(z) + zu(z) = F(z) \quad \text{on } S^1, \quad \text{for} \quad F \in A(\Delta). \tag{14}$$

It is easy to derive the proposition from formula (14) as follows. Define $G \in A(\Delta)$ by $F(z) = F(0) + zG(z)$. Then (14) becomes

$$u(z) + iu^*(z) = G(z) + F(0)\bar{z} \quad \text{on } S^1. \tag{15}$$

We therefore have

$$\begin{aligned} u(z) &= \operatorname{Re}(G(z) + F(0)\bar{z}) \quad \text{on } S^1 \\ &= \operatorname{Re}(G(z) + \overline{F(0)}z) \quad \text{on } S^1. \end{aligned} \tag{16}$$

Of course, $D(z) = G(z) + \overline{F(0)}z$ is also in $A(\Delta)$, and the above calculation shows that $D(z)$ solves the "holomorphic Dirichlet problem" for the real boundary values u on S^1.

Equation (15) now allows us to relate u^* with u as desired:

$$\begin{aligned} u^*(z) &= \operatorname{Im}(G(z) + F(0)\bar{z}), \quad \text{on } S^1 \\ &= \operatorname{Im}(D(z) - \overline{F(0)}z + F(0)\bar{z}), \quad \text{on } S^1 \\ &= \operatorname{Im}(D(z)) + kz + \bar{k}\bar{z}, \quad z = e^{i\theta}. \end{aligned} \tag{17}$$

Note that our $D(z)$ must be of the form

$$D(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta)\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) d\theta + ib, \quad b \in \mathbb{R},$$

(the "Schwarz kernel" formula for the Dirichlet problem). The fact that u in Hölder is well known (see, for example, Gakhov [9]) to guarantee that D is in $A(\Delta)$, and therefore $\operatorname{Im}(D(z))$ on S^1 is well-defined.

The Proposition is fully proved. □

Remark. Notice that the normalization of w_μ (to fix $+1, -1$ and $-i$) implies that the vector fields $u(\theta), u^*(\theta)$ must vanish at these three points. The constants b, c occurring in formula (10) can be related at least partly to the enforcement of this normalization.

To prove our Theorem we need to show that J and $\tilde{J}$ act identically on the smooth real vector fields ϑ of S^1. Let, $\vartheta = u(\theta)\partial/\partial\theta$, with Fourier expansion

$$u(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \tag{18}$$

Then,

$$D(z) = a_0 + 2\sum_{n=1}^{\infty} a_n z^n \tag{19}$$

is in $A(\Delta)$ with $\operatorname{Re} D = u$ on S^1 clearly.

By the Proposition we obtain therefore: $J\vartheta = u^*(\theta)(\partial/\partial\theta)$ where, for certain b, c,

$$\begin{aligned} u^*(\theta) &= \operatorname{Im}(D(e^{i\theta})) + (b + ce^{i\theta} + \bar{c}e^{-i\theta}) \\ &= \sum_{n=2}^{\infty} (-ia_n)e^{in\theta} + \sum_{n=2}^{\infty} \overline{(-ia_n)}e^{-in\theta} \\ &\quad + (\beta + \gamma e^{i\theta} + \bar{\gamma}e^{-i\theta}). \end{aligned} \tag{20}$$

The β and γ get normalized to zero via the $SL(2,\mathbb{R})$ normalization. Thus, comparing (20) with formula (3) for $\tilde{J}$, we see that $J\vartheta \equiv \tilde{J}\vartheta$. The theorem is proved. □

Part II: The Kähler Structures

II.1. The Kähler Metric g on M. Recall from Sect. I.1 that the Lie algebra of the Fréchet Lie group $\mathrm{Diff}(S^1)$ consists of $\mathrm{Vect}^\infty(S^1)$ (= smooth real vector fields on S^1). The homogeneous space of (right-) cosets $M = \mathrm{Diff}(S^1)/SL(2,\mathbb{R}))$ has, as its tangent space at the origin, those smooth real vector fields v which are of the form

$$v = v(\theta)\frac{\partial}{\partial\theta} = \sum_{m\in\mathbb{Z}-\{-1,0,1\}} v_m L_m, \quad v_{-m} = \overline{v_m}. \tag{1}$$

