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Intertwining Operators for Solving Differential Equations, with Applications to Symmetric Spaces

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Abstract. The use of intertwining operators to solve both ordinary and partial differential equations is developed. Classes of intertwining operators are constructed which transform between Laplacians which are self-adjoint with respect to different non-trivial measures. In the two-dimensional case, the intertwining operator transforms a non-separable partial differential operator to a separable one. As an application, the heat kernels on the rank 1 and rank 2 symmetric spaces are constructed.

It has long been appreciated that one of the nice properties of the special functions is that there exist differential operators which transform between functions of the same type, changing the indices of the functions by integral amounts. This property is the source of formulae which provide a compact expression for certain special functions in terms of powers of a differential operator applied to elementary functions. It has also been known for some time that the use of fractional differential operators, or pseudo-differential operators, extends the set of transformations between functions, allowing the indices to be changed by non-integral amounts.

Most of this common knowledge is for orthogonal polynomials and other special functions in one dimension. Similar results have been found for orthogonal polynomials in two dimensions [1, 2]. For particular coefficients, these two-dimensional orthogonal polynomials correspond to eigenfunctions of the radial part of the Laplace–Beltrami operator on certain rank 2 symmetric spaces. We present here a new approach to the construction and transformation among eigenfunctions of differential equations based on the construction of intertwining operators. This approach generalizes the classical operator transformations among special functions and is naturally applied in higher dimensions.

The intertwining operator approach to eigenfunctions was motivated by the method used by Dowker [3] to find the heat kernel for a free particle propagating on a Lie group manifold. It was developed by one of the authors [4] to find the heat kernel of a free-particle propagating on an n-dimensional sphere. Intertwining operators have also been used to solve non-linear integrable systems [5]. The intent here is to construct a few general classes of intertwining operators for one and two-dimensional Laplace operators. To illustrate the usefulness of the results, the heat kernel on the rank 1 and rank 2 symmetric spaces will be calculated.

The quantum problem of a free particle propagating on a symmetric space is important as an exactly soluble model of quantum field theory in curved space and because it arises in a heat kernel approach to the Kaluza–Klein program of particle physics. Furthermore, the problem of a free particle on a symmetric space is equivalent [6] to the integrable quantum problem of n particles on a line interacting through certain potentials, e.g. $\sin^{-2}(x_i - x_j)$. To understand each of these more fully, it would be useful to have a general procedure through which the heat kernel and eigenfunctions on any symmetric space could be derived. This problem has been studied independently from a different perspective by others [7–10].

Using the intertwining operator construction, it is found that the radial Laplacian of a symmetric space can be transformed to the ordinary Laplacian on its maximal torus. Let each set of roots in the reduced root space which are connected by the action of the Weyl group be called a Weyl set. To have Weyl invariance, the multiplicity of each root in a Weyl set must be the same, though multiplicities in disjoint Weyl sets can differ. The transformation from the radial Laplacian of a symmetric space to the ordinary Laplacian takes the form of multiplicity reduction, in which a differential intertwining operator reduces the multiplicity of the roots in a Weyl set by two in each transformation. The multiplicities of restricted roots in a symmetric space are not all even, except in the special cases that the space is split rank or a Lie group. Pseudodifferential operators are needed to reduce the multiplicity of the roots in a Weyl set by one. Explicit integral representations of these operators are available in one-dimension. In the higher rank case, if only one simple restricted root has odd multiplicity, after the multiplicity of the other roots is fully reduced, the problem separates into a product of rank one problems which can then be solved. At this time, we are still working on the cases in which more than one restricted root has odd multiplicity.

The outline of this paper will be to consider first some general properties of intertwining operators and their application. This will serve to give some insight into their usefulness. Next, transformations in one-dimensional problems will be treated and two general classes of first order differential intertwining operators constructed. Following a brief remark on pseudo-differential operators in one dimension, this formal result will be applied to find the heat kernel on all of the rank one symmetric spaces. Two-dimensional problems will then be considered formally and a class of differential intertwining operators constructed. These will be used to find the heat kernel on the rank 2 symmetric space whose Dynkin diagram has a double link. The heat kernel on the rank 2 symmetric spaces whose Dynkin diagram has a single link will be found after constructing the intertwining operator specific to that problem. In the final section, the extension of these results to arbitrary rank symmetric spaces will be discussed briefly.

1. Intertwining Operators and Formal Applications

An operator D is said to be an *intertwining operator* if it relates operators, L and

$\bar{L}$, by

$$LD = D(\bar{L} + c), \tag{1}$$

where c is a constant removed from $\bar{L}$ for convenience. By applying this relation to the eigenfunctions of $\bar{L}$, one obtains the following proposition:

Proposition 1.1. *If $\bar{\phi}_n$ is an eigenfunction of $\bar{L}$ with eigenvalue $\bar{\lambda}_n$, then $\phi_n = D\bar{\phi}_n$ is an (unnormalized) eigenfunction of L with eigenvalue $\lambda_n = \bar{\lambda}_n + c$.*

This simple fact is at the heart of the usefulness of intertwining operators. If an operator L can be reduced to an operator $\bar{L}$ with known spectrum, its spectrum is then known as well.

To show the connection to the classical operator transformations among special functions, consider two examples. The Hermite polynomials are eigenfunctions of $\bar{L} = \partial^2 - 2x\partial$, where $\partial = d/dx$. Taking $D = \partial$, one has

$$\bar{L}D = D(\bar{L} + 2).$$

If H_n is an eigenfunction of $\bar{L}$ with eigenvalue $\bar{\lambda}_n = -2n$, then $DH_n = \partial H_n$ is an eigenfunction of $\bar{L}$ with eigenvalue $\lambda_n = -2(n-1)$. This is the unnormalized expression of the relation

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x). \tag{2}$$

The Bessel function $\bar{\phi}_\alpha = x^\gamma J_\gamma(\alpha x)$ is an eigenfunction of

$$\bar{L}_\gamma = \partial^2 - \frac{2\gamma - 1}{x}\partial \tag{3}$$

with (continuous) eigenvalue $\bar{\lambda}_\alpha = -\alpha^2$. Taking $D = x^{-1}\partial$, one has

$$LD = D\bar{L}_\gamma,$$

where $L = \bar{L}_{\gamma-1}$ and, consequently, the familiar formula

$$\frac{1}{x}\partial(x^\gamma J_\gamma(\alpha x)) = x^{\gamma-1}J_{\gamma-1}(\alpha x).$$

Note that in this case the entire spectrum has been transformed to solve a new differential equation while, in the previous case, one eigenfunction was transformed to another of the same operator.

One way of understanding the existence of these transformations is in terms of the factorization of the operator $\bar{L}$ into a product of "raising" and "lowering" operators. This approach is developed in detail by Infeld and Hull [11] for a wide collection of the classical special functions. The operators for the Hermite polynomials and the Bessel functions factor as

$$\bar{L} = (\partial - 2x)\partial$$

and

$$\bar{L}_\gamma = (x^{2\gamma-1}\partial x^{-2\gamma+2})\left(\frac{1}{x}\partial\right).$$

Infeld and Hull transform all operators to *potential-form*,

$$\tilde{L} = \partial^2 - V(x) \tag{4}$$

before factoring. The approach here is to work with operators in *measure-form*

$$L = \partial^2 + \frac{\mu'(x)}{\mu(x)}\partial, \tag{5}$$

where the prime indicates differentiation with respect to x. This operator is self-adjoint in the measure $\mu(x)dx$, hence the name. Using the measure density, one can transform from L to $\tilde{L}$ by

$$\mu^{1/2}L\mu^{-1/2} = \tilde{L}. \tag{6}$$

Transforming the other direction is more difficult *a priori* because it requires an eigenfunction of $\tilde{L}$. The square of this eigenfunction becomes the measure density. The higher dimensional analogs of the measure- and potential-forms are obvious.

The significance of using the measure-form of the differential equation is that the intertwining operator acts to transform the measure density. Let

$$D = f^{-1}\hat{D} \tag{7}$$

be the ansatz for the form of a differential intertwining operator, where $\hat{D}$ is an operator whose leading term has constant coefficient and f is a function to be determined. It will be found that this operator intertwines two measure-form operators whose measure densities are related by

$$\mu = \kappa f^2 \bar{\mu}, \tag{8}$$

where κ is a possible constant normalizing the measure densities. It should be emphasized that f is not arbitrary, but will be constructed later (see Theorems 2.1, 2.2, 4.1 and 6.1).

