Commun. Math. Phys. 127, 195–224 (1990)

Communications in
Mathematical
Physics
© Springer-Verlag 1990

Braiding Matrices, Modular Transformations and Topological Field Theories in 2+1 Dimensions

Miao Li[1,]* and Ming Yu[2]

[1] International School for Advanced Studies, I-34014 Trieste, Italy
[2] International Center for Theoretical Physics, I-34100 Trieste, Italy

Abstract. Relations between $3D$ topological field theories and rational conformal field theories are discussed. In the former framework, we can define the generalized Verlinde operators. Using these operators, we find modular transformations for conformal blocks of one point functions and two point functions on the torus. The result is generalized to higher genus. The correctness of our formulae is illustrated by some examples. We also emphasize the importance of the fusion algebra.

1. Introduction

The classification of conformal field theories (CFT's) is an important issue in recent research [1–24]. There is evidence suggesting that several areas, including CFT's, integrable lattice models, three dimensional topological field theories and the link polynomials, are related by an underlying link, namely, the quantum groups. Witten [26] conjectured that quantum groups should arise naturally in the topological Chern-Simons theories (TCST). Alvarez-Gaumé et al. [22] found that the polynomial equations [15] can be neatly encoded in the structure of a quantum group [32, 33]. However, some important questions remain unanswered. For example, why should quantum group symmetries emerge from the RCFT's only as a secondary phenomenon rather than the first principle in the theory? The profound meaning of the quantum group symmetry can only be understood when such issues are settled.

Further studies on the structure of CFT's may help us catch the key point. In our previous paper we showed that one can express modular transformations in higher genus in terms of braiding matrices. The tool is the TCST. In this paper, we do this explicitly for modular transformations $S(j)$'s for one-point functions on the torus. We find that the $S(j)$'s are not independent data, and that the polynomial equations involving $S(j)$ put further restrictions on braiding matrices. This result

---

* Addresses after October 1, 1989, Institute of Theoretical Physics, Academia Sinica, Beijing, P. R. China

can also be obtained in the framework of RCFT. With $S(j)$, it is possible to relate our formulas in [31] to those obtained by Moore and Seiberg [15]. The correctness of our formula for $S(j)$ is illustrated by some examples. By studying polynomial equations involving $S(j)$, we can get new constraints on conformal field theories. We show explicitly that for the fusion algebra $\phi \cdot \phi = 1 + n\phi$, $n \geqq 2$, there are no corresponding modular invariant theories.

Our paper consists of two parts. The first is concerned with the finite topological field theories in $2+1$ dimensions. Giving an appropriate definition for FTFT, we derive many familiar concepts in RCFT. The second part is devoted to the study of modular transformations. Here we note that if we work with the help of FTFT, everything becomes simpler.

Our work on $S(j)$ may provide some incentive for further studies of the polynomial equations, namely, the proof of the reconstruction theorem. Indeed there are two conjectures related to this problem. The first one states that any RCFT can be obtained from some topological Chern-Simons theory [29]. Another is that all three dimensional topological field theories (TFT's) are TCST's [27]. These two conjectures are not independent, provided one believes that there is a one-to-one relationship between the set of RCFT's and that of three dimensional TFT's. In the next section, we give an appropriate definition of finite TFT's and show how this definition leads to all familiar concepts in RCFT's. In Sect. 3, the generalized Verlinde operators are defined. By the use of the Verlinde operators, we derive the modular transformations for one point functions and two point functions on the torus. The formula for $S(j)$ can also be obtained in the framework of RCFT. The fact that the same formula can be derived in both FTFT and RCFT again suggests that there is a one to one relationship between FTFT and RCFT. We stress that the polynomial equations concerning $S(j)$ put further restrictions on braiding matrices. Modular transformations in higher genus are discussed in Sect. 4. We derive a generalized fusion algebra, which is not commutative. This algebra should be related to some quantum group. However, we postpone our study on this for the future. We also discuss the automorphism among the braiding matrices. This guarantees the modular invariance of the off-diagonal theories. Some examples are given in Sect. 5 to demonstrate that our formula actually works. It can also be shown that the fusion algebra plays an important role in the classification of the RCFT's. We give our conclusion for this paper in Sect. 6.

2. Topological Field Theories and RCFT's

In this section, we discuss the relations between the TFT's and the RCFT's. This section is independent of the following sections.

Witten's study on the $2+1$ dimensional Chern-Simons theories [26] shows that one can relate these theories to the corresponding WZW models. The argument is the following. When one quantizes the Chern-Simons action canonically on the space Σ, a Riemann surface, one finds a compact phase space. The symplectic form, together with the complex structure endowed from a conformal structure on Σ, gives a Kähler polarization. Thus, the wave functions in the Hilbert space depend on the moduli parameters explicitly [28]. All the Hilbert spaces for different Kähler polarization constitute a projectively flat vector bundle over the moduli space of Σ. The flatness is due to the fact that the expectation values of the observables are topologically invariant, as we shall see later. We conclude that this bundle is in fact the bundle of the conformal blocks in the WZW model on the Riemann surface Σ. Recently, this has been shown explicitly in [28].

There are several axiom systems for a RCFT. The original one can be found in the classic work of Belavin, Polyakov, and Zamolodchikov [1]. Another is by Segal [24], which we will not follow here. The pure geometric one is by Friedan and Shenker [10], in which one deals with the flat bundles over compactified moduli spaces. Most recently, Moore and Seiberg [15] formulated the fourth, which is based on the polynomial equations. This axiom system may be referred to as a pure algebraic one. To establish a one-to-one correspondence between the set of the RCFT's and that of the FTFT's, one starts from a certain axiom system or a mixing of them. Hereafter we shall outline the idea which will be proven useful in studying the relations between the two categories of the RCFT's and FTFT's. The main point can be proven working in the TCST's [28], although for a general FTFT, one needs more precise definitions for what we mean here. We start from the geometric definition of a FTFT, and show that it leads to the algebraic axiom system by Moore and Seiberg. This means that a FTFT provides a bridge between the two axiom systems.

Inspired by the Chern-Simons theory, we define a finite topological quantum field theory by three axioms. 1. Upon canonical quantization on manifold $\Sigma \times R$, the phase space is compact and of finite dimension. This ensures that the Hilbert space is finite dimensional. Since there is no natural polarization, we assume that *there is always a Kähler polarization associated to a conformal structure on* Σ. The polarization depends holomorphically on the conformal structure. 2. The topologically invariant observables are in one-to-one correspondence with the states in the Hilbert spaces. 3. The expectation values of the observables are topologically invariant. These three axioms are strong enough to define the theory completely.

Let us start from the genus one case. Here we have a finite dimensional Hilbert space in which one can choose a basis corresponding to the characters of the primary fields in the CFT. The wave functions are functions of holomorphic parameters of the Teichmüller space of the torus. This is demonstrated explicitly in [28] for the TCST. For the holomorphic vector bundle over the Teichmüller space we define a parallel transportation. Consider a three manifold bounded by two copies in Σ_1, the torus. For simplicity let $M_3 = \Sigma_1 \times I$, where I is the interval [0, 1]. Suppose that the two boundaries are endowed with different conformal structures τ and τ' (Fig. 1). In the canonical quantization, we have two Hilbert spaces $\mathscr{H}_\tau$ and $\mathscr{H}_{\tau'}$ associated to the corresponding boundaries. Now the parameter t in I characterizes a series of conformal structures interpolated between τ and τ'. This

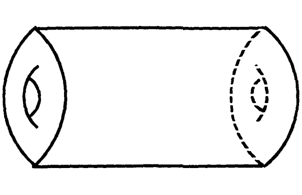


Fig. 1. The manifold $\Sigma_1 \times I$. We assign to each boundary a conformal structure. The intermediate slices constitute a series of conformal structures interpolating between the two conformal structures on the boundaries

describes a path in the Teichmüller space connecting τ and τ'. The symplectic form resulting from the action depends on t, as we assumed in Axiom 1. Thus the path integral depends on the series of conformal structures which we choose. Given a state $\psi(\tau)$ in $\mathscr{H}_\tau$, by the path integral, we obtain a state $\psi(\tau')$ in $\mathscr{H}_{\tau'}$. If we denote collectively the fields in the theory by ϕ, then

$$\psi(\tau') = \int \exp(iS)\,[d\phi]\psi(\tau)\,. \tag{2.1}$$

This was discussed for the TCST in [28] and will be exploited further in [30]. By invoking the geometric quantization and its relation to the path integral, we can use Eq. (2.1) to define a connection of the holomorphic flat bundle on the Teichmüller space. Elitzur et al. [28] showed that this connection corresponds precisely to the connection defined by the stress tensor for the Chern-Simons theory. Let us compactify the Teichmüller space. When $\tau \to 0$, the torus tends to be a cycle and the wave function can be replaced by a loop vertex [26, 27]. This is just the Wilson line operator in the TCST (the fact that in FTFT's we can define these line-like vertices suggests that any FTFT is a TCST). Thus, we extend the vector bundle to the compactified Teichmüller space. In taking the limit, we have to do some regularizations. If, instead of letting $\tau \to 0$, we take $\tau \to 1$, we will obtain another vertex by the regularization. These two limits differ by a Dehn twist. So by the regularization, the two vertices differ by a phase factor. We will show this for the TCST in [30].

In the above we showed that there is an operator associated to each state. According to our second axiom, we assume that they are topologically invariant operators. The flatness of the vector bundle is a consequence of this assumption. Suppose we have two paths from the boundary of the Teichmüller space to a conformal structure τ. Each path gives a vector parallelly transported from the boundary. We can take the inner product of the two vectors. The result is given by a path integral over a compact three manifold with the insertion of two loops. (The two loops may link each other.) Continuous deformation of one path does not change the topology of the three manifold. Thus, the inner product remains unchanged. This shows that the connection is flat.

Consider now the modular group acting on the bundle. Let g be an element of the modular group. Under its action $\tau \to g\tau$. By (2.1), a vector at τ is transported to a vector at $g\tau$. Because τ and $g\tau$ represent the same conformal structure, the new vector can be treated as a vector in $\mathscr{H}_\tau$. We can define a vector bundle over the moduli space as the vector bundle over the Teichmüller space modulo the action of the modular group. Usually, the representation of the modular group on the Hilbert space is projective, the bundle over the moduli space is a projectively flat bundle. Let S be the modular transformation under which $\tau \to -\frac{1}{\tau}$. If we choose W_i as a basis of the operators, then S_{ij} is given by the path integral with two linked operators inserted in S^3, as shown in Fig. 2 [26]. Here we assume W_i are normalized such that two parallel lines embedded in $S^2 \times S^1$ along the direction S^1 give a delta function (Fig. 3). Considering Fig. 3, we quantize the system by taking the time along S^1. The space is the sphere with two punctures. The Hilbert space is either one dimensional or zero dimensional. In the former case, the charge on the punctures are conjugate [26].