Here the $L_m = e^{im\theta}(\partial/\partial\theta)$, $m\in\mathbb{Z}$, generate the complexification of $\mathrm{Vect}^\infty(S^1)$. Note that:

$$[L_m, L_n] = i(n-m)L_{m+n}. \tag{2}$$

There exists a *unique homogeneous Kähler metric g* on M ([6–8]). By homogeneity, one needs to determine the Kähler form ω only at the origin of M. The requirement $d\omega = 0$ forces (see [7]):

$$\omega([L_m, L_n], L_p) + \omega([L_n, L_p], L_m) + \omega([L_p, L_m], L_n) = 0. \tag{3}$$

Also, ω must vanish whenever one of its arguments is L_{-1}, L_0 or L_1—since these vector fields (which generate $sl(2,\mathbb{R})$) give the zero tangent vector on M. From these conditions one finds that *the only possible homogeneous Kähler form ω is given at the origin by*:

$$\omega(L_m, L_n) = a(m^3 - m)\delta_{m,-n}, \quad m,n\in\mathbb{Z}-\{\pm 1, 0\}. \tag{4}$$

(a is as yet any non-zero complex constant.) Elsewhere on M, of course, ω is transported by translations. This Kähler form ω is precisely the Kirillov–Kostant symplectic form that exists on M since M is a coadjoint orbit of $\widehat{\mathrm{Diff}(S^1)}$. Compare Witten [16]. Note that the metric exists on each holomorphic leaf of the foliation of $T(1)$ by M and its coset-translates.

Remark. Interestingly, the formula (4) has occurred in more than one other context. Segal ([15], p. 321) in calculating $H^2(\mathrm{Vect}^\infty(S^1), \mathbb{C})$ gets (4), where (3) becomes interpreted as a cocycle condition. Again, (4) occurs in conformal field theory where the interpretation of (3) is basically as the Jacobi identity.

Proposition. *Let $v = \Sigma v_m L_m$ and $w = \Sigma w_m L_m$ (of the form (1)) represent two real tangent vectors to M at the origin. Then the Kähler metric g, whose Kähler form ω was determined above, assigns the inner product*

$$g(v,w) = -2ia\,\mathrm{Re}\left[\sum_{m=2}^{\infty} v_m\overline{w_m}(m^3 - m)\right]. \tag{5}$$

The infinite series in (5) converges absolutely whenever the vector fields v and w are $C^{3/2+\varepsilon}$ on S^1 (any $\varepsilon > 0$). In particular C^2 vector fields produce convergence.

Proof. Recall that the Kähler 2-form ω is related to the corresponding pairing g

by $\omega(v,w) = g(v, \tilde{J}w)$. So $g(v,w) = -\omega(v, \tilde{J}w)$. It is therefore trivial to deduce (5) from (4). The convergence assertion follows from the fact that the Fourier coefficients of a $C^{k+\varepsilon}$ function on S^1 decay at least as fast as $1/n^{k+\varepsilon}$. (See Katznelson [11], p. 24–25.) Indeed, apply the Cauchy–Schwarz inequality to (5). Convergence is guaranteed if $\{v_m m^{3/2}\}$ and $\{w_m m^{3/2}\}$ are in l^2, and this happens when the vector fields are $C^{3/2+\varepsilon}$ smooth. □

Note: Since we want $g(v,v) > 0$ we must have $a = ib$ with $b > 0$.

II.2. The Teichmüller Spaces. We recall the basic facts we need from Teichmüller theory. See Nag [14] for complete proofs. The section is necessarily a trifle long since we have to understand the relation between various equivalent definitions of Teichmüller space.