Since many problems are more familiar in their potential-forms (4), it is useful to give the transformation of the potential that a differential intertwining operator (7) induces. This result will hold in arbitrary dimension.

Proposition 1.2. *If $D = f^{-1}\hat{D}$ intertwines two measure-form operators L and $\bar{L}$, then $\tilde{D} = \kappa^{1/2}\bar{\mu}^{1/2}\hat{D}\bar{\mu}^{-1/2}$ intertwines their potential-forms. The potentials are related by*

$$V = \frac{1}{f}\bar{L}f + \bar{V}. \tag{9}$$

Proof. From (6), one finds that

$$L = \mu^{-1/2}\tilde{L}\mu^{1/2}.$$

Substituting this in the definition of the intertwining operator (1), one finds $\tilde{D} = \mu^{1/2}f^{-1}\hat{D}\bar{\mu}^{-1/2}$ intertwines the potential forms. The desired form of $\tilde{D}$ follows from using (8). Using (8) again in evaluating the transformation from L to $\tilde{L}$ gives the transformation of the potential. ■

Using factorization to construct the intertwining operators is special to one-dimension, but the nature of factorization and its generalization to higher dimensions can be understood from the following proposition.

Proposition 1.3. *The operator* $\bar{M} = \bar{\mu}^{-1} D^{\dagger} \mu D$ *commutes with* $\bar{L}$.

Proof. From Proposition 1.1, let the normalized eigenfunctions of L be $\phi_n = N_n D \bar{\phi}_n$, where N_n is an as yet undetermined normalization constant. From the orthonormality of the eigenfunctions of L, we have

$$\begin{aligned}\lambda_n \delta_{nm} &= N_n \int \phi_m(x)(LD\bar{\phi}_n(x))\mu(x)dx \\ &= N_n \int \phi_m(x)(D(\bar{L} + \lambda_n - \bar{\lambda}_n)\bar{\phi}_n(x))\mu(x)dx.\end{aligned}$$

Integrating by parts, respectively, to form the adjoint,

$$\begin{aligned}\lambda_n \delta_{nm} &= N_n \int \left(\frac{1}{\bar{\mu}} D^{\dagger} \mu L \phi_m \right) \bar{\phi}_n \bar{\mu} dx \\ &= N_n \int \left((\bar{L} + \lambda_n - \bar{\lambda}_n) \frac{1}{\bar{\mu}} D^{\dagger} \mu \phi_m \right) \bar{\phi}_n(x) \bar{\mu}(x) dx.\end{aligned}$$

From this, using orthonormality of the eigenfunctions of $\bar{L}$, one concludes that

$$\frac{1}{\bar{\mu}} D^{\dagger} \mu L = (\bar{L} + \lambda_n - \bar{\lambda}_n) \frac{1}{\bar{\mu}} D^{\dagger} \mu.$$

Acting on this equation from the right by D, one has

$$\begin{aligned}\frac{1}{\bar{\mu}} D^{\dagger} \mu (LD) &= \frac{1}{\bar{\mu}} D^{\dagger} \mu D (\bar{L} + \lambda_n - \bar{\lambda}_n) \\ &= (\bar{L} + \lambda_n - \bar{\lambda}_n) \frac{1}{\bar{\mu}} D^{\dagger} \mu D,\end{aligned}$$

from which one concludes that $\bar{M} = \bar{\mu}^{-1} D^{\dagger} \mu D$ commutes with $\bar{L}$. ■

In one-dimension, the only operators which commute with $\bar{L}$ are the identity operator and powers of $\bar{L}$. If one takes $\bar{M} = \bar{L}$, this leads to the factorization of Infeld and Hull [11]. In higher dimensions, there are independent operators of higher order which commute with $\bar{L}$. In the example of symmetric spaces, one knows that the rank of the symmetric space corresponds to the number of independent commuting Casimir operators, of which the Laplace–Beltrami operator is one. $\bar{M}$ will in general be formed from a combination of these operators.

Using $\bar{M}$, one can normalize the eigenfunctions of L produced by Proposition 1.1. Since $\bar{M}$ commutes with $\bar{L}$, they have the same eigenfunctions though in general different eigenvalues. Denoting the eigenvalues of $\bar{M}$ by $\bar{\nu}_n$, one finds

Proposition 1.4. *The normalized eigenfunctions of* L *are* $\phi_n = N_n D \bar{\phi}_n$, *where* $N_n = (\bar{\nu}_n)^{-1/2}$.

Proof. Assume orthonormality of the ϕ_n and express this in terms of the $\bar{\phi}_n$,

$$\delta_{nm} = N_n N_m \int D\bar{\phi}_m D\bar{\phi}_n \mu dx.$$

Integrate by parts to obtain the result from

$$\delta_{nm} = N_n N_m \int \bar{\phi}_m \bar{M} \bar{\phi}_n \bar{\mu} dx = N_n^2 \bar{v}_n \delta_{nm}. \quad \blacksquare$$

Similarly, one can obtain the normalized eigenfunctions of the potential form of the operator. Denote the eigenfunctions of the potential form operator $\tilde{\bar{L}}$ by $\tilde{\bar{\phi}}_n = \bar{\mu}^{1/2}\bar{\phi}_n$. Using Proposition 1.2 and Proposition 1.4, one finds

Proposition 1.5. *The normalized eigenfunctions of $\tilde{L}$ are*

$$\tilde{\phi}_n = \left(\frac{\kappa}{\bar{v}_n}\right)^{1/2} \bar{\mu}^{1/2} \hat{D} \bar{\mu}^{-1/2} \tilde{\bar{\phi}}_n. \tag{10}$$

Using Proposition 1.4 one can go partway to relating the heat kernel of $i\partial_t + L$ to that of $i\partial_t + \bar{L}$. Assume for convenience a compact manifold so that the spectrum is discrete. An analogous argument holds in the non-compact case. From the eigenfunction expansion of the heat kernel, one finds

$$\begin{aligned} K(x, y; t) &= \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)e^{i\lambda_n t} \\ &= D(x)D(y) \sum_{n=0}^{\infty} \bar{\phi}_n(x)\bar{\phi}_n(y)\frac{e^{i\lambda_n t}}{\bar{v}_n}. \end{aligned}$$

If an operator $\mathcal{O}(t)$ can be found such that

$$\frac{e^{i\lambda_n t}}{\bar{v}_n} = \mathcal{O}(t)\, e^{i\bar{\lambda}_n t}, \tag{11}$$

for all n, then the heat kernels of $i\partial_t + L$ and $i\partial_t + \bar{L}$ will be related by

$$K(x, y; t) = D(x)D(y)\mathcal{O}(t)\bar{K}(x, y; t). \tag{12}$$

In one-dimension, this is straightforward (see Proposition 2.1). In higher dimensions, it is not clear that in general there will be any functional relation between $\bar{v}_n$ and $\bar{\lambda}_n$.

There is another approach to finding the heat kernel which works when one endpoint is at zero and the measure density vanishes there. This is the case for the symmetric spaces because the reduction from the full Laplacian to the radial Laplacian is made by translating one endpoint to the identity and then recognizing that since all maximal tori are conjugate under the action of the isotropy group, the heat kernel depends only on distance in a chosen maximal tours (i.e. the heat kernel is a class function). If the intertwining operator D in (1) relates the delta functions defined with respect to the measures in which L and $\bar{L}$ are self-adjoint,

$$\delta(x, 0) = ND\bar{\delta}(x, 0), \tag{13}$$

where N is a normalization constant, then one has

Proposition 1.6. *The heat kernels $K(x, 0; t)$ and $\bar{K}(x, 0; t)$ of the heat operators, $i\partial_t + L$ and $i\partial_t + \bar{L}$, are related by*

$$K(x, 0; t) = e^{ict}ND\bar{K}(x, 0; t). \tag{14}$$

The relation (13) between the delta functions can be verified by substituting into the defining equation

$$f(0) = \int f(x)\delta(x,0)\mu(x)dx \tag{15}$$

and integrating by parts. One can also use the formal definition of the delta function in terms of the eigenfunction expansion to derive a general condition for the relation to hold. One finds

Proposition 1.7. *A necessary and sufficient condition for* $\delta(x,0) = ND\bar{\delta}(x,0)$ *is that*

$$N = \frac{D\bar{\phi}_n(0)}{\bar{v}_n\bar{\phi}_n(0)},$$

independent of n.

