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Slowly Rotating Drops

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Abstract. We study the existence of gyrostatic equilibria of slowly rotating liquid masses subject only to the force of surface tension. We give a rigorous proof of the existence of non-axisymmetric equilibria. The shape of such an equilibrium approximates a number of spherical lobes connected by thin necks and symmetrically arranged around the axis of rotation.

1. Introduction

The study of the gyrostatic equilibria of rotating liquid masses and their stability is an old problem in Mathematical Physics. The subject probably starts with Newton's discussion of the shape of the Earth in 1687. He modeled the Earth as a homogeneous gravitating liquid and he proposed the figure of an oblate spheroid as a gyrostatic equilibrium. The subject was developed by Maclaurin (1740), Jacobi (1843), and Poincaré (1885) who studied more complicated equilibria and their stability.

In this century Lyttleton [Ly] and Chandrasekhar [Ch] made further contributions. Also models with more forces were allowed, for example surface tension, electrostatic forces produced by uniformly distributed electric charge, and others. This gave crude models for heavy nuclei [B–W]. The interested reader is referred to [Sw, C–P–S, Ly, and Ch], where he can find further references.

We are interested in the case when the only non-negligible force involved (besides friction) is the surface tension. This subject started with the experiments of J. A. F. Plateau [Pl] who, although blind, produced an impressive amount of work. To our knowledge, the most interesting recent work in this field is a numerical study of the shapes and stability of rotating drops by Brown and Scriven [B–S], and to which we will often refer in this paper. Experimental work in the space shuttle has been carried out [W–T–C–E]. Tsamopoulos and Brown [T–B] have further studied stability questions. The case of planar drops has also been studied, see [B, L–M–R], where further references can be found.

We limit ourselves in this paper to studying slowly rotating drops, that is gyrostatic equilibria which would be plotted close to the origin in Figs. 3 or 4 of

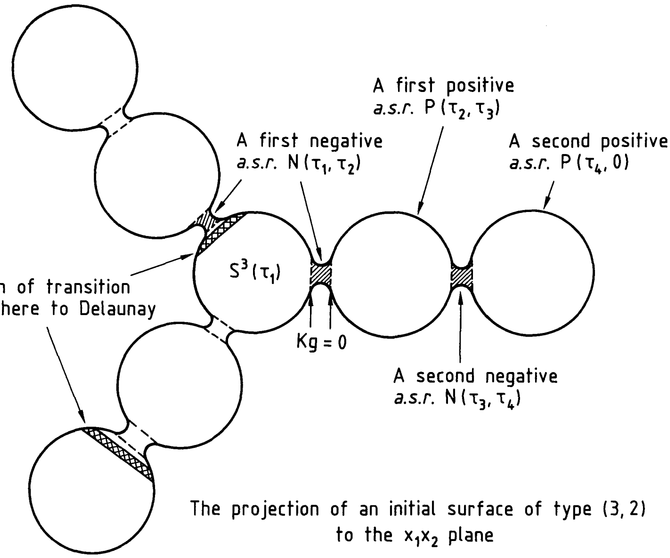

Fig. 1

[B–S]. Such equilibria have been systematically studied before only in the axisymmetric case. In [B–S] the authors expect that the two-lobed family should be extendable to the origin in Fig. 4, and the shapes corresponding to points close to the origin should look like two spherical lobes connected by a thin neck. In this paper we give a rigorous proof of the existence of such and of much more complicated equilibria also. These shapes are symmetric with respect to a plane orthogonal to the axis of rotation and look like a central spherical lobe around which m $(2 \leqq m \leqq 6)$ strings of n $(n \geqq 1)$ spherical lobes each, are symmetrically arranged. Each lobe is attached to its neighbors by thin necks. All lobes are to 0^{th} order (in the angular velocity of rotation or in the angular momentum) of the same size. We call such a shape of type (m, n). We also have shapes which have a central thin neck to which two symmetric "strings" of n $(n \geqq 1)$ lobes each are attached. We call these shapes of type $(1, n)$. So the two lobed examples anticipated by Brown and Scriven are of type $(1, 1)$. All these shapes have maximum possible symmetry. More precisely we prove the following theorem which is a less technical restatement of the main theorem of the paper, Theorem 4.7.

Theorem 1.1. *Given $1 \leqq m \leqq 6$, $n \geqq 1$, a volume $\mathscr{V}$ of homogeneous liquid of density $\mathscr{D}$ and surface tension $\mathscr{T}$, and an angular momentum $\mathscr{L}$ smaller than a constant depending only on the above, there is a shape of type (m, n) which allows this liquid mass to rotate with angular momentum $\mathscr{L}$ in unstable gyrostatic equilibrium.*

A similar theorem could be proved for driven rotating drops (see [B–S]) with a prescribed small angular velocity instead of momentum. The proof is almost identical to that of Theorem 1.1 and so the interested reader should have no difficulty in adapting the proof we present to the angular velocity case.

The instability of the drops in Theorem 1.1 is proved in the sense of [B–S] (see also [Ly], p. 25). We expect these drops to be unstable in any reasonable sense but we will not attempt to discuss such questions in this paper.

It would be interesting to carry out a numerical study of these new drops and follow the evolution of each family away from the origin of Fig. 4 of [B–S].

We give now an outline of the proof and we describe intuitively the main ideas involved. The approach is similar to the one in [K 1] and [K 2]. We will always assume that we have an orthonormal coordinate system (x_1, x_2, x_3), where the x_3-axis is the axis of rotation and the x_1x_2-plane is a plane of symmetry. Then (see the proof of 4.7 or [B–S]) having a rotating drop amounts to having an embedded surface on which the Young-Laplace equation of interface configuration is satisfied

$$2H\mathscr{T} = \hat{\mathscr{K}} + \frac{\mathscr{L}^2}{2\mathscr{I}^2\mathscr{D}} R^2, \tag{1.2}$$

where H is the mean curvature of the surface, R is the distance from the x_3-axis, $\hat{\mathscr{K}}$ is a constant which physically is the pressure difference between the two sides of the surface where the x_3-axis intersects the surface, $\mathscr{I} = \int R^2$ is the moment of inertia of the enclosed volume, and the remaining constants are as in Theorem 1.1.

Now if we homothetically expand by a suitable factor we can reduce this equation to

$$H = 1 + \tilde{\tau} R^2, \tag{1.3}$$

where $\tilde{\tau}$ is a small constant proportional to $\mathscr{L}^2$. To construct a surface of type (m, n) satisfying (1.3) we first construct a surface M which approximately satisfies (1.3) and then perturb it to correct the error. The surface which approximately satisfies the equation is constructed by using as "building blocks" pieces of the unit sphere and necks which are pieces of Delaunay surfaces. (See Lemma 2.1 for a description of the Delaunay surfaces.) The transition from one piece to another is done on annuli which support $\delta H \equiv H - 1$. (We adopt the convention that $H \equiv 1$ on the unit sphere $S^2(1)$ and on the Delaunay surfaces.) Each Delaunay surface is characterized by a parameter τ (see 2.1 again) and choosing the correct τ for each neck is a major part of the problem. So we would like to find $u \in C^\infty(M)$ so that if $X: M \to E^3$ is the immersion of M in E^3 and $\nu: M \to S^2(1)$ its Gauss map, then $X + u\nu: M \to E^3$ gives the immersion of a surface M_u which satisfies (1.3). This leads to an elliptic PDE for u. The linearization jointly in $\tilde{\tau}$ and u of this PDE turns out to be

$$\Delta_g u + |A|^2 u = 2\tilde{\tau} R^2 - 2\delta H, \tag{1.4}$$

where g is the first fundamental form of M, and $|A|^2$ is the square of the length of the second fundamental form of M. As in [K 1] we define a new metric $h = \frac{|A|^2}{2} g$ and then (1.4) becomes

$$(\Delta_h + 2)u = \frac{4\tilde{\tau}R^2 - 4\delta H}{|A|^2}. \tag{1.5}$$

Let K_g be the Gauss curvature of M with respect to the induced metric g. The zero set of $\{K_g = 0\}$ is a union of circles separating the necks of M from the lobes of M. In the h-metric these circles have neighborhoods which look like thin necks. If we remove these h-necks we get connected components on which the Gauss map is close to being an isometry (with respect to the h-metric). We call each component of $M \backslash \{K_g = 0\}$ an almost spherical region of M, a.s.r. for short. So an a.s.r. in the g metric is a lobe or a neck. If we substitute each a.s.r. of M with an $S^2(1)$ we get $M_{S(1)}$, a disjoint union of $S^2(1)$'s. We carry over the symmetries of M to $M_{S(1)}$ in the obvious way and functions of M to functions on $M_{S(1)}$ by using the Gauss map.

It turns out that the spectrum and the eigenfunctions of $\Delta_h + 2$ on M can be approximated in a suitable way by those of $\Delta + 2$ on $M_{S(1)}$. This means that we get small eigenvalues for $\Delta_h + 2$ on M. We call the span of the corresponding eigenfunctions the approximate kernel. We need the right-hand side of (1.5) to be nearly orthogonal to the approximate kernel. This determines the τ-parameters of the Delaunay necks used (see Lemma 3.8 and the definition of $\zeta[M]$, we have been discussing now the M with $\zeta[M]=0$). The proof of Eq. (2) in the proof of 3.8 was suggested to us by Kusner ([Ku], [K–K–S]) and this replaced a less conceptual explicit calculation we had initially. The reader familiar with [K 1] will also notice that we use the $\tilde{\tau}R^2$ term to "balance" the approximate kernel created by attaching a single neck to the outermost spherical pieces of M. This is the crucial difference which allows one to do this construction while it is impossible to have topological spheres of constant mean curvature which are not round [H].

Working orthogonally to the kernel and doing higher order estimates in a new metric χ (Lemmas 3.9 and 4.1) we can succeed now to find u such that M_u satisfies (1.3) modulo a small element of the approximate kernel. This step is somewhat complicated, mainly due to the fact that we need better estimates on the necks than what we can get on the lobes. This aside, to correct for the approximate kernel we construct a family of initial surfaces by perturbing M suitably. Each of these surfaces has the right-hand side of (1.5) arranged to have an approximately predetermined (by $\zeta[M]$) projection to the approximate kernel (see 3.8). A topological argument (Schauder fixed point theorem) then allows us to prove that one of the initial surfaces M in the family has to give an M_u on which (1.3) is satisfied precisely.

