Commun. Math. Phys. 128, 509–520 (1990)

Communications in Mathematical Physics

Moduli Space of $SU(2)$ Monopoles and Complex Cyclic–Toda Hierarchy

Yoshimasa Nakamura
Department of Mathematics, Gifu University, Yanagido, Gifu 501–11, Japan

Abstract. We study the problem of complete parametrization of the moduli space of $SU(2)$ Yang–Mills–Higgs monopoles in terms of a nonlinear integrable system. It is shown that the moduli space is homeomorphic to the solution space of a new generalization of finite nonperiodic Toda equation called the complex cyclic-Toda hierarchy.

1. Introduction

All $SU(2)$ Yang–Mills–Higgs monopoles, namely, finite energy static solutions of the $SU(2)$ Yang–Mills–Higgs equations, are given as solutions of Bogomolny's first-order equations over $\mathbb{R}^3$ satisfying appropriate boundary conditions. Monopole solutions of (magnetic) charge k are parametrized by the *moduli space* $\mathscr{K}(k)$, where $k \in \mathbb{N} \cup \{0\}$. The integer k is called the monopole number. It is also known that by the ADHM-Nahm construction [8, 9, 14] all k-monopoles can be produced from solutions of $k \times k$ matrix ODEs called the Nahm equations satisfying certain boundary and symmetry conditions. Such a solution is converted into the Nahm complex [4], $\{\alpha, \beta, v\}$, i.e. $k \times k$ matrix valued functions $\alpha(s)$ and $\beta(s)$, for $s \in (0, 2)$, and vector $v \in \mathbb{C}^k$ satisfying

(i) $(d\beta(s)/ds) + 2[\alpha(s), \beta(s)] = 0$,
(ii) $\alpha(2-s) = \alpha^T(s)$, $\quad \beta(2-s) = \beta^T(s)$,
(iii) α and β are analytic over $(0, 2)$, with simple poles at 0, 2, and residues a and b at $s = 0$,
(iv) $\mathrm{Tr}(a) = 0$, and v is an eigenvector of norm 1 associated with the eigenvalue $-(k-1)/4$ of a such that $\mathrm{rank}\,(v\, bv\, b^2v \cdots b^{k-1}v) = k$.

Donaldson [4] succeeded in giving a description of the space of $SU(2)$ k-monopoles by the space $\mathrm{rat}^{\mathbb{C}}(k)$ of rational functions of the form

$$f(z) = \frac{p(z)}{q(z)} = \frac{p_{k-1}z^{k-1} + \cdots + p_0}{z^k + q_{k-1}z^{k-1} + \cdots + q_0}, \tag{1}$$

where $p(z)$ and $q(z)$ are coprime polynomials over $\mathbb{C}$. He proved that there is a one-to-one correspondence between the equivalence classes under $GL(k, \mathbb{C})$ of Nahm complexes and the rational functions (1) of degree k such that $f(\infty)=0$. Combining this result with those of [8, 9, 14] it is concluded that a circle bundle $\tilde{\mathcal{K}}(k)$ over the moduli space $\mathcal{K}(k)$ of $SU(2)$ monopoles can be identified with the space $\mathrm{rat}^{\mathbb{C}}(k)$. Moreover, the correspondence is a *homeomorphism* under the natural topologies on $\tilde{\mathcal{K}}(k)$ and $\mathrm{rat}^{\mathbb{C}}(k)$, namely, $\tilde{\mathcal{K}}(k) \cong \mathrm{rat}^{\mathbb{C}}(k)$ (Nahm–Hitchin–Donaldson theorem). See [3, Sect. 7] and [23, p. 476]. Note that the moduli space itself can be regarded as the quotient space $\mathrm{rat}^{\mathbb{C}}(k)/\sim$ under the relation $f(z) \sim e^{i\theta} f(z)$, $\theta \in \mathbb{R}$, $i=\sqrt{-1}$. Set

$$\mathrm{Rat}^{\mathbb{C}}(k) = \mathrm{rat}^{\mathbb{C}}(k)/\sim . \tag{2}$$

The Nahm–Hitchin–Donaldson theorem asserts that $\mathcal{K}(k)$ is homeomorphic to $\mathrm{Rat}^{\mathbb{C}}(k)$, namely

$$\mathcal{K}(k) \cong \mathrm{Rat}^{\mathbb{C}}(k). \tag{3}$$

It should be remarked that $SU(2)$ monopoles can be associated with *linear flows* on the Jacobian of a spectral curve in $T\mathbb{C}P^1$ through the Nahm equations. This interesting fact was observed by Hitchin [9] and Griffiths [7]. From an alternative point of view, in [16], the author and Duncan show how the linearized Toda flow acts on the space of Taubes' generic $SU(2)$ monopoles [21] through the Nahm–Hitchin–Donaldson theorem. Here the generic k-monopoles correspond to the rational functions $f(z) = \sum_{j=1}^{k} \exp(\gamma_j)/(z-\zeta_j)$ of $\mathrm{rat}^{\mathbb{C}}(k)$, where the points ζ_j in $\mathbb{C}$ are sufficiently far from each other. It is shown [16] that one-parameter group action $f(z) \to \sum_{j=1}^{k} \exp(\gamma_j + t\zeta_j)/(z-\zeta_j)$, $t \in \mathbb{R}$, is congruent to the complex version of linearized finite nonperiodic Toda flow by Moser [13]. This will be related to the previous results concerning a reduction of the Bogomolny's equations to the real Toda equation under the spherical symmetry [5, 11] and generalizes a reduction of the Nahm equations for the generic and axially symmetric configurations to the Toda equation [14].

