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Integration on the n^{th} Power of a Hyperbolic Space in Terms of Invariants Under Diagonal Action of Isometries (Lorentz Transformations)

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Abstract. The integral of a function over the n'th power of hyperbolic d-dimensional space H is decomposed into integration along each orbit under diagonal action on H^n of the isometry group G on H, followed by integration over the orbit space, parametrized in terms of a complete set of invariants. The Jacobian entering in this last integral is expressed explicitly in terms of certain determinants. When viewing H as a half-hyperboloid in $\mathbb{R}^{d+1}$, G is induced by the homogeneous Lorentz group $O^{\uparrow}(1,d)$ acting on $\mathbb{R}^{d+1}$.

Contents

Introduction

We shall consider a problem which enters in connection with the study of tensor products of certain representations of the Poincaré group and the Lorentz group. These particular representations occur for example in the study of a free scalar quantum field in Fock space, see e.g. Bogoliubov, Logunov and Todorov [1].

The problem is to give an explicit formula for the decomposition of an integral over the n^{th} power of hyperbolic d-dimensional space into integrals along the orbits under diagonal action of the isometry group G on that space followed by an integral over the orbit space, parametrized in terms of a complete set of invariants.

It can be shown that there is an integral formula of this kind in the case of isometric action on any Riemannian manifold. Our main purpose, however, is to determine explicitly the relevant measure $d\mu$ on the orbit space via the stated parametrization. For $n>1$ this measure seems to be known explicitly only in the simple case $n=2$, cf. [1, Sect. 8.2]. In the present paper we shall treat the case of an arbitrary number of "particles" n.

The explicit determination of the stated measure $d\mu$ does not seem to be facilitated appreciably by drawing on the general integral formula alluded to above. We have therefore preferred to give a self-contained presentation, including a discussion of the invariant theory of the diagonal action in question (Theorem 1.1).

It turns out that the orbit space is a smooth manifold also in the non-trivial case $n>d+1$ (Theorem 1.2). This result is not essential for the determination of the measure $d\mu$ in suitable local coordinates (Theorem 2 and 3), but it enters naturally in a unified and permutation invariant presentation (Theorem 4).

Taking, e.g., $d=3$ we use the well-known representation of hyperbolic 3-space as the half-hyperboloid

$$H=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1^2-x_2^2-x_3^2-x_4^2=1,\ x_1>0\},$$

endowed with its natural hyperbolic Riemannian metric as induced by the pseudo-Riemannian metric $-dx_1^2+dx_2^2+dx_3^2+dx_4^2$ on $\mathbb{R}^4$, and the corresponding volume measure dH. The isometry group G is then induced by the homogeneous Lorentz group $O^{\uparrow}(1,3)$ acting on $\mathbb{R}^4$. The corresponding diagonal action on H^n is defined by

$$g(p_1,\dots,p_n)=(gp_1,\dots,gp_n),\quad g\in G, \tag{1}$$

as $(p_1,\dots,p_n)$ ranges over H^n. For this action we obviously have the invariants

$$a_{ij}=\cosh \operatorname{dist}(p_i,p_j)=p_i\cdot p_j, \tag{2}$$

where $p_1,\dots,p_n\in H$ denote the positions of the n particles, dist denotes hyperbolic distance, and $p_i\cdot p_j$ is the Lorentz inner product of p_i and p_j as vectors in $\mathbb{R}^4$. This set of invariants is *complete.* Since $a_{ii}=1$ and $a_{ji}=a_{ij}$ $(\geqq 1)$, it suffices to let $i<j$, but even so there are relations among the a_{ij} if $n>4$.

Let H_*^n denote the open subset of H^n consisting of all n-tuples of *maximal rank*

$$\operatorname{rk}(p_1,\dots,p_n)=r:=\min\{n,4\}. \tag{3}$$

Then $H^n\backslash H_*^n$ has measure 0 with respect to the product measure dH^n on H^n. To each n-tuple $(p_1,\dots,p_n)\in H_*^n$ we assign the symmetric $n\times n$ matrix $A=(a_{ij})$ given by (2). The range of the mapping $(p_1,\dots,p_n)\mapsto A$ is the class $\mathscr{A}_{n,4}$ of all symmetric $n\times n$ matrices $A=(a_{ij})$ with $a_{ii}=1$, $a_{ij}\geqq 1$, positivity index: $\operatorname{ind}_+ A=1$, and rank: $\operatorname{rk} A=r$ from (3). The pre-image of each $A\in\mathscr{A}_{n,4}$ is an orbit under diagonal action of G on H_*^n; this expresses the completeness of the set of invariants (2). The diagonal action is bijective on each fibre if and only if $n\geqq 4$. When $n\leqq 3$ one must factor out from G the orthogonal group $O(4-n)$ in order to obtain bijective action on the fibres. (Theorem 1.1.)

The matrix class $\mathscr{A}_{n,4}$, thus representing the space of orbits in H^n under diagonal action of G, is a (real-)analytic manifold of dimension

$$N=(r-1)(n-\tfrac{1}{2}r)$$

($= 3n - 6$ if $n > 2$) embedded in the Euclidean space $\mathscr{M}_n$ ($\cong \mathbb{R}^{n(n-1)/2}$) of *all* symmetric $n \times n$ matrices with diagonal entries 1. When $n \leqq 4$, $\mathscr{A}_{n,4}$ is simply an open set in $\mathscr{M}_n$, but when $n > 4$ the manifold $\mathscr{A}_{n,4}$ has codimension $\frac{1}{2}(n-3)(n-4)$ in $\mathscr{M}_n$; this codimension is the number of relations between the invariants $a_{ij}, i < j$. (Theorem 1.2.)

The main result may now be stated as follows, using a suitable normalization (see Sect. 1) of the invariant measure on G (Theorems 2, 3, and 4).

Theorem. *There exists a unique positive measure $d\mu$ on the N-dimensional manifold $\mathscr{A}_{n,4}$ (representing the orbit space) with the property that, for any integrable function f on H^n,*

$$\int_{H^n} f\, dH^n = \int_{\mathscr{A}_{n,4}} d\mu(A) \int_G f(gp_1(A), \ldots, gp_n(A))dg,$$

where $A \mapsto (p_1(A), \ldots, p_n(A))$ denotes an arbitrary selection of n-tuples of points of H such that, for any $A = (a_{ij}) \in \mathscr{A}_{n,4}$,

$$p_i(A) \cdot p_j(A) = a_{ij}.$$

For $n \leqq 4$, $d\mu$ is given as follows in terms of Lebesgue measure $d\lambda$ on the open set $\mathscr{A}_{n,4}$:

$$d\mu = c_n |\det A|^{(3-n)/2} d\lambda, \quad d\lambda = dA = \prod_{i<j} da_{ij}. \tag{4}$$

In the present case of $d = 3$ space dimensions we have (denoting by ω_k the "surface area of the unit sphere" in $\mathbb{R}^k$)

$$c_1 = 1, \quad c_2 = \omega_3 = 4\pi, \quad c_3 = \omega_3\omega_2 = 8\pi^2, \quad c_4 = \omega_3\omega_2\omega_1 = 16\pi^2.$$

For $n \geqq 4$, $d\mu$ is given as follows when using as local coordinates in the manifold $\mathscr{A}_{n,4}$ the entries to the right of the diagonal in the first 3 rows of matrices $A \in \mathscr{A}_{n,4}$:

$$d\mu = c_4 \prod_{k=4}^{n} \frac{1}{\sqrt{|\det A_{3;k}|}} \prod_{\substack{i \leqq 3 \\ i<j \leqq n}} da_{ij}, \tag{5}$$

where

$$A_{3;k} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1k} \\ a_{21} & a_{22} & a_{23} & a_{2k} \\ a_{31} & a_{32} & a_{33} & a_{3k} \\ a_{k1} & a_{k2} & a_{k3} & a_{kk} \end{pmatrix}$$

with $a_{ik} = a_{ki}$, $a_{kk} = 1$.

The measure $d\mu$ of the theorem is of course invariant under simultaneous permutation of rows and columns of matrices A of class $\mathscr{A}_{n,4}$ (corresponding to permutation of the particles $p_1, \ldots, p_n$). Note that the two expressions for $d\mu$ coalesce if $n = 4$, and that, in any case, only determinants of order $r = \min\{n, 4\}$ occur. For $n = 2$ we obtain $d\mu = 4\pi\sqrt{1 - a_{12}^2}\, da_{12}$, in agreement with [1, Sect. 8.2]. For $n = 3$ we get $d\mu = 8\pi^2 dA = 8\pi^2 da_{12} da_{13} da_{23}$.

The key to this result is the determination in Lemma 2.1 of the hyperbolic volume element $dH(p)$ in a half-space H^+ in H (and similarly in lower dimensions), using as parameters for the variable point $p = p_4$ the numbers $a_{i4} = p_i \cdot p_4$, where $p_i, i = 1, 2, 3$, denote 3 prescribed points of the bounding hyperplane ∂H^+:

$$dH(p) = \frac{1}{\sqrt{|\det A|}} da_{14} da_{24} da_{34},$$

the symmetric matrix A of class $\mathscr{A}_{4,4}$ having the entries

$$a_{ij} = p_i \cdot p_j \quad (i, j = 1, \ldots, 4).$$

A further ingredient in the case $n \leqq 4$ is the following known formula (and its lower-dimensional analogues), entering in the proof of Lemma 2.2:

$$\sinh \varrho_4 = \frac{\sqrt{|\det A|}}{\sqrt{|\det A_3|}}, \quad A_3 = (a_{ij})_{i,j=1,2,3},$$

where ϱ_4 denotes the hyperbolic distance between $p = p_4$ and the hyperbolic hyperplane through p_1, p_2, p_3.

The rest of the proof of the above theorem is largely a matter of putting things together, using Fubini's theorem, invariance of Haar measure, and performing induction with respect to n. Actually, the whole analysis will be carried out in arbitrary dimension $d = 1, 2, \ldots$ in place of $d = 3$ as above.

In Sect. 5 we mention the completely analogous case in which the hyperbolic d-space H in $\mathbb{R}^{d+1}$ is replaced by the unit sphere S in $\mathbb{R}^{d+1}$. And in Sect. 6 we treat the intermediate case where H is replaced by Euclidean d-space E (endowed of course with the group of Euclidean isometries). In this latter case the invariant parameters $a_{ij} = a_{ji}$ are defined in terms of Euclidean distance, dist, by

$$a_{ij} = \tfrac{1}{2} \operatorname{dist}(p_i, p_j)^2,$$

cf. Menger [4] and Schoenberg [5]. In (4) and (5) above, each determinant should be replaced, in the Euclidean case, by the sum of all its minors.

1. Parametrization in the Hyperbolic Case

As a model of d-dimensional hyperbolic space $H = H_d$ $(d \geqq 1)$ we shall take the upper sheet of the hyperboloid in $\mathbb{R}^{d+1}$ with equation $x \cdot x = 1$:

$$H = \{x \in \mathbb{R}^{d+1} \mid x \cdot x = 1,\ x_1 > 0\},$$

where

$$x \cdot y = x_1 y_1 - \sum_{k=2}^{d+1} x_k y_k$$

is the Lorentz inner product of two vectors $x = (x_1, \ldots, x_{d+1})$ and $y = (y_1, \ldots, y_{d+1})$ of $\mathbb{R}^{d+1}$. For $x, y \in H$ we have

$$x \cdot y = \cosh \operatorname{dist}(x, y),$$

where dist refers to hyperbolic distance.

In this model a hyperbolic k-plane is the intersection (if non-empty) of H and a $(k+1)$-dimensional linear subspace of $\mathbb{R}^{d+1}$. Points of H are said to be *independent* if they are linearly independent as vectors in $\mathbb{R}^{d+1}$.

The group G of isometries of H is induced by the Lorentz group $O^{\uparrow}(1,d)$, that is, the linear self-mappings of $\mathbb{R}^{d+1}$ which preserve $x\cdot x$ and map the upper light-cone $\{x\in\mathbb{R}^{d+1} | x\cdot x=0 \wedge x_1>0\}$ onto itself.

For any $p\in H$ consider the fixed-point subgroup

$$G[p]=\{g\in G | gp=p\}.$$

Then $G[p]\cong O(d)$ (clear for $p=(1,0,\dots,0)$, and hence for any $p\in H$ via a transformation from G). The mapping which takes a left coset $gG[p]$ into gp is a bijection of $G/G[p]$ onto H.

More generally we consider for any independent points $p_1,\dots,p_k\in H$ the following subgroup of G (isomorphic to $O(d+1-k)$ if $k\geqq 1$):

$$G[p_1,\dots,p_k]:=\{g\in G | gp_1=p_1,\dots,gp_k=p_k\}. \tag{6}$$

On every compact subgroup of G we use normalized Haar measure (total mass equals 1). On G itself (which is likewise unimodular) we normalize Haar measure so that the above bijection $G/G[p]\to H$ becomes measure preserving for one and hence for any choice of $p\in H$. The invariant measure on H in question is the volume measure derived in the usual way from the (hyperbolic) Riemannian metric on H, which is induced by $-ds^2$, where

$$ds^2=dx_1^2-\sum_{k=2}^{d+1}dx_k^2$$

is the pseudo-riemannian Lorentz metric on $\mathbb{R}^{d+1}$.