Finally, one would notice that if $\mathscr{V}$, $\mathscr{D}$, and $\mathscr{T}$ are specified precisely, then the above method cannot specify $\mathscr{L}$ precisely. We remedy this by playing the above game with a range of $\tilde{\tau}$'s and then choosing the correct one by a topological argument again. All these choices of M, u, and $\tilde{\tau}$ have been incorporated in a single argument, namely the proof of Theorem 4.7.

The paper has three sections besides the introduction. In Sect. 2 we define the families of initial surfaces, the various metrics used, and study their properties. In Sect. 3 we study the linearized equation and produce the linear estimates we need. In Sect. 4 we estimate the quadratic error and we prove the main theorem by using the Schauder fixed point theorem.

2. The Initial Surfaces

In this section we construct the families of initial surfaces we need. We start by recalling the properties of the embedded Delaunay surfaces [K 1, Appendix A].

Proposition 2.1. *For each $\tau \in (0, \frac{1}{4})$ there is a unique embedded surface $DS(\tau)$ in E^3 which has the following properties:*

(i) *$DS(\tau)$ has constant mean curvature $H\equiv 1$.*
(ii) *$DS(\tau)$ possesses rotational symmetry: We write r for the function whose graph $x_2=r(x_1)$ rotated around the x_1-axis gives $DS(\tau)$ and r attains its minimum at $x_1=0$.*
(iii) *There is $p(\tau)>0$ such that $x_1=n(1+p(\tau))$ is a plane of reflectional symmetry of $DS(\tau)$ (any integer n) and r is increasing on $(0,1+p(\tau))$.*
(iv) *There is a map $X_\tau:\mathbb{R}^2\to DS(\tau)$ given by*

$$(u,v)\xrightarrow{X_r}\left(x_1(u),r(u)\cos\frac{v}{2\sqrt{\tau}},\ r(u)\sin\frac{v}{2\sqrt{\tau}}\right),$$

where $r(u)=\sqrt{\tau}e^w$, $\dfrac{dx_1}{du}=\sqrt{\tau}e^w\cosh w$, $x_1(-A_\tau)=0$, and $w=w(u)$ is determined by

$$\frac{d^2w}{du^2}(u)+\frac{\sinh 2w}{2}=0,\qquad \left(\frac{dw}{du}\right)^2+\cosh^2 w=\frac{1}{4\tau}.$$

w is an odd periodic function, even around A_τ, and hence periodic with period $4A_\tau$. w is increasing on $(0,A_\tau)$ and positive on $(0,2A_\tau)$.
(v) *The first and second fundamental form of $DS(\tau)$ (pulled back by X_τ) are:*

$$g=\frac{e^{2w}}{4}(du^2+dv^2),\qquad A=\frac{e^{2w}-1}{4}du^2+\frac{e^{2w}+1}{4}dv^2.$$

We have $|A|^2=2+2e^{-4w}$ and the Gauss map is

$$\nu=2\sqrt{\tau}\left(w_u,\ -\cosh w\cos\frac{v}{2\sqrt{\tau}},\ -\cosh w\sin\frac{v}{2\sqrt{\tau}}\right).$$

(vi) *$DS(\tau)$ depends continuously on τ in the smooth topology.*
(vii) *As $\tau\to 0$, $x_1(0)\to 0$, $x_1(A_\tau)\to 1$, $\dfrac{x_1(0)}{\tau}\to\infty$,*

$$\frac{1}{\log\tau}\frac{dp}{d\tau}\to -1,\qquad \frac{p(\tau)}{\tau\log\tau}\to -1.$$

Proof. All this is well known, a possible reference is [K 1]. □

We write $DS^+(\tau)$ and $DS^-(\tau)$ for

$$\{X\in DS(\tau):K_g(X)\geqq 0\}\quad\text{and}\quad\{X\in DS(\tau):K_g(X)\leqq 0\}$$

respectively, where $K_g(X)$ is the Gauss curvature of $DS(\tau)$ at X with respect to g. All the connected components of $DS^+(\tau)$ are identical up to translation and we write $P(\tau)$ for one of them. Similarly the components of $DS^-(\tau)$ will be denoted $N(\tau)$. We write $\tilde{N}(\tau)$ for $N(\tau)$ enlarged homothetically by a factor of $\frac{1}{\tau}$. Let $r_\tau:[-a_\tau,a_\tau]\to\mathbb{R}$ and $\tilde{r}_\tau:[-b_\tau,b_\tau]\to\mathbb{R}$ be functions whose graphs rotated around the x_1-axis give $P(\tau)$ and $\tilde{N}(\tau)$ respectively. Notice that $a_\tau=x_1(A_\tau)-x_1(0)$ and $b_\tau=\dfrac{x_1(0)}{\tau}$ in the notation of 2.1. Also $r_0:[-1,1]\to\mathbb{R}$ and $\tilde{r}_0:\mathbb{R}\to\mathbb{R}$ are defined by $r_0(x_1)=\sqrt{1-x_1^2}$

and $\tilde{r}_0(x_1)=\cosh x_1$. We define also

$$S^2_\delta(1)=\{(x_1,x_2,x_3): x_1^2+x_2^2+x_3^2=1 \text{ and } x_2^2+x_3^2 \geqq \delta^2\},$$

a subset of the unit sphere. We define $P_\delta(\tau)$, $N_\delta(\tau)$, and $\tilde{N}_\delta(\tau)$ to be the preimages of $S^2_\delta(1)$ by the Gauss map v on $P(\tau)$, $N(\tau)$, and $\tilde{N}(\tau)$ respectively. Notice also that all the connected components $DS(\tau)\backslash v^{-1}(S^2_\delta(1))$ are identical up to translation. We write $A_\delta(\tau)$ for one of them. $A^+_\delta(\tau)$ and $A^-_\delta(\tau)$ are defined to be $DS^+(\tau)\cap A_\delta(\tau)$ and $DS^-(\tau)\cap A_\delta(\tau)$ respectively.

We define now some more complicated building blocks we need for our surfaces. Define $\hat{r}_\tau: [\sqrt{1-9\delta_1^2}, a_\tau] \to \mathbb{R}$ by

$$\hat{r}_\tau(x_1)=\psi_1(x_1)r_0(x_1)+(1-\psi_1(x_1))r_\tau(x_1),$$

where $\psi_1:\mathbb{R}\to\mathbb{R}$ is a fixed smooth non-increasing function with $\psi_1(x_1)\equiv 1$ on $(-\infty, \sqrt{1-4\delta_1^2})$ and $\psi_1(x_1)\equiv 0$ on $(\sqrt{1-\delta_1^2}, \infty)$; where δ_1 is some fixed constant such that $3\delta_1 < \sin\frac{\pi}{6}=\frac{1}{2}$. We write $A(\tau)$ for the surface obtained by rotating the graph of $\hat{r}_\tau$ around the x_1-axis. We define $P(\tau_1,\tau_2)$ to be the connected smooth surface which is the union of $S^2_{3\delta_1}(1)$, $A(\tau_1)$, and $A(\tau_2)$. In other words $P(\tau_1,\tau_2)$ is a spherical piece with $A(\tau_1)$ and $A(\tau_2)$ attached to it. $P(\tau,0)$ is defined as the union of $S^2_{3\delta_1}(1)$, $A(\tau)$ and a geodesic disc of $S^2(1)$, so that $P(\tau,0)$ has boundary a single circle. So $P(\tau,0)$ is a spherical piece with one $A(\tau)$ attached. If $2\leqq m\leqq 6$, we write $S^m(\tau)$ for the connected smooth surface which is the union of the following:

(i) A subset of $S^2(1)$ which is $S^2(1)$ minus m geodesic discs. The centers of these discs lie on an equator and they form the vertices of a regular m-gon. The circles making up the boundary have all radius (in E^3) equal to $3\delta_1$.
(ii) m $A(\tau)$'s.

In other words $S^m(\tau)$ is a spherical piece with m $A(\tau)$'s attached symmetrically around its equator.

We define now $\tilde{r}_{\tau_1,\tau_2}: \left[-b_{\tau_1}, \frac{\tau_2}{\tau_1}b_{\tau_2}\right] \to \mathbb{R}$ by

$$\tilde{r}_{\tau_1,\tau_2}(x_1)=\psi_2(x_1)\tilde{r}_{\tau_1}(x_1)+\frac{\tau_2}{\tau_1}(1-\psi_2(x_1))\tilde{r}_{\tau_2}\left(\frac{\tau_1}{\tau_2}x_1\right),$$

where $\psi:\mathbb{R}\mapsto[0,1]$ is a smooth nonincreasing function such that $\psi\equiv 1$ on $(-\infty,-\frac{1}{4}]$ and $\psi\equiv 0$ on $[\frac{1}{4},\infty)$.

We call the surface obtained by rotating the graph of $\tilde{r}_{\tau_1,\tau_2}$ around the x_1-axis $\tilde{N}(\tau_1,\tau_2)$. $N(\tau_1,\tau_2)$ is defined to be $\tilde{N}(\tau_1,\tau_2)$ reduced homothetically by a factor of τ_1. Notice that $N(\tau_1,\tau_2)$ has a neighborhood of one of its boundary circles identical to a subset of $DS(\tau_1)$ and a neighborhood of the other boundary circle identical to a subset of $DS(\tau_2)$. [So does $P(\tau_1,\tau_2)$.] We define $P_\delta(\tau_1,\tau_2)$ and $N_\delta(\tau_1,\tau_2)$ to be the preimages of $S^2_\delta(1)$ by the Gauss map on $P(\tau_1,\tau_2)$ and $N(\tau_1,\tau_2)$ respectively. $S^m_\delta(\tau)$ is defined to be the closure of $S^m(\tau)\backslash A$, where A is the neighborhood of the boundary of $S^m(\tau)$ which consists of m $A^+_\delta(\tau)$'s. We have the following lemma:

Lemma 2.2. *Given b large enough there is $T(b)>0$ such that if $\tau,\tau'\in(0,T(b))$ then*

(i) $\|\hat{r}_\tau - r_0\|_{C^k([\sqrt{1-9\delta_1^2},\,1-1/b])} \leqq C(k,b)\tau$,

(ii) $\|\tilde{r}_{\tau,\tau'} - \tilde{r}_0\|_{C^k([-b,b])} \leqq C(k,b)\left(\tau + \dfrac{|\tau-\tau|}{\tau}\right)$.