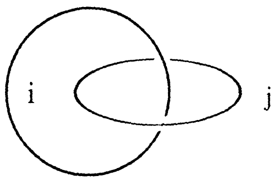


Fig. 2. A link with two components in S^3, the path integral at the presence of such a link is given by the entry $S_{i,j}$ of the modular transformation for the characters on the torus

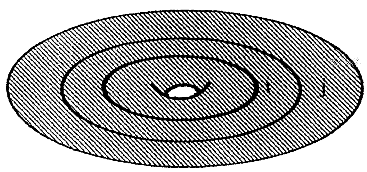


Fig. 3. Two parallel Wilson lines inserted in manifold $S^2 \times S^1$, along the direction of S^1

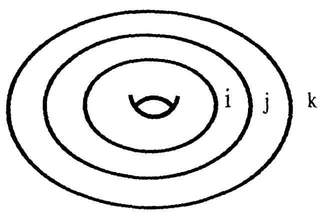


Fig. 4. Three parallel Wilson lines reside in $S^2 \times S^1$. The path integral gives the fusion rule

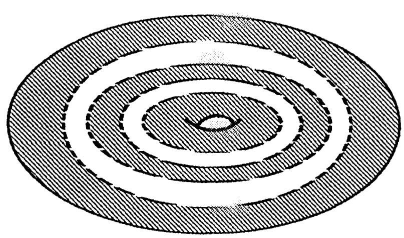


Fig. 5. Three tori removed from $S^2 \times S^1$. Closing this manifold amounts to inserting three Wilson lines along the direction of S^1

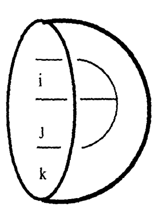


Fig. 6. The basis for the Hilbert space on the three punctured sphere. These are the basic building blocks in the construction of the baryon graphs

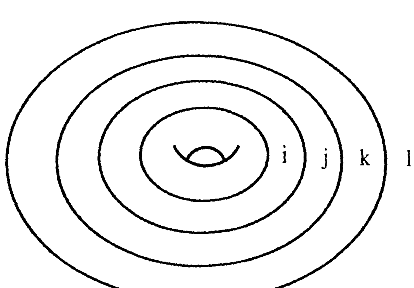


Fig. 7. Four Wilson lines inserted in $S^2 \times S^1$

We are ready to define the fusion and chiral vertices. Consider three line operators embedded in $S^2 \times S^1$ without linking (Fig. 4). The path integral is given by the dimension of the Hilbert space of the sphere with three punctures. We denote it by N_{ij}^k. We thicken each line to a "tubular neighbourhood," namely a solid torus. Removing the solid tori, we have a manifold with these boundaries with each component being a torus. This manifold can be constructed by removing two solid tori from the third one, as shown in Fig. 5. Attaching states $|i\rangle$, $|j\rangle$, and $|k\rangle$ to these boundaries, respectively, the path integral gives N_{ijk}. If we attach states $|i\rangle$ and $|j\rangle$ to the inner boundaries only, the state generated on the outer boundary is $\sum_k N_{ijk}|k\rangle$. (Attaching the state $|k\rangle$ on the outer boundary amounts to taking the inner product $|k\rangle$ with the state $\sum_k N_{ijk}|k\rangle$.) In other words, this defines a map $\mathscr{H} \otimes \mathscr{H} \to \mathscr{H}$.

The vectors in the Hilbert space on the sphere with three punctures correspond to the chiral vertices. We show how the notations introduced in [27] about the baryon diagrams emerge here naturally. Suppose we have three lines in a three manifold which is not closed. We cut a ball in this manifold which meets these three lines. Thus by the path integral we obtain a vector in H_{ijk}, the Hilbert space on the sphere with three punctures. Let $\{|a\rangle\}$ be a basis for H_{ijk}; we chose the manifold by attaching a state $|a\rangle$. We denote abstractly the effect of this operation by a vertex as shown in Fig. 6. We will show later that we can actually denote this by a vertex, namely three lines meet at a point.

Consider four lines embedded in $S^2 \times S^1$ without linking (Fig. 7). We denote the state generated by two lines W_i and W_j inside the solid torus by $|ij\rangle$, which is actually $\sum N_{ij}^k|k\rangle$. The expectation value for Fig. 7 is $\langle l^*k^* \mid ij\rangle = \sum N_{lk}^m N_{ijm}$, where we use l^* to denote the line oriented in the opposite direction of l. Thus we find the dimension of the Hilbert space for the sphere with four punctures. By the vertices constructed before, we can represent the states in H_{ijlk} by diagrams in Fig. 8 or Fig. 9. In fact, by counting the number, we find the right dimension.

We prove now $S_{0,i} > 0$. Let $|a\rangle$ be a state in the Hilbert space H_{ijl}. We normalize this state by requiring $\langle a \mid b\rangle = S_{0,0}\sqrt{n_i n_j n_l}\,\delta_{ab}$, where $n_i = S_{0,i}/S_{0,0}$. This is the

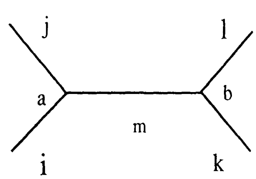


Fig. 8

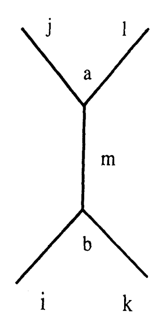


Fig. 9

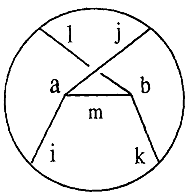
Fig. 8. A basis for the Hilbert space of the four punctured sphere. The vertices are given by the insertions of states on the three punctured sphere

Fig. 9. Another possible basis for the Hilbert space on the sphere with four punctures

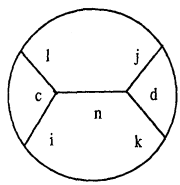
convention used by Witten [27]. Let $l=0$ and the corresponding state be denoted by $|i\rangle$. We have $\langle a\,|\,b\rangle/\langle 0\,|\,0\rangle = S_{0,i}$. If our theory is unitary, namely, every Hilbert space has a positive inner product, then $S_{0,i}>0$. In the same way we can normalize states in H_{ijkl}. If we denote $|m,ab\rangle$ the state in H_{ijkl} given by Fig. 8, then

$$\langle m,a,b\,|\,n,c,d\rangle = \delta_{mn}\delta_{ac}\delta_{bd}\sqrt{n_i n_j n_l n_k}\,S_{0,0}\,.$$

We define the braiding matrices and the fusion matrices [12–15] in accordance with our normalizations.



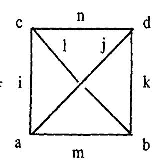
$$= \sum B_{mn}\begin{bmatrix} j & l \\ i & k \end{bmatrix}_{ab}^{cd}$$



, (2.2)

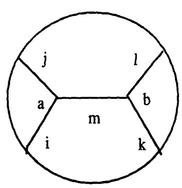
so

$$B_{mn}\begin{bmatrix} j & l \\ i & k \end{bmatrix}_{ab}^{cd} = \frac{1}{S_{0,0}\sqrt{n_i n_j n_l n_k}}$$

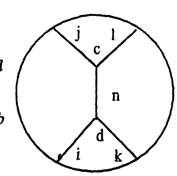


. (2.3)

Next we define the fusion matrices as



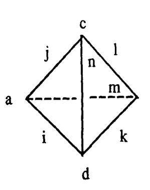
$$= \sum F_{mn}\begin{bmatrix} j & l \\ i & k \end{bmatrix}_{ab}^{cd}$$



, (2.4)

so

$$F_{mn}\begin{bmatrix} j & l \\ i & k \end{bmatrix}_{ab}^{cd} = \frac{1}{S_{0,0}\sqrt{n_i n_j n_l n_k}}$$



, (2.5)

The dotted line indicates that the diagram is not projected to the plane.

We see immediately that B matrices and F matrices are not independent, we deduce directly from comparing the diagrams

$$F_{mn}\begin{bmatrix} j & l \\ i & k \end{bmatrix}_{ab}^{cd} = B_{mn}\begin{bmatrix} j & k \\ i & l \end{bmatrix}_{ab}^{dc} e^{\pi i[h_i+h_l-h_m-h_n]}\varepsilon_{kl}^{m}\varepsilon_{nk}^{i}\,, \tag{2.6}$$

where ε is a sign depending on the coupling type of the chiral vertex. By the symmetry of the tetrahedron, we can obtain more relations.

The polynomial equations about B matrices and F matrices can be obtained by considering the sphere with 5 punctures. These equations are the consistent conditions among the different bases. Consider the pentagon equations as an

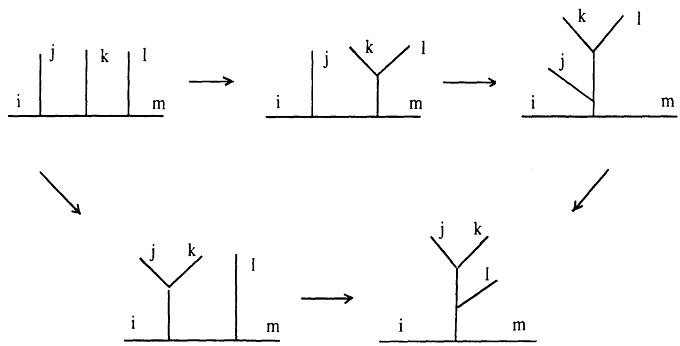


Fig. 10. The pentagon diagram. In the graph preceding an arrow, we remove a ball containing the basis graph corresponding to the four punctured sphere. In the graph after the same arrow, we glue back a ball containing another basis graph

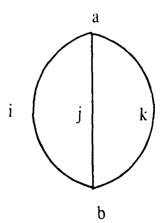


Fig. 11. The baryon graphs, which generate a complete basis for the conformal blocks on genus two Riemann surface

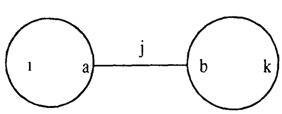


Fig. 12. Another possible basis for the conformal blocks on the genus two Riemann surface

example. In Fig. 10, along each arrow we change the bases for the four punctured sphere accordingly. More precisely, we remove a ball which contains the graph corresponding to the old basis, and glueing back the ball which contains the new graph.