Proof. From the eigenfunction expansion of the heat kernel at $t = 0$ in terms of both ϕ_n and $\bar{\phi}_n$,

$$\delta(x,0) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(0)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\bar{v}_n} D\bar{\phi}_n(x)D\bar{\phi}_n(0).$$

(Note that the normalization factor N for D cancels out in the result for the normalized eigenfunctions of L given in Proposition 1.4. This is because changing D to ND changes $\bar{v}_n$ to $N^2\bar{v}_n$.) Requiring $\delta = ND\bar{\delta}$ implies

$$\delta(x,0) = \sum_{n=0}^{\infty} ND\bar{\phi}_n(x)\bar{\phi}_n(0).$$

Comparing these expressions gives

$$N = \frac{D\bar{\phi}_n(0)}{\bar{v}_n\bar{\phi}_n(0)},$$

independent of n. The converse follows by using this expression to substitute for $D\bar{\phi}_n(0)$ in the eigenfunction expansion above of $\delta(x,0)$ in terms of $\bar{\phi}_n$. ■

Since simple computations of the eigenvalues $\bar{v}_n$ and of the value of $D\bar{\phi}_n(0)$ are not known at this time, it is generally easiest to verify the relationship between the delta functions directly.

This completes the list of results that can be obtained without reference to the structure of the intertwining operator. The next step is to consider intertwining operators in one-dimension.

2. One-Dimension

Two classes of differential intertwining operators can be constructed in one-dimension. The approach is to propose an ansatz for D and to require that it intertwine two measure-form operators by

$$LD = D(\bar{L} + c). \tag{16}$$

The general ansatz for D is $D=f^{-1}\hat{D}$, where $\hat{D}$ is an operator to be determined whose leading term has constant coefficient. Let L and $\bar{L}$ be operators of measure-form

$$L=\partial^2+m(x)\partial,$$
$$\bar{L}=\partial^2+\bar{m}(x)\partial, \tag{17}$$

where $m=\mu'/\mu$ and $\bar{m}=\bar{\mu}'/\bar{\mu}$. Let $\bar{\phi}_n$ and $\bar{\lambda}_n$ be the (normalized) eigenfunctions and eigenvalues of $\bar{L}$.

The first class of one-dimensional intertwining operators is given by

Theorem 2.1. *$D=f^{-1}\hat{D}$ is an intertwining operator of the first class between L and $\bar{L}$ when $\hat{D}$ commutes with $\bar{L}$ and f is an eigenfunction of $\bar{L}$ with eigenvalue $\bar{\lambda}_k$. Then $\mu=\kappa f^2\bar{\mu}$ (κ a constant) and $c=-\bar{\lambda}_k$.*

Proof. Substituting the ansatz $D=f^{-1}\hat{D}$ in (16) and allowing the operators in L to act on f^{-1}, one obtains three equations

$$m-2(f'/f)=\bar{m},$$
$$-(f''/f)-\bar{m}(f'/f)=c, \tag{18}$$
$$\bar{L}\hat{D}=\hat{D}\bar{L},$$

where the first equation has been used in obtaining the second two. The theorem follows from the solution of these three equations. Since $m=\mu'/\mu$, a constant κ normalizing the measure density μ relative to $\bar{\mu}$ may be present in the solution of the first equation. ■

In one-dimension, the only operators which commute with $\bar{L}$ are the identity operator and powers of $\bar{L}$. Without loss of generality, in intertwining operators of the first class, one can take $\hat{D}=1$ (other choices merely change the coefficient of the unnormalized eigenfunctions of L). Then $\bar{M}=\bar{\mu}^{-1}f^{-2}\mu=\kappa$ with eigenvalues $\nu_n=\kappa$. The normalized eigenfunctions of L are

$$\phi_n=f^{-1}\kappa^{-1/2}\bar{\phi}_n. \tag{19}$$

The second class of differential intertwining operators takes the form $D=f^{-1}\partial$.

Theorem 2.2. *$D=f^{-1}\partial$ is an intertwining operator of the second class relating the measure-form operator $\bar{L}$ to the measure-form operator L, with measure density $\mu=\kappa f^2\bar{\mu}$ (κ a constant) when $f=\bar{\phi}_k'$ and $c=-\bar{\lambda}_k$, or when $f=\bar{\mu}^{-1}\int\bar{\mu}\,dx$ and $c=0$.*

Proof. Let $D=f^{-1}\partial$ and L have measure density μ. Write out $LD=D(\bar{L}+c)$ with all of the (unevaluated) differential operators on the right. Collecting like powers of ∂, one obtains two equations

$$-2f'/f+m=\bar{m},$$
$$-(f'/f^2)'-m(f'/f^2)=f^{-1}\bar{m}'+cf^{-1}.$$

The first may be solved to find $\mu=\kappa f^2\bar{\mu}$, where κ is a possible overall constant normalizing the measures. This can be used in the second to obtain

$$f''+\bar{m}f'+\bar{m}'f=-cf.$$

Letting $F = \partial^2 + \bar{m}\partial + \bar{m}'$, this equation is $Ff = -cf$. But it is clear that $\partial\bar{L} = F\partial$, so if $\bar{\phi}_k$ is an eigenfunction of $\bar{L}$ with eigenvalue $\bar{\lambda}_k$, then one has $f = \bar{\phi}'_k$ with $c = -\bar{\lambda}_k$. On the other hand, if $\bar{L}g = a$ where a is constant, then $f = \partial g$ satisfies $Ff = 0$. Writing $\bar{L} = \bar{\mu}^{-1}\partial\bar{\mu}\partial$ and taking $a = 1$, $\bar{L}g = 1$ can be solved by integration to give $f = \bar{\mu}^{-1}\int\bar{\mu}\,dx$. ■

One finds $\bar{M} = -\kappa\bar{L}$, so $\bar{\nu}_n = -\kappa\bar{\lambda}_n$. This can be used in (11) and (12) to prove

Proposition 2.1. *In one-dimension for intertwining operators of the second class, the heat kernel $K(x, y; t)$ of $i\partial_t + L$ is given in terms of the heat kernel $\bar{K}(x, y; t)$ of $i\partial_t + \bar{L}$ by*

$$K(x, y; t) = i\kappa^{-1}e^{ict}\int_t^{t+i\infty} dt D(x)D(y)\bar{K}(x, y; t).$$

An application of Theorem 2.2 to the Bessel equation (3) with $\bar{\mu} = x^{1-2\gamma}$ in the case $c = 0$ gives $f = x$ (up to a constant which will be determined by normalization). After normalization, this gives the classical transformation $x^{\gamma-1}J_{\gamma-1}(\alpha x) = x^{-1}\partial(x^{\gamma}J_{\gamma}(\alpha x))$.

This completes the general construction of intertwining operators in one dimension. These results may now be used to find the eigenfunctions and heat kernel for a particle propagating on a rank one symmetric space.

3. Rank-One Symmetric Spaces

The quantum problem of a free particle propagating on a rank-one symmetric space is a straightforward application of the intertwining operators. The problem on the n-dimensional spheres was discussed previously by one of the authors [4]. The result on the rank 1 spaces has also been obtained independently by Debiard and Gaveau [8].

To be concrete, the compact (positive curvature) symmetric spaces will be considered, but the solution on the non-compact (negative and zero curvature) symmetric spaces follows directly. The first step is to recognize that the heat kernel (propagator) for a particle on the symmetric space only depends on the separation of the endpoints in a maximal torus. This is evident because without loss of generality one endpoint can be taken at the identity of the symmetric space while the other endpoint lies in a maximal torus. The action of the isotropy group "rotates" the other endpoint about the identity. All maximal tori are conjugate under the action of the isotropy group and one concludes that the propagator is only a function of distance in the maximal torus. Restricting the full Laplace–Beltrami operator on the symmetric space to the maximal torus gives the radial part of the operator [13]. In a compact rank one space, this is a one dimensional operator on the circle.