Proof. This follows by considering the ODE's satisfied by r_τ and $\tilde{r}_\tau$ to deduce the smooth dependence of the various functions on their parameters. See the similar proof of [K 1, A.2.1] for more details. □

This lemma allows us to compare the fundamental forms of our building blocks with those of $S^2(1)$ and the catenoid.

Given now $2\leqq m\leqq 6$ and a sequence $\{\tau_i\}_{i=1}^{2n}$ ($n\geqq 1$, τ_i's small enough), we define a smooth connected surface M which we call an initial surface of type (m,n) and defining sequence $\{\tau_i\}$. M has the symmetries which $S^m(\tau_1)$ has, and it contains one $S^m(\tau_1)$ which if removed leaves m rotationally invariant identical connected components each of which is the union of the following: $N(\tau_1,\tau_2)$ attached to $P(\tau_2,\tau_3)$ attached to $N(\tau_3,\tau_4)$ and so on up to $P(\tau_{2n},0)$. We call $N(\tau_1,\tau_2)$ the first negative almost spherical region, $N(\tau_3,\tau_4)$ the second negative almost spherical region and so on. (This name is justified by Lemma 2.4.)

We write a.s.r. for short for "almost spherical region" from now on. We call $P(\tau_2,\tau_3)$ first positive a.s.r., and so on.

Given $m=1$ and $\{\tau_i\}_{i=1}^{2n-1}$ we define as above a surface M which is rotationally invariant around the x_1-axis and contains an $N(\tau_1)$ whose removal gives two identical connected components each of which is the union of the following: $P(\tau_1,\tau_2)$ (first positive a.s.r.) to which $N(\tau_2,\tau_3)$ (first negative a.s.r.) is attached and so on up to $P(\tau_{2n-1},0)$. The x_2x_3-plane is a plane of symmetry for M.

Now suppose M is as above with $m=1$ or $m\neq 1$. We call each of the $P(\tau,\tau')$, $N(\tau,\tau')$ or $S^m(\tau)$ contained in M an a.s.r. of M. We define

$$M^+=\{X\in M: K_g(X)\geqq 0\}\quad\text{and}\quad M^-=\{X\in M: K_g(X)\leqq 0\},$$

then M^+ is the union of the positive a.s.r.'s [which are $P(\tau,\tau')$'s and $S^m(\tau)$]. M^- is the union of the negative a.s.r.'s of M which are $N(\tau,\tau')$'s.

If S is an a.s.r. contained in M, we define $S_\delta\subset S$ to be $S_\delta^m(\tau)$, $P_\delta(\tau,\tau')$, $N_\delta(\tau,\tau')$ respectively if S is $S^m(\tau)$, or $P(\tau,\tau')$ or $N(\tau,\tau')$. M_δ stands for the union of all S_δ's where S is any a.s.r. of M. M_δ^+ and M_δ^- are $M_\delta\cap M^+$ and $M_\delta\cap M^-$ respectively. M_δ^0 is the closure of $M\backslash M_\delta$, $M_\delta^{+,0}$, and $M_\delta^{-,0}$ are $M_\delta^0\cup M^+$ and $M_\delta^0\cup M^-$ respectively. Notice that if $\delta<\delta_1$, then M_δ^0 is the union of $\Lambda_\delta(\tau_i)$'s, where τ_i belongs to the defining sequence of M.

Let $V_{m,n}$ be the Banach space of sequences $(\{\tilde{\lambda}_i\},\{\lambda_i\})$ which assign to an i^{th} positive a.s.r. of some initial surface M of type (m,n) the number $\tilde{\lambda}_i$, and to an i^{th} negative a.s.r. the number λ_i. The norm of $V_{m,n}$ is the supremum norm. Clearly $V_{m,n}$ is just $\mathbb{R}^{2n}(m\neq 1)$ or $\mathbb{R}^{2n-1}(m=1)$ with a special interpretation. We define

$$\Xi(\tilde{q},q)\equiv\{(\{\tilde{\lambda}_i\},\{\lambda_i\}):|\tilde{\lambda}_i|<\tilde{q},|\lambda_i|<q\}.$$

We proceed now to describe the families of initial surfaces we will need. We define F to be a parametrized family of initial surfaces of type (m,n) and parameters

$(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$ if the following are true:

(i) $\tilde{\tau} \in \left(0, \frac{1}{10}\right)$, $0 < \Sigma < \tilde{\Sigma} < \frac{1}{10n}$.

(ii) If M is an element of F, then it is a surface of type (m, n). To each $M \in F$ we associate a $\tilde{\tau}[M] \in (\frac{1}{2}\tilde{\tau}, 2\tilde{\tau})$ and $\zeta[M] \in V_{m,n}$. We call $\tilde{\tau}[M]$ the τ parameter of M and $\zeta[M]$ the configuration of M. If $\{\tau_i\}$ is the defining sequence of M then $\zeta[M]$ is defined as follows:

To an i^{th} positive a.s.r. we associate $\tilde{\zeta}_i$ defined by:

$$\tilde{\zeta}_i \tilde{\tau} = 2\pi(\tau_{2i+1} - \tau_{2i}) + 2\tilde{\tau} \int_{S^2(1)} x_1((x_1 + 2i)^2 + x_2^2) \quad (m \geqq 2),$$

$$\tilde{\zeta}_i \tilde{\tau} = 2\pi(\tau_{2i} - \tau_{2i-1}) + 2\tilde{\tau} \int_{S^2(1)} x_1((x_1 + 2i - 1)^2 + x_2^2) \quad (m = 1),$$

where $\tilde{\tau}$ stands for $\tilde{\tau}[M]$, $S^2(1) = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, and the integral is taken with respect to the standard metric. ($\tau_{2n+1} = 0$ if $m \geqq 2$ and $\tau_{2n} = 0$ if $m = 1$ of course). To an i^{th} negative a.s.r. we associate ζ_i defined by

$$\zeta_i \tilde{\tau} = 2\pi(\tau_{2i} - \tau_{2i-1}) \quad (m \geqq 2), \qquad \zeta_i \tilde{\tau} = 2\pi(\tau_{2i+1} - \tau_{2i}) \quad (m = 1).$$

Then we require that the map

$$M \to (\tilde{\tau}[M], \zeta[M])$$

is a bijection from F onto $[\frac{1}{2}\tilde{\tau}, 2\tilde{\tau}] \times \Xi(\tilde{\Sigma}, \Sigma)$. We write $M[\tilde{\tau}, \zeta]$ for the surface corresponding to $(\tilde{\tau}, \zeta)$ under this correspondence. Notice that there are $C_1 = C_1(m, n)$ and $C_2 = C_2(m, n)$ such that, if $M \in F$ contains in its defining sequence some τ_i, then

$$C_1 \tilde{\tau} < \tau_i < C_2 \tilde{\tau}.$$

The existence question for such families is settled by the following lemma:

Lemma 2.3. *Given integers m, n with $1 \leqq m \leqq 6$ and $n \geqq 1$ there are $T(m, n)$, $\tilde{\Sigma} = \tilde{\Sigma}(m, n)$, $\Sigma = \Sigma(m, n)$ positive constants such that if $\tilde{\tau} \in (0, T(m, n))$, there is a family F of initial surfaces of type (m, n) and parameters $(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$.*

Proof. This is a straightforward construction. To construct each M we calculate $\tau_{2n} (m \geqq 2)$ or $\tau_{2n-1} (m = 1)$ first and then proceed inductively to τ_{2n-1}, τ_{2n-2}, etc. $\square$

From now on we fix the type (m, n) of initial surfaces for the rest of the paper unless otherwise stated. Suppose M is an initial surface in F with a family of parameters $(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$. Assuming that the parameters are small enough for the definitions to make sense, we define two new metrices on M by

$$h = \frac{|A|^2}{2} g \quad \text{and} \quad \chi = \varrho^2 g = (1 - \psi) h + \psi \chi',$$

where ψ and χ' are defined as follows:

Fix some absolute constant $\delta_2 < \frac{1}{10}\delta_1$ to guarantee that $M^0_{4\delta_2}$ is a union of $\Lambda_{4\delta_2}(\tau_i)$'s. Consider one of them and assume without loss of generality that its axis is the x_1-axis. ψ on this $\Lambda_{4\delta_2}(\tau_i)$ is a smooth nonincreasing function of $v' = \sqrt{v_2^2 + v_3^2}$

only, where $v=(v_1, v_2, v_3)$ is the Gauss map. Furthermore $\psi(v')\equiv 1$ for $v'<3\delta_2$ and $\psi(v')\equiv 0$ for $v'>4\delta_2$. On $M_{4\delta_2}$ define $\psi\equiv 1$. χ' is defined on $DS(\tau)$ [and hence on $\Lambda_{4\delta_2}(\tau_i)$] to be the push-forward by X_τ of $\frac{1}{\tau}(du^2+dv^2)$ in the notation of 2.1. So χ transits smoothly from h on $M_{4\delta_2}$ to χ' on $M^0_{\delta_2}$.

We also define a metric $\bar{g}$ on M_δ by $\bar{g}=g$ on M_δ^+ and $\bar{g}=\tilde{\tau}^{-2}g$ on M_δ^-.