The surface area of the Euclidean unit sphere in $\mathbb{R}^k$ is denoted by

$$\omega_k=\frac{2\pi^{k/2}}{\Gamma(k/2)},$$

and we write $\tilde{\omega}_0=1$ and

$$\tilde{\omega}_k=\omega_k\omega_{k-1}\cdots\omega_1. \tag{7}$$

(This number $\tilde{\omega}_k$ arises as the total mass of Haar measure on $O(k)$ in an alternative and geometrically perhaps more natural normalization of that measure, in view of the fact that the homogeneous space $O(k)/O(k-1)$ can be identified with the unit sphere in $\mathbb{R}^k$.)

Definition 1. For any $n\in\mathbb{N}$ we denote by $\mathscr{A}_n$ the class of all symmetric matrices $A=(a_{ij})$, $i,j=1,\dots,n$, for which $a_{ii}=1$, $a_{ij}\geqq 0$, and

$$\operatorname{ind}_+ A=1,\quad \operatorname{ind}_- A=n-1,$$

and hence $\operatorname{rk} A=n$. Replacing the condition $\operatorname{ind}_- A=n-1$ by

$$\operatorname{ind}_- A=\min\{n-1,d\},$$

we obtain instead a class which we denote by $\mathscr{A}_{n,d+1}$.

Thus $\mathscr{A}_{n,d+1} = \mathscr{A}_n$ if $n \leqq d+1$. Here $\operatorname{ind}_+, \operatorname{ind}_-$, and rk stand for positivity index, negativity index, and rank, respectively. For brevity write

$$r = \min\{n, d+1\}. \tag{8}$$

For any $A \in \mathscr{A}_{n,d+1}$ we clearly have $\operatorname{rk} A = r$ and $a_{ij} \geqq 1$, the latter because $a_{ii}a_{jj} - a_{ij}^2 \leqq 0$ as a consequence of $\operatorname{ind}_+ A = 1$.

Any principal $k \times k$ submatrix B of a matrix A of class $\mathscr{A}_n$ is of class $\mathscr{A}_k$. In fact, $\operatorname{ind}_+ B \geqq 1$ because the diagonal entries of B are 1, and $\operatorname{ind}_- B \geqq k-1$ because a hyperplane of negativity for A intersects $\mathbb{R}^k$ $(\subset \mathbb{R}^n)$ in a hyperplane of $\mathbb{R}^k$ or in all of $\mathbb{R}^k$.

For any $A \in \mathscr{A}_n$ we clearly have

$$(-1)^{n-1} \det A > 0, \tag{9}$$

and hence $a_{ii}a_{jj} - a_{ij}^2 < 0$, that is

$$a_{ij} > 1 \quad \text{for} \quad i \neq j. \tag{10}$$

Lemma 1. *Consider a matrix $A \in \mathscr{A}_n$. In order that the bordered $(n+1) \times (n+1)$ matrix*

$$A(x) := \begin{pmatrix} A & x \\ x^t & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n,$$

be of class $\mathscr{A}_{n+1}$, it is necessary and sufficient that x belongs to the solid, open half-hyperboloid

$$\begin{aligned} \mathscr{H}(A) &= \{x \in \mathbb{R}^n \,|\, (-1)^n \det A(x) > 0 \text{ and } x > 0\} \\ &= \{x \in \mathbb{R}^n \,|\, x^t A^{-1} x > 1 \text{ and } x > 0\}, \end{aligned}$$

where $x > 0$ means $x_i > 0$ for $i = 1, \ldots, n$; and superscript t denotes transposition.

Proof. An elementary calculation shows that, for any $n \times n$ matrix A,

$$\det A(x) = (1 - x^t A^{-1} x) \det A,$$

whence the two expressions for $\mathscr{H}(A)$ represent the same set in view of (9). If $A(x) \in \mathscr{A}_{n+1}$ then each $x_j > 1$ by (10), and $(-1)^n \det A(x) > 0$ by (9). Conversely, this latter inequality implies $\operatorname{ind}_- A(x) = n$ since the alternative $\operatorname{ind}_- A(x) = n-1$ $(= \operatorname{ind}_- A)$ would lead to the contradiction $(-1)^{n-1} \det A(x) > 0$. ∎

The boundary $\partial\mathscr{H}(A)$ is one of the sheets of a hyperboloid centered at 0 and having the tangent hyperplane $x_i = 1$ at the point $a_i \in \partial\mathscr{H}(A)$ given by the i^{th} column of A (recall that $a_{ii} = 1$). It can be shown that the half-hyperboloid $\partial\mathscr{H}(A)$ is characterized by these properties.

Corollary. *In order that a symmetric $n \times n$ matrix $A = (a_{ij})$ with entries > 0 and diagonal entries $= 1$ be of class $\mathscr{A}_n$ it is necessary and sufficient that*

$$(-1)^{k-1} \det A_k > 0, \quad k = 1, \ldots, n,$$

where A_k denotes the submatrix of A formed by its first k rows columns.

Let H_*^n denote the subset of H^n consisting of all n-tuples $(p_1,\dots,p_n)$ of points of $H = H_d$ of *full rank*, that is,

$$\operatorname{rk}(p_1,\dots,p_n) = r(=\min\{n, d+1\}).$$

If $n \leqq d+1$, H_*^n consists of all *independent* n-tuples $(p_1,\dots,p_n)$ of points of H.

We proceed to show how the matrices of class $\mathscr{A}_{n,d+1}$ serve as invariants of H_*^n under diagonal action of G. In terms of the $(d+1)\times(d+1)$ matrix

$$E = E_{d+1} = \operatorname{diag}(1, -1,\dots, -1)$$

we associate with each n-tuple $(p_1,\dots,p_n)\in H_*^n$ (identified with the $(d+1)\times n$ matrix $P=(p_{ij})$ having $p_1,\dots,p_n$ as columns) its "hyperbolic Gramian"

$$A = P^tEP$$

with entries (invariant under diagonal action of G)

$$a_{ij} = p_i\cdot p_j = \cosh\operatorname{dist}(p_i, p_j) \geqq 1, \tag{11}$$

where dist denotes hyperbolic distance in H. In particular, $a_{ii}=1$.

The former part of the following theorem implies that the set of invariants (11) is *complete*: Every function $f(p_1,\dots,p_n)$, defined on H_*^n and invariant under diagonal action of G, is a function of the invariants $p_i\cdot p_j$, $i<j$. (See also the corollary to Theorem 1.2.)

Theorem 1.1. *For any n the map $(p_1,\dots,p_n)\mapsto(p_i\cdot p_j)_{i,j=1,\dots,n}$ is a surjection of H_*^n onto $\mathscr{A}_{n,d+1}$. Each fibre (= pre-image of some $A\in\mathscr{A}_{n,d+1}$) is an orbit in H_*^n under diagonal action of G*, cf. (1).

If $n\geqq d+1$, the diagonal action of G is bijective on each fibre. If $n\leqq d$, the diagonal action of $G/O(d+1-n)$ is bijective on each fibre.

Proof. We begin by showing that the symmetric matrix $A=P^tEP$ is of class $\mathscr{A}_{n,d+1}$ for any $P=(p_1,\dots,p_n)\in H_*^n$. Consider first the case $n\geqq d+1$, i.e., $r=d+1$ (cf. (8)), and write

$$E_{n,r} = \operatorname{diag}(1, -1,\dots, -1, 0,\dots, 0)$$

with $r-1$ entries -1 and $n-r$ diagonal entries 0. If $P=(p_1,\dots,p_n)$ belongs to H_*^n we may adjoin $n-r$ further rows to P so as to obtain a non-singular $n\times n$ matrix U. Clearly

$$A = P^tEP = U^tE_{n,r}U$$

has $\operatorname{ind}_+A=\operatorname{ind}_+E_{n,r}=1$, $\operatorname{ind}_-A=\operatorname{ind}_-E_{n,r}=r-1=\min\{n-1,d\}$, and so $A\in\mathscr{A}_{n,d+1}$, invoking also (11). The case $n\leqq d+1$, where $r=n$, reduces immediately to the case $n=d+1$ because we may assume via a Lorentz transformation that $p_1,\dots,p_n$ lie in the $x_1,\dots,x_n$-space in $\mathbb{R}^{d+1}$ and hence equally well on the hyperboloid H_{n-1} in $\mathbb{R}^n$.

Conversely, if $A\in\mathscr{A}_{n,d+1}$, the above argument may be reversed. In the case $n\geqq d+1$ there is a non-singular $n\times n$ matrix U such that $U^tE_{n,r}U=A$, and we merely have to cancel the last $n-r$ rows of U to be left with a $(d+1)\times n$ matrix $P=(p_1,\dots,p_n)$ satisfying $p_i\cdot p_j=a_{ij}$. Since $a_{ii}=1$, each p_i belongs to $H\cup(-H)$; and since $a_{ij}>0$, $p_1,\dots,p_n$ all lie on the same sheet H or $-H$. If they are on $-H$ just replace P by $-P$. The case $n \leqq d+1$ reduces to the case $n=d+1$ via the identification of $\mathbb{R}^n$ with a subspace of $\mathbb{R}^{d+1}$ as above.

Next we note that the pre-image of any $A \in \mathscr{A}_{n,d+1}$ under the map $P \mapsto P^t E P$ of H^n_* onto $\mathscr{A}_{n,d+1}$ is an orbit in H^n_* under diagonal action of G (or just of $G/O(d+1-n)$ if $n \leqq d$, cf. below). This expresses the well known property that the hyperbolic space H_d is n-point homogeneous for every $n \in \mathbb{N}$. For any two n-tuples $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ in H^n_* we must show that there exists $g \in G$ satisfying $g p_j = q_j$ for $j = 1, \dots, n$ if (and only if) $p_i \cdot p_j = q_i \cdot q_j$ for all $i, j = 1, \dots, n$. First choose $h_1 \in G$ so that $h_1 p_1 = q_1$, next (if $r > 1$) $h_2 \in G[q_1]$ ($\cong O(d)$) so that $h_2 h_1 p_2 = q_2$, etc., and finally $h_r \in G[q_1, \dots, q_{r-1}]$ so that $g := h_r h_{r-1} \cdots h_1$ takes p_r to q_r. We have now achieved that $g p_j = q_j$ for $j = 1, \dots, r$. If $n \leqq d+1$, i.e., $n = r$, we are done. If $n > d+1$, then $r = d+1$, and $g \in G$, as constructed, satisfies automatically the remaining conditions $g p_j = q_j$, $j = d+2, \dots, n$ in view of the unique determination of a point q of H from the numbers $q \cdot q_j$ when q_j, $j = 1, \dots, d+1$ are given independent points of H.

As to the bijective action on fibres, suppose that the n-tuples $(p_1, \dots, p_n) \in H^n_*$ and $(q_1, \dots, q_n) \in H^*_n$ belong to the same fibre, and that, e.g., $p_1, \dots, p_r$ are independent, hence likewise $q_1, \dots, q_r$. Choose $g \in G$ so that $g p_j = q_j$ for $j = 1, \dots, n$ (possible as shown above). For any $\tilde{g} \in G$ we then have $\tilde{g} p_j = q_j$ for $j \leqq r$ (and hence for all $j \leqq n$) if and only $\tilde{g}$ belongs to the coset $g G[p_1, \dots, p_r]$. cf. (6). If $n \geqq d+1$ then $G[p_1, \dots, p_r] = \{\mathrm{id}\}$, and the action of G itself is therefore bijective on each fibre. If $n \leqq d$ we must factor out the fixed point subgroup $G[p_1, \dots, p_n] \cong O(d+1-n)$ in order to obtain bijective action on the fibres. ■

Remark 1. In the case $n \leqq d+1$ it is easy to impose on $(p_1, \dots, p_n) \in H^n_*$ suitable additional conditions designed to make $(p_1, \dots, p_r)$ depend *uniquely* and *analytically* (even algebraically) on $A \in \mathscr{A}_{n,d+1}$ ($= \mathscr{A}_n$). One may require for example (in addition to each p_j being in H) that the coordinates p_{ij} of p_j satisfy $p_{ij} = 0$ for $i > j$ and $p_{jj} > 0$. (This is well known from linear algebra and also geometrically obvious.) This particular choice begins as follows:

$$
\begin{aligned}
&p_{11} = 1, \\
&p_{12} = a_{12}, \quad p_{22} = \sqrt{[1,2]} \quad \text{if} \quad n > 1, \\
&p_{13} = a_{13}, \quad p_{23} = (a_{23} - a_{12} a_{13}) / \sqrt{[1,2]}, \quad p_{33} = \sqrt{[1,2,3]} / \sqrt{[1,2]}
\end{aligned}
$$

if $n > 2$, whereby we write, for brevity,

$$[1, \dots, k] = (-1)^{k-1} \det (a_{ij})_{i,j=1,\dots,k} \quad (>0).$$

It is easy to write down a scheme that permits a recursive determination of the explicit expression for $p_1, \dots, p_n$ as functions of $A \in \mathscr{A}_n$, $n \leqq d+1$.