Let us recall the well-known fact that the infinite Toda hierarchy [25] is viewed as a discrete form of the KP hierarchy from which every nonlinear integrable system of infinite dimensions (soliton equation) can be derived by a periodic reduction [17]. For example, the periodic Toda equation having a Kac–Moody symmetry is derived from the infinite Toda hierarchy. Thus systems of Toda type seem to hold the key of the problem of generalization and classification of nonlinear integrable systems. Note that the infinite Toda hierarchy induces a linear flow on an infinite dimensional Grassmann manifold. However, not so much is known about the totality of finite dimensional systems, such as the Nahm equations and the finite nonperiodic Toda equation. In [16], it is shown that the complex finite nonperiodic Toda equation induces a flow on the moduli space of generic $SU(2)$ monopoles, which is the first step in the existence of integrable flows on the moduli space. But it has not yet been known how to completely parametrize the whole moduli space $\mathcal{K}(k)$ of $SU(2)$ k-monopoles in terms of an integrable system of

Toda–Lax type. The existence of such completely integrable flow will make it possible to investigate the topology of the moduli space in terms of the invariant-tori theorem in classical mechanics. Moreover, the Toda flow will have much information about various cell decompositions of the moduli space.

Let $\mathrm{rat}_q^{\mathbb{C}}(k)$ be the space of rational functions of $\mathrm{rat}^{\mathbb{C}}(k)$ having the fixed denominator $q(z) = z^k + q_{k-1}z^{k-1} + \cdots + q_0$, where $q = (q_0, \ldots, q_{k-1}) \in \mathbb{C}^k$. The space $\mathrm{rat}^{\mathbb{C}}(k)$ is decomposed into $\mathrm{rat}^{\mathbb{C}}(k) = \bigcup_{q \in \mathbb{C}^k} \mathrm{rat}_q^{\mathbb{C}}(k)$. By identifying $f(z)$ with $e^{i\theta} f(z)$, $\theta \in \mathbb{R}$, we obtain a decomposition of the moduli space

$$\mathscr{K}(k) = \bigcup_{q \in \mathbb{C}^k} \mathscr{K}_q(k), \tag{4}$$

where $\mathscr{K}_q(k) \cong \mathrm{Rat}_q^{\mathbb{C}}(k) = \mathrm{rat}_q^{\mathbb{C}}(k)/\sim$. The main purpose of this paper is to present a new generalization of the finite nonperiodic Toda equation whose solution space can be identified with $\mathscr{K}(k)$ via the decomposition (4). The nonlinear system we use will be called the *complex cyclic–Toda hierarchy*. Such $q = (q_0, \ldots, q_{k-1})$ is determined according to the choice of equivalence classes of initial values. The original Toda equation and the cyclic–Toda hierarchy can be viewed as isospectral deformation equations of Jacobi (real tridiagonal) matrices and general cyclic matrices, respectively. The result is then:

Theorem. *The moduli space $\mathscr{K}(k)$ of $SU(2)$ k-monopoles is homeomorphic to the solution space of the complex cyclic–Toda hierarchy.*

This implies that the integrable cyclic–Toda flow fills up the whole moduli space $\mathscr{K}(k)$. Thus we shall obtain a new and complete parametrization of $\mathscr{K}(k)$. This theorem hints that the cyclic–Toda hierarchy is more natural than the Nahm complexes (i)–(iv), in the sense in which the Nahm complexes parametrize the circle bundle $\tilde{\mathscr{K}}(k)$ over $\mathscr{K}(k)$.

We now turn to the organization of this paper. In Sect. 2, we introduce the complex cyclic–Toda hierarchy which is a finite set of compatible nonlinear PDEs of Lax type depending on a finite number of time variables. It is shown that the hierarchy can be solved by the QR decomposition of the exponential of matrices. In Sect. 3, we show how the complex cyclic–Toda hierarchy completely parametrizes the space $\mathrm{Rat}_q^{\mathbb{C}}(k)$ of rational functions and consequently the moduli space $\mathscr{K}(k)$ of $SU(2)$ monopoles (Theorem). This result is also of interest in connection with Segal's homeomorphism [18] from $\mathrm{rat}^{\mathbb{C}}(k)$ to a space of rational functions over $\mathbb{R}$ of degree $2k$ and in Sect. 4 we discuss this aspect. It is also shown that the cyclic–Toda hierarchy can be linearized on a space of cyclic vectors.

2. Preliminaries

Let $f(z) = (p(z)/q(z))$ be an element of $\mathrm{rat}_q^{\mathbb{C}}(k)$ of the form (1). It is worth noting that $f(z)$ can be written uniquely as

$$f(z) = C_0^*(zI - A_0)^{-1}B_0, \tag{5-a}$$

$$A_0=\begin{pmatrix}0 & 0\cdots & & -q_0\\ 1 & 0 & & \vdots\\ \vdots & \ddots & 0 & -q_{k-2}\\ 0 & \cdots & 1 & -q_{k-1}\end{pmatrix},\quad B_0=\begin{pmatrix}p_0\\ \vdots\\ p_{k-2}\\ p_{k-1}\end{pmatrix},\quad C_0=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}. \tag{5-b}$$

where $*$ denotes the Hermitian conjugate. We note that $\det(zI-A_0)=q(z)$ and $\operatorname{rank}(C_0\,A_0^*C_0\cdots(A_0^*)^{k-1}C_0)=k$. The factorization (5) of $\operatorname{rat}^{\mathbb{C}}(k)$ is called the observable canonical form in linear systems theory [10]. Since the denominator and the numerator of $f(z)\in\operatorname{rat}^{\mathbb{C}}(k)$ have no common factor,

$$\operatorname{rank}(B_0\,A_0B_0\cdots A_0^{k-1}B_0)=k. \tag{6}$$

This condition guarantees the controllability of the linear dynamical system $(dx(t)/dt)=A_0x(t)+B_0u(t)$, $y(t)=C_0^*x(t)$ realized by the factorization (5-a), where $x(t)\in\mathbb{C}^k$, $u(t)$, $y(t)\in\mathbb{C}$ and $t\in\mathbb{R}$. The vectors B_0 and C_0 are cyclic vectors of A_0. We call $\{A_0,B_0,C_0\}$ a *cyclic triplet*. Let us introduce an equivalence relation on the space of cyclic triplets by

$$\{A_0,B_0,C_0\}\sim\{A_0,\exp(i\theta I)B_0,C_0\},\quad \theta\in\mathbb{R}. \tag{7}$$

It is easy to see there is a one-to-one correspondence between the space of equivalence classes $[\{A_0,B_0,C_0\}]$ under (7) and the space $\operatorname{Rat}^{\mathbb{C}}(k)$ through the unique factorization (5-a).