We define two metrics g_1 and g_2 on some domain Ω to be equivalent by a sequence of constants $\{C_k\}_{k=1}^\infty$ if and only if

$$\|g_2\|_{C^k(\Omega, g_1)} \leqq C_k, \qquad \|g_1\|_{C^k(\Omega, g_2)} \leqq C_k.$$

The following lemma describes the relation between the various metrics, and their properties, we will need later:

Lemma 2.4. *There is a positive absolute constant such that if the parameters $(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$ of a family are smaller than it, then the following are true:*

(i) *For each $(\tau, \zeta)\in[\frac{1}{2}\tilde{\tau}, 2\tilde{\tau}]\times \Xi(\tilde{\Sigma}, \Sigma)$ there is a smooth diffeomorphism $D_{\tilde{\tau},\zeta}: M[\tilde{\tau}, 0] \to M[\tilde{\tau}, \zeta]$ which depends continuously on $(\tilde{\tau}, \zeta)$ and has the property that χ and $D^*_{\tilde{\tau},\zeta}\chi$ on $M[\tilde{\tau}, 0]$ are equivalent by a sequence of universal constants.*

(ii) *If $M\in F$, then the area of M_δ with respect to the h-metric is bounded by a universal constant times δ^2.*

(iii) *If $\Omega_1\subset\Omega_2$ are domains in $M\in F$ such that the h or χ-distance of Ω_1 from $\partial\Omega_2$ is at least δ, then if $u\in C^{2,\alpha}(\Omega_2)$, $f\in C^{0,\alpha}(\Omega_1)$ and $\left(\Delta_\chi+\frac{|A|^2}{\varrho^2}\right)u=f$, then*

$$\|u\|_{C^{2,\alpha}(\Omega_1,\chi)} \leqq C(\delta)\left(\|u\|_{C^0(\Omega_2)}+\|f\|_{C^{0,\alpha}(\Omega_2,\chi)}\right).$$

(This is a Schauder estimate for the particular case we will be interested in.)

(iv) *If $M\in F$ and $f\in C^1(M)$, then we have a Sobolev estimate*

$$\|f\|_{L^2(M,h)} \leqq C(\|\nabla f\|_{L^1(M,h)}+\|f\|_{L^1(M,h)}).$$

Given $\delta>0$ there is $T(\delta)>0$ such that if $\tilde{\tau}<T(\delta)$, then the following are true as well:

(v) *If $M\in F$, then the metrics h, $\bar{g}$, and χ on M_δ are all equivalent by a sequence of constants depending only on δ.*

(vi) *If $M\in F$ then*

$$\|h-v^*g\|_{C^k(M_\delta, h)} \leqq C(\delta, k)\tilde{\tau}+C(k)\Sigma, \qquad \|h-v^*g\|_{C^k(M_\delta^+, h)} \leqq C(\delta, k)\tilde{\tau},$$

where v^*g is the pullback of the standard metric of $S^2(1)$ by the Gauss map.

Proof. We refer the reader to [K 1]. □

3. The Linearized Equation

Suppose $X: M\to E^3$ is the smooth immersion of some initial surface M and $v: M\to S^2(1)$ its Gauss map. It turns out (see Sect. 4) that if $X+uv$ is an immersion for some $u\in C^2(M)$, then

$$H(u)=H+\tfrac{1}{2}(\Delta u+|A|^2u)+Q(u),$$

where $H(u)$ is the mean curvature of $X+u\nu$ pulled back to M and $Q(u)$ is the linearization error involving quadratic and higher order terms in u and its derivatives.

The linearized in u (and $\bar{\tau}$) equation for our problem (properly rescaled, see Sect. 1) can be written in one of the following forms:

$$\Delta_g u+|A|^2 u=2\bar{\tau}R^2-2\delta H, \tag{3.1}$$

$$\Delta_h u+2u=\frac{4\bar{\tau}R^2-4\delta H}{|A|^2}, \tag{3.2}$$

$$\Delta_\chi u+\frac{|A|^2}{\varrho^2}u=\frac{2\bar{\tau}R^2-2\delta H}{\varrho^2}, \tag{3.3}$$

where $\bar{\tau}=\bar{\tau}[M]$, M is an initial surface.

The second of these forms turns out to be useful for C^0 estimates on u, while the third one for $C^{2,\alpha}$ estimates. Before we can estimate anything we need to understand the spectrum of the linearized operator. We will need some definitions we provide now. We write L_h for Δ_h+2 and L_χ for $\Delta_\chi+\frac{|A|^2}{\varrho^2}$. We adopt the convention that all the functions we consider on any initial surface M have to respect the symmetries of M. In other words we can think of these functions as defined on M/G rather than on M, where G is the finite subgroup of $O(3)$ which preserves M and the x_3-axis (the axis of rotation). We define the approximate kernel E of L_h on M to be the span of those eigenfunctions for L_h on M which correspond to eigenvalues in $[-1,1]$. We write P for the orthogonal projection of $L^2(M,h)$ onto E. Remove now from M the circles where $K_g=0$. Consider the Dirichlet problem for L_h on the resulting (disconnected) domain. We define $\tilde{E}$ to be the span of the eigenfunctions for this Dirichlet problem whose eigenvalues lie in $[-1,1]$. We write $\tilde{P}$ for the orthogonal projection of $L^2(M,h)$ onto $\tilde{E}$.

Consider now a collection of unit spheres in one to one correspondence with the a.s.r.'s of M, we call their union $M_{S(1)}$. We define $\bar{\nu}: M\to M_{S(1)}$ by demanding that if S is an a.s.r. of M, then the restriction of $\bar{\nu}$ to S is the Gauss map of S mapping to the sphere of $M_{S(1)}$ corresponding to S. We let G act on $M_{S(1)}$ by isometries and so that $\bar{\nu}$ is equivariant under the action of G on M and $M_{S(1)}$. Let $\hat{E}'$ be the kernel of $\Delta+2$ on $M_{S(1)}$, or rather, according to our convention, on $M_{S(1)}/G$. Let $\hat{E}\equiv\{f\circ\bar{\nu}: f\in\hat{E}'\}$. $\hat{E}$ has dimension $2n$ if $m>1$ or $2n-1$ if $m=1$. Define Π to be the orthogonal projection of $L^2(M,h)$ onto $\hat{E}$.

We have now the following lemmas whose proofs we omit, the reader is referred again to [K 1] where similar statements are proved. In the following M is some surface in F, and F is some family of initial surfaces of parameters $(\tilde{\tau},\tilde{\Sigma},\Sigma)$.

Lemma 3.4. *There is* $\Sigma_1>0$ *such that given* $\varepsilon>0$ *there is* $T_1(\varepsilon)>0$ *such that if* $\tilde{\tau}<T_1(\varepsilon)$, $\Sigma<\Sigma_1$, $f\in C^{0,\alpha}(M)$ *orthogonal to the approximate kernel* E, *then there is* $u\in C^{2,\alpha}(M)$ *such that* u *is also orthogonal to* E, $L_h u=f$, *and*

$$\|u\|_{L^2(S,h)}\leqq\|f\|_{L^2(S,h)}+\varepsilon\|f\|_{L^2(M,h)}$$

for any almost spherical region S *of* M.

Lemma 3.5. *There is $\Sigma_2>0$, and given $\varepsilon>0$, $T_2(\varepsilon)>0$, such that if $\Sigma<\Sigma_2$ and $\tilde{\tau}<T_2(\varepsilon)$, then*

(i) $\|P-\tilde{P}\|<\varepsilon$.
(ii) $\|\tilde{P}^+-\Pi^+\|<\varepsilon$, *where $\tilde{P}^+$ and Π^+ are the restrictions of $\tilde{P}$, Π to the space of functions supported on M^+.*
(iii) *The approximate kernel of L_h on M has dimension $2n-1$ $(m=1)$ or $2n$ $(m\geqq 2)$.*

Lemma 3.6. *Given $\varepsilon>0$ there are $\Sigma_3(\varepsilon)>0$, $T_3(\varepsilon)>0$ such that if $\tilde{\tau}<T_3(\varepsilon)$, $\Sigma<\Sigma_3(\varepsilon)$, then*

(i) $\|P-\Pi\|<\varepsilon$.
(ii) *L_h on M has no eigenvalues in $[-1,-\varepsilon]\cup[\varepsilon,1]$.*

The above lemmas enable us to understand the spectrum of L_h well enough for our purposes. Notice that Σ_3 depends on ε while Σ_2 does not. In the next lemma we produce the estimates for the right-hand side of (3.2) and (3.3) we need. We adopt some new notation: $(\delta H)^+$ is $\equiv\delta H$ on M^+ and $\equiv 0$ on M^-, $(\delta H)^-$ is $\equiv\delta H$ on M^- and $\equiv 0$ on M^+. $\delta H=\delta H^++\delta H^-$. δH^+ and δH^- are smooth functions because $\delta H\equiv 0$ on M_δ^0 for $\tilde{\tau}$ and δ small enough. We define also

$$f_h^1=\frac{4\tilde{\tau}[M]R^2-4(\delta H)^+}{|A|^2},\qquad f_\chi^1=\frac{2\tilde{\tau}[M]R^2-2(\delta H)^+}{\varrho^2},$$
$$f_h^2=-4|A|^{-2}(\delta H)^-,\qquad f_\chi^2=-2\varrho^{-2}(\delta H)^-.$$

Notice that $f_h^1+f_h^2$ and $f_\chi^1+f_\chi^2$ are the right-hand sides of Eqs. (3.2) and (3.3) respectively.

Lemma 3.7. *There are $b_1,\Sigma_4,T_4>0$ such that if $\Sigma<\Sigma_4$, $\tilde{\tau}<T_4$, then the following are true:*

(i) $\|f_h^1\|_{L^2(M,h)}<b_1\tilde{\tau}$, $\|f_\chi^1\|_{C^1(M,\chi)}<b_1\tilde{\tau}$,
(ii) $\|f_h^1\|_{L^2(M^-,h)}<b_1\tilde{\tau}^{3/2}$, $\|f_\chi^1\|_{C^1(M^-,\chi)}<b_1\tilde{\tau}^2$,
(iii) $\|f_h^2\|_{L^2(M,h)}<b_1\Sigma\tilde{\tau}$, $\|f_\chi^2\|_{C^1(M,\chi)}<b_1\Sigma\tilde{\tau}$.

Proof. The required estimates for $(\delta H)^-$ and $(\delta H)^+$ follow from Lemmas 2.2 and 2.4 (v). (See [K 1] for more details.) We produce now the required estimates for R^2. Recall the definition of ϱ and χ. We subdivide M into the following pieces and we estimate on each of them: On $M_{\delta_2}^+$ it is clear from 2.2 (i), 2.4 (v), and (vi), that we can assume

$$\||A|^{-2}R^2\|_{L^2(M_{\delta_2}^+,h)}\leqq C,\qquad \|\varrho^{-2}R^2\|_{C^1(M_{\delta_2}^+,\chi)}\leqq C,\tag{1}$$

where we used also that if $\{\tau_i\}$ is the defining sequence of M, then $C_1\tilde{\tau}<\tau_i<C_2\tilde{\tau}$. Now let $\Lambda_{\delta_2}(\tau)$ be one of the components of $M_{\delta_2}^0$. Then using the notation of 2.1 (iv) we have

$$4\varrho^{-2}R^2=\tau e^{2w}\left(x_1^2(u)+\tau e^{2w}\cos^2\frac{v}{2\sqrt{\tau}}\right),$$

where $\varrho^{-2}R^2$ has been pulled back to a subset of $\mathbb{R}^2$ by X and $\Lambda_{\delta_2}(\tau)$ is assumed without loss of generality to be rotationally invariant around the x_1-axis: X is X_τ up to a translation in the direction of the x_1-axis.