We show that the vertices defined above can be represented by three lines in a ball meeting at a point. To achieve this, we consider Riemann surfaces of higher genus. Without loss of generality, take the genus two case as an example. In (2.1), we replace the states on the torus by the ones on the genus two Riemann surface. Taking a limit of the moduli parameters, one can always shrink the surface to the baryon diagrams in Fig. 11 or Fig. 12. The state in the right-hand side of (2.1) corresponds to an operator associated to the diagram. The state in the left-hand side of (2.1) is obtained by the path integral over the handlebody with the insertion of the operator (Fig. 13). We argue that each line corresponds to a certain W_i known before. If we have now a new line, we can patch two segments of this line to obtain a new loop. Inserting this new loop in the solid torus, we obtain a new state on the torus. But this contradicts our assumption: any state can be generated by the known loops. Thus, the lines in Figs. 11 and 12 can be identified as the ones already presented in genus one case. Now in Fig. 12, we have the vertices which are described by three lines joining at a point. By removing a ball, as shown in Fig. 14,

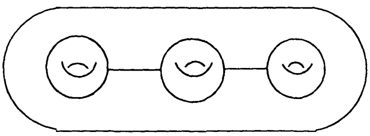


Fig. 13. A basis of graphs, after inserted into the handlebody, generates the basis of the Hilbert space on the Riemann surface

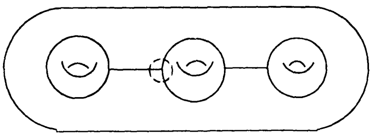


Fig. 14. A ball containing the vertex is removed. The vertices in this way generate all states on the three punctured sphere

we obtain a state on the sphere defined by the vertex, which must be a linear combination of the known states. Conversely, we can glue back a ball which contains a known vertex and obtain a state on the double torus. The completeness of the baryon diagrams then indicate that all the vertices are given by the states on the sphere with three punctures. This proves our statement made before.

In this way we have obtained all the notations appearing in the TCST for a FTFT defined by our three axioms. This formalism needs to be rigorously formulated. We believe that this is possible. Starting from the geometric formulation, we deduce the polynomial equations as the consistent conditions. Therefore the FTFT plays the role as a bridge between different approaches to RCFT's. The fact that the Wilson lines can be defined for every FTFT supports the conjecture that every FTFT is a TCST. If one can prove the statement that any RCFT gives a FTFT (this is very plausible, as can be felt by our discussion), then the conjecture made by Moore and Seiberg is proved. The most important problem is to find all consistent conditions for a FTFT, then compare these to those for a RCFT. And last, we stress that some argument made for FTFT can be applied to the infinite TFT, for example, to the $2+1$ gravity.

3. Modular Transformations for One Point and Two Point Conformal Blocks on the Torus

In our previous paper [31], we derived the expressions of the modular transformations in higher genus in terms of braiding matrices and the modular transformation S of the characters on the torus, where we used the method in TCST theory and the basis shown in Fig. 15. Moore and Seiberg [15] already found the similar expressions in terms of braiding matrices and the modular transformation $S(j)$'s for one point conformal blocks on the torus, where the basis in Fig. 16 was used. Our results indicate that $S(j)$ can be expressed by braiding matrices. We derive $S(j)$ and modular transformations for two point conformal blocks on the torus in this section.

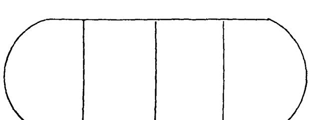


Fig. 15. A basis of the graphs, which can be viewed as a basis of conformal blocks on the Riemann surface

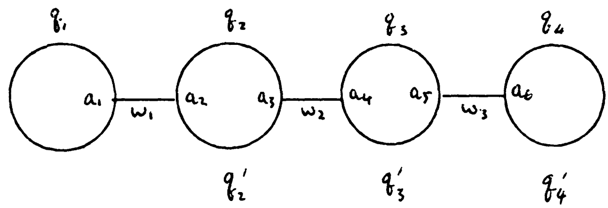


Fig. 16. Another basis for the conformal blocks

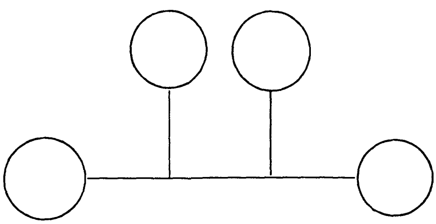


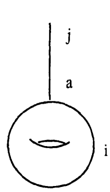
Fig. 17. The third basis for the conformal blocks on the Riemann surface. The modular transformations in this basis can be constructed from the ones for the one-point conformal blocks on the torus and simple duality moves

When our argument given in the last section is applied to the Riemann surface of higher genus, one can construct the generalized characters by the path integrals over the handlebody with insertion of the vertices of the diagrams in Fig. 15 or Fig. 16.

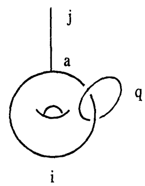
These diagrams can be explicitly constructed in the Chern-Simons theory [27]. One can also use the basis of diagrams in Fig. 17. This set can be constructed by attaching the diagrams for one point conformal blocks on the torus to a tree diagram. Thus, duality and modular invariance of one point functions guarantee the modular invariance in higher genus. The basis in Fig. 16 are constructed by sewing one-point functions and two-point functions. So we derive first the modular transformations for the corresponding conformal blocks.

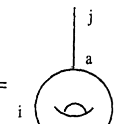
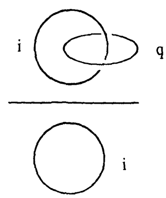
We start by defining the generalized Verlinde operators [11, 31]. Given any closed loop C on the Riemann surface, we associate it with a W_i. Continuously send it inside the handlebody without breaking any line that already exists. The resulting state is denoted as $T_i(C)|\chi\rangle$, where $|\chi\rangle$ is the state generated by the original diagram. The framing of the loop is fixed by taking the outward vectors along the basic cycles a_i and b_i tangent to the Riemann surface. If a loop is a composition in these basic cycles, the framing is given by smoothly connecting the framing of the basic cycles.

The same operators can be defined for Riemann surfaces with punctures. Let us consider the torus with one puncture. On this surface there are two basic cycles a and b. The cycle surrounding the puncture is homotopic to $aba^{-1}b^{-1}$. We first consider the Verlinde operators associated to a. Consider the following basis of the states:

$|i,a\rangle_j =$  . (3.1)

This Hilbert space is denoted by Moore and Seiberg as $\bigoplus_i V^i_{ji}$. Acting the operator $T_q(a)$ on the state results in

$T_q(a)\,|\,i, a\rangle =$ 

$=$   $= \dfrac{S_{q,i}}{S_{0,i}}\,|i, a\rangle_j .$ (3.2)

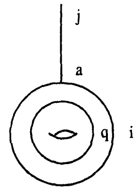
We have used the fact that the expectation values of two linked Wilson loops W_q and W_i in the manifold S^3 is $S_{q,i}$ [26]. Thus, we find the operators $T_q(a)$ diagonalized in this basis. The Dehn twist along the cycle a^{-1} is also diagonalized. The eigenvalues are $\exp(-2\pi i\tilde{h}_i)$, where $\tilde{h}_i = h_i - c/24$ and c is the central charge. We have the following remarkable identity:

$$S(a^{-1}) = \sum_q e^{2\pi i(\tilde{h}_q + \tilde{h}_0)} S_{0,q} T_q(a) . \tag{3.3}$$

To prove this formula we need relation $(ST)^3 = S^2$, namely $STS = T^{-1}ST^{-1}$. Write it explicitly

$$\sum_j S_{i,j} e^{2\pi i \tilde{h}_j} S_{j,k} = S_{i,k} e^{-2\pi i(\tilde{h}_i + \tilde{h}_k)} . \tag{3.4}$$

What is important here is that (3.3) can be generalized to the case of b cycles. Let $S(j)$ be the modular transformation under which $\tau \to -1/\tau$ and $\log z \to \log z/\tau$. Under this transformation, the state $|i, a\rangle_j$ is transformed to $S(j)\,|i, a\rangle_j$. Accordingly, $T_q(b) = ST_q(a)S^{-1}$ and $S(b^{-1}) = SS(a^{-1})S^{-1}$. So (3.3) is also valid when a is replaced by b. To calculate $S(b^{-1})$ we first calculate $T_q(b)$. The action of it on the state $|i, a\rangle_j$ is given by

$T_q(b)\,|i, a\rangle_j =$  (3.5)

The diagram in the above equation can be deformed by using duality transformations as in Fig. 18. At last we get

$$T_q(b)\,|i, a\rangle_j = \sum_{l,b,c} F_{il} \begin{bmatrix} j & l \\ i & q \end{bmatrix}^{bc}_{ac} |l, b\rangle_j , \tag{3.6}$$

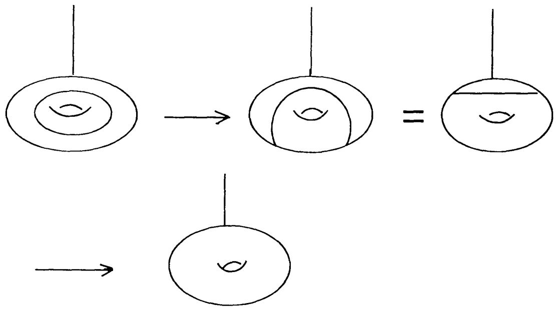
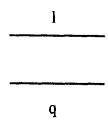
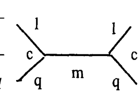


Fig. 18. The Verlinde operator acts on the basis of one-point conformal blocks on the torus. The corresponding graph can be transformed by simple duality moves into a linear combination of the simpler ones

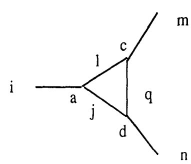
where for simplicity, we have restricted ourselves to the self-conjugate theories. In Fig. 18, we have used the following formulae

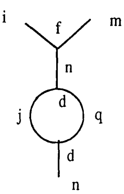


$= \sum_{m,c} \sqrt{\frac{n_m}{n_l n_q}}$


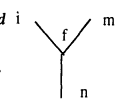
(3.7)

and



$= \sum_{f,n} F_{ln} \begin{bmatrix} i & m \\ j & q \end{bmatrix}_{ac}^{fd}$


(3.8)

$= \sum_{f,n} \sqrt{\frac{n_j n_q}{n_n}}\, F_{ln} \begin{bmatrix} i & m \\ j & q \end{bmatrix}_{ac}^{fd}$


.

We remark that the above equation encodes what is called X operations by the authors of [22], which were used to generate the quantum group.

By using the relations among B and F matrices, we can also write $T_q(b)$ in terms of B matrices. The Dehn twist along b^{-1} is

$$S(b^{-1})_{ia,lb} = \sum_{q,c} e^{2\pi i(\tilde{h}_q + \tilde{h}_0)} S_{0,q} B_{il} \begin{bmatrix} j & q \\ i & l \end{bmatrix}_{ac}^{cb}. \tag{3.9}$$

Knowing $S(b^{-1})$, it is easy to calculate $S(j)$. Since

$$T^{-1}S(b^{-1})T^{-1} = T^{-1}SS(a^{-1})S^{-1}T^{-1} = T^{-1}ST^{-1}S^{-1}T^{-1} = S,$$

so the entries of $S(j)$ are

$$S(j)_{ia,lb} = \sum_{q,c} e^{2\pi i(h_q - h_i - h_l)} S_{0,q} B_{il} \begin{bmatrix} j & q \\ i & l \end{bmatrix}_{ac}^{cb}. \tag{3.10}$$

Now let us consider the constraint on $S(j)$. First, it should satisfy $(S(j)T)^3 = S(j)^2$. Next we find that under $S(j)^2$, the coordinates around the puncture are transformed by ahlf monodromy, so the result under the $S(j)^2$ is to multiply the states by the charge conjugation matrix and a phase $\exp(-\pi i h_j)$. But here because of the coupling type of the vertices in V^i_{ji}, there is a sign ε^i_{ji}. These together give the operator $\Theta(-)^i_{ji}$, as defined in [15].