Next we give a description of the cyclic–Toda hierarchy. Let

$$\tau=(\tau_0,\tau_1,\ldots,\tau_{k-1})\in\mathbb{C}^k \tag{8}$$

be a set of k complex parameters. The complex cyclic-Toda hierarchy is given by the system ($0\leqq j\leqq k-1$) of nonlinear PDEs of Lax type

$$\frac{\partial A(\tau)}{\partial\tau_j}=[A^j(\tau)_L^*-A^j(\tau)_L,A(\tau)] \tag{9}$$

with the supplementary linear PDEs

$$\frac{\partial B(\tau)}{\partial\tau_j}=(A^j(\tau)+A^j(\tau)_L^*-A^j(\tau)_L)B(\tau),$$
$$\frac{\partial C(\tau)}{\partial\tau_j}=(A^j(\tau)_L^*-A^j(\tau)_L)C(\tau), \tag{10}$$

where M_L denotes the strictly lower triangular part of M and $M_L^*=(M_L)^*$. Though the matrix $A(\tau)$ and the vector $C(\tau)$ do not depend on τ_0, the parameter τ_0 is important in our parametrization of rational functions. In this paper we consider the initial value problem of (9) and (10) for the initial cyclic triplet

$$\{A(\mathbf{0}),B(\mathbf{0}),C(\mathbf{0})\}=\{A_0,B_0,C_0\}, \tag{11}$$

where $\mathbf{0}=(0,\ldots,0)\in\mathbb{C}^k$. Note that if $A(\tau)$ is similar to a Jacobi matrix, then the system (9) for $\tau_j\in\mathbb{R}$, $1\leqq j\leqq k-1$, is the system of usual finite nonperiodic Toda equations given by Moser [13]. Let us remark that

Proposition 1. *The cyclic–Toda hierarchy* (9) *and* (10) *is compatible.*

Proof. We shall prove that (9) is integrable if and only if (10) is integrable. Set for $0 \leqq j \leqq k-1$,

$$F_j = A^j(\tau)_L^* - A^j(\tau)_L. \tag{12}$$

The integrability condition $(\partial^2 A/\partial\tau_i\partial\tau_j) = (\partial^2 A/\partial\tau_j\partial\tau_i)$ of the system (9) leads to the system $[(\partial F_j/\partial\tau_i) - (\partial F_i/\partial\tau_j) + [F_j, F_i], A] = 0$. If F_j solve the system $(0 \leqq i \leqq k-1)$ of Zakharov–Shabat equations,

$$\frac{\partial F_j}{\partial\tau_i} - \frac{\partial F_i}{\partial\tau_j} + [F_j, F_i] = 0, \tag{13}$$

then the system (9) is clearly integrable. Conversely, let $A(\tau)$ solve (9). We consider the Lax pair $AY = \lambda Y$, $(\partial Y/\partial\tau_j) = F_j Y$, where $Y = Y(\tau) \in \mathbb{C}^k$, $\lambda = \lambda(\tau) \in \mathbb{C}$. It is well-known that $(\partial A/\partial\tau_j) = [F_j, A]$ is equivalent to $(\partial\lambda/\partial\tau_j) = 0$. Thus the system (9) leads to $(\partial\lambda/\partial\tau_j) = 0$ for $0 \leqq j \leqq k-1$, namely, $\operatorname{Spec} A(\tau) = \operatorname{Spec} A_0$. Let $\operatorname{ord}(M)$ denote an integer associated with the matrix $M = (m_{ij})_{1 \leqq i,j \leqq k}$ which is defined as follows: If $m_{ij} = 0$ for every $i - j > n$ but some m_{ij} is not equal to zero for $i - j = n$, then $\operatorname{ord}(M) = n$. Thus $-k \leqq \operatorname{ord}(M) < k$. We call the integer the *order* of the matrix M. Since if $\operatorname{Spec} A(\tau) = \operatorname{Spec} A_0$, then there exists a nonsingular upper-triangular matrix $U(\tau)$ which satisfies the matrix equation $A(\tau)U(\tau) = U(\tau)A_0$ [6, p. 219]. Here $U(\tau)$ is unique up to a diagonal factor. Since A_0 is upper-Hessenberg (5-b), $A(\tau)$ is so, namely, $\operatorname{ord}(A(\tau)) = 1$. Thus the matrices defined by (12) are band matrices such that $\operatorname{ord}(F_j) = \operatorname{ord}(F_j^*) = j$. Along the same line as in [25, p. 8], we can derive for $0 \leqq i \leqq k-1$,

$$\frac{\partial A^i}{\partial\tau_j} - [F_j, A^i] = \left(\frac{\partial G_i}{\partial\tau_j} - \frac{\partial F_j}{\partial\tau_i} - [F_j, G_i]\right) + \left(\frac{\partial F_j}{\partial\tau_i} - \frac{\partial F_i}{\partial\tau_j} + [F_j, F_i]\right), \tag{14}$$

where $G_i = A^i + F_i$. Note that $(\partial A^i/\partial\tau_j) - [F_j, A^i] = 0$ from (9). Since $\operatorname{ord}(G_i) = 0$, the order of the first term of the right-hand side of (14) is not greater than j and does not depend on i. Whereas the order of the second is $i + j$. This implies that F_j solve (13) for $0 \leqq i \leqq k-1$. It is proved that the system (9) is equivalent to the system (13). The systems (9) and (13) also guarantee the integrability of (10). Thus the complex cyclic–Toda hierarchy is a compatible system. □