We denote (for $n \leqq d+1$) by $\varphi_n : \mathscr{A}_n \to H^n_*$ the above particular analytic right inverse ($=$ analytic selection in the fibres) of the map $H^n_* \to \mathscr{A}_n$ given by $(p_1, \dots, p_n) \mapsto (p_i \cdot p_j)_{i,j \leqq n}$. Clearly φ_n is an analytic diffeomorphism of $\mathscr{A}_n$ onto a submanifold of H^n_* of dimension $\frac{1}{2} n(n-1)$.

Returning to the case n arbitrary we denote by $\mathscr{M}_n$ the $\frac{1}{2} n(n-1)$-dimensional Euclidean space of all symmetric $n \times n$ matrices with diagonal entries 1, and we endow $\mathscr{M}_n$ with the obvious Euclidean metric and analytic structure.

Theorem 1.2. *$\mathscr{A}_{n,d+1}$ is a (real-)analytic submanifold of $\mathscr{M}_n$ of dimension*

$$N = (r-1)(n - \tfrac{1}{2}r), \quad r = \min\{n, d+1\}. \tag{12}$$

Proof. For $n \leqq d+1$ this is obvious because $\mathscr{A}_{n,d+1} = \mathscr{A}_n$ is then an open set in $\mathscr{M}_n$. In the case $n > d+1$ we shall specify below suitable local coordinates for $\mathscr{A}_{n,d+1}$ among the entries a_{ij}. Let $i_1 < \cdots < i_{d+1}$ denote $d+1$ distinct subscripts among $1, \ldots, n$, and consider any map

$$\imath: \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{d+1}\} \to \{i_1, \ldots, i_{d+1}\}.$$

A matrix $A \in \mathscr{A}_{n,d+1}$ is said to be of class $\mathscr{A}(i_1, \ldots, i_{d+1}; \imath)$ if

$$(-1)^d \det(a_{ij})_{i,j=i_1,\ldots,i_{d+1}} > 0,$$

and for each $k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{d+1}\}$,

$$(-1)^d \det(a_{ij})_{i,j=i_1,\ldots,\widehat{\imath(k)},\ldots,i_{d+1},k} > 0,$$

the "hat" over $\imath(k)$ indicating that this number among $i_1, \ldots, i_{d+1}$ should be omitted. The first of these two conditions means that, if $(p_1, \ldots, p_n)$ is in the fibre of A (cf. Theorem 1.1), then the $(d+1)$-tuple $(p_{i_1}, \ldots, p_{i_{d+1}})$ is independent; and the last condition translates similarly. The classes $\mathscr{A}(i_1, \ldots, i_{d+1}; \imath)$ are relatively open subsets of $\mathscr{A}_{n,d+1}$ and form together a covering of $\mathscr{A}_{n,d+1}$ because a non-zero vector p_k cannot belong to every subspace of $\mathbb{R}^{d+1}$ which is spanned by d among $d+1$ given linearly independent vectors $p_{i_1}, \ldots, p_{i_{d+1}}$.

As local coordinates for $\mathscr{A}_{n,d+1}$ in $\mathscr{A}(i_1, \ldots, i_{d+1}; \imath)$ we take the entries a_{ik}, $i < k$, such that either $i, k \in \{i_1, \ldots, i_{d+1}\}$ or else $k \notin \{i_1, \ldots, i_{d+1}\}$ and $i \in \{i_1, \ldots, \widehat{\imath(k)}, \ldots, i_{d+1}\}$. The number of these entries is

$$\tfrac{1}{2}d(d+1) + d(n-d-1) = d(n - \tfrac{1}{2}(d+1)) = N.$$

For the proof that $\mathscr{A}(i_1, \ldots, i_{d+1}; \imath)$ is an analytic manifold and that the stated entries do serve as local coordinates we consider the typical case

$$(i_1, \ldots, i_{d+1}) = (1, \ldots, d+1),$$

where we are given a map $\imath: \{d+2, \ldots, n\} \to \{1, \ldots, d+1\}$. In this case we write briefly $\mathscr{A}(\imath)$ for $\mathscr{A}(1, \ldots, d+1; \imath)$. Let J denote the set (ordered in some fixed way) of those N ordered pairs (i,k) for which either $i, k \in \{1, \ldots, d+1\}$ or else $k \in \{d+2, \ldots, n\}$ and $i \in \{1, \ldots, \widehat{\imath(k)}, \ldots, d+1\}$. Let $\tilde{\mathscr{A}}(\imath)$ denote the open set in $\mathbb{R}^N$ consisting of all N-tuples $\tilde{A} = (a_{ik})_{(i,k) \in J}$ for which the $(d+1) \times (d+1)$ submatrix

$$\tilde{A}_{d+1} = (a_{ij})_{i,j \leqq d+1}$$

and the $n-(d+1)$ analogous submatrices

$$(a_{ik})_{i,k=1,\ldots,\widehat{\imath(k)},\ldots,d+1,k}, \quad k = d+2, \ldots, n,$$

all belong to $\mathscr{A}_{d+1}$.

Consider the restriction map, or projection, $\pi: \mathscr{M}_n \to \mathbb{R}^N$ defined by

$$\pi(A) = (a_{ik})_{(i,k) \in J}, \quad A = (a_{ik})_{i,k \leqq n} \in \mathscr{M}_n.$$

If $A \in \mathscr{A}(\imath)$ then $\pi(A)$ is the N-tuple of local coordinates of A as specified above. Clearly $\pi(A(\imath)) \subset \tilde{\mathscr{A}}(\imath)$. We proceed to show that $\pi(\mathscr{A}(\imath)) = \tilde{\mathscr{A}}(\imath)$. For any $\tilde{A} = (a_{ik})_{(i,k)\in J} \in \tilde{\mathscr{A}}(\imath)$ we define, using the notation of Remark 1,

$$(p_1, \ldots, p_{d+1}) = \varphi_{d+1}(\tilde{A}_{d+1}) \in H_*^{d+1}$$

in terms of the above submatrix $\tilde{A}_{d+1}$. Then $\det(p_1, \ldots, p_{d+1}) > 0$. For each $k = d+2, \ldots, n$ and each sign $\varepsilon_k = 1$ for -1 there is a unique point $p_k(\varepsilon_k)$ of H such that

$$p_i \cdot p_k(\varepsilon_k) = a_{ik}, \quad i = 1, \ldots, \widehat{\imath(k)}, \ldots, d+1$$

and further that

$$\varepsilon_k \det(p_1, \ldots, p_{\imath(k)}, \ldots, p_{d+1}, p_k(\varepsilon_k)) > 0.$$

For each $(n-d-1)$-tuple

$$\varepsilon = (\varepsilon_{d+2}, \ldots, \varepsilon_n) \in \{1, -1\}^{n-d-1}$$

we have thus defined altogether a mapping $\varphi(\varepsilon): \tilde{A}(\imath) \to H_*^n$ by

$$\varphi(\varepsilon)(\tilde{A}) = (p_1, \ldots, p_{d+1}, p_{d+2}(\varepsilon_{d+2}), \ldots, p_n(\varepsilon_n)), \quad \tilde{A} \in \tilde{\mathscr{A}}(\imath).$$

When composed with the analytic map $\psi: H_*^n \to \mathscr{A}_{n,d+1} (\subset \mathscr{M}_n)$ from Theorem 1.1, given by $\psi(p_1, \ldots, p_n) = (p_i \cdot p_j)_{i,j \leq n}$, this leads to the map

$$\psi \circ \varphi(\varepsilon): \tilde{\mathscr{A}}(\imath) \to \mathscr{A}(\imath)$$

having the projection π as left inverse:

$$\pi \circ \psi \circ \varphi(\varepsilon) = \mathrm{id},$$

and so indeed $\pi(\mathscr{A}(\imath)) = \tilde{\mathscr{A}}(\imath)$.

Next we show that each of the maps $\varphi(\varepsilon): \tilde{\mathscr{A}}(\imath) \to H_*^n (\subset (\mathbb{R}^{d+1})^n)$ is analytic. According to Remark 1, $(p_1, \ldots, p_{d+1}) = \varphi_{d+1}(\tilde{A}_{d+1})$ depends analytically on $\tilde{A}_{d+1}$ and hence on $\tilde{A} \in \tilde{\mathscr{A}}(\imath)$, and so it remains to establish the analyticity of the map $\tilde{A} \mapsto p_k(\varepsilon_k)$ ($= p_k$, for brevity) for any fixed $k \in \{d+2, \ldots, n\}$ and $\varepsilon_k = \pm 1$. It is convenient to write

$$(1, \ldots, \widehat{\imath(k)}, \ldots, d+1, k) = (\tau(1), \ldots, \tau(d+1)).$$

With any $\tilde{A} \in \tilde{\mathscr{A}}(\imath)$ we have associated in the definition of $\tilde{\mathscr{A}}(\imath)$ the "submatrix"

$$B = (b_{ij})_{i,j \leq d+1} = (a_{\tau(i),\tau(j)})_{i,j \leq d+1} \in \mathscr{A}_{d+1}.$$

Invoking once more Remark 1, we write

$$(q_1, \ldots, q_{d+1}) = \varphi_{d+1}(B),$$

which depends analytically on B and hence on $\tilde{A}$ and satisfies

$$q_i \cdot q_j = b_{ij} = a_{\tau(i),\tau(j)} = p_{\tau(i)} \cdot p_{\tau(j)}, \quad i,j \leq d+1.$$

Consequently, there exists a unique isometry $g \in G$ such that

$$g q_j = p_{\tau(j)}, \quad j = 1, \ldots, d+1,$$

whence $\det g = \varepsilon_k$, the sign of $\det(p_{\tau(1)}, \ldots, p_{\tau(d+1)})$. If $e_1 \ldots, e_{d+1}$ denote the columns of the $(d+1)\times(d+1)$ unit matrix, we note (cf. Remark 1) for $h=1,\dots,d$ that e_h is a linear combination of $q_1,\dots,q_h$ with coefficients that are analytic functions of the b_{ij} with $i,j\leq h(\leq d)$, hence of the a_{ij} with $i,j\leq d+1$. Viewing g as a Lorentz transformation of $\mathbb{R}^{d+1}$, it follows that ge_h is the corresponding linear combination of the $gq_j=p_{\tau(j)}$, $j=1,\dots,d$. Since $p_1,\dots,p_{d+1}$ are analytic functions of $\tilde{A}_{d+1}$, so are therefore the first d columns ge_h, $h\leq d$, of the Lorentz matrix g. Each entry of the last column of g is plus or minus a minor which is a polynomial in the entries of the first d columns (because $g^t=Eg^{-1}E$ and $\det g=\varepsilon_k$), in particular an analytic function of $\tilde{A}_{d+1}$ for $\tilde{A}\in\tilde{\mathscr{A}}(\iota)$). We conclude that indeed $p_k(\varepsilon_k)=p_k=p_{\tau(d+1)}=gq_{d+1}$ depends analytically on $\tilde{A}\in\tilde{\mathscr{A}}(\iota)$, that is, on the proposed local coordinates of $A\in\mathscr{A}(\iota)$.

Each of the analytic maps $\psi\circ\varphi(\varepsilon):\tilde{\mathscr{A}}(\iota)\to\mathscr{A}(\iota)$ is injective and has a relatively closed range because of the continuous left inverse π. The ranges of these maps corresponding to distinct $(n-d-1)$-tuples ε are disjoint. To see this, choose $k\in\{d+2,\dots,n\}$ and note that $p_{\iota(k)}\cdot p_k(1)\neq p_{\iota(k)}\cdot p_k(-1)$ because the bisecting hyperplane

$$\{x\in\mathbb{R}^{d+1}\,|\,p_k(1)\cdot x=p_k(-1)\cdot x\}$$

contains, hence equals, the hyperplane spanned linearly by $p_1,\dots,\widehat{p_{\iota(k)}},\dots,p_{d+1}$, and this latter hyperplane does not contain $p_{\iota(k)}$. Summing up, $\mathscr{A}(\iota)$ is the union of the 2^{n-d-1} disjoint, relatively closed ranges $(\psi\circ\varphi(\varepsilon))(\tilde{\mathscr{A}}(\iota))$, each of which is therefore open in $\mathscr{A}(\iota)$. It thus suffices to show that each such range is an analytic manifold, embedded in $\mathscr{M}_n$ and admitting the entries of $\tilde{A}=\pi(A)$ as local coordinates of $A\in(\psi\circ\varphi(\varepsilon))(\tilde{\mathscr{A}}(\iota))$. But this is clear because the map $\psi\circ\varphi(\varepsilon)$ defined on $\tilde{\mathscr{A}}(\iota)$ is analytic and has the analytic left inverse π. ■

As a corollary we have the following result, somewhat in the spirit of Hall and Wightman [3], though dealing with real-analytic functions and altogether more elementary.

Corollary. *Every analytic function f which is defined in the open subset H^n_* of full measure in H^n and is invariant under diagonal action of G has the form*

$$f(p_1,\dots,p_n)=\tilde{f}((p_i\cdot p_j)_{i,j=1,\dots,n})$$

for a unique and analytic function $\tilde{f}$ defined on the manifold $\mathscr{A}_{n,d+1}$.