What is remarkable is that $S(j)$ can be expressed in terms of B matrices. Then all the polynomial equations concerning $S(j)$ in turn put more restrictions on the B matrices.

We note that the $T_q(a)$'s satisfy the fusion algebra

$$T_q(a)T_{q'}(a)=\sum_m N^m_{qq'}T_m(a), \tag{3.11}$$

hence $T_q(b)$ also satisfy the fusion algebra. This can be proved directly by using the B matrices and the duality. What is important here is that $T_q(b)$, when expressed in terms of the B matrices, can be simultaneously diagonalized by $S(j)$. This fact, together with the polynomial equations about $S(j)$, put a strong restriction on the B matrices involved. We shall show this explicitly by giving examples in Sect. 5. Moreover, using these conditions, we can prove that some fusion algebras do not correspond to RCFT.

Many properties enjoyed by $T_q(a)$ are also enjoyed by $T_q(b)$. For example, we have $T_q^\dagger(a)=T_{q^*}(a)$, where q^* is the charge conjugate of q. Thus, $T_q^\dagger(b)=T_{q^*}(b)$, which is a restriction on some of the B matrices. In the case of zero point functions, this statement is equivalent to $N_{ijk}=N_{i^*j^*k^*}$.

It is easy to see that knowing $S(j)$, the modular transformations for Riemann surfaces of higher genus acting on the basis in Fig. 17 can be constructed. To construct modular transformations acting on the basis in Fig. 16, we need to know the modular transformations for two point conformal blocks on the torus.

On the torus with two punctures, there are more nontrivial cycles. The Dehn twist along the cycle that encompasses a puncture is diagonalized. If we use the basis in Fig. 19, the Dehn twists along the up a cycle and the down a cycle are also diagonalized. We call the up a as a and the down a as a'. Both Verlinde operators associated to these cycles are transformed to the one associated to the b cycle by some modular transformations. The relations among all the Dehn twists will be studied in a coming paper [40]. Here we merely consider the Dehn twist along the cycle b^{-1}, which is related to the modular transformations in higher genus when

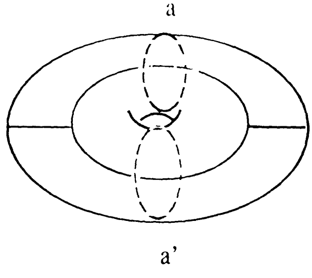


Fig. 19. A basis for the two point conformal blocks on the torus. Note that here we have two a cycles, which are inequivalent at the presence of the graph inside the solid torus

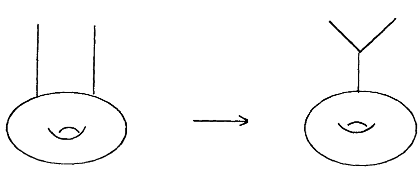


Fig. 20. The two external legs are fused. We obtain in this way another basis for the two point conformal blocks on the torus

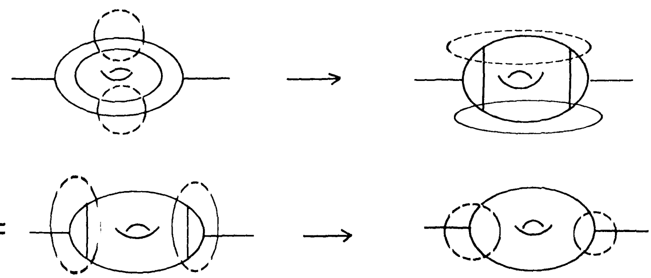
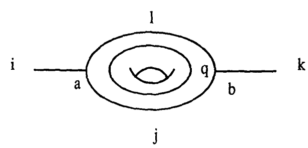


Fig. 21. The graph, which representing the action of the Verlinde operator, can be transformed into a linear combination of the graphs in the basis for the two point functions on the torus

we use the basis in Fig. 16. If we use a fusion indicated in Fig. 20, then $T_q(b)$ can be expressed in terms of $T_q(b)$ for the one-point conformal blocks.

Using the basis in Fig. 19, by some operations shown in Fig. 21, we find

$$T_q(b)\,|l,j,a,b\rangle_{i,k} =$$

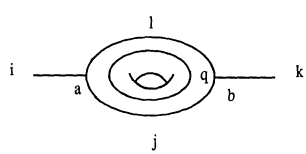


$$= \sum_{l',j',c,d,e,f} F_{lj'}\begin{bmatrix} i & l' \\ j & q \end{bmatrix}_{ac}^{ed} F_{jl'}\begin{bmatrix} k & j' \\ l & q \end{bmatrix}_{bd}^{fc} |l',j',e,f\rangle_{i,k}, \quad (3.12)$$

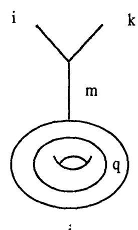
where Eqs. (3.7) and (3.8) have been used. Similar to Eq. (3.9), $S(b^{-1})$ can be written out

$$S(b^{-1})\,|l,j,a,b\rangle_{i,k} = \sum_{q,l',j',c,d,e,f} e^{2\pi i(\tilde{h}_q+\tilde{h}_0)} S_{0,q}, \quad (3.13)$$
$$F_{lj'}\begin{bmatrix} i & l' \\ j & q \end{bmatrix}_{ac}^{ed} F_{jl'}\begin{bmatrix} k & j' \\ l & q \end{bmatrix}_{bd}^{fc} |l',j',e,f\rangle_{i,k}.$$

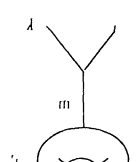
However, we remark that Eq. (3.12) can also be obtained from Eq. (3.6) by some simple moves



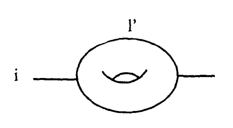
$$k = \sum_m F_{lm}\begin{bmatrix} i & k \\ j & j \end{bmatrix}$$



$$= \sum_{m,j'} F_{lm}\begin{bmatrix} i & k \\ j & j \end{bmatrix} F_{jj'}\begin{bmatrix} m & j' \\ j & q \end{bmatrix}$$



$$= \sum_{m,j',l'} F_{lm}\begin{bmatrix} i & k \\ j & j \end{bmatrix} F_{jj'}\begin{bmatrix} m & j' \\ j & q \end{bmatrix} F_{ml'}\begin{bmatrix} k & j' \\ i & j' \end{bmatrix}$$



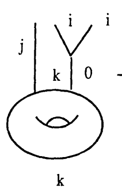
$$. \quad (3.14)$$

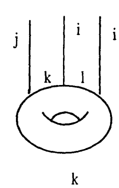
By the symmetry property of the F matrices, we can rearrange the indices in the F matrices in the above equation. To prove it is equivalent to (3.12), we then need the following pentagon identities:

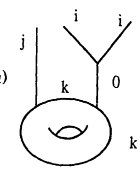
$$\sum_m F_{lm}\begin{bmatrix} k & i \\ j & j \end{bmatrix} F_{jj'}\begin{bmatrix} j' & m \\ q & j \end{bmatrix} F_{ml'}\begin{bmatrix} j' & k \\ j' & i \end{bmatrix} = F_{jl'}\begin{bmatrix} j' & k \\ q & l \end{bmatrix} F_{lj'}\begin{bmatrix} l' & i \\ q & j \end{bmatrix}. \quad (3.15)$$

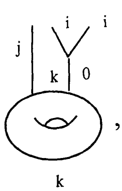
We would like to stress that $T_q(b)$ can be diagonalized both to $T_q(a)$ and $T_q(a')$. This means that the eigenvalues of $T_q(a)$ and $T_q(a')$ are the same but arranged in different orders.

Now we show that one can derive (3.6) using the naive Verlinde operation in RCFT. By the definition, we insert the identity operator into the trace which gives the conformal blocks of the one-point function. Split the identity operator to the product of the primary field q and its conjugate q^*. Move one of them around the cycle under consideration, then fuse them again to the identity operator. The resulting function is just what is obtained by acting on the Verlinde operator. We first consider $T_q(a)$.

$T_q(a): n_q$ 

$\to n_q \sum_l F_{0l}\begin{bmatrix} q & k \\ q & k \end{bmatrix}$ 

$\to \sum_l F_{0l}\begin{bmatrix} q & k \\ q & k \end{bmatrix} F_{l0}\begin{bmatrix} q & q \\ k & k \end{bmatrix} e^{2\pi i(h_q + h_k - h_l)}$ 

$= \lambda_q^{(k)}$ 

, (3.16)

where we normalized $T_q(a)$ by adding a factor n_q. In the above we need the formula

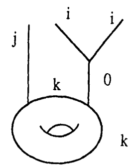
$$\sum_l n_q F_{0l}\begin{bmatrix} q & k \\ q & k \end{bmatrix} F_{l0}\begin{bmatrix} q & q \\ k & k \end{bmatrix} e^{2\pi i(h_q + h_k - h_l)} = \lambda_q^{(k)} = \frac{S_{q,k}}{S_{0,k}}. \quad (3.17)$$

This can easily be proved when we substitute

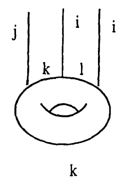
$$F_{0l}\begin{bmatrix} q & k \\ q & k \end{bmatrix} = \sqrt{\frac{S_{0,l}S_{0,0}}{S_{0,k}S_{0,q}}}\, N_{qjk} = F_{l0}\begin{bmatrix} q & q \\ k & k \end{bmatrix},$$
$$n_q^{-1} = F_{00}\begin{bmatrix} q & q \\ q & q \end{bmatrix} = \frac{S_{0,0}}{S_{0,q}}. \quad (3.18)$$

We see that this operation is exactly the same as defined by the help of FTFT. The action of the operator $T_q(b)$ is demonstrated as follows:

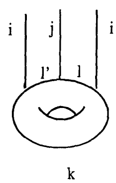
$$T_q(b): n_q$$



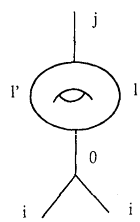
$$= \sum_l n_q F_{0,l}\begin{bmatrix} q & k \\ q & k \end{bmatrix}$$



$$= n_q \sum_{l,l'} F_{0,l}\begin{bmatrix} q & k \\ q & k \end{bmatrix} B_{kl'}\begin{bmatrix} j & q \\ k & l \end{bmatrix}$$



$$\to n_q \sum_{l,l'} F_{0l}\begin{bmatrix} q & k \\ q & k \end{bmatrix} B_{kl'}\begin{bmatrix} j & q \\ k & l \end{bmatrix} F^{-1}_{k0}\begin{bmatrix} q & l' \\ q & l \end{bmatrix}$$



. (3.19)

We see that in the above equation l' must equal l. If we have the following identity:

$$n_q F_{0l}\begin{bmatrix} q & k \\ q & k \end{bmatrix} F^{-1}_{k,0}\begin{bmatrix} q & l \\ q & l \end{bmatrix} = 1\,, \tag{3.20}$$

we have the formula (3.6) from (3.19). It is easy to prove (3.20), using the explicit expressions of the simple F matrices.