The pullback of χ to $\mathbb{R}^2$ is $\tau^{-1}(du^2+dv^2)$ and so abusing slightly the notation we have

$$\left\| x_1^2(u)+\tau e^{2w}\cos^2\frac{v}{2\sqrt{\tau}}\right\|_{C^1(\chi)} \leqq C,$$

where 2.1 was used. Similarly on $\Lambda_{\delta_2}(\tau)$ we have $\|\tau e^{2w}\|_{C^1(\chi)} \leqq C$ while on $\Lambda_{\delta_2}^-(\tau)$ where $w \leqq 0$ we have $\|e^{2w}\|_{C^1(\chi)} \leqq C$. Combining with (1) we get

$$\|\varrho^{-2}R^2\|_{C^1(M^+,\chi)} \leqq C, \qquad \|\varrho^{-2}R^2\|_{C^1(M^-\setminus M_{\delta_2},\chi)} \leqq C\tilde{\tau}. \tag{2}$$

By (1), 2.1 (v), and 2.4 (ii),

$$\| |A|^{-2}R^2\|_{L^2(M^+,h)} \leqq C. \tag{3}$$

On $\Lambda_{\delta_2}^-(\tau)$ we similarly estimate by using 2.1 that

$$\begin{aligned} \| |A|^{-2}R^2\|_{L^2(h)}^2 &= \int_0^{2\sqrt{\tau}\cosh\omega=\delta_2} 4\pi\sqrt{\tau}R^4(2+2e^{-4w})^{-2}\frac{\cosh 2w}{2}\,du \\ &\leqq C\tilde{\tau}\left(C+\int_1^{v=\delta_2/2\sqrt{\tau}} v^{-7}dv\right) \leqq C\tilde{\tau}. \end{aligned} \tag{4}$$

Now let $N_{\delta_2}(\tau,\tau')$ be one of the components of $M_{\delta_2}^-$. By homothetically enlarging the picture by a factor of τ we will have $\tilde{R}=R\tau^{-1}$ to be the distance from the x_3-axis on the enlarged $N_{\delta_2}(\tau,\tau')$, that is on a $\tilde{N}_{\delta_2}(\tau,\tau')$. There we have

$$\|\tilde{R}^2\|_{C^1(\tilde{N}_{\delta_2}(\tau,\tau'),\tilde{g})} \leqq C\tau^{-2}$$

by 2.2 (ii), where $\tilde{g}=\tau^{-2}g$ is the induced metric on $\tilde{N}_{\delta_2}(\tau,\tau')$. But then (2.4) (v) and (vi) allow us to conclude that

$$\|\varrho^{-2}R^2\|_{C^1(M_{\delta_2},\chi)}\tilde{\tau}^2, \qquad \| |A|^{-2}R^2\|_{L^2(M_{\delta_2},h)} \leqq C\tilde{\tau}^2. \tag{5}$$

Combining (2), (3), (4), and (5) we conclude the proof. □

We define now on the unit sphere $x_1^2+x_2^2+x_3^2=1$ a smooth function η. η is some fixed function of the x_1 coordinate only, supported on $|x_1|<\frac{1}{10}$, and satisfies

$$\langle \eta, x_1\rangle_{L^2(S^2(1))} = -1.$$

Suppose now S is an i^{th} positive a.s.r. of M, without loss of generality S is rotationally symmetric around the x_1-axis. We define $\tilde{w}^i$ to be $\eta\circ\nu$ on S and to be supported on S and its images under G (where it is defined by our convention that all functions are invariant under G). Similarly we define w^i to be $\eta\circ\nu$ on an i^{th} negative a.s.r. of M (whose axis is the x_1-axis) and to be supported on the i^{th} negative a.s.r.'s of M. Notice that by 2.4 (v) and (vi) the $C^n(M,\chi)$ and $L^2(M,h)$ norms of the w^i's and the $\tilde{w}^i$'s are bounded uniformly. From now on we drop the indices m,n from $V_{m,n}$ and we write V instead. We define $\bar{\Theta}_M: V\to C^\infty(M)$ by $(\{\tilde{\lambda}_i\},\{\lambda_i\}) \to \sum\tilde{\lambda}^i\tilde{w}^i+\sum\lambda^i w^i$. If $\lambda=(\{\tilde{\lambda}_i\},\{\lambda_i\})\in V$, we write λ^+ and λ^- for $(\{\tilde{\lambda}_i\},\{0\})$, $(\{0\},\{\lambda_i\})$ respectively. We also write $\zeta^+[M]$ and $\zeta^-[M]$ for $(\zeta[M])^+$ and $(\zeta[M])^-$ respectively. The following lemma allows us to understand (implicitly) the approximate kernel content of the right-hand side of 3.2.

Lemma 3.8. (i) *Given* $\varepsilon>0$ *there is* $T(\tilde{\Sigma},\varepsilon)$ *such that if* $\tilde{\tau}<T(\tilde{\Sigma},\varepsilon)$, *then*

$$\|\Pi(f_h^1-\tilde{\tau}\bar{\Theta}_M(\zeta^+[M]))\|_{L^2(M^+,h)}<\varepsilon\tilde{\Sigma}\tilde{\tau}.$$

(ii) *Given* $\varepsilon>0$ *there are* $\Sigma_5(\varepsilon)$ *and* $T_5(\Sigma,\varepsilon)$ *such that if* $\Sigma<\Sigma_5$ *and* $\tilde{\tau}<T_5$, *then*

$$\|\Pi(f_h^2-\tilde{\tau}\bar{\Theta}_M(\zeta^-[M]))\|_{L^2(M,h)}<\varepsilon\Sigma\tilde{\tau}.$$

Proof. (i) Assume $m\neq 1$, the other case being similar. Because of 3.7 (ii), 2.2 (i), and 2.4 (ii) and (vi), it is enough to prove that if S is an i^{th} positive a.s.r., then

$$|\langle f_h^1, v_1\rangle_{L^2(S,h)}+\tilde{\tau}\tilde{\zeta}_i|<\tfrac{1}{2}\varepsilon\tilde{\Sigma}\tilde{\tau}, \tag{1}$$

where without loss of generality the axis of S is the x_1-axis and $v=(v_1,v_2,v_3)$ is the Gauss map of S.

2.2 (i) and 2.4 (ii) and (vi) imply that

$$\tilde{\tau}\left|2\int_{S^2(1)} x_1((x_1+2i)^2+x_2^2)+4\int_S dh\frac{R^2}{|A|^2}v_1\right|<\tfrac{1}{4}\varepsilon\tilde{\Sigma}\tilde{\tau}. \tag{2}$$

Now extend S to $\hat{S}$ by attaching to it subsets of $DS(\tau_{2i+1})$ and $DS(\tau_{2i})$ so that $\partial\hat{S}$ consists of circles of minimum radius perpendicular to the axes. (We assume $m\geqq 2$, the case $m=1$ differs only in notation.) By Lemma 2.1 (iv) one of the circles has radius $\tau_{2i+1}+O(\tilde{\tau}^2)$ and the other one $\tau_{2i}+O(\tilde{\tau}^2)$. Now we have

$$\int_S\frac{\delta H}{|A|^2}v_1dh=\frac{1}{2}\int_{\hat{S}}\delta Hv_1dg=\frac{1}{4}\int_{\hat{S}}(\Delta_gX_1-2v_1)dg,$$

where dg is the measure with respect to the induced metric g by the immersion $X=(X_1,X_2,X_3)$ of $\hat{S}$ into E^3. Then an integration by parts gives

$$\int_{\hat{S}}\Delta_gX_1dg=2\pi(\tau_{2i+1}-\tau_{2i})+O(\tilde{\tau}^2).$$

Also by extending suitably $\hat{S}$ to a closed surface and using the divergence theorem we conclude that

$$\int_{\hat{S}}v_1dg=O(\tilde{\tau}^2).$$

Then we conclude

$$\left|2\pi(\tau_{2i+1}-\tau_{2i})-4\int_S\frac{\delta H}{|A|^2}v_1dh\right|<\frac{1}{4}\varepsilon\tilde{\Sigma}\tilde{\tau}. \tag{3}$$

(2) and (3) imply (1).

(ii) This is similar to the proof of (3) in (i). □

We fix now once and for all real numbers $0<\tilde{\alpha}<\alpha<1$. We will write $\hat{M}$ for $M[\tilde{\tau},0]$. We have the following lemma in which estimates are provided for the solution of the linearized problem.

Lemma 3.9. *There is* $b_2>0$ *such that given* $\varepsilon>0$ *there is* $\Sigma_6(\varepsilon)>0$ *such that if* $\Sigma<\Sigma_6(\varepsilon)$ *and* $\delta>0$, *there is* $T_6(\varepsilon,\delta,\tilde{\Sigma},\Sigma)>0$ *such that if* $\tilde{\tau}<T_6$, *then there are continuous maps:*

$$\lambda:[\tfrac{1}{2}\tilde{\tau},2\tilde{\tau}]\times\Xi(\tilde{\Sigma},\Sigma)\to\Xi(\varepsilon\tilde{\Sigma},\varepsilon\Sigma),\qquad \hat{\phi}:[\tfrac{1}{2}\tilde{\tau},2\tilde{\tau}]\times\Xi(\tilde{\Sigma},\Sigma)\to C^{2,\tilde{\alpha}}(\hat{M}),$$

such that the following are true:

(i) $L_h u = f_h^1 + f_h^2 - \tilde{\tau}\bar{\Theta}_M(\zeta + \lambda(\bar{\tau},\zeta))$ *for any* $\zeta \in \Xi(\tilde{\Sigma},\Sigma)$, $\bar{\tau} \in [\frac{1}{2}\tilde{\tau}, 2\tilde{\tau}]$, *where* $u = \hat{\phi}(\bar{\tau},\zeta) \circ D_{\bar{\tau},\zeta}^{-1}$, *$M$ stands for* $M[\bar{\tau},\zeta]$ *and* f_h^1, f_h^2, *and* L_h *are defined on* M.
(ii) $\|u\|_{C^{2,\alpha}(M,\chi)} \leqq b_2\tilde{\tau}$.
(iii) *The* $C^{2,\alpha}$ *norm on* M_δ^- *with respect to* χ *satisfies* $\|u\|_{C^{2,\alpha}(M_\delta^-,\chi)} \leqq b_2\Sigma\tilde{\tau}$.