If $n\leq d+1$ we have in fact $\tilde{f}=f\circ\varphi_n$, with φ_n as defined in Remark 1. And if $n>d+1$ we similarly have $\tilde{f}=f\circ\varphi(\varepsilon)\circ\pi$ in each of the 2^{n-d-1} relatively open sets $\psi\circ\varphi(\varepsilon)(\tilde{\mathscr{A}}(\iota))$, $\varepsilon\in\{1,-1\}^{n-d-1}$, and their analogues, which together cover $\mathscr{A}_{n,d+1}$, cf. the proof of Theorem 1.2. Clearly $\tilde{f}$ is well defined globally on $\mathscr{A}_{n,d+1}$ by $f=\tilde{f}\circ\psi$ because f is constant on each orbit, that is, on each fibre of the map $\psi:H^n_*\to\mathscr{A}_{n,d+1}$ from Theorem 1.1.

2. Integration in the Hyperbolic Case with $n\leq d+1$

We proceed to establish, for the case $n\leq d+1$, the formula (suitably interpreted)

$$dH^n=\frac{\tilde{\omega}_d}{\tilde{\omega}_{d+1-n}}|\det A|^{(d-n)/2}dA\,dg$$

in terms of the mapping $(p_1,\dots,p_n)\mapsto A=(p_i\cdot p_j)$ of H^n_* onto $\mathscr{A}_n$ and the diagonal action of $G/O(d+1-n)$, cf. Theorem 1.1. The notation is as follows (see (7) as to the constants $\tilde{\omega}_k$):

dH^n refers to the product measure on H^n. Note that $H^n\backslash H^n_*$ has measure 0.

dA refers to Lebesgue measure on the Euclidean space $\mathscr{M}_n(\cong\mathbb{R}^{n(n-1)/2})$ of all $n\times n$ symmetric matrices $A=(a_{ij})$ with $a_{ii}=1$:

$$dA=\prod_{i<j} da_{ij}.$$

(For $n=1$, $\mathscr{M}_n=\mathscr{A}_n=\{1\}$ has measure 1.)

dg refers to Haar measure on G, normalized as described in Sect. 1, and also to the induced invariant measure on $G/O(d+1-n)$.

A complete formulation of the indicated result is given in the following theorem.

Theorem 2. *Suppose* $1\leqq n\leqq d+1$. *For any integrable function* f *on* H^n *we have*

$$\int_{H^n} f\,dH^n=\frac{\tilde{\omega}_d}{\tilde{\omega}_{d+1-n}}\int_{\mathscr{A}_n}|\det A|^{(d-n)/2}dA\int_G f(gp(A))dg,$$

where

$$p(A)=(p_1(A),\dots,p_n(A)),\quad A=(a_{ij})\in\mathscr{A}_n,$$

denotes an arbitrary selection of n-tuples of points of H such that

$$p_i(A)\cdot p_j(A)=a_{ij},\quad i,j=1,\dots,n,$$

cf. Theorem 1.1, and where we write (conforming with (1))

$$gp(A)=(gp_1(A),\dots,gp_n(A)).$$

Remark 2.1. In view of the last statement in Theorem 1.1 it is more instructive to rewrite the inner integral in the above formula as follows:

$$\int_G f(gp(A))dg=\int_{G/G[p(A)]} f(\tilde{g}p(A))d\tilde{g},$$

where

$$G[p(A)]:=G[p_1(A),\dots,p_n(A)](\cong O(d+1-n))$$

denotes the subgroup of G consisting of those $g\in G$ which leave $p_1(A),\dots,p_n(A)$ fixed, while $\tilde{g}=gG[p(A)]$ stands for a left coset in G modulo $G[p(A)]$, and $\tilde{g}p(A):=gp(A)$.

Remark 2.2. It is clear from the beginning that the inner integral on the right in the formula of Theorem 2 is independent of the particular choice of $p(A)$. In fact, if $q(A)=(q_1(A),\dots,q_n(A))$ is another such choice, we have $q_i(A)=g(A)p_i(A)$ for some $g(A)\in G$, cf. Theorem 1.1, whence the assertion by invariance of dg.

Remark 2.3. For $n=1$ the integration over $\mathscr{A}_n$ drops out and the formula reduces to $\int_H f\,dH=\int_G f(gp)dg$ for any $p\in H$. This equation is obvious in view of the chosen normalization of Haar measure dg on G, cf. Sect. 1.

For the proof of Theorem 2 we prepare two lemmas.

Lemma 2.1. *For any integrable function f on H and any prescribed d-tuple of independent points $p_1,\dots,p_d$ of H we have for $\varepsilon=1$ or -1,*

$$\int_{H^+} f(p)dH(p)=\int_{\mathscr{K}(A_d)} f(p_{d+1}(A))\frac{1}{\sqrt{|\det A|}}da_{1,d+1}\cdots da_{d,d+1},$$

where H^+ denotes the hyperbolic half-space

$$H^+=H^+(p_1,\dots,p_d):=\{p\in H\,|\,\varepsilon\det(p_1,\dots,p_d,p)>0\},$$

while A_d is the matrix $(a_{ij})=(p_i\cdot p_j)_{i,j\leqq d}$ of class $\mathscr{A}_d$, and $\mathscr{K}(A_d)$ denotes the solid, open half-hyperboloid in $\mathbb{R}^d$ consisting all points $(a_{1,d+1},\dots,a_{d,d+1})$ such that (writing $a_{d+1,i}=a_{i,d+1}$ and $a_{d+1,d+1}=1$) the bordered matrix

$$A(a_{ij})_{i,j\leqq d+1}$$

is of class $\mathscr{A}_{d+1}$ (cf. Lemma 1). For each such d-tuple $(a_{1,d+1},\dots,a_{d,d+1})$ in $\mathscr{K}(A_d)$, $p_{d+1}(A)$ denotes the unique point of H^+ for which

$$p_i\cdot p_{d+1}(A)=a_{i,d+1},\quad i=1,\dots,d.$$

Proof. For each $(a_{1,d+1},\dots,a_{d,d+1})\in\mathscr{K}(A_d)$ the bordered matrix $A=(a_{ij})_{i,j\leqq d+1}\in\mathscr{A}_{d+1}$ has, by Theorem 1.1, the form $a_{ij}=q_i\cdot q_j$ for some independent $(d+1)$-tuple $(q_1,\dots,q_{d+1})$ of points of H. Because $q_i\cdot q_j=p_i\cdot p_j=a_{ij}$ for $i,j\leqq d$ we may arrange via a transformation from G that $q_i=p_i$ for $i\leqq d$, and further, if necessary, by reflection in the hyperbolic hyperplane passing through $p_1,\dots,p_d$ that $q_{d+1}\in H^+$; and then $p_{d+1}(A):=q_{d+1}$ is uniquely determined. The smooth mapping $(a_{1,d+1},\dots,a_{d,d+1})\mapsto p_{d+1}(A)$ is a bijection of $\mathscr{K}(A_d)$ onto H^+ because each point $p_{d+1}\in H^+$ together with the given points $p_1,\dots,p_d$ determine a unique matrix $A=(a_{ij})_{i,j\leqq d+1}$ of class $\mathscr{A}_{d+1}$ such that $p_{d+1}=p_{d+1}(A)$.

In the rest of the proof we write, for brevity, p_{d+1} for $p_{d+1}(A)$. By differentiation of $p_i\cdot p_{d+1}=a_{i,d+1}$ we obtain (since $p_1,\dots,p_d$ are fixed)

$$p_i\cdot\frac{\partial p_{d+1}}{\partial a_{j,d+1}}=\delta_{ij},\quad i\leqq d+1, j\leqq d. \tag{13}$$

When combined with $p_i\cdot p_{d+1}=a_{i,d+1}$ and $a_{d+1,d+1}=1$, this leads to the following relation between $(d+1)\times(d+1)$ matrices:

$$P^tE\left(\frac{\partial p_{d+1}}{\partial a_{1,d+1}},\dots,\frac{\partial p_{d+1}}{\partial a_{d,d+1}},p_{d+1}\right)=J:=\begin{pmatrix}1&\cdots&0&a_{1,d+1}\\ \vdots&\ddots&\vdots&\vdots\\ 0&\cdots&1&a_{d,d+1}\\ 0&\cdots&0&1\end{pmatrix},$$

where $E=E_{d+1}=\operatorname{diag}(1,-1,\dots,-1)$ while the columns of P are $p_1,\dots,p_{d+1}$. Equivalently,

$$\left(\frac{\partial p_{d+1}}{\partial a_{1,d+1}},\dots,\frac{\partial p_{d+1}}{\partial a_{d,d+1}},p_{d+1}\right)=Q:=E(P^t)^{-1}J. \tag{14}$$

The pseudo-riemanninan metric

$$-ds^2=-dx_1^2+dx_2^2+\cdots+dx_{d+1}^2$$

on $\mathbb{R}^{d+1}$ induces the hyperbolic Riemannian metric $d\sigma^2$ on H and in particular on the hyperbolic half-space H^+, the image of $\mathscr{K}(A_d)$ under the diffeomorphism $(a_{1,d+1},\dots,a_{d,d+1})\mapsto x=p_{d+1}=p_{d+1}(A)$ whose differential $dx=dp_{d+1}\in\mathbb{R}^{d+1}$ is given in coordinates by

$$dx_k = dp_{k,d+1} = \sum_{i=1}^{d} q_{kj}da_{j,d+1}, \quad k \leqq d+1,$$

according to (14). We thus find

$$\begin{aligned} d\sigma^2 &= -\left(\sum_{j=1}^{d} q_{1j}da_{j,d+1}\right)^2 + \sum_{k=2}^{d+1}\left(\sum_{j=1}^{d} q_{kj}da_{j,d+1}\right)^2 \\ &= \sum_{i,j=1}^{d} (-q_{1i}q_{1j}+q_{2i}q_{2j}+\cdots+q_{d+1,i}q_{d+1,j})da_{i,d+1}da_{j,d+1} \\ &= \sum_{i,j=1}^{d} (-r_{ij})da_{i,d+1}da_{j,d+1}, \end{aligned}$$

where

$$r_{ij} = q_i\cdot q_j = q_i^tEq_j, \quad i,j \leqq d, \tag{15}$$

in terms of the first d columns $q_1,\dots,q_d$ of Q, cf. (14). The last column of Q is $q_{d+1}=p_{d+1}$, and we have from (13) with $i=d+1$,

$$q_{d+1}\cdot q_j = p_{d+1}\cdot\frac{\partial p_{d+1}}{\partial a_{j,d+1}} = 0, \quad j \leqq d.$$

Finally, $q_{d+1}\cdot q_{d+1} = p_{d+1}\cdot p_{d+1} = 1$, and consequently (r_{ij}) from (15) extends to the bordered matrix

$$Q^tEQ = (q_i\cdot q_j)_{i,j\leqq d+1} = \begin{pmatrix} r_{11} & \cdots & r_{1d} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ r_{d1} & \cdots & r_{dd} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It follows that

$$\det(r_{ij}) = \det(Q^tEQ) = (-1)^d(\det Q)^2 = (-1)^d(\det P)^{-2} = (-1)^d|\det A|^{-1}$$

in view of the last equation in (14) together with $\det J = 1$ and $P^tEP = A$, viz. $p_i\cdot p_j = a_{ij}$.

For the volume element in H^+ we therefore have the following expression in terms of the parameters $a_{1,d+1},\dots,a_{d,d+1}$:

$$\sqrt{|\det(r_{ij})|}\,da_{1,d+1}\cdots da_{d,d+1} = \frac{1}{\sqrt{|\det A|}}da_{1,d+1}\cdots da_{d,d+1}. \quad \blacksquare$$

Lemma 2.2. *Suppose $2 \leqq n \leqq d+1$. For any integrable function f on H and any prescribed $(n-1)$-tuple of independent points $p_1,\dots,p_{n-1}$ of H we have* (cf. (6))

$$\int_H f\,dH = \omega_{d+2-n}\int_{\mathscr{K}(A_{n-1})}\frac{|\det A_n|^{(d-n)/2}}{|\det A_{n-1}|^{(d+1-n)/2}}\prod_{i=1}^{n-1}da_{in}\int_{G[p_1,\dots,p_{n-1}]}f(hp_n(A_n))dh,$$

where

$$A_{n-1} = (a_{ij})_{i,j \leqq n-1} = (p_i \cdot p_j)_{i,j \leqq n-1}.$$

The domain of integration $\mathcal{K}(A_{n-1})$ in the outer integral is the solid, open half-hyperboloid in $\mathbb{R}^{n-1}$ consisting of all points $(a_{1n}, \ldots, a_{n-1,n}) \in \mathbb{R}_+^{n-1}$ such that (writing $a_{ni} = a_{in}$ and $a_{nn} = 1$) the bordered matrix

$$A_n = (a_{ij})_{i,j=1,\ldots,n}$$

is of class $\mathscr{A}_n$. For each such $(a_{1n}, \ldots, a_{n-1,n}) \in \mathcal{K}(A_{n-1})$, $p_n(A_n)$ denotes any point of H such that

$$p_i \cdot p_n(A_n) = a_{in}, \quad i = 1, \ldots, n-1.$$

Remark 2.4. For $n = 2$ we have $A_{n-1} = (1)$, and the half-hyperboloid $\mathcal{K}(A_{n-1})$ is simply the interval $1 < a_{12} < \infty$. For any a_{12} $(= a_{21})$ in this interval we have $A_n = \begin{pmatrix} 1 & a_{12} \\ a_{21} & 1 \end{pmatrix} \in \mathscr{A}_n$, and $p_n(A_n)$ denotes any point of the hyperbolic sphere centered at p_1 and of hyperbolic radius ρ given by $\cosh \rho = a_{12}$. The integral over $G[p_1]$ in the formula is the mean value of f over the stated hyperbolic sphere.