From a standard theory in Toda equations [20], the initial value problem of the cyclic–Toda hierarchy can be solved by the QR decomposition

$$\exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right) = Q^{-1}(\tau)R(\tau), \quad Q(\mathbf{0}) = I, \quad R(\mathbf{0}) = I, \tag{15}$$

where Q is unitary and R is upper-triangular. Let us suppose that $Q(\tau)$ does not depend on τ_0. Since $\exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right)$ is nonsingular, there always exist unique C^∞-factors Q and R. Flow of the hierarchy on $\mathbb{C}^{k\times k+2k}$ is given by the formula

$$A(\tau) = Q(\tau)A_0Q^{-1}(\tau), \quad B(\tau) = R(\tau)B_0, \quad C(\tau) = Q(\tau)C_0. \tag{16}$$

In order to prove this, observe that $(\partial Q/\partial \tau_j)\cdot Q^{-1}$ are skew-Hermitian which appear in derivatives of both (15) and (16). Note that $A(\tau)$ is an isospectral deformation of A_0, namely,

$$\det(zI - A(\tau)) = q(z) \tag{17}$$

for every $\tau \in \mathbb{C}^k$. The next proposition plays a crucial role in our parametrization of $\mathscr{K}_q(k)$ in terms of the flow (16) on $\mathbb{C}^{k\times k+2k}$.

Proposition 2. *If* $\{A_0, B_0, C_0\}$ *is a cyclic triplet, then* $\{A(\tau), B(\tau), C(\tau)\}$ *given by* (16) *is also a cyclic triplet for every* $\tau \in \mathbb{C}^k$.

Proof is carried out by a straightforward calculation as follows;

$$(B\,AB \cdots A^{k-1}B) = Q\exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right)\cdot\left(B_0\,A_0B_0 \cdots A_0^{k-1}B_0\right),$$

$$(C\,A^*C \cdots (A^*)^{k-1}C) = Q(C_0\,A_0^*C_0 \cdots (A_0^*)^{k-1}C_0). \quad \square$$

It is shown from Proposition 2 that the complex cyclic–Toda hierarchy induces a flow on the space $\mathrm{rat}^{\mathbb{C}}(k)$ defined by

$$f(z) = C_0^*(zI - A_0)^{-1}B_0 \to f(z;\tau) = C^*(\tau)(zI - A(\tau))^{-1}B(\tau). \tag{18}$$

Furthermore, we see from (17) that if $f(z) \in \mathrm{rat}_q^{\mathbb{C}}(k)$, then $f(z;\tau) \in \mathrm{rat}_q^{\mathbb{C}}(k)$ for every $\tau \in \mathbb{C}^k$. Define a space of cyclic vectors B_0 of A_0

$$\omega_{A_0} = \{W \mid W \in \mathbb{C}^{k\times 1}, \mathrm{rank}\,(W\,A_0W \cdots A_0^{k-1}W) = k\}. \tag{19}$$

Since $f(z;\tau) = C_0^*(zI - A_0)^{-1}\left\{\exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right)\right\}B_0$ and $\left\{\exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right)\right\}B_0 \in \omega_{A_0}$, the cyclic–Toda hierarchy induces the linear flow

$$B_0 \to \left\{\exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right)\right\}B_0 \tag{20}$$

on the space ω_{A_0}. This aspect will be discussed again in the end of this paper.

Finally in this section, we define the space $\mathrm{Sol}(k)$ of C^∞-solutions of the $k \times k$ complex cyclic–Toda hierarchy (9) and (10). For a given $q(z)$ let $\{A_0, B_0, C_0\}$ be a set of matrices of the form (5-b) such that (6) and $\det(zI - A_0) = q(z)$. Let $\mathrm{sol}_q(k)$ be the space of matrices $\{A(\tau), B(\tau), C(\tau)\} \in \mathbb{C}^{k\times k} \times \mathbb{C}^{k\times 1} \times \mathbb{C}^{k\times 1}$, C^∞ in τ, satisfying (9) and (10) for any such initial values $\{A_0, B_0, C_0\}$. We write the quotient of $\mathrm{sol}_q(k)$ by the free $U(1)$ action $\{A_0, B_0, C_0\} \to \{A_0, \exp(i\theta I)B_0, C_0\}$ $(\theta \in \mathbb{R})$ on the space of initial values as $\mathrm{Sol}_q(k) = \mathrm{sol}_q(k)/\sim$. The space Sol_q is parametrized by 2k − 1 real parameters. Now set

$$\mathrm{Sol}(k) = \bigcup_{q \in \mathbb{C}^k} \mathrm{Sol}_q(k), \tag{21}$$

where the right-hand side denotes the union with respect to every $q = (q_0, \ldots, q_{k-1})$ of $\mathbb{C}^k$.

3. Proof of Main Theorem

Let $q(z)$ be a fixed monic polynomial over $\mathbb{C}$ of degree k. As it was said in Sect. 2, we can express any element $f(z)$ of $\mathrm{rat}_q^{\mathbb{C}}(k)$ as $f(z) = C_0^*(zI - A_0)^{-1}B_0$, where $\det(zI - A_0) = q(z)$. If we identify B_0 with $\exp(i\theta I)B_0$, $\theta \in \mathbb{R}$, we obtain an element $f(z)$ of $\mathrm{Rat}_q^{\mathbb{C}}(k) = \mathrm{rat}_q^{\mathbb{C}}(k)/\sim$ which corresponds to an equivalence class $[\{A_0, B_0, C_0\}]$ of cyclic triplets. The fact that both the space $\mathscr{K}_q(k) \cong \mathrm{Rat}_q^{\mathbb{C}}(k)$ of $SU(2)$ k-monopoles and the solution space $\mathrm{Sol}_q(k)$ of $k \times k$ cyclic–Toda hierarchy are described by $2k-1$ real parameters is not accidental. As we shall see, $\mathscr{K}_q(k)$ can be identified with $\mathrm{Sol}_q(k)$.