Proof. We fix some $(\bar{\tau},\zeta) \in [\frac{1}{2}\tilde{\tau}, 2\tilde{\tau}] \times \Xi(\tilde{\Sigma},\Sigma)$ and we write M for $M[\bar{\tau},\zeta]$. We define $\Theta_M \equiv P \circ \bar{\Theta}_M$. We will specify b_2, T_6, and Σ_6 as we carry out the proof. We claim we can choose Σ_6 and T_6 small enough to ensure that Θ_M is an invertible map onto the approximate kernel of L_h on M. By Lemma 3.5 (iii) it is enough to check that Θ_M has trivial kernel.

2.4 (vi) implies that $\|\bar{\Theta}_M\| < C$ and $\|(\Pi \circ \bar{\Theta}_M)^{-1}\| < C$, where the inverse of $\Pi \circ \bar{\Theta}_M$ (as a map onto its range) exists. But then by Lemma 3.6 (i) if $\mu \in V$,

$$\|\Theta_M(\mu)\| \geqq \|\Pi \circ \bar{\Theta}_M(\mu)\| - \|(\Pi - P) \circ \bar{\Theta}_M(\mu)\| \geqq C\|\mu\|, \tag{1}$$

provided we choose T_6 and Σ_6 small enough. This proves our claim.

We can define now the maps λ_1, λ_2, and λ from $[\tilde{\tau}, \frac{1}{2}\tilde{\tau}] \times \Xi(\tilde{\Sigma},\Sigma)$ into V by:

$$\tilde{\tau}\lambda_1(\bar{\tau},\zeta) = \Theta_M^{-1} \circ P(f_h^1 - \tilde{\tau}\bar{\Theta}_M(\zeta^+)),$$

$$\tilde{\tau}\lambda_2(\bar{\tau},\zeta) = \Theta_M^{-1} \circ P(f_h^2 - \tilde{\tau}\bar{\Theta}_M(\zeta^-)).$$

Then clearly $f_1 = f_h^1 - \tilde{\tau}\bar{\Theta}_M(\zeta^+ + \lambda_1(\bar{\tau},\zeta))$ and $f_2 = f_h^2 - \tilde{\tau}\bar{\Theta}_M(\zeta^- + \lambda_2(\bar{\tau},\zeta))$ are orthogonal to the approximate kernel of L_h on M.

(1) shows that $\|\Theta_M^{-1}\| < C$, so Lemma 3.7 and the definitions allow us to conclude

$$\|\lambda_1\| < C, \qquad \|\lambda_2\| < C\Sigma, \tag{2}$$

where we abuse the notation by writing λ_1 and λ_2 instead of $\lambda_1(\bar{\tau},\zeta)$ and $\lambda_2(\bar{\tau},\zeta)$. (2) implies then that

$$\|f_1\|_{L^2(M,h)} < C\tilde{\tau}, \qquad \|f_2\|_{L^2(M,h)} < C\Sigma\tilde{\tau}. \tag{3}$$

Since $Pf_1 = 0$ and $Pf_2 = 0$, Lemma 3.5 (i) allows us to choose T_6 and Σ_6 so that to guarantee

$$\|\tilde{P}f_1\|_{L^2(M,h)} < \varepsilon_1\tilde{\tau}, \tag{4}$$

where $\varepsilon_1 = \varepsilon_1(\tilde{\Sigma},\Sigma,\delta,\varepsilon) > 0$ is to be determined later. Let f_1^+ and f_1^- be defined by $f_1^+ = f_1$ on M^+, $f_1^- = f_1$ on M^-, and $f_1 = f_1^+ + f_1^-$. Lemmas 2.4 (vi) and 3.5 (ii), allow us to assume that $\|\tilde{P}\tilde{w}_i\|_{L^2(M,h)} > C$, so together with 3.7 (ii) we conclude

$$C\tilde{\tau}\|\lambda_1^-\| - C\tilde{\tau}^{3/2} < \|\tilde{P}f_1^-\|.$$

Then (4) and 3.7 (ii) allow us to conclude

$$\|\lambda_1^-\| < C\varepsilon_1, \qquad \|f_1\|_{L^2(M^-,h)} < C\varepsilon_1\tilde{\tau}. \tag{5}$$

(3), (4) and Lemma 3.5 (ii) allow us to conclude by choosing T_6 and Σ_6 suitably that

$$\|\Pi f_1^+\| < \varepsilon_2\tilde{\Sigma}\tilde{\tau},$$

where $\varepsilon_2=\varepsilon_2(\varepsilon)>0$ is to be determined shortly. Referring to Lemma 3.8 (i) we can conclude

$$\|\Pi\circ\bar{\Theta}_M(\lambda_1^+)\|<2\varepsilon_2\tilde{\Sigma},$$

which implies

$$\|\lambda_1^+\|<C\varepsilon_2\tilde{\Sigma}. \tag{6}$$

Now (3) and Lemmas 3.6 (i) and 3.8 (ii) allow us to conclude that

$$\|\lambda_2\|<C\varepsilon_2\Sigma. \tag{7}$$

By choosing ε_1 and ε_2 suitably, (5), (6), and (7) allow us to conclude that $\lambda=\lambda_1+\lambda_2$ maps into $\Xi(\varepsilon\tilde{\Sigma},\varepsilon\Sigma)$.

There are u_1 and u_2 is $C^{2,\alpha}(M)$ which are orthogonal to the approximate kernel and satisfy $L_h u_1=f_1$, $L_h u_2=f_2$. We define $u=u_1+u_2$, then the equation in (i) is satisfied. (3) implies

$$\|u_1\|_{L^2(M,h)}\leqq C\tilde{\tau}, \qquad \|u_2\|_{L^2(M,h)}\leqq C\Sigma\tilde{\tau}. \tag{8}$$

Also (3), (5) and Lemma 3.4 allow us to conclude

$$\|u_1\|_{L^2(M^-,h)}\leqq C\varepsilon_1\tilde{\tau}. \tag{9}$$

Standard elliptic theory (Theorem 8.17 of [G–T] modified to a manifold setting where use of the uniformity of the constant in 2.4 (iv) is made) allows us to conclude

$$\|u_1\|_{C^0(M)}\leqq C\tilde{\tau}, \qquad \|u_2\|_{C^0(M)}\leqq C\Sigma\tilde{\tau}, \qquad \|u_1\|_{C^0(M^-_{\delta/2})}\leqq C(\delta)\varepsilon_1\tilde{\tau}, \tag{10}$$

where we know that the h-distance of $M^-_{\delta/2}$ from ∂M^- is larger than $C(\delta)$ by referring to Lemma 2.4 (vi). We have now $L_\chi u_i=\dfrac{|A|^2}{2\varrho^2}f_i$ $(i=1,2)$, and by Lemmas 3.7 and 2.4 (v), and (5), (6), and (7), we can assume

$$\left\|\frac{|A|^2}{2\varrho^2}f_1\right\|_{C^{0,\alpha}(M,\chi)}\leqq C\tilde{\tau}, \qquad \left\|\frac{|A|^2}{2\varrho^2}f_2\right\|_{C^{0,\alpha}(M,\chi)}\leqq C\Sigma\tilde{\tau},$$
$$\left\|\frac{|A|^2}{2\varrho^2}f_1\right\|_{C^{0,\alpha}(M^-,\chi)}\leqq C\varepsilon_1\tilde{\tau}.$$

Then Lemma 2.4 (iii) allows us to conclude

$$\|u_1\|_{C^{2,\alpha}(M,\chi)}\leqq C\tilde{\tau}, \qquad \|u_2\|_{C^{2,\alpha}(M,\chi)}\leqq C\Sigma\tilde{\tau}, \qquad \|u_1\|_{C^{2,\alpha}(M^-_\delta,\chi)}\leqq C(\delta)\varepsilon_1\tilde{\tau}.$$

By choosing ε_1 small enough we conclude from these estimates parts (ii) and (iii) of the lemma.

It remains only to prove the continuity of λ and $\hat{\phi}$. The continuity of λ follows from the fact that P depends continuously on L_h and h, and the continuity of the various constructions. (We pull back everything to $\hat{M}$ by $D_{\bar{\tau},\zeta}$.) This implies that $(f_1+f_2)\circ D_{\bar{\tau},\zeta}$ depends continuously on $\bar{\tau},\zeta$ as an element of $C^{0,\bar{\alpha}}(\hat{M})$. We define projections $P_{\bar{\tau},\zeta;0}$ and $P_{\bar{\tau},\zeta;2}$ in $C^{0,\bar{\alpha}}(\hat{M})$ and $C^{2,\bar{\alpha}}(\hat{M})$ respectively by:

$$u\to u-\bar{\Theta}_M(\Theta_M^{-1}\circ P(u\circ D_{\bar{\tau},\zeta}^{-1}))\circ D_{\bar{\tau},\zeta}.$$

These projections depend continuously on $\tilde{\tau}, \zeta$. Then define $L_{\tilde{\tau},\zeta}$ from the range of $P_{\tilde{\tau},\zeta;2}$ to the range of $P_{\tilde{\tau},\zeta;0}$ by

$$u \to [L_h(u \circ D_{\tilde{\tau},\zeta}^{-1})] \circ D_{\tilde{\tau},\zeta}.$$

Notice that the ranges of $P_{\tilde{\tau},\zeta;2}$ and $P_{\tilde{\tau},\zeta;0}$ correspond to the orthogonal [with respect to the $L^2(M,h)$ inner product] complements of the approximate kernel of L_h on $M[\tilde{\tau},\zeta]$ in $C^{2,\tilde{\alpha}}(M[\tilde{\tau},\zeta])$ and $C^{0,\tilde{\alpha}}(M[\tilde{\tau},\zeta])$ respectively. So standard linear theory implies that $L_{\tilde{\tau},\zeta}$ has a bounded inverse. Since $L_{\tilde{\tau},\zeta}$ depends continuously on $\tilde{\tau}, \zeta$, and $P_{\tilde{\tau},\zeta;0}, P_{\tilde{\tau},\zeta;2}$ are the identity on their ranges, we conclude that $L_{\tilde{\tau},\zeta}^{-1} \circ P_{\tilde{\tau},\zeta;0}$ depends continuously on $\tilde{\tau}, \zeta$. Since $\hat{\phi}(\tilde{\tau},\zeta) = L_{\tilde{\tau},\zeta}^{-1} \circ P_{\tilde{\tau},\zeta;0}(f_1+f_2) \circ D_{\tilde{\tau},\zeta}$, $\hat{\phi}$ is continuous and the proof is complete. $\square$

4. The Main Theorem

In this section, we first provide estimates for the linearization error (Lemma 4.1) and then we prove the main theorem by combining all the estimates we have.