4. Modular Transformations in Higher Genus

We have derived the modular transformations for one point functions and two-point functions in the last section. Based on the bases in Figs. 16 and 17, we see that by knowing these we can find modular transformations associated to b cycles. Dehn twist along a cycle connecting two genera can also be found by duality moves. In this section we derive modular transformations acting on the basis in Fig. 16.

In the last section, we defined the Verlinde operators associated to a nontrivial cycle on the Riemann surface. It is obvious that the operators $T_q(a_i)$ are diagonal, and satisfy the fusion algebra:

$$T_q(a_i)T_{q'}(a_i) = \sum_m N^m_{qq'}T_m(a_i)\,. \tag{4.1}$$

This is because the diagonal entries of $T_q(a_i)$ are of the form $\lambda^{(n)}_q = S_{q,n}/S_{0,n}$. For a fixed n, $\lambda^{(n)}_q$ form a one-dimensional representation of the fusion algebra.

Consider the modular transformation S under which $a_i \to b_i$ and $b_i \to b_i^{-1}a_i^{-1}b_i$. If under this modular transformation, the state $|\chi\rangle$ is transformed into $S|\chi\rangle$, then [31]

$$T_q(b_i) = ST_q(a_i)S^{-1}\,. \tag{4.2}$$

By this formula, we know that the $T_q(b_i)$'s also form the fusion algebra. In fact, given a nontrivial cycle C, $T_q(C)$ form the fusion algebra, provided there exists a modular transformation under which an a-cycle is mapped to C. Since these operators can be written out in terms of the braiding matrices, we have many equations about the braiding matrices coming from the fusion algebra. These provide no more restrictions other than the polynomial equations. By concrete calculation, we can prove that these operators form the fusion algebra by the use of the fundamental duality moves.

Since the Dehn twist $S(a_i^{-1})$ along a_i^{-1} is diagonal, and given by $\exp(-2\pi i\tilde{h}_{q_i})$ when it is acting on the basis of Fig. 16, we have the following formula similar to (3.3):

$$S(a_i^{-1}) = \sum_q \exp[2\pi i(h_q - c/12)]S_{0,q}T_q(a_i), \tag{4.3}$$

where $\tilde{h}_q$ denotes $h_q - c/24$.

We also have a formula for $S(b_i^{-1})$ similar to (4.3). Using (4.2) and (4.3), we find

$$S(b_i^{-1}) = \sum_q \exp[2\pi i(h_q - c/12)S_{0,q}T_q(b_i). \tag{4.4}$$

If we know all the $T_q(b_i)$'s, all the Dehn twists along b_i and b_i^{-1} can be calculated by using (4.4). A similar formula holds for the Dehn twist along $a_i a_{i+1}^{-1}$.

We know that all the modular transformations are generated by the Dehn twists along a_i, b_i, and $a_i a_{i+1}^{-1}$. Knowing the operators T_q associated to these cycles is enough to help us calculate all modular transformations.

We work for the basis Fig. 16. By observation we find that the calculation of $T_q(b_i)$ is similar to the calculation of $T_q(b)$ for the one-point functions on the torus, when $i = 1, g$. Otherwise the calculation is similar to the calculation for the two-point functions on the torus.

We give an explicit expression for the $T_q(b_i)$ when $1 < q < g$ in the basis of Fig. 16:

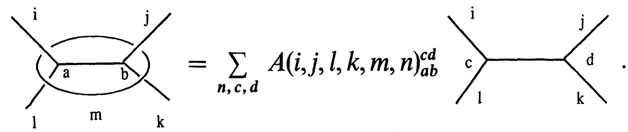
$$\begin{aligned} &\frac{\langle q'_1 \ldots, \tilde{q}'_2 \ldots, \omega'_1 \ldots, a'_1 \ldots | T_q(b_i) | q_1 \ldots, \tilde{q}_2 \ldots, \omega_1 \ldots, a_1 \ldots \rangle}{\langle q_1 \ldots, \tilde{q}_2 \ldots, \omega_1 \ldots, a_1 \ldots | q_1 \ldots, \tilde{q}_2 \ldots, \omega_1 \ldots, a_1 \ldots \rangle} \\ &= {\prod_j}' \delta_{q_j, q'_j} \delta_{\tilde{q}_j, \tilde{q}'_j} \delta_{\omega_j, \omega'_j} \delta_{a_j, a'_j} \sum_{a,b} F_{q_i \tilde{q}'_i} \begin{bmatrix} \omega_{i-1} & q'_i \\ \tilde{q}_i & q \end{bmatrix}_{a_{2i-2a}}^{a'_{2i-2b}} F_{\tilde{q}_i q'_i} \begin{bmatrix} \omega_i & \tilde{q}'_i \\ q_i & q \end{bmatrix}_{a_{2i-1b}}^{a'_{2i-1a}}, \end{aligned} \tag{4.5}$$

where the superscript prime in the product means that the indices appearing in the braiding matrices should be omitted in the product. The calculation is demonstrated in (3.14). We can also write down an expression similar to Eq. (3.6). It is easy to check that Eq. (4.5) is equivalent to the formula in our previous paper [31] by the symmetries of the fusion and braiding matrices.

Next we calculate the entries of $T_q(a_i a_{i+1}^{-1})$,

$$\begin{aligned} &\frac{\langle q'_1 \ldots, \tilde{q}'_2 \ldots, \omega'_1 \ldots, a'_1 \ldots | T_q(a_i a_{i+1}^{-1}) | q_1 \ldots, \tilde{q}_2 \ldots, \omega_1 \ldots, a_1 \ldots \rangle}{\langle q_1 \ldots, \tilde{q}_2 \ldots, \omega_1 \ldots, a_1 \ldots | q_1 \ldots, \tilde{q}_2 \ldots, \omega_1 \ldots, a_1 \ldots \rangle} \\ &= \prod_j \delta_{q_j, q'_j} \delta_{\tilde{q}_j, \tilde{q}'_j} \delta_{\omega_j, \omega'_j} \delta_{a_j, a'_j} A(q_i, q_{i+1}, \tilde{q}_i, \tilde{q}_{i+1}, q, \omega_i, \omega'_i)_{a_{2i-1} a_{2i}}^{a'_{2i-1} a'_{2i}}, \end{aligned} \tag{4.6}$$

where the matrix A is defined by the following equation



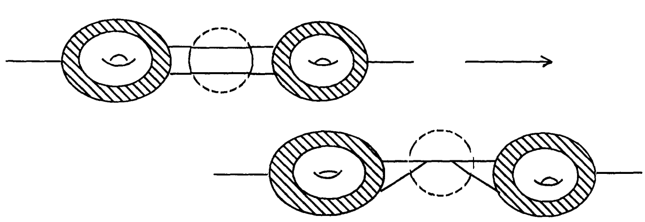
The A matrix can be written out in terms of the fusion matrices and the eigenvalues of the Wilson line operators,

$$A(i,j,l,k,m,n)_{ab}^{cd} = \sum_{m',a',b'} F_{mm'} \begin{bmatrix} i & j \\ l & k \end{bmatrix}_{ab}^{a',b'} \lambda_q^{(m')} F_{m'n}^{-1} \begin{bmatrix} i & j \\ l & k \end{bmatrix}_{a',b'}^{cd} . \tag{4.7}$$

As for the basis shown in Fig. 17, $T_q(b_i)$ is given by a similar expression for one-point functions on the torus. $T_q(a_i a_{i+1}^{-1})$ is quite complicated, we will not do the calculation here.

We would like to stress here that practically all modular transformations for multi-point functions can be derived. Our key point is that operators $T_q(b_i)$ can be simultaneously diagonalized and obey the fusion algebra. In fact, these operators can be generalized to those associated to any graph as in Figs. 15–17. Geometrically, we can construct an operator associated to a graph as follows. Consider a manifold $\Sigma \times I$ with boundary (Σ, Σ). Embedding the graph into this manifold as we do for the handlebody, we find that the path integral over this manifold behaves like an operator: Gluing this manifold to the handlebody along the boundary Σ to form another handlebody amounts to continuously sending the graph into the original handlebody. We denote this operator as $T(\Gamma)$; Γ is the corresponding graph. A typical operator is shown in Fig. 23. From the definition we see that these operators are linear operators. The state obtained by embedding the graph Γ into the handlebody without the presence of the other graphs is generated by the operator $T(\Gamma)$, namely $|\Gamma\rangle = T(\Gamma)|0\rangle$. The vacuum $|0\rangle$ is given by the path integral over the handlebody without any graph. This demonstrates that there is a one-one correspondence between the set of planar operators and the Hilbert space of the states. By planar operators we mean the operators associated to the planar graphs as in Figs. 15–17. Of course we have non-planar operators, e.g. $T_q(a_i)$.

We show that for a given basis of graphs, the operators form an algebra. Consider the product $T(\Gamma_I)T(\Gamma_J)$; this operator is formed by embedding Γ_J and Γ_I into $\Sigma \times I$ subsequently. These two graphs can be viewed as a single graph. We can use the simple duality moves to reduce this complicated graph to a linear combination of the basis graphs. To see this, we demonstrate for an example

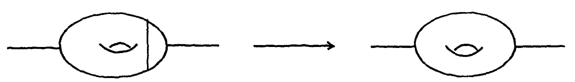


(4.8)

The dark marked rings denote some complicated parts of the graph. These complicated graphs can be also simplified, for example

(4.9)

and



(4.10)

Thus we find that the basis of the operators corresponding to the basis of the graphs are complete in the sense that we consider only operators associated to the planar graphs. Given as basis of graphs $\{\Gamma_I\}$, we have the following algebra:

$$T(\Gamma_I)T(\Gamma_J)=\sum_K C_{IJ}^K T(\Gamma_K)\,. \tag{4.11}$$

This algebra is associative. Unlike the fusion algebra, usually it is not commutative. Since the operators $T_q(b_i)$ belong to this algebra, we see that this algebra contains many copies of the fusion algebra. Given a Riemann surface of genus g, we have such an algebra, called $\mathscr{A}_g$. The fusion algebra is $\mathscr{A}_1$. It is easy to see that $\mathscr{A}_g \subset \mathscr{A}_{g+1}$. Thus, we have an infinite chain of algebras. This chain of algebras is closely related to a quantum group, since both the quantum group and the algebras arise from the duality.

Finally, we discuss the automorphism of the set of B matrices. Dijkgraaf and Verlinde in [11] discussed the automorphism of the fusion algebra. Starting from a non-diagonal theory, modular invariance of the partition function on the torus implies that there is a nontrivial automorphism among the operators $T_q(b)$. The argument is the following. Suppose we have a non-diagonal theory; the Hilbert space of the vertex operators is

$$\mathscr{H}=\oplus_{i,\bar{i}}[\phi_i]\otimes[\phi_{\bar{i}}]\,, \tag{4.12}$$

where $[\phi_i]$ and $[\phi_{\bar{i}}]$ are the irreducible representations of the left chiral algebra and the right chiral algebra, respectively. Let Π be a map under which $i\to\bar{i}$; the partition function on the torus can be written as

$$Z-\bar{\chi}\cdot\Pi\cdot\chi\,. \tag{4.13}$$

The modular invariance of this partition function implies that S as a matrix commutes with Π. Note that here Π is symmetric.