The rank one symmetric spaces have one root $\hat{x}$ of multiplicity $2m_1$ and a half root $\hat{x}/2$ of multiplicity $2m_2$. The correspondences between the multiplicities and the explicit quotient of Lie groups for the compact symmetric spaces is given in Table 1.

The measure density is given by [13]

$$\mu(x) = \Omega_{2m_1+2m_2}2^{2m_2}\sin^{2m_1}(x)\sin^{2m_2}(x/2), \tag{20}$$

Table 1. Compact rank-one symmetric spaces ($n \geqq 2$) [12]

G/H	$2m_1$	$2m_2$
$S^n = SO(n+1)/SO(n)$	$n-1$	0
$P^{2n}(C) = SU(n+1)/S(U_n \times U_1)$	1	$2(n-1)$
$P^{4n}(H) = Sp(n+1)/Sp(n) \times Sp(1)$	3	$4(n-1)$
$P^{16}(Cay) = F_4/SO(9)$	7	8

where

$$\Omega_N = 2\pi^{(N+1)/2}/\Gamma((N+1)/2)$$

is the surface area of the unit N-sphere. One-half the sum of the positive roots is

$$\rho_{m_1,m_2} = (m_1 + m_2/2)\hat{x}. \tag{21}$$

The radial Laplacian is

$$L_{m_1,m_2} = \partial^2 + \left(2m_1 \cot x + m_2 \cot\frac{x}{2}\right)\partial. \tag{22}$$

By using a sequence of transformations based on eigenfunctions of the radial Laplacian at each stage, L_{m_1,m_2} will be built up from the ordinary Laplacian on the circle (the rank one torus) $L_{0,0} = \partial^2$ using Theorem 2.2. After normalizing the D operators so that they properly relate the delta functions on each space, Theorem 1.6 will be used to relate the heat kernel K_{m_1,m_2} on the rank one symmetric space to the heat kernel of the ordinary Laplacian on the circle, $K_T(x,0;t)$.

Proposition 3.1. *$D_2 = (\sin(x/2))^{-1}\partial$ intertwines L_{0,m_2+1} and L_{0,m_2} with $c = (2m_2+1)/4$.*

Proof. Observe that $f = \cos(x/2)$ is an eigenfunction of L_{0,m_2} with eigenvalue $-(2m_2+1)/4$ and apply Theorem 2.2. ∎

Proposition 3.2. *$D_1 = (\sin x)^{-1}\partial$ intertwines L_{m_1+1,m_2} and L_{m_1,m_2} with $c = 2m_1 + m_2 + 1$.*

Proof. Observe that $f = \cos(x) + m_2/(2m_1 + m_2 + 1)$ is an eigenfunction of L_{m_1,m_2} with eigenvalue $-(2m_1 + m_2 + 1)$ and apply Theorem 2.2. ∎

In each of these propositions, the differential intertwining operator changes m_i by one and hence the multiplicity by two. To change the multiplicity by one, one must use a pseudo-differential operator. Since the intertwining operator which changes the multiplicity is independent of the multiplicity, the n^{th} power of D_i changes m_i by n. This makes it reasonable to propose that the 1/2 power of D_i changes m_i by 1/2. Care must be taken however because the integral representation of a fractional derivative involves a boundary term [14]

$$\partial^{1/2}_{x-a} f(x) = \frac{f(a)}{(\pi(x-a))^{1/2}} + \frac{1}{\pi^{1/2}}\int_a^x \frac{f'(y)dy}{(x-y)^{1/2}}. \tag{23}$$

Usually the boundary point is chosen so that the boundary term vanishes. A fractional differential operator which differentiates with respect to a function can also be defined [15]

$$\partial^{1/2}_{g(x)} f(x) = \frac{f(g^{-1}(0))}{(\pi g(x))^{1/2}} + \frac{1}{\pi^{1/2}} \int_{g^{-1}(0)}^{x} \frac{f'(t)dt}{(g(x)-g(t))^{1/2}}. \tag{24}$$

Here, a fractional derivative will be applied to the differential operator L_{m_1,m_2} to intertwine to give $L_{m_1+1/2,m_2}$. Boundary terms will arise and these must vanish for the intertwining to work.

Proposition 3.3. $D_1^{1/2} = \partial^{1/2}_{\cos(x)+1}$ *intertwines* $L_{m_1+1/2,m_2}$ *and* L_{m_1,m_2},

$$L_{m_1+1/2,m_2} D_1^{1/2} f = D_1^{1/2}(L_{m_1,m_2} + c) f,$$

with $c = m_1 + m_2/2 + 1/4$ *when applied to functions* f *which vanish at* $x = \pi$.

Proof. It is easiest to change variables, $y = \cos(x) + 1$, in L_{m_1,m_2} and apply $\partial_y^{1/2}$ to it. Using the identities [14],

$$\partial_y^{1/2}(y^n f) = \frac{\pi^{1/2}}{2} \sum_{j=0}^{n} \binom{n}{j} \frac{y^{n-j}\partial_y^{1/2-j} f}{\Gamma(3/2-j)}, \tag{25}$$

$$\partial_y^{(2k+1)/2}(\partial_y^n f) = \partial_y^{n+(2k+1)/2} f + \sum_{j=0}^{n-1} \frac{y^{j-n-(2k+1)/2} f^{(j)}(0)}{\Gamma(j-n+1-(2k+1)/2)}, \tag{26}$$

one verifies that $\partial_y^{1/2}$ performs the desired intertwining with $c = m_1 + m_2/2 + 1/4$. A boundary term arises which must vanish at $y = 0$ or $x = \pi$. ■

The intertwining operators are normalized by requiring that they relate the delta functions (with one endpoint at 0) on the spaces whose radial Laplacians they intertwine.

Proposition 3.4. *The intertwining operators transform between the delta functions of each measure density*

$$\delta_{0,m_2+1}(x,0) = -(4\pi)^{-1} D_2 \delta_{0,m_2}(x,0), \tag{27}$$

$$\delta_{m_1+1,m_2}(x,0) = -(2\pi)^{-1} D_1 \delta_{m_1,m_2}(x,0). \tag{28}$$

Proof. Integrate by parts in the defining integral for the delta function (15). ■

Proposition 3.5.

$$\delta_{1/2,m_2}(x,0) = (2\pi)^{-1/2} D_1^{1/2} \delta_{0,m_2}(x,0). \tag{29}$$

This is easily verified by using the properties of the Jacobi polynomials, but we do not have a more direct proof.

Using these propositions, one can apply Theorem 1.6 to construct the heat kernel of $i\partial_t + L_{m_1,m_2}$ in terms of the heat kernel of $i\partial_t + \partial^2$ on the circle.

Theorem 3.1. *The heat kernel of* $i\partial_t + L_{m_1,m_2}$ *on the rank one symmetric spaces is given by*

$$K_{m_1,m_2}(x,0;t) = e^{i\rho^2_{m_1,m_2} t} \left(\frac{-1}{2\pi \sin x}\frac{d}{dx}\right)^{m_1} \left(\frac{-1}{4\pi \sin x/2}\frac{d}{dx}\right)^{m_2} K_T(x,0;t), \tag{30}$$

where $K_T(x,0;t)$ is the heat kernel of $i\partial_t+\partial^2$ on the circle (with periodic boundary conditions when ρ_{m_1,m_2} is an integer and with antiperiodic boundary conditions when it is a half integer). When m_1 is a half-integer,

$$\left(\frac{-1}{2\pi\sin x}\frac{d}{dx}\right)^{m_1}=\left(\frac{-1}{2\pi\sin x}\frac{d}{dx}\right)^{m_1-1/2}(2\pi)^{-1/2}\partial^{1/2}_{\cos(x)+1}.$$