We come to the basic result.

Proposition 3. *The following two spaces are homeomorphic to each other;*

(a) *the subspace* $\mathscr{K}_q(k)$ *of the moduli space of* $SU(2)$ *k-monopoles,*
(b) *the solution space* $\mathrm{Sol}_q(k)$ *of* $k \times k$ *complex cyclic–Toda hierarchy for initial value* $[\{A_0, B_0, C_0\}]$ *such that* $\det(zI - A_0) = q(z)$.

Proof of (a)$\to$(b). Recall that $\mathscr{K}_q(k) \cong \mathrm{rat}_q^{\mathbb{C}}(k)/\sim$. Any rational function $f(z) = (p(z)/q(z))$ of $\mathrm{rat}_q^{\mathbb{C}}(k)$ always admits a unique factorization (5-a), where $\det(zI - A_0) = q(z)$. Set $\tau \in \mathbb{C}^k$ and $G(\tau) = \exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right)$. Since $G(\tau)$ is nonsingular and in class C^∞, there always exist the unique C^∞-factors $Q(\tau)$ and $R(\tau)$ of the QR decomposition (15), $G(\tau) = Q^{-1}(\tau)R(\tau)$, $Q(\mathbf{0}) = I$ and $\mathbb{R}(\mathbf{0}) = I$. The parameter τ_0 appears only in $R(\tau)$. The factor $Q(\tau)$ gives rise to a C^∞-solution $A(\tau)$ such that $\det(zI - A(\tau)) = q(z)$ and $A(\mathbf{0}) = A_0$ via the first of (16). The remainders of (16) give C^∞-solutions of (10), where $B(\mathbf{0}) = B_0$ and $C(\mathbf{0}) = C_0$. Thus we have a surjection from $\mathrm{rat}_q^{\mathbb{C}}(k)$ to $\mathrm{sol}_q(k)$. Let us introduce the equivalence relation $f(z) \sim e^{i\theta} f(z)$, $\theta \in \mathbb{R}$. Then the resulting equivalence class denoted by $[f(z)]$ determines an initial triplet $[\{A_0, B_0, C_0\}]$. Then for any $[f(z)] \in \mathrm{Rat}_q^{\mathbb{C}}(k)$ there exists a flow of the cyclic–Toda hierarchy corresponding to the initial value $[\{A_0, B_0, C_0\}]$. We have obtained a surjection α_q from $\mathrm{Rat}_q^{\mathbb{C}}(k)$ to $\mathrm{Sol}_q(k)$ of class C^∞. It is to be noted that the flow induces a flow

$$[f(z)] \to [f(z;\tau)] = C_0^*(zI - A_0)^{-1} \exp\left(\mathrm{Re}(\tau_0)I + \sum_{j=1}^{k-1} \tau_j A_0^j\right) B_0 \tag{22}$$

on $\mathrm{Rat}_q^{\mathbb{C}}(k)$. Clearly, $[f(z;\mathbf{0})] = [f(z)]$.

Proof of (b)$\to$(a). Let $A(\tau)$ be an arbitrary C^∞-solution of (9) for an initial triplet $[\{A_0, B_0, C_0\}]$ of the form (5-b) satisfying (6) and $\det(zI - A_0) = q(z)$. Let us consider the system $(0 \leqq j \leqq k-1)$ of PDEs

$$\frac{\partial Q(\tau)}{\partial \tau_j} = F_j Q(\tau), \quad Q(\mathbf{0}) = I, \tag{23}$$

where F_j are defined in (12). This system is integrable from Proposition 1. Noting that F_j are analytic and bounded under the standard norm for (12), we see that there exists a unique C^∞-solution $Q(\tau)$ for every $\tau \in \mathbb{C}^k$. Since F_j are skew–Hermitian and $F_0 = 0$, $Q(\tau)$ is unitary and does not depend on τ_0. Multiplying $A(\tau)$ to (23) from the left and using $(\partial A/\partial \tau_j) = [F_j, A]$, we obtain $(\partial(AQ)/\partial \tau_j) = F_j AQ$. On the other hand, $(\partial Q/\partial \tau_j)A_0 = F_j Q A_0$. Since $A_0 Q(\mathbf{0}) = Q(\mathbf{0})A_0$, these equations yield

$$A(\tau)Q(\tau) = Q(\tau)A_0 \tag{24}$$

for every $\tau \in \mathbb{C}^k$. We have shown that for any $A(\tau)$ there exists a unique C^∞-solution $Q(\tau)$ of (23) satisfying (24).

Next let us consider the integrable system $(0 \leqq j \leqq k-1)$

$$(\partial R(\tau)/\partial \tau_j) = (A^j + F_j)R(\tau), \quad R(\mathbf{0}) = I, \tag{25}$$

which have a unique solution being nonsingular and upper-triangular. Along the same way as above, we see that the initial condition $R(\mathbf{0}) = I$ guarantees

$$A(\tau)R(\tau) = R(\tau)A_0. \tag{26}$$

Note that $R(\tau)$ takes the form $R(\tau) = \exp(\tau_0 I)U(\tau)$, where $U(\tau)$ is a nonsingular and upper-triangular matrix, independent of τ_0, such that $A(\tau)U(\tau) = U(\tau)A_0$ and $U(\mathbf{0}) = I$. Here $U(\tau)$ is a unique solution of the matrix equation which appears in the proof of Proposition 1.