Proof of Lemma 2.2. As in Remark 2.2 it is clear that the inner integral on the right in the above formula is independent of the choice of $p_n(A_n)$. In fact, if $q_n(A_n)$ denotes another such choice, we have $q_n(A_n) = h(A_n) p_n(A_n)$ for some $h(A_n) \in G[p_1, \ldots, p_{n-1}]$, by Theorem 1.1. It is known that the hyperbolic distance $\varrho(p_n)$ between a point $p_n \in H$ and the hyperbolic $(n-2)$-plane $H \cap \operatorname{span}(p_1, \ldots, p_{n-1})$ through $p_1, \ldots, p_{n-1}$ is given by

$$\sinh \varrho(p_n) = \frac{|\det A_n|^{1/2}}{|\det A_{n-1}|^{1/2}},$$

where A_n now denotes the extension of A_{n-1} by the entries $a_{ni} = a_{in} = p_i \cdot p_n$ and of course $a_{nn} = 1$. For the case $d = 3$ this formula is found, e.g., in Fenchel [2, (17), p. 168].

Via a transformation from G we may arrange that the $n-1$ given points $p_1, \ldots, p_{n-1} \in H$ all lie in the $x_1, \ldots, x_{n-1}$-space of $\mathbb{R}^{d+1}$ and that their determinant then is positive. Furthermore, for each n-tuple $(a_{1n}, \ldots, a_{n-1,n}) \in \mathcal{K}(A_{n-1})$ we may choose $p_n(A_n)$ in the $x_1, \ldots, x_n$-space of $\mathbb{R}^{d+1}$ so that the n^{th} coordinate $p_{nn}(A_n)$ of $p_n(A_n)$ is positive. In fact, replacing $p_1, \ldots, p_{n-1}$ by $gp_1, \ldots, gp_{n-1}$ for some $g \in G$, and accordingly $p_n(A)$ by $gp_n(A)$, transforms the formula in Lemma 2.2 into the similar formula for the function $p \mapsto f(g^{-1}p)$.

Via the natural identification of the $x_1, \ldots, x_n$-space of $\mathbb{R}^{d+1}$ with $\mathbb{R}^n$, and hence of the part of H in the $x_1, \ldots, x_n$-space with H_{n-1}, we may now apply Lemma 2.1 (writing $n-1$ for d there) to $\varepsilon = 1$, the above $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$, and the function

$$p_n \mapsto f(hp_n) \sinh^{d+1-n} \varrho(p_n)$$

on H_{n-1} $(\subset H)$ for fixed $h \in G[p_1, \ldots, p_{n-1}]$. The hyperbolic half-space H^+ in H_{n-1}

in our application of Lemma 2.1 is

$$H_{n-1}^{+}=\{p_n\in H_{n-1}\,|\det(p_1,\dots,p_n)>0\}=\{p_n\in H_{n-1}\,|\,p_{nn}>0\},$$

that is, those $p_n\in H_{n-1}$ for which the last coordinate is >0. The matrix $A_n\in\mathscr{A}_n$ and the point $p_n(A_n)\in H_{n-1}^{+}$ serve as the matrix and the point denoted, in Lemma 2.1, by A and $p_{d+1}(A)$, respectively. We obtain

$$\int\limits_{H_{n-1}^{+}} f(hp_n)\sinh^{d+1-n}\varrho(p_n)dH_{n-1}(p_n)$$

$$=\int\limits_{\mathscr{K}(A_{n-1})} f(hp_n(A_n))\left(\frac{|\det A_n|}{|\det A_{n-1}|}\right)^{(d+1-n)/2}\frac{1}{\sqrt{|\det A_n|}}\prod_{i=1}^{n-1}da_{in},$$

from which the stated result follows after multiplication by dh and integration over $G[p_1,\dots,p_{n-1}]$, while inverting the order of integrations. In fact, the orbit of $p_n(A_n)$ under the action of the transformations $h\in G[p_1,\dots,p_{n-1}]$ is a hyperbolic $(d+1-n)$-sphere of radius $\varrho(p_n(A_n))$. This sphere is situated in the hyperbolic $(d+2-n)$-plane passing through $p_n(A_n)$ and perpendicular to the hyperbolic $(n-2)$-plane H_{n-2} through $p_1,\dots,p_{n-1}$, the centre of the sphere being the (hyperbolic) projection of $p_n(A_n)$ on H_{n-2}. And the hyperbolic, or Riemannian, surface measure on this sphere $G[p_1,\dots,p_{n-1}]p_n(A_n)$ is $\omega_{d+2-n}\sinh^{d+1-n}\varrho(p_n(A))$ times the image of normalized Haar measure dh on $G[p_1,\dots,p_{n-1}]\cong O(d+2-n)$ under the map $h\mapsto hp_n(A_n)$, whence the result. ■

Proof of Theorem 2. As to the trivial case $n=1$ see Remark 2.3. Let therefore $n\geqq 2$, and take

$$p(A)=(p_1(A),\dots,p_n(A))=\varphi_n(A),$$

cf. Remark 1. Apply Lemma 2.2 to the function $p_n\mapsto f(p_1,\dots,p_{n-1},p_n)$ on H for $n-1$ given, independent points $p_1,\dots,p_{n-1}$ of H yields

$$F(p_1,\dots,p_{n-1}):=\int\limits_H f(p_1,\dots,p_{n-1})dH(p_n)=\omega_{d+2-n}$$

$$\int\limits_{\mathscr{K}(A_{n-1})}\frac{|\det A_n|^{(d-n)/2}}{|\det A_{n-1}|^{(d+1-n)/2}}\prod_{i=1}^{n-1}da_{in}\int\limits_{G[p_1,\dots,p_{n-1}]} f(p_1,\dots,p_{n-1},hp_n(A_n))dh \quad (16)$$

in the notation of Lemma 2.2, cf. (6). Suppose the theorem holds with $n-1$ in place of n, and apply it to the function F in (16) and the $(n-1)$-tuple $(p_1(A_{n-1}),\dots,p_{n-1}(A_{n-1})=\varphi_{n-1}(A_{n-1}))$ described in Remark 1 (taking $n-1$ in place of n and writing A_{n-1} in place of A). We obtain

$$\int\limits_{H^n} f\,dH^n=\int\limits_{H^{n-1}} F\,dH^{n-1}$$

$$=\frac{\tilde{\omega}_d}{\tilde{\omega}_{d+2-n}}\int\limits_{\mathscr{A}_{n-1}}|\det A_{n-1}|^{(d+1-n)/2}\prod_{i<j<n}da_{ij}\int\limits_G F(g\varphi_{n-1}(A_{n-1}))dg. \quad (17)$$

(In the case $n=2$ the integration over $\mathscr{A}_{n-1}(=\mathscr{A}_1=\{1\})$ drops out and (17) becomes, by Remark 2.3, $\int\limits_{H^2} f\,dH^2=\int\limits_H F\,dH=\int\limits_G F(gp_1)dg$, valid for any $p_1\in H$, e.g. for $p_1=(1,0,\dots,0)$ as in Remark 1.)

$$G[p_1(A_{n-1}),\dots,p_{n-1}(A_{n-1})]=G_{n-1},$$

where G_{n-1} denotes the subgroup of G consisting of all $g\in G$ such that $gx=x$ for every $x\in H$ having its last $d+2-n$ coordinates $x_n,\dots,x_{d+1}$ equal to 0. Note that G_{n-1} does not depend on A_{n-1}.

With a view at (17) we now take for $(p_1,\dots,p_{n-1})$ in (16) the $(n-1)$-tuple

$$g\varphi_{n-1}(A_{n-1})=(gp_1(A_{n-1}),\dots,gp_{n-1}(A_{n-1}))$$

(cf. Remark 1) for given $g\in G$ and $A_{n-1}=(a_{ij})_{i,j\leqq n-1}\in\mathscr{A}_{n-1}$. Note that then $gp_i(A_{n-1})\cdot gp_j(A_{n-1})=a_{in}$, $i=1,\dots,n-1$, as it should be. In place of $p_n(A_n)$ in (16) take $gp_n(A_n)$, where $p_n(A_n)$ was specified in the beginning of the proof with reference to Remark 1, and so

$$(\varphi_{n-1}(A_{n-1}),p_n(A_n))=\varphi_n(A_n).$$

We thus comply with the requirement

$$gp_i(A_{n-1})\cdot gp_n(A_n)=a_{in},\quad i=1,\dots,n-1.$$

Because $G[gp_1(A_{n-1}),\dots,gp_{n-1}(A_{n-1})]=gG_{n-1}g^{-1}$, we now obtain

$$\begin{aligned}&\frac{1}{\omega_{d+2-n}}F(g\varphi_{n-1}(A_{n-1}))\\&=\int_{\mathscr{K}(A_{n-1})}\frac{|\det A_n|^{(d-n)/2}}{|\det A_{n-1}|^{(d+1-n)/2}}\prod_{i=1}^{n-1}da_{in}\int_{gG_{n-1}g^{-1}}f(g\varphi_{n-1}(A_{n-1}),hgp_n(A_n))dh\\&=\int_{\mathscr{K}(A_{n-1})}\frac{|\det A_n|^{(d-n)/2}}{|\det A_{n-1}|^{(d+1-n)/2}}\prod_{i=1}^{n-1}da_{in}\int_{G_{n-1}}f(g\tilde{h}\varphi_{n-1}(A_{n-1}),g\tilde{h}p_n(A_n))d\tilde{h},\end{aligned}$$

noting also that $\varphi_{n-1}(A_{n-1})=\tilde{h}\varphi_{n-1}(A_{n-1})$ when $\tilde{h}\in G_{n-1}$, and that Haar measure dh on $gG_{n-1}g^{-1}$ is the image of Haar measure $d\tilde{h}$ on G_{n-1} under the conjugation $h=g\tilde{h}g^{-1}$.

In the rest of the proof we prefer to write A in place of A_n. By the continuity of φ_n the integrand

$$f(g\tilde{h}\varphi_{n-1}(A_{n-1}),g\tilde{h}p_n(A_n))=f(g\tilde{h}\varphi_n(A))$$

in the above integral over G_{n-1} is a measurable function of $(g,\tilde{h},A)\in G\times G_{n-1}\times\mathscr{A}_n$. When integrating $F(g\varphi_{n-1}(A_{n-1}))$ over G to obtain the inner integral in (17) we may therefore invert the order of integrations so as to obtain from the above

$$\frac{1}{\omega_{d+2-n}}\int_G F(g\varphi_{n-1}(A_{n-1}))dg=\int_{\mathscr{K}(A_{n-1})}\frac{|\det A|^{(d-n)/2}}{|\det A_{n-1}|^{(d+1-n)/2}}\prod_{i=1}^{n-1}da_{in}\int_G f(g\varphi_n(A))dg \tag{18}$$

by right invariance of Haar measure dg on the unimodular group G, together with the normalization of Haar measure $d\tilde{h}$ on the compact group $G_{n-1}\cong O(d+2-n)$ (with $d+2-n\leqq d$ since $n\geqq 2$).

Finally insert (18) in (17) and interchange integrations. This leads to the formula stated in the theorem (with $p(A)=\varphi_n(A)$) when we note that $\tilde{\omega}_{d+2-n}=\omega_{d+2-n}\tilde{\omega}_{d+1-n}$ according to (7) and further that

$$\prod_{i<j<n}da_{ij}\prod_{i=1}^{n-1}da_{in}=\prod_{i<j\leqq n}da_{ij}=dA.\quad\blacksquare$$

3. Integration in the Hyperbolic Case with $n \geqq d+1$

In this section we use, as parameters for H^n modulo diagonal action of G, the entries to the right of the diagonal in the first d rows of matrices of class $\mathscr{A}_{n,d+1}$, noting that this gives the correct number

$$N = dn - \tfrac{1}{2}d(d+1)$$

of degrees of freedom, cf. (12), where now $r = d+1$.

Definition 3.1. When $n \geqq d+1$ we denote by $\mathscr{A}_{d\times n}$ the class of all $d\times n$ matrices $A = (a_{ij})$ such that, writing

$$A_d = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix},$$

the bordered $(d+1)\times(d+1)$ matrix

$$A_{d;k} := A_d(a_{1k},\ldots,a_{dk})$$

(cf. Lemma 1) is of class $\mathscr{A}_{d+1}$ for each $k = d+1,\ldots,n$.