Proof. If m_1 is an integer (and $m_2=0$), apply Proposition 3.2 repeatedly to intertwine $L_{m_1,0}$ and $L_{0,0}$ with

$$c_1=\sum_{j=0}^{m_1-1}2j+1=m_1^2.$$

From (21), $c_1=\rho^2_{m_1,0}$. If m_1 is a half-integer, apply Proposition 3.1 repeatedly to intertwine L_{0,m_2} and $L_{0,0}$ with

$$c_2=\sum_{j=0}^{m_2-1}\frac{2j+1}{4}=\frac{m_2^2}{4}.$$

From (21), $c_2=\rho^2_{0,m_2}$. Apply Proposition 3.3 to intertwine $L_{1/2,m_2}$ and L_{0,m_2} with $c_{1/2}=m_2/2+1/4$. Here, $c_{1/2}+c_2=\rho^2_{1/2,m_2}$. Now apply Proposition 3.2 to intertwine L_{m_1,m_2} and $L_{1/2,m_2}$ with

$$c_1=\sum_{j=0}^{[m_1]-1}2j+m_2+2=[m_1]^2+[m_1](m_2+1)$$

(where $[m_1]$ is the integral part of m_1). Here, $c_{1/2}+c_1+c_2=\rho^2_{m_1,m_2}$. Combining these results with Proposition 3.4, Proposition 3.5 and Theorem 1.6, one finds for integral m_2 and for integral or half-integral m_1 the desired result for the heat kernel for $i\partial_t+L_{m_1,m_2}$. ■

The heat kernel on the circle can be expressed in geometric form using the method of images to project the heat kernel from the universal covering space to the circle. Boundary conditions on the symmetric space require

$$K_T(x+2\pi n,0;t)=\exp(2\pi i\rho n)K_T(x,0;t).$$

Starting from the geometric form, the heat kernel can be written as a Jacobi theta function

$$\begin{aligned}K_T(x,0;t)&=(4\pi it)^{-1/2}\sum_{n=-\infty}^{\infty}e^{i(x+2\pi n)^2/4t-2\pi i\rho n}\\&=(4\pi it)^{-1/2}e^{ix^2/4t}\theta_3(x\pi/2t-\pi\rho,\pi/t).\end{aligned}\tag{31}$$

Using the inversion formula for Jacobi theta functions,

$$\theta_3(z,t)=(-it)^{-1/2}e^{-iz^2/\pi t}\theta_3(-z/t,-1/t),$$

$K_T(x,0;t)$ can be rewritten as an eigenfunction expansion

$$K_T(x,0;t)=\frac{1}{2\pi}e^{-i\rho^2t+i\rho x}\theta_3(x/2-\rho t,-t/\pi)$$

$$= \frac{1}{2\pi} \sum_{\lambda=-\infty}^{\infty} e^{i(\rho-\lambda)x - i(\lambda-\rho)^2 t}. \tag{32}$$

If ρ is an integer, the index of the sum can be shifted and the expression simplifies to

$$K_T(x, 0; t) = \frac{1}{2\pi}\left(2 \sum_{\lambda=1}^{\infty} \cos(\lambda x) e^{-i\lambda^2 t} + 1\right). \tag{33}$$

If ρ is a half-integer, the index of the sum can be shifted and the expression simplifies to

$$K_T(x, 0; t) = \frac{1}{\pi} \sum_{\lambda=0}^{\infty} \cos((\lambda + 1/2)x) e^{-i(\lambda+1/2)^2 t}. \tag{34}$$

It is important to emphasize that the intertwining operators relate the eigenfunctions of the operators (22) even when the values of m_1 and m_2 are not consistent with the existence of a symmetric space. In this more general case, it is not as useful to have the heat kernel, but it is the eigenfunctions themselves which are of interest. These are the Jacobi polynomials $P_n^{(a,b)}$, where $a = m_1 + m_2 - 1/2$, $b = m_1 - 1/2$. The intertwining operators are the shift operators which change the indices of the Jacobi polynomials. By reducing the indices down to zero, one obtains the Mehler–Dirichlet representation.

Proposition 3.6. *For a and b positive integer or half-integer, $a > b$, $\rho = (a + b + 1)/2$,*

$$P_n^{(a,b)}(\cos(x)) = N \partial_{\cos(x)+1}^{b+1/2} \partial_{\cos(x/2)+1}^{a-b} \frac{\cos((n+\rho)x)}{2n + 2\rho}, \tag{35}$$

where

$$N = \frac{2^{2b-a+3/2}\Gamma(n+b+1)}{\pi^{1/2}\Gamma(n+2\rho)}.$$

If $c_n^m(\cos(x/2))$ are the Gegenbauer polynomials, then one also has

Proposition 3.7. *For a arbitrary and b integer or half-integer, $a - b$ not a negative integer or zero,*

$$P_n^{(a,b)}(\cos(x)) = N \partial_{\cos(x)+1}^{b+1/2} c_{2n+2b+1}^{a-b}(\cos(x/2)), \tag{36}$$

where

$$N = \frac{2^{b+1/2}\Gamma(n+b+1)\Gamma(a-b)}{\pi^{1/2}\Gamma(n+a+b+1)}.$$

The above has concentrated on the compact case. The results in the noncompact case follow by repeating the same arguments after one replaces $\sin(x)$ in the measure density by either x or $\sinh(x)$. One finds that $f = x$ or $f = \cosh(x)$ are the appropriate eigenfunctions to apply Theorem 2.2. The intertwining operator is then either $D = \partial$ or $D = (\sinh x)^{-1}\partial$. The fractional differential operator is modified with the endpoint a becoming infinity which eliminates the boundary term and the $\cos(x)$ changing as one would expect. Theorem 2.1 then follows with these simple changes when the heat kernel on the circle is replaced by the heat kernel on the line. This kind of modification to obtain the non-compact results works in higher rank as well and will not be repeated.

4. Two-Dimensional Intertwining Operators

Intertwining operators for partial differential operators in two dimensions can be constructed by similar means to those in one dimension. The first observation is that Theorem 2.1 applies equally in any dimension, so that $D = f^{-1}\hat{D}$, where $\hat{D}$ commutes with $\bar{L}$ and f is an eigenfunction of $\bar{L}$ in the first class of 2-d intertwining operators. The higher the dimension, the larger the space of operators which commute with $\bar{L}$. In the symmetric space case, these will always be formed from linear combinations of the independent G-invariant differential operators.

To go further, we begin with two soluble one-dimensional measure-form operators. Let $\bar{L}_x = \partial_x^2 + \bar{m}\partial_x$ and $\bar{L}_y = \partial_y^2 + \bar{n}\partial_y$ denote these. The operator $\bar{L}_0 = \bar{L}_x + \bar{L}_y$ is separable and has two kinds of eigenfunctions. The first kind are separable products of the eigenfunctions $\psi_m(x)$ and $\chi_n(y)$ of $\bar{L}_x$ and $\bar{L}_y$,

$$\bar{\phi}_{mn}(x, y) = \psi_m(x)\chi_n(y). \tag{37}$$

The second kind are formed from linear combinations of eigenfunctions each having the same eigenvalue, e.g.

$$\bar{\phi}_n(x, y) = \alpha\psi_n(x) + \beta\chi_n(y). \tag{38}$$

Letting a comma indicate differentiation with respect to the subscripts which follow, the second class of 2-d intertwining operator is given by

Theorem 4.1. *Given an operator* $\bar{L} = \bar{L}_x + \bar{L}_y + m_1\partial_x + n_1\partial_y$ *whose coefficients satisfy*

$$m_{1,x} = n_{1,y}, \tag{39}$$

$$m_{1,y} = n_{1,x}, \tag{40}$$

$$(\bar{m}m_1 + \bar{n}n_1)_{,x} = 0, \tag{41}$$

$$(\bar{m}m_1 + \bar{n}n_1)_{,y} = 0, \tag{42}$$

then $D = f^{-1}(\bar{L}_x - \bar{L}_y)$ *intertwines* $\bar{L}$ *and* $L = \bar{L} + m_2\partial_x + n_2\partial_y$ *with constant* c *where*

$$m_2 = \frac{2f_{,x}}{f}, \tag{43}$$

$$n_2 = \frac{2f_{,y}}{f}, \tag{44}$$

and f *satisfies*

$$\bar{L}f + 2m_{1,x}f = -cf. \tag{45}$$

Proof. Insert the ansatz for D in the intertwining equation (1) and collect like powers of derivatives. Using (39) and (40), one shows that $m_{1,xx} = m_{1,yy}$ and $n_{1,xx} = n_{1,yy}$ and these are used with the fact that $\bar{m}$ and $\bar{n}$ are independent of y and x, respectively, to obtain (41) and (42). ■

A useful example is when $m_1 = af_{,x}/f$ and $n_1 = af_{,y}/f$, where a is a constant. In this instance, (40) is trivial and (39) simplifies to

$$\left(\frac{f_{,x}}{f}\right)_{,x}=\left(\frac{f_{,y}}{f}\right)_{,y}. \tag{46}$$

If f is taken to be an eigenfunction of $\bar{L}_x+\bar{L}_y$ with eigenvalue k, then the condition (45) on f becomes

$$2a(f_{,xx}+f_{,yy})=-(c+k)f. \tag{47}$$

So, f must also be an eigenfunction of the ordinary Laplacian.