Now set

$$G(\tau) = Q^{-1}(\tau)R(\tau). \tag{27}$$

Taking the derivatives of (27) and using (23)–(25), we have

$$Q(\tau)\frac{\partial G(\tau)}{\partial \tau_j}R^{-1}(\tau) = A^j(\tau) = Q(\tau)A_0^j Q^{-1}(\tau).$$

This implies that $G(\tau)$ holds $(\partial G/\partial \tau_j) = A_0^j G$ with $G(\mathbf{0}) = I$. Consequently, we derive

$$G(\tau) = \exp\left(\sum_{j=0}^{k-1} \tau_j A_0^j\right). \tag{28}$$

Set $B(\tau) = R(\tau)B_0$ and $C(\tau) = Q(\tau)C_0$. It is easy to see that $B(\tau)$ and $C(\tau)$ satisfy (10) for the initial value $B(\mathbf{0}) = B_0$ and $C(\mathbf{0}) = C_0$, respectively. Let $f(z;\tau)$ be a rational function defined by $f(z;\tau) = C^*(\tau)(zI - A(\tau))^{-1}B(\tau)$. Then from Proposition 2 we see $f(z;\tau) \in \mathrm{rat}_q^{\mathbb{C}}(k)$ for every $\tau \in \mathbb{C}^k$. Let us introduce the equivalence class $[\{A_0, B_0, C_0\}]$ of initial values by $\{A_0, B_0, C_0\} \sim \{A_0, \exp(i\theta I)B_0, C_0\}$ for $\theta \in \mathbb{R}$. This induces the relation $\{A(\tau), B(\tau), C(\tau)\} \sim \{A(\tau), \exp(i\theta I)B(\tau), C(\tau)\}$ for the cyclic–Toda flow. Equivalently, this amounts to $R(\tau) \sim \exp(i\theta I)R(\tau)$ for fixed B_0. For each $\{A(\tau), B(\tau), C(\tau)\}$, $\tau \in \mathbb{C}^k$, we then obtain a unique rational function

$$[f(z;\tau)] = C_0^*(zI - A_0)^{-1}\exp\left(\mathrm{Re}(\tau_0)I + \sum_{j=1}^{k-1}\tau_j A_0^j\right)B_0$$

of $\mathrm{Rat}_q^{\mathbb{C}}(k)$. See (22). It is proved that the mapping $\alpha_q: \mathrm{Rat}_q^{\mathbb{C}}(k) \to \mathrm{Sol}_q(k)$ is an injection of class C^∞.

We have observed that there exists a bijection $\alpha_q: \mathrm{Rat}_q^{\mathbb{C}}(k) \to \mathrm{Sol}_q(k)$ which is C^∞ in τ, namely, $\mathrm{Rat}_q^{\mathbb{C}}(k)$ is diffeomorphic to $\mathrm{Sol}_q(k)$. Moreover, it follows that $\mathrm{Rat}_q^{\mathbb{C}}(k)$ is homeomorphic to the subspace $\mathscr{K}_q(k)$ of $SU(2)$ k-monopoles from the Nahm–Hitchin–Donaldson theorem. These facts imply Proposition 3. □

We now come to a position to prove the Theorem. The moduli space $\mathscr{K}(k) \cong \mathrm{Rat}^{\mathbb{C}}(k)$ is decomposed into a union of disjoint connected subsets $\mathscr{K}_q(k) \cong \mathrm{Rat}_q^{\mathbb{C}}(k)$, where $q \in \mathbb{C}^k$. Indeed, it is easy to see $\pi_0(\mathrm{Rat}_q^{\mathbb{C}}(k)) \simeq \{0\}$. Thus $\mathscr{K}(k)$ admits a foliation of codimension $2k$ with the leaves $\mathscr{K}_q(k)$. The solution space $\mathrm{Sol}(k)$ of cyclic–Toda hierarchy also admits codimension-$2k$ foliation whose leaves are $\mathrm{Sol}_q(k)$. Proposition 3 asserts that each $\mathscr{K}_q(k)$ is homeomorphic to $\mathrm{Sol}_q(k)$. Thus there exists a mapping $\alpha \circ \beta: \mathscr{K}(k) \to \mathrm{Sol}(k)$ which preserves the foliated structures, where the homeomorphism $\beta: \mathscr{K}(k) \to \mathrm{Rat}(k)$ is due to the Nahm–Hitchin–Donaldson theorem. Hence $\mathscr{K}(k)$ is homeomorphic to $\mathrm{Sol}(k)$. This proves the main theorem. □

Finally in this section we consider the generic case where $q(z)$ has k distinct roots ζ_j. Any rational function $f(z)$ of this class admits the expansion $f(z) = \sum_{j=1}^{k} \exp(\gamma_j)/(z - \zeta_j)$. If the points ζ_j in $\mathbb{C}$ are very far apart, then $f(z)$ parametrizes Taubes' generic k-monopoles [21, 22] based on the points $(\zeta_j, -\mathrm{Re}(\gamma_j)/2)$ on $\mathbb{C} \times \mathbb{R}$. In this case, the space $\mathrm{rat}_q^{\mathbb{C}}(k)$ is clearly diffeomorphic to $\mathbb{C}^k$ and the foliation is a trivial fibration. We have observed from (22) that the cyclic–Toda flow leaves invariant the position ζ_j on $\mathbb{C}$ and moves the position $-\mathrm{Re}(\gamma_j)/2$ on $\mathbb{R}$ and the phase angle $\mathrm{Im}(\gamma_j)$.

Manton [12] pointed out that a geodesic flow on the moduli space describes the low-energy scattering of monopoles. To carry out this program Atiyah and Hitchin introduced a metric on the space $\mathscr{K}^0(2) = \tilde{\mathscr{K}}(2)/(S^1 \times \mathbb{R}^3)$ [1] and investigated the dynamics of 2-monopoles [2]. This geodesic flow moves the position of poles of rational functions in $\mathrm{rat}^{\mathbb{C}}(2)$ and is not integrable [24].