The universal Teichmüller space, $T(1)$, is a holomorphically homogeneous complex Banach manifold which contains within itself, as complex submanifolds, all the Teichmüller spaces $T(G)$, (for arbitrary Fuchsian groups G operating on Δ). $T(G)$ parametrizes the various complex structures on the Riemann surface $X = \Delta/G$. The ball of proper Beltrami differentials, $L^\infty(\Delta)_1$, is fundamental: it is the open unit ball in the complex Banach space of L^∞ functions on Δ. The G-invariant Beltrami differentials constitute the closed complex subspace:

$$L^\infty(G) = \{\mu \in L^\infty(\Delta) : \mu(gz)\overline{g'(z)}/g'(z) = \mu(z) \text{ a.e. on } \Delta \text{ for all } g \text{ in } G\}. \tag{6}$$

The unit ball $L^\infty(G) \cap L^\infty(\Delta)_1$ is denoted $L^\infty(G)_1$.

The chief construction is to solve the Beltrami equation

$$w_{\bar{z}} = \mu w_z \tag{7}$$

for any $\mu \in L^\infty(\Delta)_1$. One needs to look at two solutions of (7), namely,

$[w_\mu]$: The quasiconformal homeomorphism of $\mathbb{C}$ which is μ-conformal (i.e. solves (7)) in Δ, fixes ± 1 and $-i$, and keeps Δ and Δ^* (= exterior of Δ) both invariant. This w_μ is obtained by applying the existence and uniqueness theorem of Ahlfors–Bers (for (7)) to the Beltrami coefficient which is μ on Δ and extended to Δ^* by reflection ($\tilde{\mu}(1/\bar{z}) = \overline{\mu(z)}z^2/\bar{z}^2$ for $z \in \Delta$).

$[w^\mu]$: The quasiconformal homeomorphism on $\mathbb{C}$ which is μ-conformal on Δ and conformal on Δ^* (fixing ± 1 and $-i$ again, say). w^μ is obtained by applying the Ahlfors–Bers theorem to the Beltrami coefficient which is μ on Δ and zero on Δ^*.

Now one defines, for any Fuchsian group G including $G = \{1\}$, the *Teichmüller space*,

$$T(G) = L^\infty(G)_1/\sim, \tag{8}$$

where $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on $\partial\Delta = S^1$, which happens if and only if $w^\mu = w^\nu$ on $\Delta^* \cup S^1$. The space $T(G)$ carries uniquely the structure of a complex Banach manifold induced from the complex structure of the open ball $L^\infty(G)_1$—i.e., such that the quotient projection

$$\Phi : L^\infty(G)_1 \to T(G) \tag{9}$$

becomes a *holomorphic submersion.*

How does this relate to moduli? If $\mu \in L^\infty(G)_1$ then w_μ conjugates G to another Fuchsian group

$$G_\mu = w_\mu G w_\mu^{-1}. \tag{10}$$

The equivalence class of μ in $T(G)$ represents the Riemann surface $X_\mu = \Delta/G_\mu$.

Alternatively, one can utilize w^μ to conjugate G to a quasi-Fuchsian group

$$G^\mu = w^\mu G (w^\mu)^{-1} \tag{11}$$

so that G^μ operates discontinuously on the quasidisks $\Delta^\mu = w^\mu(\Delta)$ and its exterior $\Delta^{*\mu} = w^\mu(\Delta^*)$. X_μ is represented by Δ^μ/G^μ (whereas $\Delta^{*\mu}/G^\mu$ is the fixed Riemann surface Δ^*/G—since w^μ was conformal on Δ^*).

From the w_μ picture it is clear that

$$T(G) = \left\{ \begin{matrix} \text{quasisymmetric homeomorphisms of } S^1 \\ \text{compatible with } G \end{matrix} \right\} \Big/ \text{Möb}(S^1). \tag{12}$$

Here a homeomorphism f of S^1 is called quasisymmetric if it has some quasiconformal extension into Δ, and f is compatible with G if fGf^{-1} is again a group of (restrictions to S^1 of) Möbius transformations. The relation between (8) and (12) is by associating to $\mu \in L^\infty(G)_1$ the homeomorphism w_μ restricted to S^1.