Using eigenfunctions of $\bar{L}_x+\bar{L}_y$ of the first kind (37) only generates transformations on $\bar{L}_x$ and $\bar{L}_y$ separately. This is evident because the measure remains separable in terms of x and y. These transformations are essentially one-dimensional. To generate measures which are not separable in x and y, one must use eigenfunctions of the second kind (38). The rank two symmetric spaces with Dynkin diagram $\circ\!\Rightarrow\!\circ$ provide an example of this.

5. Rank 2 $\circ\!\Rightarrow\!\circ$

There are essentially two types of rank-two symmetric spaces with at most one restricted root of odd multiplicity. These have the three Dynkin diagrams [12] $\circ-\circ$, $\circ\!\Rightarrow\!\circ$ and $\circ\!\Leftarrow\!\circ$. (The arrow points to the shorter root.) The latter two differ in whether double roots are present.

Again we will consider the compact case. The noncompact case follows immediately. Some of the non-compact rank 2 spaces have been considered previously by other authors [7,9,10].

Let $\hat{x}_1$ and $\hat{x}_2$ be unit basis vectors on the maximal torus.

In the examples with the Dynkin diagrams $\circ\!\Rightarrow\!\circ$ and $\circ\!\Leftarrow\!\circ$, there are six positive roots which may be labelled $\alpha_2=\hat{x}_2/2$, $a_2=(\hat{x}_1-\hat{x}_2)/2$, $\alpha_1=\alpha_2+a_2$, $a_1=2\alpha_2+\alpha_2$, $2\alpha_1$, and $2\alpha_2$. The root pairs α_1 and α_2, a_1 and a_2, and $2\alpha_1$ and $2\alpha_2$ each form Weyl sets. The multiplicity of roots within a Weyl set must be the same, but different sets can have different multiplicities. The multiplicities of the roots in these sets will be denoted, respectively, $2m_\alpha$, $2m_a$ and $2m_{2\alpha}$. The allowed (compact) symmetric spaces are classified by the multiplicities of the restricted roots and these are given in Table 2.

Table 2. Rank 2 symmetric spaces $\circ\!\Rightarrow\!\circ$ with at most one restricted root of odd multiplicity [12]

G/H	$2m_a$	$2m_\alpha$	$2m_{2\alpha}$
$SU(n+3)/S(U(n+1)\times U(2))$	2	$2(n-1)$	1
$Sp(n+2)/Sp(n)\times Sp(2)$	4	$4(n-2)$	3
$SO(2n+2)/SO(2n)\times SO(2)$	1	$2(n-1)$	0
$SO(8)/U(4)$	4	0	1
$SO(10)/U(5)$	4	4	1
$E_6/SO(10)\times R$	6	8	1

The density is given by

$$\mu_{m_a,m_\alpha,m_{2\alpha}} = V_{m_a,m_\alpha,m_{2\alpha}} \prod_{\gamma \in R_+} (\sin(\gamma \cdot x))^{2m_\gamma},$$

where R_+ is the set of positive roots. One-half the sum of the positive roots is

$$\rho_{m_a,m_\alpha,m_{2\alpha}} = \frac{1}{2} \sum_{\gamma \in R_+} 2m_\gamma \gamma$$
$$= (m_\alpha + 2m_{2\alpha} + 2m_a)\frac{\hat{x}_1}{2} + (m_\alpha + 2m_{2\alpha})\frac{\hat{x}_2}{2}.$$

The radial Laplacian for these symmetric spaces is

$$L_{m_a,m_\alpha,m_{2\alpha}} = \partial^s \partial_s + \sum_i 2m_a \cot(a_i \cdot x) a_i \cdot \partial$$
$$+ \sum_i (2m_\alpha \cot(\alpha_i \cdot x) + 4m_{2\alpha} \cot(2\alpha_i \cdot x))\alpha_i \cdot \partial, \tag{48}$$

where s runs from 1 to 2 and i from 1 to 2. The notation $a_i \cdot \partial$ means to form the scalar product of the a_i root with the gradient operator, so for example, $a_1 \cdot \partial = (\partial_1 + \partial_2)/2$.

The approach will be to eliminate the Weyl set of roots a_1 and a_2 by reducing the multiplicity $2m_a$ to zero using Theorem 4.1. This will leave a separable product of rank one problems involving α_1, $2\alpha_1$ and α_2, $2\alpha_2$. The results of Sect. 3 can then be used to complete the reduction.

Let

$$\begin{aligned} \bar{L}_1 &= \partial_1^2 + (m_\alpha \cot(\alpha_1 \cdot x) + 2m_{2\alpha} \cot(2\alpha_1 \cdot x))\partial_1, \\ \bar{L}_2 &= \partial_2^2 + (m_\alpha \cot(\alpha_2 \cdot x) + 2m_{2\alpha} \cot(2\alpha_2 \cdot x))\partial_2 \end{aligned} \tag{49}$$

be the operators for the rank one problems. The intertwining operator used in reducing the rank 2 radial Laplacian to the sum of these two operators is given by

Theorem 5.1. *The operator $D = f^{-1}(\bar{L}_1 - \bar{L}_2)$ intertwines $L_{m_a+1,m_\alpha,m_{2\alpha}}$ and $L_{m_a,m_\alpha,m_{2\alpha}}$ with constant $c = |\rho_{m_a+1,m_\alpha,m_{2\alpha}}|^2 - |\rho_{m_a,m_\alpha,m_{2\alpha}}|^2$ when $f = \cos(x_1) - \cos(x_2)$.*

Proof. One identifies

$$m_1 = \frac{-2m_a \sin(x_1)}{\cos(x_1) - \cos(x_2)},$$
$$n_1 = \frac{2m_a \sin(x_2)}{\cos(x_1) - \cos(x_2)}, \tag{50}$$

so that

$$L_{m_a,m_\alpha,m_{2\alpha}} = \bar{L}_1 + \bar{L}_2 + 2m_a \cot(a_i \cdot x) a_i \cdot \partial$$
$$= \bar{L}_1 + \bar{L}_2 + m_1 \partial_1 + n_1 \partial_2. \tag{51}$$

The conditions of Theorem 4.1 are satisfied and it may be applied. The function $f = \cos(x_1) - \cos(x_2)$ is both an eigenfunction of $\bar{L}_1 - \bar{L}_2$ with eigenvalue

$-(m_\alpha + 2m_{2\alpha} + 1)$ and of the ordinary Laplacian with eigenvalue -1. Using this, one finds $c = 3 + m_\alpha + 2m_{2\alpha} + 2m_a$ which equals $|\rho_{m_a+1,m_\alpha,m_{2\alpha}}|^2 - |\rho_{m_a,m_\alpha,m_{2\alpha}}|^2$. The shift in measure density produced from f increments m_a by 1, and the desired result is obtained. ■

The intertwining operator also shifts the covariant delta function having one endpoint at the identity.

Proposition 5.1.

$$\delta_{m_a,m_\alpha,m_{2\alpha}}(\vec{x},0) = ND\delta_{m_a-1,m_\alpha,m_{2\alpha}}(\vec{x},0), \tag{52}$$

where

$$N_{m_a} = \frac{-2V_{m_a-1,m_\alpha,m_{2\alpha}}}{V_{m_a,m_\alpha,m_{2\alpha}}(2m_a-1)(2m_a+2m_\alpha+2m_{2\alpha}-1)}. \tag{53}$$

Proof. Integrate the defining expression (15) for the delta function by parts. ■

Using this, one can use Theorem 1.6 to relate the heat kernel on the rank 2 symmetric space to the separable product of heat kernels of rank 1 spaces.