On the other hand, the cyclic–Toda hierarchy is (completely) integrable and describes mainly a phase shift of k-monopoles. The cyclic–Toda hierarchy is an isospectral deformation equation of cyclic matrices which flows 'downhill' toward upper-triangular matrices with diagonal entries consisting of eigenvalues. In this sense the cyclic–Toda hierarchy can be regarded as a gradient flow on the space of cyclic matrices. This aspect of the usual Toda equation and the cyclic–Toda hierarchy is due to [13] and [15], respectively. Thus the cyclic–Toda hierarchy is in sharp contrast with the geodesic flow which has been studied by many authors.

4. Discussion

The topology of rational functions over $\mathbb{C}$ has been studied originally by Segal [18]. He also proved that $\mathrm{rat}^{\mathbb{C}}(k)$ is homeomorphic to the space $\mathrm{rat}^{\mathbb{R}}(k,k)$ of rational functions over $\mathbb{R}$ of degree $2k$ and the Cauchy index 0, namely,

$$\mathrm{rat}^{\mathbb{C}}(k) \cong \mathrm{rat}^{\mathbb{R}}(k,k). \tag{29}$$

The space $\mathrm{rat}^{\mathbb{R}}(k,k)$ is one of the $2k+1$ connected components of $\mathrm{rat}^{\mathbb{R}}(2k)$ being labelled by the Cauchy index $\{-2k, -2k+2, \ldots, 2k\}$.

Recently, the author [15] solves the problem of parametrization of the space $\mathrm{rat}_q^{\mathbb{R}}(n)$ of rational functions of degree n with fixed denominator $q(z)$ in terms of $n \times n$ real cyclic–Toda hierarchy. If $q(z)$ has r real distinct real roots, then there exist 2^r connected components of $\mathrm{rat}_q^{\mathbb{R}}(n)$. It is proved that the flow of real cyclic–Toda hierarchy can be identified with one of the connected components distinguished by the choice of initial values, and then each component is diffeomorphic to a cylinder $\mathbb{R}^{k-m} \times T^m$ $(0 \leqq m \leqq k-1)$ in terms of the invariant-tori theorem. Here the flow is parametrized by $\tau \in \mathbb{R}^n$ and leaves invariant the Cauchy index. It should be remarked that the original (finite nonperiodic) Toda equation is a special member of the $n \times n$ real cyclic–Toda hierarchy where the denominator has n real distinct roots. In this case the flow describes an isospectral deformation of $n \times n$ Jacobi matrices and should be called the Jacobi–Toda flow. The associated rational functions have the Cauchy index $-n$ and are in the connected component $\mathrm{rat}^{\mathbb{R}}(n, 0)$ being diffeomorphic to $\mathbb{R}^{2n}$.

Segal's homeomorphism (29) enables us to "embed" the $k \times k$ complex cyclic–Toda hierarchy into the $2k \times 2k$ real one. The resulting space $\mathrm{rat}^{\mathbb{R}}(k, k)$ should have rather different topological properties than the space $\mathrm{rat}^{\mathbb{R}}(2k, 0)$ of the Jacobi–Toda hierarchy. For example, $\mathrm{rat}^{\mathbb{R}}(1,1) \cong \mathbb{R}^3 \times S^1$ and $\mathrm{rat}^{\mathbb{R}}(2,0) \cong \mathbb{R}^4$. Since $\tilde{\mathscr{K}}(1) \cong \mathrm{rat}^{\mathbb{C}}(1)$, this implies that a 1-monopole is determined by a point in $\mathbb{R}^3$ and a phase angle.

Finally let us consider a linearization of the complex cyclic–Toda hierarchy (9) and (10). As was shown in Sect. 2, the hierarchy induces the linear flow (20) on the space w_{A_0} of cyclic vectors. Define $\mathscr{W}_{A_0} = w_{A_0}/\sim$, where $W \sim \exp(i\theta I)W$ for $\theta \in \mathbb{R}$ and

$$\mathscr{H}_{A_0} = \hbar_{A_0}/\sim, \quad \hbar_{A_0} = \left\{ H_p | H_p = \sum_{j=0}^{k-1} p_j A_0^j \colon \text{nonsingular}, \ p \in \mathbb{C}^k \right\}, \tag{30}$$

where $H_p \sim \exp(i\theta I)H_p$ for $\theta \in \mathbb{R}$. It is easy to see that $\mathscr{H}_{A_0}$ acts freely on $\mathscr{W}_{A_0}$; $\mathscr{H}_{A_0} \times \mathscr{W}_{A_0} \to \mathscr{W}_{A_0}$ by $(H_p, W) \to H_p W$. Thus $\mathscr{W}_{A_0}$ is diffeomorphic to $\mathscr{H}_{A_0}$. On the other hand, by the spectral mapping theorem [6, p. 115] $\sum_{j=0}^{k-1} p_j A_0^j$ is nonsingular if and only if $p(\zeta_j) \neq 0$ for any root ζ_j of $\det(zI - A_0) = q(z)$, where $p(z) = p_{k-1} z^{k-1} + \dots + p_0$. Thus each $H_p \in \hbar_{A_0}$ uniquely gives a rational function $(p(z)/q(z))$ of $\mathrm{rat}_q^{\mathbb{C}}(k)$ and vice versa. Moreover, we can see that $\mathscr{H}_{A_0}$ is diffeomorphic to $\mathrm{Rat}_q^{\mathbb{C}}(k)$. It is concluded that $SU(2)$ monopoles can be associated with the linear flow

$$[B_0] \to \left\{ \exp\left(\mathrm{Re}(\tau_0) I + \sum_{j=1}^{k-1} \tau_j A_0^j \right) \right\} B_0 \tag{31}$$

on the space $\mathscr{W}_{A_0}$ of cyclic vectors through the cyclic–Toda hierarchy.