Remark. We want to draw attention to the fundamental fact that the moduli of two-dimensional Riemann surfaces are hereby determined by homeomorphisms of just the unit circle. *Thus reparametrization of the closed string* (S^1) *are intimately related to moduli of complex structures.* Indeed, the holomorphic embedding of M in $T(1)$ exhibited in Part I is induced simply by the inclusion $\text{Diff}(S^1) \hookrightarrow \text{Homeo}_{qs}(S_1)$.

Utilizing the w^μ picture, one can associate to the Teichmüller class of μ the quasidisk Δ^μ. Consequently:

$$T(1) = \left\{ \begin{matrix} \text{quasidisks on the complex sphere} \\ \text{normalized by Möbius transformations} \end{matrix} \right\}. \tag{13}$$

$T(G)$ comprises those quasidisks on which some quasi-Fuchsian conjugate of G acts discontinuously.

How does one pass back and forth between the descriptions (12) and (13) of Teichmüller space? Given a quasisymmetric homeomorphism f on S^1, extend it to *any* quasiconformal homeomorphism F of $\Delta \cup S^1$. Then, if μ is the Beltrami coefficient $\bar{\partial}F/\partial F$ of F on Δ, the quasidisk corresponding to f is $D = \Delta^\mu$. D does *not* depend on the choice of the extension F.

Conversely, note that w_μ and w^μ are both μ-conformal on Δ—and consequently $\rho^\mu = w^\mu \circ w_\mu^{-1}$ is a (normalized) Riemann mapping of Δ onto Δ^μ. It follows that if a quasidisk D is supplied the corresponding quasisymmetric homeomorphism f on S^1 is the "welding" homeomorphism:

$$f = \rho^{-1} \circ \sigma, \tag{14}$$

where ρ and σ are respectively the (normalized) Riemann mappings of Δ onto D and Δ^* onto D^* (= exterior of D).

We recall now Teichmüller's Lemma, which identifies the tangent space to any $T(G)$ at the origin. The fundamental pairing $L^\infty(G) \times A_2(G) \to \mathbb{C}$ between L^∞ Beltrami coefficients and integrable holomorphic quadratic differentials for G is given by

$$\langle \mu, \varphi \rangle = \iint_{\Delta/G} \mu\varphi. \tag{15}$$

Teichmüller's Lemma. *Let $\Phi: L^\infty(G)_1 \to T(G)$ be the defining quotient projection (9). The kernel of the derivative of Φ at 0 is the subspace*

$$N(G) = \{\mu \in L^\infty(G): \langle \mu, \varphi \rangle = 0, \text{ for all } \varphi \in A_2(G)\}. \tag{16}$$

Consequently, the tangent space at origin of $T(G)$ is $L^\infty(G)/N(G)$. □

We are equipped now to define the *Weil–Petersson* (W–P) inner product on $T(G)$. Given $\mu, \nu \in L^\infty(G)$ one sets

$$\text{W–P}(\mu, \nu) = \iint_{\Delta/G} \times \iint_{\Delta} \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4} d\xi \, d\eta \cdot dx \, dy. \tag{17}$$

We want to explain this formula. From Ahlfors [2] one recalls the fundamental map

$$L^\infty(G) \to B_2(G)$$

given by

$$\nu \mapsto \varphi[\nu](z) = \iint_{\Delta} \frac{\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4} d\xi \, d\eta. \tag{18}$$

Here $B_2(G)$ are all the "Nehari-bounded" holomorphic quadratic differentials for G (i.e. $\| \varphi(z)(1 - |z|^2)^2 \|_\infty < \infty$ on Δ). The kernel of (18) is known to be precisely $N(G)$, and consequently the W–P inner product (17) becomes:

$$\text{W–P}(\mu, \nu) = \langle \mu, \varphi[\nu] \rangle. \tag{19}$$

However, $\varphi[\nu] (\in B_2(G))$ is not necessarily in $A_2(G)$ for general Fuchsian group G. So the W–P formula is usually defined only for the finite dimensional $T(G)$—when Δ/G is on finite conformal type—because for such G one knows $B_2(G) \equiv A_2(G)$. For these G the formula (17), equivalently (19), converges.