Theorem 5.2. *The heat kernel of $i\partial_t + L_{m_a,m_\alpha,m_{2\alpha}}$ (m_a an integer) is given in terms of those of $i\partial_t + \bar{L}_1$ and $i\partial_t + \bar{L}_2$ by*

$$K_{m_a,m_\alpha,m_{2\alpha}}(\vec{x},0;t) = \left(\prod_{j=1}^{m_a} N_j\right) \exp[i(|\rho_{m_a,m_\alpha,m_{2\alpha}}|^2 - |\rho_{0,m_\alpha,m_{2\alpha}}|^2)t]$$
$$\cdot D^{m_a}(K_{m_\alpha,m_{2\alpha}}(x_1,0;t)K_{m_\alpha,m_{2\alpha}}(x_2,0;t)). \tag{54}$$

In the case of $SO(2n+2)/SO(2n) \times SO(2)$ in which $m_a = 1/2$, one merely reduces the α roots first. This is equivalent to rotating the root diagram so that the roles of the a and α roots are reversed.

The eigenfunctions of the Laplacian (48) are obtained by the intertwining operator from a product of eigenfunctions of one-dimensional Laplacians (22) even when the multiplicities do not correspond to symmetric spaces. These are two-dimensional orthogonal polynomials [1, 2].

6. Rank 2 $\circ - \circ$

A second class of rank 2 symmetric spaces which can be solved are those with the Dynkin diagram $\circ - \circ$. In this case, there are three positive roots which may be labelled $\alpha_1 = (\hat{x}_1 - \sqrt{3}\hat{x}_2)/2$, $\alpha_2 = (\hat{x}_1 + \sqrt{3}\hat{x}_2)/2$, and $\alpha_3 = \alpha_1 + \alpha_2 = \hat{x}_1$. All three of these roots have the same multiplicity $2m$ and $\rho_m = \frac{1}{2}\sum_{i=1}^{3} 2m\alpha_i$. The density appearing in the measure is $\mu_m(x) = V_m \prod_{i=1}^{3} \sin^{2m}(\alpha_i \cdot x)$. The Lie group $SU(3)$ has the same root structure with $m = 1$ but the density is scaled differently, $\tilde{\mu}(x) = \tilde{V}_m \prod_{i=1}^{3} 4\sin^2(\alpha_i \cdot x/2)$, so the volumes are different. To distinguish between them, the re-sized Lie group here will be denoted $SU'(3)$.

Four values of m are consistent with (compact) symmetric spaces: $m = 1/2$ corresponds to $SU(3)/SO(3)$, $m = 1$ is the re-sized Lie group $SU'(3)$, $m = 2$ corresponds to $SU(6)/Sp(3)$, and $m = 4$ corresponds to E_6/F_4 (where E_6 is the compact real form of E_6^C). The space with $m = 1/2$ has two simple restricted roots of unit multiplicity and we cannot handle this at this time. For the others, the multiplicities $2m$ are all even so these symmetric spaces are of split rank. Since there are no non-trivial Weyl sets, all of the root multiplicities must be reduced together. This will result in a chain of reductions

$$\frac{E_6}{F_4} \xrightarrow{D_4 D_3} \frac{SU(6)}{Sp(3)} \xrightarrow{D_2} SU'(3) \xrightarrow{D_1} T^2.$$

The radial Laplacian for these symmetric spaces is

$$L_m = \partial^s \partial_s + \sum_i 2m \cot(\alpha_i \cdot x) \alpha_i \cdot \partial, \tag{55}$$

where s runs from 1 to 2 and i from 1 to 3.

Theorem 6.1. *The operator D_m intertwines L_m and L_{m-1} with constant $c = |\rho_m|^2 - |\rho_{m-1}|^2$, where D_m is given by*

$$\begin{aligned} D_m = f\bigg(& \tfrac{1}{3}(\alpha_1 \cdot \partial)^3 + \frac{m-1}{2} \cot(\alpha_1 \cdot x)(\alpha_1 \cdot \partial)^2 - \frac{m-1}{2} \cot^2(\alpha_1 \cdot x) \alpha_1 \cdot \partial \\ & + \tfrac{1}{3}(\alpha_2 \cdot \partial)^3 + \frac{m-1}{2} \cot(\alpha_2 \cdot x)(\alpha_2 \cdot \partial)^2 - \frac{m-1}{2} \cot^2(\alpha_2 \cdot x) \alpha_2 \cdot \partial \\ & - \tfrac{1}{3}(\alpha_3 \cdot \partial)^3 - \frac{m-1}{2} \cot(\alpha_3 \cdot x)(\alpha_3 \cdot \partial)^2 + \frac{m-1}{2} \cot^2(\alpha_3 \cdot x) \alpha_3 \cdot \partial \bigg) \end{aligned} \tag{56}$$

with

$$f = \left(\prod_{i=1}^{3} \sin(\alpha_i \cdot x) \right)^{-1}. \tag{57}$$

The proof of this is a long computation. An abstract form for this intertwining operator can be derived, but it is more direct to simply compute D_m for this particular case. Assume the following form for D:

$$\begin{aligned} D_m = f(& \alpha_1 \cdot \partial \alpha_2 \cdot \partial \alpha_3 \cdot \partial + G_1 \alpha_2 \cdot \partial \alpha_3 \cdot \partial + G_2 \alpha_1 \cdot \partial \alpha_3 \cdot \partial \\ & + G_3 \alpha_1 \cdot \partial \alpha_2 \cdot \partial + H_{12} \alpha_3 \cdot \partial + H_{13} \alpha_2 \cdot \partial + H_{23} \alpha_1 \cdot \partial). \end{aligned} \tag{58}$$

Substituting into the intertwining relation and collecting like powers of derivatives gives four non-trivial equations. The first three may be solved successively for f, G_i and H_{ij} giving the desired result for f and

$$\begin{aligned} G_1 &= \tfrac{1}{2}(m-1)(-\cot(\alpha_3 \cdot x) + \cot(\alpha_2 \cdot x)), \\ G_2 &= \tfrac{1}{2}(m-1)(\cot(\alpha_1 \cdot x) - \cot(\alpha_3 \cdot x)), \\ G_3 &= -\tfrac{1}{2}(m-1)(\cot(\alpha_1 \cdot x) + \cot(\alpha_2 \cdot x)), \end{aligned}$$

$$H_{12} = -\frac{(m-1)(3m-2)}{4} - \frac{(m-1)(m-2)}{4}\cot^2(\alpha_3\cdot x)$$

$$-\frac{m(m-1)}{4}(\cot^2(\alpha_1\cdot x)+\cot^2(\alpha_2\cdot x)),$$

$$H_{13} = \frac{(m-1)(3m-2)}{4} + \frac{(m-1)(m-2)}{4}\cot^2(\alpha_2\cdot x)$$

$$+\frac{m(m-1)}{4}(\cot^2(\alpha_1\cdot x)+\cot^2(\alpha_3\cdot x)),$$

$$H_{23} = \frac{(m-1)(3m-2)}{4} + \frac{(m-1)(m-2)}{4}\cot^2(\alpha_1\cdot x)$$

$$+\frac{m(m-1)}{4}(\cot^2(\alpha_2\cdot x)+\cot^2(\alpha_3\cdot x)).$$

The fourth equation is a consistency condition. Rewriting this expression for D in terms of powers of $\alpha_i\cdot\partial$ gives the desired result. In solving the equations, the following identity and derivatives of it are helpful:

$$-\cot(\alpha_1\cdot x)\cot(\alpha_2\cdot x)+\cot(\alpha_1\cdot x)\cot(\alpha_3\cdot x)+\cot(\alpha_2\cdot x)\cot(\alpha_3\cdot x) = -1.$$

This identity (and analogous ones for other problems) come from the property of Lie groups [13] that

$$\partial^s\partial_s(\mu_1^{1/2}) = -|\rho_1|^2\mu_1^{1/2}.$$

To obtain the heat kernel, one also needs to know how the delta function transforms under D_m.