The modern theory of soliton equations has its origin in the observation in [17] that the Schur polynomials (functions) completely parametrize rational solutions of the KP hierarchy. It is also known that the tau-function (a generating function of solutions) can be expressed as an infinite linear combination of the Schur polynomials. See also [19]. In the present case, we can introduce the tau-function of the cyclic–Toda hierarchy in the form of a linear combination of Toeplitz determinants of $h_j(\tau)$, where $\sum_{j=0}^{k-1} h_j(\tau) A_0^j = \exp\left(- \sum_{j=0}^{k-1} \tau_j A_0^j \right)$, $h_0(\tau) = 1$ and

$h_j(\tau) = 0$ for $j < 0$ and $j \geqq k$. These Toeplitz determinants can be regarded as Schur polynomials for the present case. Here we use the fact that any matrix that commutes with A_0 can be expressed linearly by the matrices $\{I, A_0, \ldots, A_0^{k-1}\}$, which follows from a consequence of the Cayley–Hamilton theorem [6, p. 223]. Combining this observation with the main theorem of this paper, we see that the moduli space of $SU(2)$ monopoles can be naturally characterized by the Schur polynomials. The details of this aspect will appear elsewhere.

Acknowledgement. The author would like to thank T. Iwai for helpful conversations.

References

1. Atiyah, M. F., Hitchin, N. J.: Low energy scattering of non-Abelian monopoles. Phys. Lett. **107B**, 539–543 (1985); Low-energy scattering of non-Abelian magnetic monopoles. Phil. Trans. R. Soc. Lond. **A315**, 459–469 (1985)
2. Atiyah, M. F., Hitchin, N. J.: The geometry and dynamics of magnetic monopoles. Princeton, NJ: Princeton University Press 1988
3. Boyer, C. P., Mann, B. M.: Monopoles, non-linear σ models, and two-fold loop spaces. Commun. Math. Phys. **115**, 571–594 (1988)
4. Donaldson, S. K.: Nahm's equations and the classification of monopoles. Commun. Math. Phys. **96**, 387–407 (1984)
5. Ganoulis, N., Goddard, P., Olive, D.: Self-dual monopoles and Toda molecules. Nucl. Phys. **B205**, 601–636 (1982)
6. Gantmacher, F. R.: The theory of matrices vol. I. New York: Chelsea 1959
7. Griffiths, P. A.: Linearizing flows and a cohomological interpretation of Lax equations. Am. J. Math. **107**, 1445–1483 (1985)
8. Hitchin, N. J.: Monopoles and geodesics. Commun. Math. Phys. **80**, 579–602 (1982)
9. Hitchin, N. J.: On the construction of monopoles. Commun. Math. Phys. **89**, 145–190 (1983)
10. Kalman, R. E.: Mathamatical description of linear dynamical systems. SIMA J. Control **1**, 152–192 (1963)
11. Leznov, A. V., Saveliev, M. V.: Representation theory and integration of nonlinear spherically symmetric equations to gauge theories. Commun. Math. Phys. **74**, 111–118 (1980)
12. Manton, N. S.: A remark on the scattering of BPS monopoles. Phys. Lett. **110B**, 54–56 (1982)
13. Moser, J.: Finitely many mass points on the line under the influence of an exponential potential—An integrable system. In: Lecture Notes in Physics, Vol. **38**, pp. 467–497. Berlin, Heidelberg, New York: Springer 1975
14. Nahm, W.: All self-dual multimonopoles for arbitrary gauge groups. In: Structural elements in particle physics and statistical mechanics. Honerkamp, K. et al. (ed.). pp. 301–310. New York: Plenumm Press 1983
15. Nakamura, Y.: Geometry of rational functions and nonlinear integrable systems. Preprint, 1989
16. Nakamura, Y., Duncan, T. E.: Remarks on the moduli space of $SU(2)$ monopoles and Toda flow, to be published in Lett. Math. Phys.
17. Sato, M., Sato, Y.: Soliton equations as dynamical systems on infinite dimensional Grassmann manifold. In: Lecture Notes in Numerical and Applied Analysis, Vol. **5**, pp. 259–271. Fujita, H. et al. (eds.). Tokyo: Kinokuniya 1982
18. Segal, G.: The topology of spaces of rational functions. Acta Math. **143**, 39–72 (1979)
19. Segal, G., Wilson, G.: Loop groups and equations of KdV type. Publ. Math. I.H.E.S. **61**, 5–65 (1985)
20. Symes, W. W.: Hamiltonian group actions and integrable systems. Physica **1D**, 339–374 (1980); Systems of Toda type, inverse spectral problems, and representation theory. Invent. Math. **59**, 13–51 (1980)
21. Taubes, C. H.: The existence of a non-minimal solution to the $SU(2)$ Yang–Mills–Higgs equations on $\mathbb{R}^3$. Part I. Commun. Math. Phys. **86**, 257–298 (1982); Part II. Commun. Math. Phys. **86**, 299–320 (1982)

22. Taubes, C. H.: Monopoles and maps from S^2 to S^2; the topology of the configuration space. Commun. Math. Phys. **95**, 345–391 (1984)
23. Taubes, C. H.: Min-max theory for the Yang–Mills–Higgs equations. Commun. Math. Phys. **97**, 473–540 (1985)
24. Temple-Raston, M.: Closed 2-dyon orbits. Nucl. Phys. **B313**, 447–472 (1989)
25. Ueno, K., Takasaki, K.: Toda lattice hierarchy. In: Advanced Studies in Pure Mathematics, Vol. **4**, pp. 1–95. Tokyo: Kinokuniya 1984

Communicated by H. Araki

Received May 8, 1989; in revised form October 3, 1989