According to Lemma 1 this means that A_d should be of class $\mathscr{A}_d$ and that, for each k as stated, $(a_{1k},\ldots,a_{dk})$ should belong to the solid, open half-hyperboloid $\mathscr{K}(A_d)$ in $\mathbb{R}^d$. It follows that

$$a_{ii} = 1, \quad a_{ij} > 1 \quad \text{for} \quad i \neq j \quad (i = 1,\ldots,d;\ j = 1,\ldots,n).$$

Definition 3.2. For each $(n-d-1)$-tuple $\varepsilon = (\varepsilon_{d+2},\ldots,\varepsilon_n)$ of numbers 1 or -1 let $H^n(\varepsilon)$ denote the set of all n-tuples $(p_1,\ldots,p_n)\in H^n$ such that the numbers $\varepsilon_k \det(p_1,\ldots,p_d,p_k)$, $k = d+2,\ldots,n$, are all $\neq 0$ and have the same sign as $\det(p_1,\ldots,p_{d+1})$, likewise supposed $\neq 0$. Further write

$$\begin{aligned}\hat{H}^n &= \bigcup_{\varepsilon\in\{1,-1\}^{n-d-1}} H^n(\varepsilon) \\ &= \{(p_1,\ldots,p_n)\in H^n \mid \det(p_1,\ldots,p_d,p_k) \neq 0 \text{ for } k = d+1,\ldots,n\}.\end{aligned}$$

In geometric terms, $(p_1,\ldots,p_n)\in H^n(\varepsilon)$ means that p_{d+1} and p_k lie on the same side of the (hyperbolic) hyperplane in H passing through $p_1,\ldots,p_d$ (supposed independent) if $\varepsilon_k = 1$, and on opposite sides if $\varepsilon_k = -1$ $(k = d+2,\ldots,n)$.

These 2^{n-d-1} sets $H^n(\varepsilon)$ are disjoint, and each of them is open and invariant under diagonal action of G. For $n = d+1$ it is understood that (with ε empty) $H^{d+1}(\varepsilon) = H_*^{d+1} = \hat{H}^{d+1} =$ the set of independent $(d+1)$-tuples of points of H.

Lemma 3. *Suppose $n \geqq d+1$ and $\varepsilon\in\{1,-1\}^{n-d-1}$. The map*

$$(p_1,\ldots,p_n) \mapsto (p_i\cdot p_j)_{i\leqq d,\, j\leqq n}$$

is then a surjection of $H^n(\varepsilon)$ onto $\mathscr{A}_{d\times n}$. Each fibre is an orbit in $H^n(\varepsilon)$ under diagonal action of G, and this action is bijective on each fibre.

Proof. For $n = d+1$, this reduces trivially to Theorem 1.1 because $H^{d+1}(\varepsilon)$ equals H_*^{d+1} and $\mathscr{A}_{d\times(d+1)}$ can be identified with $\mathscr{A}_{d+1}$. For $n > d+1$ and any

$k = d+1, \dots, n$, apply Theorem 1.1 (with $d+1$ in place of n) to the matrix $A_{d;k} \in \mathscr{A}_{d+1}$ (cf. Definition 3.1). This leads to an independent $(d+1)$-tuple $(p_1, \dots, p_d, p_k) \in H_*^{d+1}$ satisfying

$$p_i \cdot p_j = a_{ij} \quad \text{for} \quad i, j = 1, \dots, d,$$
$$p_i \cdot p_k = a_{ik} \quad \text{for} \quad i = 1, \dots, d.$$

By Theorem 1.1 applied to $n = d$ each fibre of the map $(p_1, \dots, p_d) \mapsto (p_i \cdot p_j)$ of H_*^d onto $\mathscr{A}_d$ is an orbit under diagonal action of G, and this allows us to choose the $(d+1)$-tuple $(p_1, \dots, p_d, p_k) \in H_*^{d+1}$ in the fibre of $A_{d;k}$ in such a way that $(p_1, \dots, p_d)$ is the same d-tuple for all $k = d+1, \dots, n$. Next, if, for some $k > d+1$, $\varepsilon_k \det(p_1, \dots, p_d, p_k)$ has the opposite sign of $\det(p_1, \dots, p_{d+1})$, we replace p_k by its image under the (hyperbolic) reflection of H in the hyperplane in H passing through $p_1, \dots, p_d$; this change does not affect the value of $p_i \cdot p_k$ $(= a_{ik})$, $i = 1, \dots, d$. We have thus proved that the map $H^n(\varepsilon) \mapsto \mathscr{A}_{d \times n}$ is surjective. The remaining assertions are likewise easily derived from Theorem 1.1 applied to $n = d+1$. ∎

We proceed to establish, for the case $n \geqq d+1$, the following formula (suitably interpreted), valid in each of the sets $H^n(\varepsilon)$:

$$dH^n = \tilde{\omega}_d \prod_{k=d+1}^{n} \frac{1}{\sqrt{|\det A_{d;k}|}} \, dA \, dg$$

in terms of the mapping $(p_1, \dots, p_n) \mapsto A = (p_i \cdot p_j)$ of $H^n(\varepsilon)$ on $\mathscr{A}_{d \times n}$ described in Lemma 3, and the diagonal action of G. The notation is as follows.

dH^n refers again to the product measure on H^n. Note that $H^n \setminus \hat{H}^n$ has measure 0.

dA refers to Lebesgue measure on the space $(\cong \mathbb{R}^{dn-(1/2)d(d+1)})$ of all $d \times n$ matrices $A = (a_{ij})$ such that $a_{ii} = 1$ and $a_{ji} = a_{ij}$ for $i, j = 1, \dots, d$:

$$dA = \prod_{i \leqq d,\, i < j \leqq n} da_{ij}.$$

dg refers to Haar measure on G, normalized as described in Sect. 1.

A precise formulation of the indicated result is given in the following theorem (recall Definitions 3.1 and 3.2).

Theorem 3. *Let $\varepsilon = (\varepsilon_{d+2}, \dots, \varepsilon_n) \in \{1, -1\}^{n-d-1}$ be given. For any integrable function f on $H^n(\varepsilon)$ we have*

$$\int_{H^n(\varepsilon)} f \, dH^n = \tilde{\omega}_d \int_{\mathscr{A}_{d \times n}} \prod_{k=d+1}^{n} \frac{1}{\sqrt{|\det A_{d;k}|}} \, dA \int_G f(gp(A)) \, dg,$$

where

$$p(A) = (p_1(A), \dots, p_n(A)) \in H^n(\varepsilon), \quad A = (a_{ij}) \in \mathscr{A}_{d \times n},$$

denotes an arbitrary selection of n-tuples in $H^n(\varepsilon)$ such that

$$p_i(A) \cdot p_j(A) = a_{ij} \quad \text{for} \quad i \leqq d, \quad j \leqq n,$$

cf. Lemma 3, and where we write

$$gp(A) = (gp_1(A), \dots, gp_n(A)).$$

Remark 3.1. Once $p_1(A),\dots,p_{d+1}(A)$ have been selected in H subject to $p_i(A)\cdot p_j(A)=a_{ij}$ for $i\leqq d$, $j\leqq d+1$, the remaining points $p_k(A)$ are uniquely determined by the remaining conditions $p_i(A)\cdot p_k(A)=a_{ik}$ for $i\leqq d, d+2\leqq k\leqq n$ because of the requirement that $p(A)\in H^n(\varepsilon)$.

Remark 3.2. It is clear from the beginning that the inner integral on the right in the formula of Theorem 3 is independent of the particular choice of $p(A)$. In fact, if $q(A)=(q_1(A),\dots,q_n(A))$ is another such choice (again in $H^n(\varepsilon)$) then we have $q_i(A)=g(A)p_i(A)$, $i=1,\dots,n$, for some $g(A)\in G$, cf. Lemma 3.

Proof of Theorem 3. In the case $n=d+1$, Theorems 2 and 3 coalesce because $H^{d+1}(\varepsilon)=H_*^{d+1}$, $\tilde{\omega}_0=1$, and $\mathscr{A}_{d\times(d+1)}$ may be identified with $\mathscr{A}_{d+1}$.

In the remaining case $n>d+1$ we suppose that the theorem holds with n replaced by $n-1$. We make a measurable selection (e.g. as in Remark 1) of $(d+1)$-tuples

$$(p_1(B),\dots,p_{d+1}(B))\in H^{d+1},\quad B=(b_{ij})\in\mathscr{A}_{d+1},\tag{19}$$

so that

$$p_i(B)\cdot p_j(B)=b_{ij}\quad\text{for}\quad i,j=1,\dots,d+1.$$

For any $A=(a_{ij})\in\mathscr{A}_{d\times n}$ define in terms of (19)

$$p_j(A)=p_j(A_{d;d+1}),\quad j=1,\dots,d+1,$$

noting that $A_{d;d+1}\in\mathscr{A}_{d+1}$, and extend this uniquely (Remark 3.1) to a likewise measurable selection of n-tuples

$$p(A)=(p_1(A),\dots,p_n(A))\in H^n(\varepsilon),\quad A=(a_{ij})\in\mathscr{A}_{d\times n},$$

of the kind stated in the theorem. Replacing n by $n-1$ ($\geqq d+1$) and ε by

$$\tilde{\varepsilon}=(\varepsilon_{d+2},\dots,\varepsilon_{n-1})\in\{1,-1\}^{n-d-2}$$

we obtain a similar unique measurable selection of $(n-1)$-tuples

$$p(\tilde{A}):=(p_1(\tilde{A}),\dots,p_{n-1}(\tilde{A}))\in H^{n-1}(\tilde{\varepsilon}),\quad \tilde{A}\in\mathscr{A}_{d\times(n-1)}.$$

In view of the uniqueness of the performed extension we have

$$p_j(\tilde{A})=p_j(A)\quad\text{for}\quad j=1,\dots,n-1\tag{20}$$

whenever A is obtained from $\tilde{A}$ by adjoining an additional n^{th} column.

By the inductive hypothesis we may apply the formula of the theorem to the following function F of $(n-1)$-tuples of class $H^{n-1}(\tilde{\varepsilon})$:

$$F(p_1,\dots,p_{n-1}):=\int_{H(p_1,\dots,p_{d+1};\varepsilon_n)} f(p_1,\dots,p_{n-1},p_n)\,dH(p_n),\tag{21}$$

where we integrate over the hyperbolic half-space

$$H(p_1,\dots,p_{d+1};\varepsilon_n)=\left\{p_n\in H\,\middle|\,\varepsilon_n\frac{\det(p_1,\dots,p_d,p_n)}{\det(p_1,\dots,p_{d+1})}>0\right\},\tag{22}$$

the set of all $p_n \in H$ such that $(p_1, \dots, p_n) \in H^n(\varepsilon)$. This leads to

$$\int_{H^n(\varepsilon)} f\,dH^n = \int_{H^{n-1}(\tilde{\varepsilon})} F\,dH^{n-1}$$

$$= \tilde{\omega}_d \int_{\mathscr{A}_{d\times(n-1)}} \prod_{k=d+1}^{n-1} \frac{1}{\sqrt{|\det \tilde{A}_{d;k}|}} d\tilde{A} \int_G F(gp(\tilde{A}))dg, \tag{23}$$

where

$$gp(\tilde{A}) := (gp_1(\tilde{A}), \dots, gp_{n-1}(\tilde{A})).$$

Inserting (21) in the inner integral on the right in (23) gives for any $\tilde{A} \in \mathscr{A}_{d\times(n-1)}$,

$$\int_G F(gp(\tilde{A}))dg = \int_G dg \int_{H(gp_1(\tilde{A}),\dots,gp_{d+1}(\tilde{A});\varepsilon_n)} f(gp(\tilde{A}), p_n)dH(p_n)$$

$$= \int_G dg \int_{H(p_1(\tilde{A}),\dots,p_{d+1}(\tilde{A});\varepsilon_n)} f(gp(\tilde{A}), gp_n)dH(p_n) \tag{24}$$

after performing the substitution $p_n \mapsto gp_n$ in the inner integral.