Proposition 6.1. *D_m transforms the delta function $\delta_{m-1}(x,0)$ defined with respect to the density $\mu_{m-1}(x)$ to the delta function $\delta_m(x,0)$ defined with respect to the density $\mu_m(x)$*

$$\delta_m(x,0) = N_m D_m \delta_{m-1}(x,0),$$

where

$$N_m = \frac{-2V_{m-1}}{3V_m(2m-1)((2m-1)^2+(m-1)^2)}. \tag{59}$$

Proof. Integrate the defining expression for the delta function by parts. ■

Applying Theorem 1.6, one has

Theorem 6.2. *The heat kernel of $i\partial_t + L_m$ is given by*

$$K_m(x,0;t) = e^{i|\rho_m|^2 t}(N_m D_m)(N_{m-1}D_{m-1})\cdots(N_1 D_1)K_T(x,0;t), \tag{60}$$

where $K_T(x,0;t)$ is the heat kernel of the $i\partial_t + \partial^s\partial_s$ on the 2-dimensional torus with periodic boundary conditions.

The heat kernel on the torus of rank r can be expressed in geometric form using the method of images to project the heat kernel from the universal covering space to the torus. If R_+ is the set of positive roots, then the set of normalized simple restricted roots $\{\hat{\alpha}_i\}$ defined by

$$\hat{\alpha}_i = \begin{cases} \vec{\alpha}_i/\vec{\alpha}_i\cdot\vec{\alpha} & \text{if } 2\alpha_i \notin R_+ \\ \vec{\alpha}_i/2\vec{\alpha}_i\cdot\vec{\alpha}_i & \text{if } 2\alpha_i \in R_+ \end{cases}, \tag{61}$$

from a basis for the torus. Boundary conditions on the symmetric space force $K_T(\vec{x}, 0; t)$ to depend on $\vec{\rho}$ (one half the sum of the positive roots on the symmetric space) through the relation:

$$K_T(\vec{x} + 2\pi\vec{n}, 0; t) = \exp(2\pi i\vec{\rho}\cdot\vec{n})K_T(\vec{x}, 0; t), \tag{62}$$

where $\vec{n} = \sum_{i=1}^{r} n_i\hat{\alpha}_i \in \Gamma/2\pi$, the unit lattice of the symmetric space.

Starting from the geometric form, the heat kernel can be written as a (multidimensional) Jacobi theta function

$$\begin{aligned} K_T(\vec{x}, 0; t) &= (4\pi i t)^{-r/2} \sum_{2\pi\vec{n}\in\Gamma} \exp[i(\vec{x} + 2\pi\vec{n})^2/4t - 2\pi i\vec{\rho}\cdot\vec{n}] \\ &= (4\pi i t)^{-r/2} \exp(i\vec{x}^2/4t)\theta_3\left(\mathbf{A}^{-1}(\vec{x}\pi/2t - \pi\vec{\rho}), \frac{\pi}{t}\mathbf{A}^{-1}\right), \end{aligned} \tag{63}$$

where $(\mathbf{A}^{-1})_{ij} = \hat{\alpha}_i\cdot\hat{\alpha}_j$. Define the dual basis $\{\vec{\pi}_i\}$ of fundamental spherical weights by

$$\vec{\pi}_i\cdot\hat{\alpha}_j = \delta_{ij}.$$

Applying the inversion formula for (multidimensional) Jacobi theta functions

$$\theta_3(\vec{z}, \mathbf{T}) = (-i)^{-r/2}(\det\mathbf{T})^{-1/2}\exp(-i\vec{z}\mathbf{T}^{-1}\vec{z}/\pi)\theta_3(-\mathbf{T}^{-1}\vec{z}, -\mathbf{T}^{-1}),$$

$K_T(\vec{x}, 0; t)$ can be rewritten as an eigenfunction expansion

$$\begin{aligned} K_T(\vec{x}, 0; t) &= \frac{1}{V_T}\exp(-i\vec{\rho}^2 t + i\vec{\rho}\cdot\vec{x})\theta_3(\vec{x}/2 - \rho t, -t/\pi\mathbf{A}) \\ &= \frac{1}{V_T}\sum_{\vec{\lambda}\in\Lambda}\exp[i(\vec{\lambda} + \vec{\rho})\cdot\vec{x} - i(\vec{\lambda} + \vec{\rho})^2 t], \end{aligned} \tag{64}$$

where $\vec{\lambda} = \sum_{i=1}^{r} n_i\vec{\pi}_i \in \Lambda$, the lattice of spherical weights, and $V_T = (2\pi)^r(\det\mathbf{A}^{-1})^{1/2}$ is the volume of the maximal torus. The inversion formula is a consequence of the Poisson summation formula on the torus. In the split rank case, $\vec{\rho}$ is a spherical weight. The phase $\exp(i2\pi\vec{\rho}\cdot\vec{n}) = 1$ and $K_T(\vec{x}, 0; t)$ is the periodic heat kernel on the torus.

The eigenfunctions of the Laplacian (55) for any integer m can be written in terms of those on torus by applying Proposition 1.1.

Proposition 6.2. *The (unnormalized) Weyl invariant eigenfunctions of L_m are given by*

$$\phi^m_{\vec{\lambda}}(\vec{x}) = D_m D_{m-1} \cdots D_1 \sum_{w \in W} \exp[iw(\vec{\lambda} + \vec{\rho})\cdot \vec{x}], \tag{65}$$

where W is the Weyl group.

7. Conclusion

The extension of the intertwining operator approach to higher rank symmetric spaces appears to be limited solely by computational detail. It is known from the work of Dowker [3] that the operator which intertwines the radial Laplacian on any compact Lie group with the ordinary Laplacian on its maximal torus is given by

$$D = \frac{1}{\prod_{+} \sin(\alpha \cdot x)} \prod_{+} \alpha \cdot \partial, \tag{66}$$

where the products are over the positive roots. In the higher rank symmetric spaces, the effort is directed at finding the lower order derivative terms which modify this D.

In the case of the symmetric spaces of type $A\ III$ [12], a natural generalization of the rank 2 result of Sect. 5 exists. The spherical functions have been known for some time [16] and more recent results have also appeared [7, 10]. The Dynkin diagram for rank l spaces of type $A\ III$ is either of the form $\circ - \circ \cdots \circ \Rightarrow \circ$ or $\circ - \circ \cdots \circ \Leftarrow \circ$, depending on the presence of double roots, where there are l circles in the diagram. From the positive simple restricted roots of the space, identify the Weyl set of l orthogonal single roots α of multiplicity $2m_\alpha$ (and possibly l double roots 2α of multiplicity $2m_{2\alpha}$) associated with the circle on the right end of the Dynkin diagram. Let $a \in A$ denote the Weyl set of $l(l-1)$ remaining roots of multiplicity $2m_a = 2$. The intertwining operator which reduces the multiplicity of the A roots in the radial Laplacian on the $A\ III$ space to leave a separable product of l rank one operators is given by

$$D = \frac{1}{\prod_{A} \sin(a \cdot x)} \prod_{i<j} \bar{L}_i - \bar{L}_j, \tag{67}$$

where $\bar{L}_i$ is the radial Laplacian for a rank one symmetric space,

$$\bar{L}_i = \partial_i^2 + (m_\alpha \cot(\alpha_i \cdot x) + 2m_{2\alpha} \cot(2\alpha_i \cdot x))\partial_i, \tag{68}$$

and i runs from 1 to l. The constant c in the intertwining relation is just the difference in ρ^2 between the $A\ III$ space and the separable collection of rank one spaces,

$$c = \frac{l(l-1)}{2}\left(\frac{2l-1}{3} + m_\alpha + 2m_{2\alpha}\right). \tag{69}$$

Using Proposition 1.1, one easily finds the spherical functions of the $A\ III$ space in terms of an antisymmetric product of the eigenfunctions on the rank one spaces.

Using these ideas on multiplicity reduction, work is in progress to extend the results here to the other higher rank symmetric spaces. Effort is also being directed at understanding the fractional partial differential operators which will necessarily appear if there is more than one simple restricted root of odd multiplicity.

The method of using intertwining operators to solve partial differential equations appears to be very fruitful. It provides a natural generalization of the operator transformations which exist between orthogonal polynomials and other special functions in one dimension. Furthermore it allows the solution of problems which do not separate in the standard collection of orthogonal coordinate systems. The method has been used here to find the heat kernel on the rank one and rank two symmetric spaces.

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