We view χ_i as a state $|i\rangle$. The matrix Π can be viewed as an operator such that $|\bar{i}\rangle=\Pi|i\rangle$. We have the following fact: $S_{\bar{i},j}/S_{0,j}=S_{i,\bar{j}}/S_{0,\bar{j}}$ [11]; this is equivalent to $S_{i,j}/S_{0,j}=S_{\bar{i},\bar{j}}/S_{0,\bar{j}}$. From this we deduce that $\Pi T_q(a)\Pi=T_q(a)$. Now $T_q(b)=ST_q(a)S^{-1}$, since S commutes with Π so $\Pi T_q(b)\Pi=T_q(b)$, namely

$$\langle\bar{i}|T_{\bar{q}}(b)|\bar{j}\rangle=\langle i|T_q(b)|j\rangle\,. \tag{4.14}$$

From this equation we find $N_{ijk}=N_{\bar{i}\bar{j}\bar{k}}$. This is what is found by Dijkgraaf and Verlinde.

Let us apply the same argument to the one-point functions. Under the map Π, the conformal blocks associated to the external line j are mapped to the conformal blocks associated with $\bar{j}$. We consider the Hilbert space which consists of all conformal blocks associated to various external lines j. Of course, in this Hilbert space the operators $T_q(b)$ are well-defined. We have $\Pi T_q(b)\Pi = T_q(b)$ again; using (3.6), we find

$$\sum_{\bar{c}} n_{\bar{q}} F_{\bar{i}\bar{l}} \begin{bmatrix} \bar{l} & \bar{j} \\ \bar{q} & \bar{i} \end{bmatrix}_{\bar{c}\bar{a}}^{\bar{b}\bar{c}} = \sum_{c} n_q F_{il} \begin{bmatrix} l & j \\ q & i \end{bmatrix}_{ca}^{bc}, \tag{4.15}$$

where we used the fact that in the non-diagonal theory, under Π, the coupling a of i, j, k must be mapped to the coupling $\bar{a}$ of $\bar{i}, \bar{j}, \bar{k}$. One can show that (4.15) is the sufficient condition that Π commutes with the modular transformation $\oplus_j S(j)$, while this is the condition of the modular invariance. Usually, under Π, 0 is mapped into 0, so $n_{\bar{q}} = n_q$. We can eliminate the factor n_q in the above equation.

We apply this argument to the generalized partition functions on Riemann surfaces of higher genus. As a result, we find in [31] that there is a automorphism among the squares of B matrices. Similarly, applying this argument to the two point functions on the torus, we will find a similar formula from (3.12).

5. Some Examples

In this section, we calculate explicitly some examples of the $S(j)$ matrices. We found our results consistent with the other authors [36, 37].

First, let us consider the cases in the minimal conformal field theories [1, 2]. The minimal models are described by the central charges and the conformal weights. For the unitary minimal models, the central charge c and the conformal weight h's are given by

$$\begin{gathered} c = 1 - \frac{6}{m(m+1)}, \qquad m = 2, 3, 4, \dots, \\ h_{r,s} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)}, \qquad 1 \leqq r \leqq m-1, \qquad 1 \leqq s \leqq m, \\ r + s = \text{even}. \end{gathered} \tag{5.1}$$

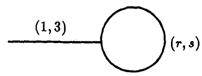
The fusion algebra is

$$\phi_{r,s}\phi_{r',s'} = \sum_{r''=|r-r'|+1}^{\min(r+r'-1,\, 2m-1-r-r')} \quad \sum_{s''=|s-s'|+1}^{\min(s+s'-1,\, 2m+1-s-s')} \phi_{r'',s''}. \tag{5.2}$$

To compare our results with the others, we also write down the corresponding formulae for the Coulomb gas approaches:

$$\begin{gathered} c = 1 - 24\alpha_0^2, \qquad \alpha_0^2 = \frac{1}{4m(m+1)}, \\ h_{r,s} = \beta(\beta - 2\alpha_0), \qquad \beta = \tfrac{1}{2}(1-r)\alpha_+ + \tfrac{1}{2}(1-s)\alpha_-, \\ \alpha_\pm(\alpha_\pm - 2\alpha_0) = 1, \\ \alpha_+ = \sqrt{\frac{m+1}{m}}, \qquad \alpha_- = -\sqrt{\frac{m}{m+1}}. \end{gathered} \tag{5.3}$$

Consider the one point function $\phi_{1,3}(\tau)$ on the torus. The corresponding conformal blocks can be represented graphically as



.

In the following we shall calculate modular transformation $S(j)$ for $m=3$ and $m=4$ theories:

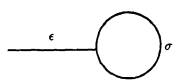
$m=3$ corresponds to the critical Ising model which is equivalent to a free Majorana fermion. There are three primary fields, $\phi_{1,1}=I$, $\phi_{2,2}=\sigma$, $\phi_{1,3}=\varepsilon$, which are called identity operator, order parameter and energy operator, respectively. The fusion algebra is

$$\begin{aligned} &I\cdot I=I, \quad I\cdot\sigma=\sigma, \quad I\cdot\varepsilon=\varepsilon, \\ &\sigma\cdot\sigma=I+\varepsilon, \quad \sigma\cdot\varepsilon=\sigma, \quad \varepsilon^2=I. \end{aligned} \tag{5.4}$$

The conformal weights are

$$h_{1,1}=0, \quad h_{2,2}=\tfrac{1}{16}, \quad h_{1,3}=\tfrac{1}{2}. \tag{5.5}$$

From the fusion algebra, we know there is only one conformal block for the one-point function $\varepsilon(\tau)$



,

and the modular transformation is given by, according to (3.10),

$$S(\varepsilon)=\sum_{q=I,\varepsilon} e^{2\pi i(h_q-2h_\sigma)} S_{0,q} F_{\sigma\sigma}\begin{bmatrix} \varepsilon & \sigma \\ \sigma & q \end{bmatrix}. \tag{5.6}$$

The F matrices have been calculated by Felder et al. in [35]. Writing

$$T_q=F_{\sigma\sigma}\begin{bmatrix} \varepsilon & \sigma \\ \sigma & q \end{bmatrix},$$

we have

$$T_I=1, \quad T_\sigma=0, \quad T_\varepsilon=-1. \tag{5.7}$$

The S transformations for the minimal model characters are given by

$$\begin{gathered} S_{rs,r's'}=\left(\frac{8}{m(m+1)}\right)^{1/2}\sin\frac{\pi rr'}{m}\sin\frac{\pi ss'}{m+1}, \\ r+s \text{ even and } r'+s' \text{ even}. \end{gathered} \tag{5.8}$$

For the critical Ising model, we have

$$S=\frac{1}{2}\begin{array}{c} \begin{array}{ccc} I & \sigma & \varepsilon \end{array} \\ \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \end{array}. \tag{5.9}$$

Substituting Eqs. (5.5), (5.7), and (5.9) into Eq. (5.6), we obtain

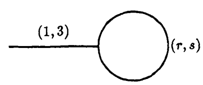
$$S(\varepsilon)=e^{-\frac{\pi i}{4}}.$$

Combining $T(\varepsilon)=\exp\left(\frac{\pi i}{12}\right)$ we have

$$(S(\varepsilon)T(\varepsilon))^3=S(\varepsilon)^2=e^{-\frac{\pi i}{2}}=e^{-\pi i h_\varepsilon}$$

as the desired result.

$m=4$ theory corresponds to the tricritical Ising model. In this model we have six primary fields. We can write them in the following order: $h_{1,1}$, $h_{1,3}$, $h_{2,2}$, $h_{2,4}$, $h_{3,1}$, $h_{3,3}$. According to the fusion algebra, (5.2), the Hilbert space of conformal blocks



is three dimensional, $(r,s)=(1,3)$, $(2,2)$, $(3,3)$. Using the method in [35], we write down the following relevant braid matrices:

$$(T_Q(b))_{rs,r's'}=B_{rs,r's'}\begin{bmatrix}13 & Q\\ rs & r's'\end{bmatrix},$$

$$\begin{gathered}T_{1,1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad T_{1,3}=-\frac{1}{2\cos\pi/5}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\\ T_{2,2}=-\frac{1}{2\cos\pi/5}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad T_{2,4}=\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\\ T_{3,1}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad T_{3,3}=-\frac{1}{2\cos\pi/5}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.\end{gathered}\tag{5.10}$$

Notice that the T_Q matrices satisfy the corresponding fusion algebra. Substituting the T_Q matrices into (3.10), and using the fact that

$$\frac{1}{2\cos\pi/5}=\frac{\sqrt{5}-1}{2},$$

we finally find

$$S(1,3)=e^{-\frac{3}{10}\pi i}\frac{1}{2}\begin{pmatrix}1&\sqrt{2}&1\\ \sqrt{2}&0&-\sqrt{2}\\ 1&-\sqrt{2}&1\end{pmatrix}\tag{5.11}$$

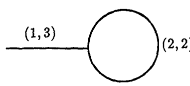
and

$$S(1,3)^2=e^{-\frac{3}{5}\pi i}I=e^{-\pi i h_{1,3}}I.$$

We remark that our result differs from that in [37] by a diagonal similarity transformation Λ, where

$$\Lambda=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

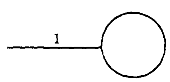
which corresponds to a different boundary condition for the conformal block



,

The correctness of our formula on this minimal model shows that our method, originally derived from the Wilson line insertion in 3D CS gauge theories, applies equally to the conformal field theories. This again suggests that to each 2D RCFT, there is a 3D CS theory as claimed by the authors of [29].

Now we could continue our check on the WZW models. Take the simplest case, $SU(2)$, for example. We check the modular transformation $S(j)$ on the conformal block

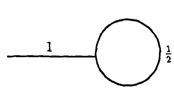


.

where we label the representation by the isospin j. For an integrable representation, j is subjected to $0 \leqq j \leqq \frac{k}{2}$, where k is the level of the corresponding Kac-Moody algebra. The fusion algebra for the $SU(2)$ WZW models is

$$\phi_i \cdot \phi_j = \sum_{l=|i-j|}^{\min(i+j,\,k-i-j)} \phi_l . \tag{5.12}$$

For $k=1$, there is no nontrivial one-point function on the torus. For $k=2$, the Hilbert space of the one-point conformal blocks on the torus is one dimensional



.