Suppose now F is a family of initial surfaces of parameters $(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$ and $M \in F$ as usual. Let $v: M \to S^2(1)$ be the Gauss map of M. Let $v \in C^2(M)$ be such that $X + vv: M \to E^3$ is an immersion of M in E^3 and $v_v: M \to S^2(1)$ is its Gauss map. We write M_v for the surface which is the image of $X + vv$. We define the following functions on M:

$$F_h^H(v) = 4|A|^{-2}(H_v - H) - (\Delta_h v + 2v),$$

$$F_h^R(v) = 4|A|^{-2}\tau[M](R^2 - R_v^2),$$

$$F_\chi^H(v) = \frac{|A|^2}{2\varrho^2} F_h^H(v), \qquad F_\chi^R(v) = \frac{|A|^2}{2\varrho^2} F_h^R(v),$$

$$F_h(v) = F_h^H(v) + F_h^R(v), \qquad F_\chi(v) = F_\chi^H(v) + F_\chi^R(v),$$

where R_v and H_v are the distance from the x_3-axis and the mean curvature function on $X + vv$ respectively, both pulled back to M by $X + vv$.

Lemma 4.1. *There is $\Sigma_7 > 0$ such that given $\varepsilon > 0$, there is $b_3(\varepsilon) > 0$ such that given $a \in (0, b_3)$ and $b > 0$, there are $\bar{\delta}(\varepsilon, a, b) > 0$ and $T_7(\varepsilon, a, b) > 0$ such that if $\Sigma < \Sigma_7$, $\tilde{\tau} < T_7$, $\phi \in C^2(M)$, and*

$$\|\phi\|_{C^{2,\alpha}(M,\chi)} \leqq b\tilde{\tau}, \qquad \|\phi\|_{C^{2,\alpha}(M_{\bar{\delta}}^-,\chi)} \leqq a\tilde{\tau},$$

then the following are true:

(i) *$X + \phi v: M \to E^3$ is an embedding.*

(ii) *$\|F_h(\phi)\|_{L^2(M,h)} \leqq \varepsilon a \tilde{\tau}$, $\|F_\chi(\phi)\|_{C^{0,\alpha}(M,\chi)} \leqq \varepsilon a \tilde{\tau}$.*

Proof. Such a statement is proved in [K 1] for F_h^H and F_χ^H instead of F_h and F_χ in (ii). That proof is straightforward but long and we omit it here. We have still to prove:

$$\|F_h^R(\phi)\|_{L^2(M,h)} \leqq \tfrac{1}{2}\varepsilon a \tilde{\tau}, \qquad \|F_\chi^R(\phi)\|_{C^{0,\alpha}(M,X)} \leqq \tfrac{1}{2}\varepsilon a \tilde{\tau}.$$

We break the estimates into estimates on $M_{\delta_2}^+$, $M_{\delta_2}^0$, and $M_{\delta_2}^-$ (δ_2 was defined in the definition of χ). Lemmas 2.2 and 2.4 (v) allow us to ensure

$$\|F_h^R(\phi)\|_{L^2(M_{\delta_2}^+,h)} \leqq C(b + b^2)\tilde{\tau}^2, \qquad \|F_\chi^R(\phi)\|_{C^{0,\alpha}(M_{\delta_2}^+,\chi)} \leqq C(b + b^2)\tilde{\tau}^2. \tag{1}$$

$M^0_{\delta_2}$ is a union of $\Lambda_{\delta_2}(\tau)$'s. Consider one of them, without loss of generality it is immersed into E^3 by a map $(x_1, x_2, x_3) = X_\tau + (a, 0, 0)$, where a is a constant [depending on the particular $\Lambda_{\delta_2}(\tau)$] and $X_\tau : \mathbb{R}^2 \to E^3$ is defined in 2.1. Recall the notation of 2.1, we calculate

$$R^2_\phi = (x_1(u) + 2\sqrt{\tau}\phi w_u)^2 + \tau(e^w - 2\phi \cosh w)^2 \cos^2 \frac{v}{2\sqrt{\tau}},$$

$$F^R_h(\phi) = 4|A|^{-2}\tilde{\tau}[M]F^R(\phi), \qquad F^R_\chi(\phi) = \frac{1}{2} e^{2w}\tau\tilde{\tau}[M]F^R(\phi),$$

where all functions have been pulled back to $\mathbb{R}^2$, and

$$-F^R(\phi) = 4\sqrt{\tau}\phi x_1 w_u + 4\tau\phi^2 w_u^2 - 4\tau\phi e^w \cosh w \cos^2 \frac{v}{2\sqrt{\tau}}$$

$$+ 4\tau\phi^2 \cosh^2 w(v) \cos^2 \frac{v}{2\sqrt{\tau}}.$$

Recall $\chi = \tau^{-1}(du^2 + dv^2)$, by referring to 2.1 we conclude that

$$\|F^R(\phi)\|_{C^1(M^0_{\delta_2}, \chi)} \leqq C(b + b^2)\tilde{\tau}.$$

This implies [recall 2.4 (ii)]

$$\|F^R_h(\phi)\|_{L^2(M^0_{\delta_2}, h)} \leqq C(b + b^2)\tilde{\tau}^2, \qquad \|F^R_\chi(\phi)\|_{C^{0,\alpha}(M^0_{\delta_2}, \chi)} \leqq C(b + b^2)\tilde{\tau}^2, \tag{2}$$

where we used $|A|^{-2} < 2$ and $\|e^{2w}\tilde{\tau}\|_{C^1(\chi)} < C$, which follow from 2.1. Now to treat the case $M^-_{\delta_2}$ we homothetically expand everything by a factor $\tilde{\tau}$. Then Lemmas 2.4 (v) and 2.2 (ii) imply that

$$\|F^R_\chi(\phi)\|_{C^1(M_{\bar{\delta}_2}, \chi)} \leqq C(\delta_2) b_3 \tilde{\tau}^4, \tag{3}$$

and by using also 2.4 (vi)

$$\|F^R_h(\phi)\|_{L^2(M_{\bar{\delta}_2}, h)} \leqq C(\delta_2) b_3 \tilde{\tau}^4. \tag{4}$$

By choosing T_7 small enough we conclude the proof by combining (1), (2), (3), and (4).

Theorem 4.7. *Suppose we are given a certain volume $\mathscr{V}$ of a liquid of surface tension $\mathscr{T}$ and density $\mathscr{D}$. Then given an angular momentum $\mathscr{L}$ smaller than a constant $\mathscr{L}(\mathscr{V}, \mathscr{D}, \mathscr{T})$, the following is true:*

There is a family F (of type (m, n)) and parameters $(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$, an initial surface $\tilde{M} \in F$, a (small) smooth function v on $\tilde{M}$, and a constant d, such that $\tilde{M}_v$ homothetically expanded by a factor d is an embedded surface enclosing volume $\mathscr{V}$ and satisfying Eq. (1.2) for a rotating in gyrostatic equilibrium liquid drop of volume $\mathscr{V}$, surface tension $\mathscr{T}$, and density $\mathscr{D}$. This rotating drop is unstable in the sense mentioned in the introduction.

Proof. We can formulate the problem of the rotating drop in terms of the total energy known as the Routhian

$$R = \mathscr{T}\mathscr{A} + \frac{\mathscr{L}^2}{2\mathscr{I}\mathscr{D}} - \hat{\mathscr{K}}(\hat{\mathscr{V}} - \mathscr{V}) \quad \text{and the constraint } \hat{\mathscr{V}} = \mathscr{V}, \tag{1}$$

where $\mathscr{A}$ is the surface area of $\partial\Omega$, $\mathscr{I}=\int_{\Omega} R^2$ is the moment of inertia of Ω around the x_3-axis (R is the distance from the x_3-axis), $\mathscr{V}$ is the volume of Ω, and Ω is the domain in E^3 occupied by the liquid mass at some time.

Then it is easy to calculate the first and second variations of (1) to conclude that:

1. Gyrostatic equilibrium amounts to having the Young-Laplace equation

$$H=\mathscr{K}+\frac{\mathscr{L}^2}{4\mathscr{T}\mathscr{D}\mathscr{I}^2}R^2 \tag{2}$$

satisfied on $\partial\Omega$, where H is its mean curvature and $\mathscr{K}=\dfrac{\hat{\mathscr{K}}}{2\mathscr{T}}$ is essentially a Lagrange multiplier.

2. Stability amounts to

$$\int_{\partial\Omega}\left\{\mathscr{T}(|\nabla\phi|^2-\phi^2|A|^2)+\frac{\mathscr{L}^2}{\mathscr{I}^2\mathscr{D}}\phi^2(X_1\nu_1+X_2\nu_2)\right\}+\frac{\mathscr{L}^2}{\mathscr{I}^3\mathscr{D}}\left(\int_{\partial\Omega}\phi R^2\right)^2>0 \tag{3}$$

for all $\phi\in H_1^2(\partial\Omega)$ with $\int_{\partial\Omega}\phi=0$, where $X=(x_1,x_2,x_3):\partial\Omega\to E^3$ and

$$\nu=(\nu_1,\nu_2,\nu_3):\partial\Omega\to S^2(1)$$

are the immersion of $\partial\Omega$ into E^3 and its Gauss map respectively.