II.3. The Kähler metric on Diff(S^1)/$SL(2, \mathbb{R})$ *is Weil–Petersson.* The main program is to calculate the physicists' Kähler metric on M in terms of Beltrami differentials. Let $\mu \in L^\infty(\Delta)$ represent any tangent vector to $T(1)$. The corresponding "quasi-symmetric" vector field on S^1 is $v = \dot{w}[\mu](z)(\partial/\partial z)$ for $z = e^{i\theta}$. Consequently,

$$v(\theta)\frac{\partial}{\partial\theta} = \frac{\dot{w}[\mu](e^{i\theta})}{ie^{i\theta}} \frac{\partial}{\partial\theta}. \tag{20}$$

Here $\dot{w}[\mu]$ is the first variation term in the solution theory of the Beltrami equation (7):

$$w_{t\mu}(z) = z + t\dot{w}[\mu](z) + o(t), \quad t \to 0. \tag{21}$$

Theorem II.1. *Let $\mu, \nu \in L^\infty(\Delta)$ represent two tangent vectors at the origin of $T(1)$.*

Then the canonical hermitian inner product g of formula (5) becomes formally:

$$g(\mu, v) = -\frac{ia}{3\pi^2} \iint_{\Delta} \times \iint_{\Delta} \frac{\mu(z)\overline{v(\zeta)}}{(1-z\bar{\zeta})^4} d\xi \, d\eta \cdot dx \, dy. \tag{22}$$

This integration is convergent whenever μ and v represent (see (20)) at least $C^{3/2+\varepsilon}$ smooth vector fields on S^1. (Thus C^2 vector fields will certainly produce convergence.)

Clearly we first need the Fourier coefficients of $v(\theta)$ so that we may apply formula (5). We have

Lemma. *$\mu \in L^\infty(\Delta)$ corresponds to the real vector field $v(\theta)(\partial/\partial\theta)$ on S^1 as per (20). The Fourier coefficients of $v(\theta)$ are:*

$$v_k = \frac{i}{\pi} \iint_{\Delta} \overline{\mu(z)} \bar{z}^{k-2} dx \, dy, \quad \text{for} \quad k \geqq 2 \tag{23}$$

and $v_k = \overline{v_{-k}}$ for $k \leqq -2$.

Proof. The explicit formula for $\dot{w}[\mu]$ is

$$\dot{w}[\mu](\zeta) = -\frac{(\zeta-1)(\zeta+1)(\zeta+i)}{\pi} \cdot \left\{ \iint_{\Delta} \frac{\mu(z) dx \, dy}{(z-1)(z+1)(z+i)(\zeta-z)} + \iint_{\Delta} \frac{i\overline{\mu(z)} dx \, dy}{(\bar{z}-1)(\bar{z}+1)(\bar{z}-i)(1-\zeta\bar{z})} \right\}. \tag{24}$$

This is the formula of Sect. I.2.12 of Nag [14] adapated to the disk. Or see formula (1.9) of Ahlfors [2]. Note that $\dot{w}[\mu]$ vanishes at ± 1 and $-i$, as it must by the normalization enforced on w_μ.

In (24) we substitute $\zeta = e^{i\theta}$ and make a straight calculation (in which some remarkable cancellations occur) to get the Fourier coefficients as desired. □

Proof of the Theorem. The coefficients v_k and w_k are determined respectively from μ and v by the preceding Lemma. To apply formula (5) for the hermitian inner product on M we calculate

$$\sum_{2}^{\infty} \bar{v}_m w_m (m^3 - m) = -\frac{1}{\pi^2} \iint_{\Delta} \times \iint_{\Delta} \mu(z)\overline{v(\zeta)} \left(\sum_{2}^{\infty} z^{m-2} \bar{\zeta}^{m-2} (m^3 - m) \right) d\xi \, d\eta \cdot dx \, dy. \tag{25}$$

We have interchanged order of sum and integral above. This is justified whenever there is absolute convergence—for example when the vector fields are $C^{3/2+\varepsilon}$ as noted before.