We now apply Lemma 2.1 to the inner integral in the last expression in (24), with $p_1(\tilde{A}), \dots, p_d(\tilde{A})$ playing the role of the d prescribed, independent points of H. Take $\varepsilon = \varepsilon_n$ if $\det(p_1(\tilde{A}), \dots, p_{d+1}(\tilde{A})) > 0$; otherwise take $\varepsilon = -\varepsilon_n$. Then H^+ from Lemma 2.1 becomes

$$H^+ = H(p_1(\tilde{A}), \dots, p_{d+1}(\tilde{A}), \varepsilon_n)$$

in the notation (22). For any point

$$(a_{1n}, \dots, a_{dn}) \in \mathscr{K}(\tilde{A}_d)$$

and associated matrix $A \in \mathscr{A}_{d\times n}$ obtained by adjoining $(a_{1n}, \dots, a_{dn})$ to $\tilde{A}$ as an additional n^{th} column, it follows from (22) and (20) that $p_n(A) \in H^+$ because $n-1 \geqq d+1$ and

$$(p_1(\tilde{A}), \dots, p_{n-1}(\tilde{A}), p_n(A)) = (p_1(A), \dots, p_{n-1}(A), p_n(A)) \in H^n(\varepsilon).$$

Altogether, $p_n(A)$ may serve as the point of H^+ denoted by $p_{d+1}(A)$ in Lemma 2.1. The inner integral at the end of (24) therefore equals

$$\int_{\mathscr{K}(\tilde{A}_d)} \frac{1}{\sqrt{|\det A_{d;n}|}} f(gp(\tilde{A}), gp_n(A)) \prod_{i=1}^{d} da_{in} = \int_{\mathscr{K}(\tilde{A}_d)} \frac{1}{\sqrt{|\det A_{d;n}|}} f(gp(A)) \prod_{i=1}^{d} da_{in}$$

in view of (20). Inserting this in (24) and inverting the order of integrations leads to

$$\int_G F(gp(\tilde{A}))dg = \int_{\mathscr{K}(\tilde{A}_d)} \frac{1}{\sqrt{|\det A_{d;n}|}} \prod_{i=1}^{d} da_{in} \int_G f(gp(A))dg.$$

Finally, the stated formula of the theorem arises from this when inserted in (23). In fact, when $\tilde{A}$ ranges over $\mathscr{A}_{d\times(n-1)}$ and for each such $\tilde{A}$ the point $(a_{1n}, \dots, a_{dn})$ ranges over $\mathscr{K}(\tilde{A}_d)$, then A as defined above ranges over $\mathscr{A}_{d\times n}$ according to Definition 3.1, and we have

$$A_d = \tilde{A}_d, \quad A_{d;k} = \tilde{A}_{d;k} \quad \text{for} \quad k = d+1, \dots, n-1,$$

and

$$dA = d\tilde{A} \prod_{i=1}^{d} da_{in}. \quad \blacksquare$$

Corollary. *Let $p(A) = (p_1(A), \dots, p_n(A)) \in H^n$, $A \in \mathscr{A}_{d \times n}$, denote any selection such that*

$$p_i(A) \cdot p_j(A) = a_{ij} \quad \textit{for} \quad i \leqq d, \quad j \leqq n,$$

and

$$\det(p_1(A), \dots, p_d(A), p_k(A)) > 0 \quad \textit{for} \quad k = d+1, \dots, n.$$

For any

$$\varepsilon = (\varepsilon_{d+2}, \dots, \varepsilon_n) \in \{1, -1\}^{n-d-1}$$

write

$$p(A, \varepsilon) = (p_1(A), \dots, p_{d+1}(A), p_{d+2}(A, \varepsilon), \dots, p_n(A, \varepsilon)),$$

where, for $k = d+2, \dots, n$,

$$p_k(A, \varepsilon) = \begin{cases} p_k(A) & \textit{if} \quad \varepsilon_k = 1 \\ h(A) p_k(A) & \textit{if} \quad \varepsilon_k = -1, \end{cases}$$

$h(A)$ denoting reflection of H in the hyperbolic hyperplane in H passing through $p_1(A), \dots, p_d(A)$. For any integrable function f on H^n we then have

$$\int_{H^n} f \, dH^n = \tilde{\omega}_d \int_{\mathscr{A}_{d \times n}} \prod_{k=d+1}^{n} \frac{1}{\sqrt{|\det A_{d;k}|}} dA \int_G \sum_{\varepsilon \in \{1, -1\}^{n-d-1}} f(gp(A, \varepsilon)) \, dg.$$

In fact, for any ε as stated and any $A \in \mathscr{A}_{d \times n}$, we have $p(A, \varepsilon) \in H^n(\varepsilon)$, and

$$p_i(A, \varepsilon) \cdot p_j(A, \varepsilon) = p_i(A) \cdot p_j(A) = a_{ij} \quad \text{for} \quad i \leqq d, \quad j \leqq n.$$

From Theorem 3, with $p(A, \varepsilon)$ in place of $p(A)$, we therefore obtain

$$\int_{H^n(\varepsilon)} f \, dH^n = \tilde{\omega}_d \int_{\mathscr{A}_{d \times n}} \prod_{k=d+1}^{n} \frac{1}{\sqrt{|\det A_{d;k}|}} dA \int_G f(gp(A, \varepsilon)) \, dg,$$

and it only remains to sum over all $\varepsilon \in \{1, -1\}^{n-d-1}$.

4. A Unified and Perturbation Invariant Form of the Result in the Hyperbolic Case

When $n > d+1$ the parameter space $\mathscr{A}_{d \times n}$ used in Sect. 3 reflects a choice of $d+1$ among the n points $p_i \in H$. To obtain a unified formulation of the result valid for all n, and permutation invariant also for $n > d+1$, we must use the full manifold $\mathscr{A}_{n,d+1}$ (cf. Definition 1 and Theorems 1.1 and 1.2) of dimension

$$\dim \mathscr{A}_{n,d+1} = N = dn - \tfrac{1}{2} d(d+1) \quad (= \dim \mathscr{A}_{d \times n})$$

and of codimension (in $\mathscr{M}_n$, the symmetric $n \times n$ matrices with diagonal entries 1)

$$\operatorname{codim} \mathscr{A}_{n,d+1} = \tfrac{1}{2}(n-d)(n-1-d).$$

Those matrices $A \in \mathscr{A}_{n,d+1}$ for which each principal $(d+1) \times (d+1)$ submatrix of the form

$$(a_{ij})_{i,j\in\{1,\dots,d,k\}}, \quad k=d+1,\dots,n,$$

is non-singular constitute the image $\hat{\mathscr{A}}_{n,d+1}$ of $\hat{H}^n$ under the map from Theorem 1.1, cf. Definition 3.2. This image $\hat{\mathscr{A}}_{n,d+1}$ is a (relatively) open subset of $\mathscr{A}_{n,d+1}$, and the rest of $\mathscr{A}_{n,d+1}$ has measure 0 with respect to the Riemannian volume measure $d\lambda$ on $\mathscr{A}_{n,d+1}$ induced by the Euclidean metric on $\mathscr{M}_n$.

As local coordinates in $\hat{\mathscr{A}}_{n,d+1}$ we may use the entries to the right of the diagonal in the matrices of class $\mathscr{A}_{d\times n}$, cf. Definition 3.1. This follows from the proof of Theorem 1.2 where we may now take $(i_1,\dots,i_{d+1})=(1,\dots,d+1)$ and $\imath(k)=d+1$ for all $k=d+2,\dots,n$.

Consider now the restriction map, or projection,

$$\hat{\mathscr{A}}_{n,d+1}\to\mathscr{A}_{d\times n}$$

consisting in deleting the last $n-d$ rows of matrices A of class $\hat{\mathscr{A}}_{n,d+1}$. This map is 2^{n-d-1}-to-one. In fact, for each $A\in\mathscr{A}_{d\times n}$ and each $\varepsilon=(\varepsilon_{d+2},\dots,\varepsilon_n)\in\{1,-1\}^{n-d-1}$ there exists, by Lemma 3, an n-tuple $(p_1,\dots,p_n)\in H^n(\varepsilon)$, uniquely determined up to diagonal action of G, such that the $n\times n$ matrix $\hat{A}=(p_i\cdot p_j)$ of class $\hat{\mathscr{A}}_{n,d+1}$ has A as a submatrix formed by the first d rows. Two such matrices $\hat{A}$ corresponding to the same $A\in\mathscr{A}_{d\times n}$, but to different choices of ε, are distinct because the associated n-tuples $(p_1,\dots,p_n)$ are not on the same fibre for the map from Theorem 1.1, by the invariance of each $H^n(\varepsilon)$ under diagonal action of G, cf. Definition 3.2 and subsequent comments.

In view of the above observations, Theorems 2 and 3 admit the following unified formulation, valid for any number n of "particles" and symmetric in these.

Theorem 4. *There exists a unique positive measure $d\mu$ on the N-dimensional manifold $\mathscr{A}_{n,d+1}$ such that, for any integrable function f on H^n,*

$$\int_{H^n} f\,dH^n=\int_{\mathscr{A}_{n,d+1}} d\mu(A)\int_G f(gp_1(A),\dots,gp_n(A))\,dg,$$

where $A\mapsto(p_1(A),\dots,p_n(A))$ denotes an arbitrary selection of n-tuples of points of H such that, for any $A=(a_{ij})\in\mathscr{A}_{n,d+1}$,

$$p_i(A)\cdot p_j(A)=a_{ij}.$$

This measure $d\mu$ is invariant under simultaneous permutation of rows and columns of matrices A of class $\mathscr{A}_{n,d+1}$.

In the case $n\leq d+1$, $d\mu$ is given as follows in terms of Lebesgue measure $d\lambda$ on the open set $\mathscr{A}_{n,d+1}=\mathscr{A}_n$ in $\mathscr{M}_n$:

$$d\mu=\frac{\tilde{\omega}_d}{\tilde{\omega}_{d+1-n}}|\det A|^{(d-n)/2}\,d\lambda,\quad d\lambda=dA=\prod_{i<j}da_{ij}.$$

In the case $n\geq d+1$, $d\mu$ is given as follows on the open submanifold $\hat{\mathscr{A}}_{n,d+1}$ of $\mathscr{A}_{n,d+1}$, using the above local coordinates $a_{ij}, i\leq d, i<j\leq n$,

$$d\mu=\tilde{\omega}_d\prod_{k=d+1}^{n}\frac{1}{\sqrt{|\det A_{d;k}|}}\prod_{\substack{i\leq d\\ i<j\leq n}}da_{ij}. \tag{25}$$

Here the principal $(d+1)\times(d+1)$ *submatrices* $A_{d;k}$ *of* $A\in\hat{\mathscr{A}}_{n,d+1}$ *are as specified in Definition 3.1. Moreover,* $\mu(\mathscr{A}_{n,d+1}\setminus\hat{\mathscr{A}}_{n,d+1})=0$, *by definition.*

Proof. The above measure $d\mu$ has the property asserted in the theorem in either case $n\leqq d+1$ or $n>d+1$, according to Theorem 2 and Theorem 3, respectively. Conversely, let $d\mu$ denote any positive measure on $\mathscr{A}_{n,d+1}$ with the stated property. Choose a Borel set $\Gamma\subset G$ of finite Haar measure $c>0$, and consider, for any Borel set $\mathscr{A}\subset\mathscr{A}_{n,d+1}$, the following subset of H_*^n:

$$E=\{(gp_1(A),\dots,gp_n(A))\,|\,g\in\Gamma,\ A\in\mathscr{A}\}.$$

The indicator function f of E then satisfies

$$f(gp_1(A),\dots,gp_n(A))=\begin{cases}1 & \text{if}\quad g\in\Gamma\quad\text{and}\quad A\in\mathscr{A}\\ 0 & \text{otherwise}\end{cases}$$

because the map $(g,A)\mapsto(gp_1(A),\dots,gp_n(A))$ of $\Gamma\times\mathscr{A}$ into H_*^n is injective according to Theorem 1.1. Consequently,

$$\int_{H^n} f\,dH^n=\int_{\mathscr{A}} d\mu\int_{\Gamma} f(gp_1(A),\dots,gp_n(A))dg=c\mu(\mathscr{A}),$$

showing that $d\mu$ is indeed uniquely determined. ■

Remark 4. The formula (25) for $d\mu$ in the case $n>d+1$ extends to certain other local coordinate systems, e.g. those considered in the proof of Theorem 1.2. In the notation of that proof we have in the open subset $\mathscr{A}(i_1,\dots,i_{d+1};\iota)$ of full $d\mu$-measure in $\mathscr{A}_{n,d+1}$,

$$d\mu=\tilde{\omega}_d\prod_{k=d+1}^{n}\frac{1}{\sqrt{|\det A(k)|}}\prod da_{ij},$$

where

$$A(k)=\begin{cases}(a_{ij})_{i,j=i_1,\dots,i_{d+1}} & \text{for}\quad k=d+1\\ (a_{ij})_{i,j=i_1,\dots,\widehat{\iota(i_k)},\dots,i_{d+1},i_k} & \text{for}\quad k=d+2,\dots,n,\end{cases}$$

while $\prod da_{ij}$ is taken over all couples (i,j), $i<j$, with either $i,j\in\{i_1,\dots,i_{d+1}\}$ or else $j\notin\{i_1,\dots,i_{d+1}\}$ and $i\in\{i_1,\dots,i_{d+1}\}\setminus\{\iota(j)\}$.

The stated formula for $d\mu$ can be obtained as in the proof of Theorem 3, or alternatively by determining the Jacobian for the transition from the above system of local coordinates to the particular system $(a_{ij})_{i\leqq d,i<j\leqq n}$ used in Theorem 3.

As a consequence of the above one finds (for any n) that $d\mu$ has an analytic density $d\mu/d\lambda$ (>0) with respect to the N-dimensional volume measure λ on the manifold $\mathscr{A}_{n,d+1}$. Writing this density in the form

$$\frac{d\mu}{d\lambda}=\frac{\tilde{\omega}_d}{\sqrt{D(A)}},$$

we have found above in the case $n=d+1$ that $D(A)=(-1)^d\det A$ for $A\in\mathscr{A}_{n,d+1}$ $(=\mathscr{A}_{d+1})$, in agreement with Theorem 2. When $n=d+2$, $\mathscr{A}_{n,d+1}$ has codimension 1 in $\mathscr{M}_n$, and $D(A)$ turns out to be the sum of the squares of all the $n(n-1)/2$

non-principal minors in A, or equally well the sum of all products of two distinct principal minors in A.

For $n-d>2$ the expression for $D(A)$ becomes increasingly complicated. For example, in the case $d=1$, $n=4$, $D(A)$ is minus the sum of all "non-cyclic" products of 3 distinct principal 2×2 subdeterminants $1-a_{ij}^2$ ($\leqq 0$), a cyclic product meaning one of the form

$$(1-a_{ij}^2)(1-a_{jk}^2)(1-a_{ki}^2) \quad \text{with} \quad i<j<k.$$

It therefore seems preferable to leave the Riemannian volume measure λ on $\mathscr{A}_{n,d+1}$ aside and stick to the above measure $d\mu$ itself, using for example one of its expressions stated above in terms of local parameters a_{ij} when it comes to computations.