The reader is referred to [B–S] for a brief discussion of the physics of (1), (2), and (3) while the calculation of the first and second variations of (1) is straightforward and the reader would have no difficulty in checking. Suppose now M is an initial surface and $v\in C^\infty(M)$ and M_v is an embedded surface satisfying

$$H_v=1+\bar{\tau}R_v^2 \tag{4}$$

and enclosing volume $\mathscr{V}(M,v)$, whose moment of inertia is $\mathscr{I}(M,v)$. Let $d=\left(\dfrac{\mathscr{V}}{\mathscr{V}(M,v)}\right)^{1/3}$. Then if we homothetically expand M_v by a factor d, we obtain a surface $\partial\Omega$ enclosing a domain Ω. This is a solution to our problem provided that

$$\bar{\tau}=\frac{\mathscr{L}^2(\mathscr{V}(M,v))^{7/3}}{4\mathscr{T}\mathscr{D}\mathscr{I}^2(M,v)\mathscr{V}^{7/3}}. \tag{5}$$

Let now Ω_0 be the union of unit balls in E^3 characterized by:

(i) Ω_0 has the symmetries which an initial surface of type (m,n) has.
(ii) It contains the unit balls with centers at the origin, $(2,0,0)$, $(4,0,0)$, …, $(2n,0,0)$ if $m\geqq 2$. If $m=1$, Ω_0 consists of the unit balls with centers at $(\pm 1,0,0)$, $(\pm 3,0,0)$, …, $(\pm(2n-1),0,0)$.
(iii) There are no other balls than the ones in (ii) modulo the symmetries of Ω_0.
In other words Ω_0 can be thought of as the limit of an initial surface M of type (m,n), as its τ parameter tends to 0. We write $\mathscr{V}_0$ and $\mathscr{I}_0$ for the volume $\int_{\Omega_0} 1$ and the moment of inertia $\int_{\Omega_0} R^2$ of Ω_0 respectively.

Then given some $\mathscr{L}$ small enough let

$$\tilde{\tau} = \frac{\mathscr{L}^2 \mathscr{V}_0^{7/3}}{4\mathscr{T}\mathscr{D}\mathscr{I}_0^2 \mathscr{V}_0^{7/3}}. \tag{6}$$

By choosing $\mathscr{L}$ small enough we can appeal to Lemma 2.3 to obtain a family of initial surfaces [of type (m, n)] and parameters $(\tilde{\tau}, \tilde{\Sigma}, \Sigma)$ ($\tilde{\Sigma}$ and Σ are uniform constants).

Recall now Lemmas 3.9 and 4.1. By choosing $\mathscr{L}$ small enough we can assume without loss of generality that $\tilde{\tau} < T_6(\frac{1}{3}, \delta, \tilde{\Sigma}, \Sigma)$, where $\delta = \bar{\delta}(\varepsilon, 2b_2\Sigma, 2b_2)$,

$$\tilde{\tau} < T_7(\varepsilon, 2b_2\Sigma, 2b_2),$$

$$\Sigma < \Sigma_6\left(\frac{1}{3}\right), \qquad \Sigma < \Sigma_7, \qquad \Sigma < \frac{b_3(\varepsilon)}{2b_2},$$

where ε is a positive absolute constant to be determined shortly. Lemmas 3.9 and 4.1 allow us to define a continuous map

$$[\tfrac{1}{2}\tilde{\tau}, 2\tilde{\tau}] \times \Xi(\tilde{\Sigma}, \Sigma) \times \{\phi \in C^{2,\bar{\alpha}}(\hat{M}) : \|\phi\|_{C^{2,\alpha}(\hat{M},\chi)} < \varepsilon_2 b_2 \Sigma \tilde{\tau}\}$$
$$\xrightarrow{\Phi} \mathbb{R} \times V \times C^{2,\bar{\alpha}}(\hat{M})),$$

as follows: Suppose $(\bar{\tau}, \zeta, \phi)$ in the domain is given. Let M stand for $M[\bar{\tau}, \zeta]$ and $u = \hat{\phi}(\zeta) \circ D_{\bar{\tau},\zeta}^{-1}$ as in the statement of Lemma 3.9. Let $v = u + \phi \circ D_{\bar{\tau},\zeta}^{-1}$. Then by choosing $\varepsilon_2 > 0$, a uniform constant, small enough, we have by Lemmas 2.4 (i) and 3.9

$$\|v\|_{C^{2,\alpha}(M,\chi)} \leqq 2b_2\tilde{\tau}, \qquad \|v\|_{C^{2,\alpha}(M_{\bar{\delta}},\chi)} \leqq 2b_2\Sigma\tilde{\tau}. \tag{7}$$

This allows us to apply Lemma 4.1 to conclude that M_v is an imbedded surface and

$$\|F_h(v)\|_{L^2(M,h)} \leqq 2\varepsilon b_2 \Sigma \tilde{\tau}, \qquad \|F_\chi(v)\|_{C^{0,\alpha}(M,\chi)} \leqq 2\varepsilon b_2 \Sigma \tilde{\tau}.$$

Arguing as in the proof of Lemma 3.9 we conclude that there are

$$\mu(\bar{\tau}, \zeta, \phi) \in \Xi(\varepsilon C\Sigma, \varepsilon C\Sigma) \quad \text{and} \quad w \in C^{2,\alpha}(M)$$

satisfying

$$\|w\|_{C^{2,\alpha}(M,\chi)} \leqq \varepsilon C \Sigma \tilde{\tau} \tag{8}$$

and

$$(\Delta_h + 2)w = F_h(v) + \tilde{\tau}\bar{\Theta}_M(\mu(\bar{\tau}, \zeta, \phi)). \tag{9}$$

Then we define

$$\begin{aligned}\Phi(\bar{\tau}, \zeta, \phi) &= (\Phi_1(\bar{\tau}, \zeta, \phi),\ \Phi_2(\bar{\tau}, \zeta, \phi),\ \Phi_3(\bar{\tau}, \zeta, \phi)) \\ &= \left(\frac{\mathscr{L}^2 \mathscr{V}(M[\bar{\tau}, \zeta], v)^{7/3}}{4\mathscr{T}\mathscr{D}\mathscr{I}^2(M[\bar{\tau}, \zeta], v)\mathscr{V}^{7/3}},\ -\lambda(\bar{\tau}, \zeta) - \mu(\bar{\tau}, \zeta, \phi),\ -w \circ D_{\bar{\tau},\zeta}\right).\end{aligned}$$

Arguing as in the proof of Lemma 3.9 it is easy to check that Φ is a continuous map. We prove now that Φ preserves its domain:

Equation (7) and Lemmas 2.1 and 2.2 allow us to conclude that by choosing $\bar{\mathscr{L}}$ small enough we can ensure that

$$|\mathscr{V}(M[\bar{\tau},\zeta],v)-\mathscr{V}_0|<\varepsilon_1\,,\qquad |\mathscr{I}(M[\bar{\tau},\zeta],v)-\mathscr{I}_0|<\varepsilon_1$$

for some $\varepsilon_1>0$ small enough to imply

$$|\Phi_1(\bar{\tau},\zeta,\phi)-\tilde{\tau}|<\tfrac{1}{4}\tilde{\tau}\,. \tag{10}$$

Clearly $\Phi_2(\bar{\tau},\zeta,\phi)\in\Xi(\tilde{\Sigma},\Sigma)$ if we choose ε small enough.

Also by (8) and 2.4 (i) we conclude

$$\|\Phi_3(\bar{\tau},\zeta,\phi)\|\leqq\varepsilon C\Sigma\tilde{\tau}\,. \tag{11}$$

Hence we can ensure that Φ preserves its domain by choosing ε small enough.

The domain of Φ is clearly compact and convex. Since it is preserved we can apply the Schauder fixed point theorem (Theorem 11.1 in [G–T]) to conclude that there is a fixed point $(\bar{\tau},\zeta,\phi)$. Lemma 3.9 and the definition of F_h imply

$$(\Delta_h+2)u=\frac{4\bar{\tau}R^2-4\delta H}{|A|^2}-\tilde{\tau}\bar{\Theta}_M(\zeta+\lambda(\bar{\tau},\zeta))\,,$$

$$F_h(v)=4|A|^{-2}(H_v-H)-(\Delta_h+2)v+4|A|^{-2}\bar{\tau}[M](R^2-R_v^2)\,.$$

Combining these with (9) we conclude that

$$\begin{aligned}4|A|^{-2}H_v=4|A|^{-2}+(\Delta_h+2)(\phi\circ D^{-1}_{\bar{\tau},\zeta}+w)-\tilde{\tau}\bar{\Theta}_M(\zeta+\lambda(\bar{\tau},\zeta)\\ +\mu(\bar{\tau},\zeta,\phi))+4|A|^{-2}\bar{\tau}R_v^2\,.\end{aligned} \tag{12}$$

Then the fact that $(\bar{\tau},\zeta,\phi)$ is a fixed point of Φ implies that (4) and (5) are satisfied. The smoothness of v and M_v then follows from standard elliptic regularity theory.

It only remains to prove instability. Let S be a positive a.s.r. of M. Then Lemmas 2.2 and 2.4 (v) and (7) imply by well-known facts (a possible reference is [K 1] again) that the lower spectrum of $S_{\delta_3,v}$ for δ_3 and $\bar{\mathscr{L}}$ small enough is close to that of the sphere. ($S_{\delta_3,v}$ stands for the image of $M_{\delta_3}\cap S$ under $X+v\nu$.) We can find $\phi\in C_0^\infty(S_{\delta_3,v})$ such that

$$\int_{M_v}|\nabla\phi|^2<\tfrac{1}{2},\qquad \sup|\phi|<2,\qquad \int_{M_v}\phi^2=1\,.$$

Combining two such ϕ's corresponding to two different positive a.s.r.'s we find a $\phi\in H_1^2(M_v)$ such that

$$\int_{M_v}\phi=0,\qquad \int_{M_v}|\nabla\phi|^2<\tfrac{1}{2},\qquad \sup|\phi|<2,\qquad \int_{M_v}\phi^2=1\,.$$

By homothetically expanding by a factor d to get our drop, and choosing $\bar{\mathscr{L}}$ small enough, we contradict (3) and the proof is complete. Notice that the more complicated the drop is, that is the larger the number of lobes is, the more linearly independent functions violating the stability we can find. So in this sense the more the lobes, the more unstable the drop is. □

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