Now,

$$\sum_{m=2}^{\infty} (m^3 - m) x^{m-2} = \frac{-1}{6(1-x)^4}, \quad \text{for} \quad |x| < 1, \tag{26}$$

(as may be seen by differentiating $\sum_{0}^{\infty} x^m = (1-x)^{-1}$ three times). Using (5), (25), (26) we get the Theorem. □

Remarks. Since the choice of a is at our disposal (up to a positive multiple) we will henceforth normalize so that the factor outside the integral in (22) becomes unity. **Notice that formula (22) is now precisely the W–P formula (17) for the case $G=\{1\}$, i.e. for universal Teichmüller space.**

Corollary. *Let G be any infinite Fuchsian group. Then $T(G)$ intersects $M=\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$* **transversely**. *Indeed, every non-null tangent vector to $T(G)$ at the origin, described by $\mu\in(L^\infty(G)-N(G))$ produces a vector field on S^1 that cannot be even $C^{3/2+\varepsilon}$ smooth (for any $\varepsilon>0$). (Thus, $\dot{w}[\mu]$ cannot be C^2 on S^1 if $\mu\in(L^\infty(G)-N(G))$, whereas the tangent vectors to M are C^∞ smooth vector fields.)*

Remark. This Corollary is an infinitesimal form of Mostow rigidity on the line. See Sect. II.4 below.

Proof. Formula (22) says that

$$g(\mu,\nu)=\iint_{\Delta}\mu\cdot\varphi[\nu], \tag{27}$$

where $\varphi[\nu]$ is defined in Ahlfors' map (18). But $\iint\mu\varphi$ over any union of N fundamental regions for G is $N\langle\mu,\varphi\rangle$, by G-invariance. Therefore, for $\mu,\nu\in(L^\infty(G)-N(G))$ formula (22) and (27) *must diverge* whenever G has infinitely many elements. All the assertions are now clear. □

Remark. If μ is a G-invariant Beltrami differential and φ is any (integrable) holomorphic function on Δ then the classic Poincaré series formula is

$$\iint_{\Delta}\mu\cdot\varphi=\iint_{\Delta/G}\mu\cdot\theta_2(\varphi), \tag{28}$$

where $\theta_2(\varphi)=\sum_{g\in G}(\varphi\circ g)(g')^2$. (See [14], p. 73, p. 174, p. 231, etc.)

But if $\nu\in L^\infty(G)$ then $\varphi[\nu]$ is *already* a G-invariant quadratic differential. So the Poincaré series is simply (order of G) $\times\,\varphi$, and the divergence of (27) is again manifest.

In spite of this divergence for $T(G)$ of the metric in formula (22), *one can still recover the* W–P *metric on every finite dimensional $T(G)$ from this formula.* The ideas is to simply regulate passage to the limit in the improper integral. For $\mu,\nu\in L^\infty(G)$ rewrite (22) as

$$g(\mu,\nu)=\lim_{r\to1^-}g_r(\mu,\nu), \tag{29}$$

where

$$g_r(\mu,\nu)=\iint_{D_r}\times\iint_{\Delta}\frac{\mu(z)\overline{\nu(\zeta)}}{(1-z\bar{\zeta})^4}d\xi\,d\eta\cdot dx\,dy, \tag{30}$$

or, equivalently (see Eqs. (19) and (27) above),

$$g_r(\mu,\nu)=\iint_{D_r}\mu\cdot\varphi[\nu], \tag{30'}$$

where $D_r=\Delta_r=\{|z|\leqq r\}$, $r<1$, is simply the disk of radius r.

Theorem II.2. *The* W–P *inner product in any finite dimensional Teichmüller space $T(G)$, for $\mu, \nu \in L^\infty(G)$, is related to the canonical inner product g by the formula*:

$$\frac{\text{W–P}(\mu,\nu)}{\text{W–P}(\nu_0,\nu_0)} = \lim_{r\to 1^-} \frac{g_r(\mu,\nu)}{g_r(\nu_0,\nu_0)}, \tag{31}$$

where $\nu_0 \in L^\infty(G) - N(G)$ represents any nonzero tangent vector to $T(G)$ at its origin.