5. The Spherical Case

In this short section we consider, instead of the hyperbolic d-space H, the unit sphere S in $\mathbb{R}^{d+1}$ given by

$$S=\{x\in\mathbb{R}^{d+1}\,|\,x\cdot x=1\}$$

in terms of the Euclidean inner product $x\cdot y$. Accordingly, G shall now denote the isometry group $O(d+1)$ of S. The hyperbolic functions cosh, sinh are of course now replaced by cos, sin.

We denote in this section by $\mathscr{A}_n$ the class of all *positive* $n\times n$ matrices with 1 in the diagonal. In Lemma 1 the solid open half-hyperboloid $\mathscr{K}(A)$ is now replaced by the solid open *ellipsoid*

$$\mathscr{K}(A)=\{x\in\mathbb{R}^n\,|\det A(x)>0\}=\{x\in\mathbb{R}^n\,|\,x^tA^{-1}x<1\}.$$

Each point x of $\mathscr{K}(A)$ satisfies $|x_i|<1$ for $i=1,\dots,n$. The boundary $\partial\mathscr{K}(A)$ is the unique ellipsoid centered at 0, passing through the columns a_i of A, and having the tangent hyperplane $x_i=1$ at $a_i, i=1,\dots,n$.

Theorem 5. *With the above changes (and their obvious consequences) all the results of Sects. 1 through 4 carry over to the spherical case when we replace throughout H by the unit sphere S.*

It is understood here that Haar measure dg on G is normalized (in analogy with Sect. 1) so that it induces the usual surface measure dS on S of total mass ω_{d+1}, when S is identified with $G/G[p]$. In this normalization the total mass of dg is

$$\int_G dg=\omega_{d+1}$$

because Haar measure on $G[p]$ ($\cong O(d)$) is taken throughout to have total mass 1. However, in the present spherical case it is more consistent to normalize the Haar measure also on $G=O(d+1)$ so that $\int_G dg=1$. *With this latter normalization the constant $\tilde{\omega}_d$ occurring in Sects. 2, 3, and 4 should be replaced throughout the present section by $\tilde{\omega}_{d+1}(=\omega_{d+1}\tilde{\omega}_d)$.*

6. The Euclidean Case

In this final section we replace the hyperbolic d-space H by Euclidean d-space E, for which $\mathbb{R}^d$ serves as a model. Accordingly, G now denotes the group of Euclidean isometries of E. Naturally, $E = \mathbb{R}^d$ is endowed with the Euclidean distance $|x-y|$ and with the standard inner product $x \cdot y$. Haar measure dg on G is normalized so that it induces Lebesgue measure dE on E when E is identified with $G/G[p]$. If g is written in the standard way

$$gp = up + v, \quad u \in O(d), \quad v \in \mathbb{R}^d,$$

then $dg = du\,dv$.

For any $n \times n$ matrix $A = (a_{ij})$ the sum of its n^2 minors will be called the *derived determinant* of A and denoted by $\det' A$. Thus

$$\det' A = \frac{d}{ds} \det(a_{ij} + s)\Big|_{s=0} = -\det\begin{pmatrix} A & e \\ e^t & 0 \end{pmatrix},$$

where the column e and the row e^t have the n entries 1.

When $a_{ii} = 0$ for all i we associate with A the following $(n-1) \times (n-1)$ matrix $B = (b_{ij})$:

$$b_{ij} = a_{ij} - a_{in} - a_{nj}, \quad i, j \neq n. \tag{26}$$

It follows that $\det(A + s) = \det A + s \det B$ ($A + s$ having the entries $a_{ij} + s$) and hence

$$\det' A = \det B. \tag{27}$$

Similarly if n in (26) is replaced by any index $1, \ldots, n-1$.

In this section $\mathscr{A}_n$ denotes the class of all symmetric $n \times n$ matrices $A = (a_{ij})$ with $a_{ii} = 0$, $a_{ij} \geqq 0$, such that the restriction of the quadratic form $\sum a_{ij} x_i x_j$ to the hyperplane $x_1 + \cdots + x_n = 0$ is *negative definite*.

Replacing here "negative definite" by *negative semidefinite and of rank* $r-1$, cf. (8), we obtain the class to be denoted now by $\mathscr{A}_{n,d+1}$. With $x_1, \ldots, x_{n-1}$ as parameters for the above hyperplane, these last two conditions for A to be of class $\mathscr{A}_{n,d+1}$ translate into the same properties of the associated $(n-1) \times (n-1)$ matrix B, cf. (26), that is,

$$\operatorname{ind}_+ B = 0, \quad \operatorname{ind}_- B = r - 1 \ \ (= \min\{n-1, d\}).$$

In the case $n \leqq d+1$, that is $r = n$, we may write $\mathscr{A}_n$ in place of $\mathscr{A}_{n,d+1}$.

We obtain counterparts to the results of Sect. 1 when we replace the map $(p_1, \ldots, p_n) \mapsto (p_i \cdot p_j)$ considered there by the mapping

$$(p_1, \ldots, p_n) \mapsto (\tfrac{1}{2}|p_i - p_j|^2)_{i,j=1,\ldots,n}$$

of the set E_*^n of all n-tuples of points of $E = \mathbb{R}^d$ spanning an affine space of maximal dimension $r-1$ into the symmetric $n \times n$ matrices. The fact that this mapping takes E_*^n onto our new class $\mathscr{A}_{n,d+1}$ is well known, cf. Schoenberg [4]. The rest of these counterparts to the statements in Theorems 1.1 and 1.2 are established mutatis mutandis.

In place of Lemma 1 we have in the Euclidean case the following lemma related to a result of Menger [3, p. 133].

Lemma 6. *Consider a matrix $A \in \mathscr{A}_n$. In order that the bordered $(n+1)\times(n+1)$ matrix*

$$A(x)=\begin{pmatrix} A & x \\ x^t & 0 \end{pmatrix}, \quad x=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n,$$

be of class $\mathscr{A}_{n+1}$ it is necessary and sufficient that x belongs to the solid, open paraboloid

$$\mathscr{K}(A)=\{x\in\mathbb{R}^n \mid (-1)^n \det' A(x)>0\} \subset \mathbb{R}^n_+ .$$

Proof. With A we associate, as above, a symmetric $(n-1)\times(n-1)$ matrix, now called $-B$, where $B=(b_{ij})_{i,j=2,\ldots,n}$ is defined this time by

$$b_{ij}=a_{i1}+a_{1j}-a_{ij}, \quad i,j=2,\ldots,n.$$

With $A(x)$ we similarly associate the symmetric $n\times n$ matrix $-B(x)$, where $B(x)=(b_{ij}(x))_{i,j=2,\ldots,n+1}$ is given by

$$b_{ij}(x)=\begin{cases} b_{ij} & \text{for} \quad i,j=2,\ldots,n \\ a_{i1}+x_1-x_i & \text{for} \quad i=2,\ldots,n, \quad j=n+1 \\ 2x_1 & \text{for} \quad i,j=n+1. \end{cases}$$

Then B is positive definite (because $A\in\mathscr{A}_n$). Similarly, $A(x)$ is of class $\mathscr{A}_{n+1}$ if and only if $x \geqq 0$ and $B(x)$ is positive definite. The latter condition (which implies $2x_1>0$) translates into $\det B(x)>0$, that is, $(-1)^n \det' A(x)>0$, by use of (27):

$$(-1)^n \det' A(x)=(-1)^n \det(-B(x))=\det B(x).$$

By the observation after (27), $(-1)^n \det' A(x)>0$ implies $x>0$, and so $\mathscr{K}(A)\subset\mathbb{R}^n_+$. To see that $\mathscr{K}(A)$ is a paraboloid we perform the substitution

$$\begin{aligned} y_1&=2x_1, \\ y_i&=a_{i1}+x_1-x_i, \quad \text{for} \quad i=2,\ldots,n, \end{aligned} \tag{28}$$

whereby

$$B(x)=\begin{pmatrix} B & \tilde{y} \\ \tilde{y}^t & y_1 \end{pmatrix}, \quad \tilde{y}=\begin{pmatrix} y_2 \\ \vdots \\ y_n \end{pmatrix},$$

and hence

$$\det B(x)=(y_1-\tilde{y}^t B^{-1}\tilde{y})\det B,$$

showing that $(-1)^n \det' A(x)\,(=\det B(x))>0$ holds if and only if $y_1>\tilde{y}^t B^{-1}\tilde{y}$. This condition means that $(y_1,\ldots,y_n)$ should belong to a certain open, solid paraboloid in $\mathbb{R}^n$, that is, x should belong to the image of that paraboloid under the inverse of the substitution (28). ∎

The axis of the paraboloid $\mathscr{K}(A)$ is parallel to the vector $(1,\ldots,1)$; this appears from the above proof. The boundary $\partial\mathscr{K}(A)$ is the unique paraboloid (surface) passing through the columns a_i of A and having at a_i the tangent hyperplane $x_i=0$.

Theorem 6. *With the above changes (and their obvious consequences) all the major results in Sects. 1 through 4 (in particular the theorems) carry over to the Euclidean case when we replace throughout H by E $(=\mathbb{R}^d)$, $p_i\cdot p_j$ by $\frac{1}{2}|p_i-p_j|^2$, (linearly) independent by affinely independent, and determinants* $\det A$ *by the associated derived determinants* $\det' A$.

(See however the last paragraph of the present section as to the quite different use of determinants to specify a half-space. Also note that the expressions for $D(A)$ in Remark 4 do not carry over in general.)

Proof. The counterparts to the remaining Theorems 2, 3, and 4 could be obtained by a limit procedure from the hyperbolic or the spherical case. We prefer, however, a direct approach. The key is the counterpart to Lemma 2.1, obtained by making the following change in the proof of that lemma. (Note that $n=d+1$ now.)

By differentiation of $\frac{1}{2}|p_i-p_{d+1}|^2=a_{i,d+1}$ for fixed $p_1,\ldots,p_d$ we get

$$da_{i,d+1}=(p_{d+1}-p_i)\cdot dp_{d+1}.$$

The $d\times d$ Jacobian matrix $(\partial a_{i,d+1}/\partial p_{j,d+1})$ thus has the rows $p_{d+1}-p_i$, and its determinant therefore has the absolute value

$$\sqrt{|\det B|}=\sqrt{|\det' A|}$$

on account of (26), (27) and the calculation

$$\begin{aligned}(p_{d+1}-p_i)\cdot(p_{d+1}-p_j)&=\tfrac{1}{2}|p_{d+1}-p_i|^2+\tfrac{1}{2}|p_{d+1}-p_j|^2-\tfrac{1}{2}|(p_{d+1}-p_i)-(p_{d+1}-p_j)|^2\\ &=a_{i,d+1}+a_{d+1,j}-a_{i,j}\\ &=-b_{ij}. \qquad (29)\end{aligned}$$

For the inverse map $(a_{1,d+1},\ldots,a_{d,d+1})\mapsto p_{d+1}$ the absolute value of the Jacobian determinant is therefore $1/\sqrt{|\det' A|}$.

The proof of the counterpart to Lemma 2.2 begins with the observation that, for n affinely independent points $p_1,\ldots,p_n$, the Euclidean distance $\varrho(p_n)$ between p_n and the affine span of $p_1,\ldots,p_{n-1}$ is given by

$$\varrho(p_n)=\frac{|\det' A_n|}{|\det' A_{n-1}|}$$

in terms of the matrix $A_n=(\frac{1}{2}|p_i-p_j|^2)_{i,j\leqq n}$ of class $\mathscr{A}_n$ and the submatrix $A_{n-1}=(\frac{1}{2}p_i-p_j|^2)_{i,j\leqq n-1}$. And this is because the $(n-1)$-dimensional volume V of the Euclidean simplex with vertices $p_1,\ldots,p_n$ is expressed in terms of the edge-lengths $|p_i-p_j|$ by the well-known formula

$$V=\frac{1}{(n-1)!}\sqrt{|\det' A_n|}.$$

In fact, $(n-1)!\,V$ is the absolute value of the determinant with columns p_i-p_n,

$i = 1, \ldots, n-1$, and hence

$$((n-1)!V)^2 = \det((p_i - p_n)\cdot(p_j - p_n)) = \det(-B) = (-1)^{n-1}\det' A,$$

cf. (29), (26), and (27) above. Thus we merely have to replace $\sinh\varrho(p_n)$ by $\varrho(p_n)$ in the rest of the proof of Lemma 2.2.

From this point on the proofs in Sects. 2, 3 and 4 carry over mutatis mutandis. Note that, in Sect. 3, $H^n(\varepsilon)$ should be replaced by $E^n(\varepsilon)$, the set of all n-tuples $(p_1, \ldots, p_n)$ of points of E such that, for each $k = d+2, \ldots, n$, the $(d+1)$-tuple $(p_1, \ldots, p_d, p_k)$ is affinely independent, and oriented like $(p_1, \ldots, p_{d+1})$ (likewise supposed affinely independent) if $\varepsilon_k = 1$, but with the opposite orientation if $\varepsilon_k = -1$. Similarly, in the corollary at the end of Sect. 3, the positivity of certain determinants should now be replaced by the positive orientation of the $(d+1)$-tuples in question. ■

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