The corresponding braiding matrices are

$$B_{\frac{1}{2}\frac{1}{2}}\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=1, \quad B_{\frac{1}{2}\frac{1}{2}}\begin{bmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=-1, \tag{5.13}$$

$$B_{\frac{1}{2}\frac{1}{2}}\begin{bmatrix}\frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=0.$$

Using these data, we find

$$\begin{aligned} S(1) &= e^{-\frac{3}{4}\pi i} = -r e^{-\frac{1}{4}\pi i},\\ T(1) &= e^{2\pi i\left(\frac{3}{16}-\frac{1}{24}\cdot\frac{3}{2}\right)} = e^{\frac{\pi i}{4}},\\ S(1)^2 &= (S(1)T(1))^3 = i = -e^{-\pi i h_1}. \end{aligned} \tag{5.14}$$

This result is correct. The minus sign in front of the exponential in the last equation is due to the antisymmetric coupling $\frac{1}{2}\otimes 1\rightarrow\frac{1}{2}$. As another example, we consider the $k=2$ $SU(2)$ WZW model. In this example we shall demonstrate that $S(j)$ can be obtained from the different approaches. The first method is to calculate the braiding matrices,

$$T_1(b)_{il}=B_{il}\begin{bmatrix}1 & 1\\ i & l\end{bmatrix},$$

$$\begin{gathered}B_{\frac{1}{2}\frac{1}{2}}\begin{bmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=B_{\frac{3}{2}\frac{3}{2}}\begin{bmatrix}1 & 1\\ \frac{3}{2} & \frac{3}{2}\end{bmatrix}=-\frac{1}{[3]},\\ B_{\frac{1}{2}\frac{3}{2}}\begin{bmatrix}1 & 1\\ \frac{1}{2} & \frac{3}{2}\end{bmatrix}=B_{\frac{3}{2}\frac{1}{2}}\begin{bmatrix}1 & 1\\ \frac{3}{2} & \frac{1}{2}\end{bmatrix}=\frac{1}{[3]}.\end{gathered}\tag{5.15}$$

So

$$T_1(b)=\frac{1}{[3]}\begin{pmatrix}-1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & -1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & -1\end{pmatrix},\tag{5.16}$$

where, [3] stands for

$$[3]=\frac{e^{\pi i\frac{3}{6}}-e^{-\pi i\frac{3}{6}}}{e^{\pi i\frac{1}{6}}-e^{-\pi i\frac{1}{6}}}=2.$$

Now

$$T_1(a)=\operatorname{diag}\left(\frac{S_{il}}{S_{0l}}\right)=\begin{pmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Of course $T_1(b)$ can be diagonalized to $T_1(a)$ by $S(1)$. But $S(1)$ is not uniquely determined this way because of the degeneracy of the $T_1(b)$ eigenvalues. To calculate $S(1)$, we need further information. Consider $T_{\frac{1}{2}}(b)$,

$$T_{\frac{1}{2}}(b)=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

The eigenvalues are 0, ± 1, which are not degenerate

$$T_{\frac{1}{2}}(a)=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

in agreement with the eigenvalues of $T_{\frac{1}{2}}(b)$.

Because the eigenvalues are non-degenerate, we can determine $S(1)$ by solving the equation

$$S^{-1}(1)T_{\frac{1}{2}}(b)S(1)=T_{\frac{1}{2}}(a)\tag{5.17}$$

up to some phases. Further using the equation

$$S(1)^2 = -e^{-\pi i h_1} = -e^{-\pi i \frac{1}{3}},$$

$S(1)$ is determined up to some $\pm$ signs. We find

$$S(1) = i\,\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} e^{-\frac{\pi i}{6}}. \tag{5.18}$$

This is in agreement with the direct calculation using (3.10). Knowing $S(1)$ and $T_q(a)$, we can calculate the other braiding matrices by the formula $T_q(b) = S(1)T_q(a)S(1)^{-1}$. We find

$$\begin{aligned} T_{3/2}(b) &= S(1)T_{3/2}(a)S(1)^{-1} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ T_2(b) &= S(1)T_2(a)S(1)^{-1} = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.19}$$

We can check that $T_{3/2}(b)$ and $T_2(b)$ found in this way agree with that calculated by $SU(2,q)$ 6-j symbols [33, 22]. We stress that here we have used $T_{1/2}(b)$ to determine $S(1)$, then to calculate the other $T_q(b)$'s. This means that some B matrices are determined by some other B matrices in this way. We shall emphasize that this method is independent of using polynomial equations.

Our second method is to use the $SU(2)$ fusion algebra. First we note that the fusion algebra together with the constraint $S^2 = C$ determine the modular transformation S for the characters on the torus. We start from the equation

$$T_q(b)_{ii'} = B_{ii'}\begin{bmatrix} 1 & q \\ i & i' \end{bmatrix} \propto N_{ii'q}\,.$$

We only need determine the matrix elements for which $N_{ii'q}$ is not zero. Using

$$\begin{aligned} &T_1^2 = I\,, \quad T_1T_2 = T_1\,, \quad T_{1/2}^2 = T_0 + T_1\,, \\ &T_{3/2}^2 = T_0 + T_1\,, \quad T_{3/2}T_{1/2} = T_1 + T_2\,, \end{aligned} \tag{5.20}$$

we find that T_q matrices can be written in such forms

$$\begin{aligned} &T_0(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{1/2}(b) = \begin{pmatrix} 0 & \frac{1}{2x} & 0 \\ x & 0 & -\varepsilon x \\ 0 & -\frac{\varepsilon}{2x} & 0 \end{pmatrix}, \quad T_1(b) = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{\varepsilon}{2} \\ 0 & 0 & 0 \\ -\frac{\varepsilon}{2} & 0 & -\frac{1}{2} \end{pmatrix}, \\ &T_{3/2}(b) = \begin{pmatrix} 0 & -\frac{1}{2x} & 0 \\ -x & 0 & \varepsilon x \\ 0 & \frac{\varepsilon}{2x} & 0 \end{pmatrix} = -T_{1/2}\,, \quad T_2(b) = \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & -1 & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \end{aligned} \tag{5.21}$$

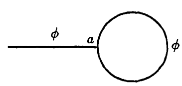
where $\varepsilon^2=1$, x is to be determined. In fact, the other equations

$$T_{1/2}T_1=T_{1/2}+T_{3/2}\,,\qquad T_{1/2}T_2=T_{3/2}\,,\qquad T_1^2=T_0+T_1+T_2\,,$$
$$T_2T_{3/2}=T_{1/2}\,,\qquad T_1T_{3/2}=T_{1/2}+T_{3/2} \tag{5.22}$$

do not provide new information. However, we can make use of the fact that $SU(2)$ theories are self-conjugate, $(T_q)^\dagger=T_{\bar{q}}=T_q$. We then solve x up to a sign $x=1/\sqrt{2}\eta$, $\eta^2=1$. Here we ignore a possible phase, because only in this case $S(j)$ could satisfy the polynomial equations. With the help of (3.10), we find $S(1)^2=-\exp(-\pi i h_1)$. The sign uncertainty in ε and η is due to a possible similarity transformation of the form

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \eta & 0\\ 0 & 0 & -\varepsilon\end{pmatrix},$$

depending on the boundary conditions of the conformal blocks. As the final example, we check the conformal blocks



of the fusion algebra

$$\phi\cdot\phi=1+n\phi\,,\qquad n\geqq 1\,,$$

which has been discussed by Verlinde in [11]. This is the general form of the fusion algebra when there are only two primary fields. We note that this has been discussed also in [19]. We can write the S modular transformation of the genus one characters as

$$S=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}. \tag{5.23}$$

Using $(ST)^3=1$, we have

$$\cos 2\pi h=-\frac{n}{2}\lambda\,,\qquad \tan\theta=\lambda=\frac{n\pm\sqrt{n^2+4}}{2}\,. \tag{5.24}$$

For $n=1$ we have $\cos 2\pi h=\dfrac{-1\pm\sqrt{5}}{4}$, so $h=\pm\dfrac{1}{5}, =\dfrac{2}{5}\,(\mathrm{mod}\,1)$. For $n\geqq 2$, we have

$$\cos 2\pi h=-\frac{n}{4}\,(n-\sqrt{n^2+4})\,. \tag{5.25}$$

Solving this equation does not look easy. However we prove $n\geqq 2$ is impossible by considering the one-point conformal blocks on the torus. Since $T_\phi(a)$ is proportional to the identity matrix, so is $T_\phi(b)$. We have

$$T_\phi(b)=T_\phi(a)=-\frac{1}{\lambda}\,.$$

Thus,

$$S(\phi)=e^{-4\pi ih}\cos\theta+e^{-2\pi ih}\sin\theta\left(-\frac{1}{\lambda}\right)$$
$$=-2i\cos\theta e^{-3\pi ih}\sin\pi h\,. \tag{5.26}$$

By the requirement $S(\phi)^2=\pm e^{-\pi ih}$, where the sign $\pm$ is due to the coupling type of (ϕ,ϕ,ϕ), we find

$$5h=0\,(\mathrm{mod}\,1)\,. \tag{5.27}$$

Our result is stronger than that obtained in [19] [1]. Substituting this result into (5.25), we find that $n\geqq 2$ is forbidden.

So far we have checked our main formula (3.10) for several examples, and showed that the fusion algebra plays a important role even in the case of the one-point functions on the torus. In the last section, we discussed the automorphism among B matrices. We shall check our formula given in [31] for some non-diagonal $SU(2)$ WZW models. Our result in [31] can be stated as: there is an automorphism of the squares of B matrices. Now in the A.D.E. classification of $SU(2)$ WZW models [5], when $k=4n-2$, there is a non-diagonal theory. Under the map Π we discussed in the last section, we have

$$\bar{j}=\tfrac{1}{2}k-j\,,\quad j\in Z+\tfrac{1}{2}\,,\quad \bar{j}=j\,,\quad j\in Z\,. \tag{5.28}$$

We use the following formula of B matrices [22]:

$$B_{ij}\begin{bmatrix} l & m\\ n & p\end{bmatrix}=(-1)^{i+j-n-p}q^{(c_n+c_p-c_i-c_j)/2}\begin{pmatrix} l & n & i\\ m & p & j\end{pmatrix},$$
$$c_i=i(i+1)\,,\qquad q=e^{\frac{2\pi i}{k+2}}\,. \tag{5.29}$$

The 6-j symbols have the following property

$$\begin{pmatrix} \frac{k}{2}-l & \frac{k}{2}-n & i\\ \frac{k}{2}-m & \frac{k}{2}-p & j\end{pmatrix}=\begin{pmatrix} l & n & i\\ m & p & j\end{pmatrix}, \tag{5.30}$$

when l, n, m, p are half integers. Using (5.29) and (5.30), we find

$$B_{\bar{i}\bar{j}}\begin{bmatrix} \bar{l} & \bar{m}\\ \bar{n} & \bar{p}\end{bmatrix}=e^{\frac{1}{2}k-n-p\ \pi i}B_{ij}\begin{bmatrix} l & m\\ n & p\end{bmatrix}, \tag{5.31}$$

when l, n, m, p are half integers. Since k is even, we see that under Π, it is possible that the B matrix differs from the original one by a sign. However the squares are

[1] In [19, 21, 22], those authors found that for the case that there are only two characters on the torus, the conformal weight should be a multiple of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$. Note that if there are two primary fields, when $n=0$, the conformal weight is a multiple of $\frac{1}{4}$. But the degeneracy of characters may occur here if there are more than two primary fields. Thus, the conformal weight of a multiple of $\frac{1}{3}$ is possible

the same: that is the property we want to check. Other cases can be checked similarly. Also, (5.30) is just (4.15) in this case.