Proof. The group G is finitely generated of the first kind. If a function f on the hyperbolic unit disk Δ is G-automorphic then it is known that

$$\iint_{\Delta/G} f\cdot dA = \lim_{r\to\infty}\left(\frac{\iint_{B_r(x)} f\cdot dA}{\iint_{B_r(x)} dA}\right) \times \text{area}\,(\Delta/G), \tag{32}$$

where dA is the hyperbolic area element and $B_r(x)$ is the closed hyperbolic ball of radius r centered at *any* $x\in\Delta$. (In particular, note that the limit involved is independent of x.) The formula (32) follows from S. J. Patterson's results on the distribution in Δ of lattice points for G,—this was kindly communicated to the first author by J. Velling.

Applying (32) to the automorphic function $f = \mu(z)\cdot\varphi[\nu](z)(1-|z|^2)^2$ we see that theorem immediately. □

Remark. The idea, of course, is that $g_r(\mu,\nu)$ is "essentially" W–P(μ,ν) times the number of copies of a fundamental domain Δ/G sitting inside Δ_r. Note that the right-hand-side of (31) is an ∞/∞ form and L'Hospital's rule can be applied to advantage.

Remark. Since the metric g was Kähler (Sect. II.1) to start with, this sheds new light on the Kählericity of the Weil–Petersson metric. That W–P is Kähler on $T(G)$ was first shown by Ahlfors in [2]. The negative curvature of W–P also fits in neatly with the corresponding fact known for g.

Utilising the natural relationship exhibited in formula (31) between the Kähler metric g on M and the usual Weil–Petersson metrics, one should now be able to *derive* the Kahlericity and the negative curvature of W–P from the corresponding facts for g.

II.4. On the Transversality of $T(G)$ with M. The corollary of Theorem II.1 of the previous section can be prettily interpreted by considering the deep question of which quasidisks correspond to points of M, and which to points of $T(G)$. We are now thinking of Teichmüller space as the set of (normalized) quasidisks—as explained in Sect. II.2.

Bowen [4] had proved that if G uniformizes a compact Riemann surface, then *every non-origin point of $T(G)$ corresponds to a quasidisk with fractal boundary* (i.e., Hausdorff dimension of the bounding quasicircle is strictly greater than unity).

On the other hand, *the quasidisks corresponding to the points of* $M = \text{Diff}(S^1)/SL(2,\mathbb{R})$ *are the ones with C^∞ boundaries.*

In fact, recall from Sect. II.2 the relationship between the two ways (12) and (13) of considering Teichmüller space. If the quasidisk is smoothly bounded then the Riemann mappings ρ and σ both extend smoothly to the boundaries. Hence, by formula (14) certainly the C^∞ quasidisks correspond to points of M. That they comprise *all* of M is shown in Kirillov [12] using an idea of Sullivan.

It is also easy to see that a G-compatible $f \in \mathrm{Homeo}_{qs}(S^1)$ must be non-smooth (if f is not itself a Möbius transformation). For example, it is known that if f is compatible with a compact surface group G, and f is C^1 at even a single point of S^1, then f must represent the origin of $T(G)$. One idea is to note that $fGf^{-1} = G_\mu$, and f assumed C^1 on the circle, implies, on direct differentiation, that the hyperbolic marked length spectra of Δ/G and Δ/G_μ are the same. Hence G_μ is not a non-trivial deformation of G—so that f represents the origin of $T(G)$, as desired. These assertions are forms of Mostow rigidity on the line. (See Agard [21].)

We thus see that the fact that each $T(G)$ projects transversely out of M is intimately related to the *non-smoothness* of the vector fields, or the quasisymmetric homeomorphisms, or the quasidisks corresponding to the (non-origin) points of $T(G)$.

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