Recently, Felder et al. [38] obtained the modular transformations of conformal blocks for any one-point functions on the torus in the minimal models [38]. We believe that, using (3.10) and the 6-j symbols we can obtain their formulae. Also, Bonora et al. discussed the conformal blocks of general multi-point functions on the Riemann surfaces of higher genus in the b-c system approach [39]. The modular transformations can be drawn from their formulae. We hope that one can check again our formula about the modular transformations for the conformal blocks of two-point functions on the torus.

6. Conclusion

In this paper we have discussed how to appropriately define the so-called 3D finite topological quantum field theory. Our definition help us to reach nearly all concepts in the rational conformal field theory. Geometric and algebraic formulations for RCFT's are unified in this framework. As Witten pointed out [27], there is a possible way to verify that a FTFT corresponds to a RCFT and vice versa. By the help of the FTFT, we are able to define the generalization of the Verlinde operators. All operations seem to be simplified. Thus, the modular transformations are easily derived.

We showed that the constraints on the modular transformations for the one point conformal blocks on the torus play an important role in the classification of RCFT's. This is verified by some examples. Since the modular transformations for two point functions contain more information about B matrices, it is expected that from the fusion rules and some constraints on the modular transformations, one can calculate more B matrices. The relation of this procedure to the polynomial equations needs to be clarified [40]. We hope that we can eventually solve the polynomial equations in this way.

Finally, we would like to remark that the quantum group symmetry will play a role in such an approach as we provided here.

Acknowledgements. M.L. would like to thank the theory group, SISSA, for their kind hospitality at the SISSA, Trieste; M.Y. would like to thank Prof. A. Salam for hospitality at the ICTP, Trieste. The authors are grateful to Z. Q. Ma and Y. S. Wu for their useful discussions on the quantum group and related topics.

References

1. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two dimensional quantum field theory. Nucl. Phys. B **241**, 33 (1984)
2. Friedan, D., Qiu, Z., Shenker, S.H.: Conformal invariance, unitarity, and critical exponents in two dimensions. Phys. Rev. Lett. **52**, 1575 (1984); Superconformal invariance in two dimensions and the tricritical Ising model. Phys. Lett. **151** B, 37 (1985)
3. Goddard, P., Kent, A., Olive, D.: Virasoro algebras and coset space models. Phys. Lett. **152** B, 88 (1985); Unitary representations of the Virasoro and super Virasoro algebra. Commun. Math. Phys. **103**, 105 (1986)
4. Cardy, J.L.: Operator content of two-dimensional conformal invariant theories. Nucl. Phys. B **270** [F216], 186 (1986)

5. Capelli, A., Itzykson, C., Zuber, J.-B.: Modular invariant partition functions in two dimensions. Nucl. Phys. B **280** [F218], 445 (1987); The A-D-E classification of two-dimensional conformal invariant theories. Commun. Math. Phys. **113**, 1 (1987)
6. Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess-Zumino model in two dimensions. Nucl. Phys. B **247**, 83 (1984)
7. Gepner, D., Witten, E.: String theory on group manifolds. Nucl. Phys. B **285**, 423 (1987)
8. Di Vecchia, P., Petersen, J.L., Yu, M.: Phys. Lett. **172** B, 211 (1986)
   Di Vecchia, P., Petersen, J.L., Yu, M., Zheng, H.B.: Explicit construction of unitary representations of the $N=2$ superconformal algebra. Phys. Lett. **174** B, 280 (1986)
   Yu, M.: The unitary constructions of the $N=4\,SU(2)$ extended superconformal algebras. Phys. Lett. **196** B, 345 (1987)
   Boucher, W., Friedan, D., Kent, A.: Phys. Lett. **172** B, 316 (1986)
9. Eguchi, T., Ooguri, H.: Conformal and current algebra on a general Riemann surface. Nucl. Phys. B **282**, 308 (1987)
10. Friedan, D., Shenker, S.H.: The analytic geometry of two-dimensional conformal field theory. Nucl. Phys. B **281**, 509 (1987)
11. Verlinde, E.: Fusion rules and modular transformations in 2-D conformal field theory. Nucl. Phys. B **300**, 360 (1988)
    Dijkgraaf, R., Verlinde, E.: Modular invariance and the fusion algebra. In the Proceedings of the Annecy Conference on Conformal Field Theory. Nucl. Phys. B (Proc. Suppl.) **5** B (1988)
    Brustein, R., Yankielowicz, S., Zuber, J.-B.: Factorization and selection rules of operator product algebras in conformal field theory. Nucl. Phys. B **313**, 321 (1989)
12. Tsuchiya, A., Kanie, Y.: Vertex operators in the conformal field theory on P^1 and monodromy representations of the braid group. Lett. Math. Phys. **13**, 303 (1987); In: Conformal field theory and solvable lattice models. Adv. Stud. Pure Math. **16**, 297 (1988)
13. Fröhlich, J.: Statistics of fields, the Yang-Baxter equation, and the theory of knots and links. Lectures at Cargese 1987
14. Rehren, K.-H., Schroer, B.: Einstein causality and artin braids. Nucl. Phys. B **312**, 715 (1989)
    Schroer, B.: Quasiprimary fields: an approach to positivity of 2-D conformal quantum field theory. Nucl. Phys. B **295**, 4 (1988)
    Rehren, K.-H.: Locality of conformal fields in two dimensions: Exchange algebra on the light cone. Commun. Math. Phys. **116**, 675 (1988)
15. Moore, G., Seiberg, N.: Polynomial equations for rational conformal field theories. Phys. Lett. **212** B, 451 (1988); Naturality in conformal field theory. Nucl. Phys. B **313**, 16 (1989); Classical and quantum conformal field theory. Commun. Math. Phys. **123**, 177 (1989)
16. Vafa, C.: Toward classification of conformal theories. Phys. Lett. **206** B, 421 (1988)
17. Harvey, J.A., Moore, G., Vafa, C.: Quasicrystalline compactification. Nucl. Phys. B **304**, 269 (1988)
18. Anderson, G., Moore, G.: Rationality in conformal field theory. Commun. Math. Phys. **117**, 144 (1988)
19. Mathur, S.D., Mukhi, S.: Correlation functions of current-algebra theories on the torus. Phys. Lett. **210** B, 133 (1988)
    Mathur, S.D., Mukhi, S., Sen, A.: On the classification of rational conformal field theories. Phys. Lett. **213** B, 303 (1988); Correlators of primary fields in the $SU(2)$ WZW theory on Riemann surfaces. Nucl. Phys. B **305** [FS23], 219 (1988); Differential equations for correlations and characters in arbitrary rational conformal field theories. Nucl. Phys. B **312**, 15 (1989); Reconstruction of conformal field theories from modular geometry on the torus. Nucl. Phys. B **318**, 483 (1989); Tata Institute preprint TIFR/TH/89-12
20. Kiritsis, E.B.: Preprint LBL-26384 (1988); LBL-26972 (1989)
21. Naculich, S.G.: Preprint BRX TH-257 (1988)
22. Alvarez-Gaumé, L., Gomez, C., Sierra, G.: Quantum group interpretation of some conformal field theories. Phys. Lett. **220** B, 142 (1989); Hidden quantum symmetries in rational conformal field theories. Nucl. Phys. B **319**, 155 (1989); Duality and quantum groups. CERN preprint, CERN-TH-5369/89

23. Cardy, J.L.: Preprint UCSBTH-89-06 (1989)
24. Segal, G.: Oxford preprint; and Lecture at the IAMP Congress, Swansea, July, 1988
25. Witten, E.: Non-Abelian bosonization. Commun. Math. Phys. **92**, 455 (1984)
26. Witten, E.: Quantum field theory and the Jones polynomials. Commun. Math. Phys. **121**, 351 (1989)
27. Witten, E.: Gauge theories and integrable lattice models. IASSNS-HEP-89/11 (1989); The search for higher symmetry in string theory. IASSNS-HEP-88/55 (1988)
28. Bos, M., Nair, V.P.: $U(1)$ Chern-Simons and $c=1$ conformal blocks. Print-89-0118
Elitzur, S., Moore, G., Schwimmer, A., Seiberg, N.: Remarks on the canonical quantization of the Chern-Simons-Witten theory. IASSNS-HEP-89/20
Li, M: Abelian Chern-Simons theory and CFT of rational torus. To appear in *Π* Nuovo Cimento B
29. Moore, G., Seiberg, N.: Taming the conformal zoo. Phys. Lett. **220** B, 422 (1989)
30. Gao, Y.-H., Li, M., Yu, M.: To appear
31. Li, M., Yu, M.: Modular transformations of conformal blocks in WZW models on Riemann surfaces of higher genus. ICTP preprint, IC/89/106 (1989)
32. Drinfel'd, V.G.: Quantum groups. In Proceedings of the International Congress of Mathematicians, Berkeley Cal. (1986)
33. Kirillov, A.N., Reshetikhin, N.Yu.: Representations of the algebra $U_q(sl(2))$. q-orthogonal polynomials and invariants of links. Preprint LOMI-E-9-88 (1988)
Hou, B.-Y., Hou, B.-Y., Ma, Z.-Q.: Preprint BIHEP-TH-89-7; BIHEP-TH-89-8 (1989)
34. Moore, G., Reshetikhin, N.: A comment on quantum group symmetry in conformal field theory. IASSNS-HEP-89/18 preprint (1989)
35. Felder, G., Fröhlich, J., Keller, G.: Braid matrices and structure constants for minimal conformal models. IAS/ETH preprint (1989)
36. Bagger, J., Nemeschansky, D., Zuber, J.-B.: Minimal model correlation functions on the torus. Preprint USC-88/009 (1988)
37. Jayaraman, T., Narain, K.S.: Correlation functions for minimal models on the torus. ICTP preprint IC/88/306
38. Felder, G., Silvotti, R.: Modular covariance of minimal model correlation functions. Commun. Math. Phys. **123**, 1 (1989)
39. Bonora, L., Matone, M., Toppan, F., Wu, K.: *b-c* system approach to minimal models. The genus-zero case. Phys. Lett. **224** B, 115 (1989); SISSA preprint ISAS/SISSA-45/89/EP
40. Li, M., Yu, M.: In preparation

Communicated by L. Alvarez-Gaumé

Received July 31, 1989

Note added in proof. Recently we found* that our formula for $S(n)$ satisfy the related polynomial equations derived by Moore and Seiberg [15]. In addition, we show that $S(n)$ is a unitary transformation, provided the duality of the model is ensured. As a consequence, the whole consistence conditions for modular invariance on Riemann surfaces of higher genus are guaranted by the duality of the theory on the sphere.

* Reference: Li, M., Yu, M., Duality ensures modular covariance, NBI preprint NBI-HE-89-47