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Continuum Analogues of Contragredient Lie Algebras

(Lie Algebras with a Cartan Operator and Nonlinear Dynamical Systems)

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Abstract. We present an axiomatic formulation of a new class of infinite-dimensional Lie algebras – the generalizations of Z-graded Lie algebras with, generally speaking, an infinite-dimensional Cartan subalgebra and a contiguous set of roots. We call such algebras "continuum Lie algebras." The simple Lie algebras of constant growth are encapsulated in our formulation. We pay particular attention to the case when the local algebra is parametrized by a commutative algebra while the Cartan operator (the generalization of the Cartan matrix) is a linear operator. Special examples of these algebras are the Kac-Moody algebras, algebras of Poisson brackets, algebras of vector fields on a manifold, current algebras, and algebras with differential or integro-differential Cartan operator. The nonlinear dynamical systems associated with the continuum contragredient Lie algebras are also considered.

Introduction

In this paper we present an axiomatic formulation and give the principal examples of continuum generalizations of Z-graded algebras with generally speaking, an infinite-dimensional Cartan subalgebra. Our construction includes the simple Lie algebras of constant growth. Very special cases of these algebras have been discussed previously[1]. However, their (more or less) precise definition, albeit rather imperfect, was given in [1]. There, the discovery of the continuum algebras (called "continual Lie algebras" there) was stimulated by an investigation of nonlinear dynamical systems. On the other hand, already in the 60's and 70's, associative

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[1] As an example note paper [5] in which dynamical systems are generated by the associative algebras of integral operators. These algebras are defined in the space of measurable functions on an arbitrary set M with a measure preserving invertible transformation $M: M \to M$. They form, in particular, some subclass of the algebras considered in Example 6 in Sect. 2

algebras related with dynamical systems were considered, which actually coincide with a special case of our continuum Lie algebras, namely that having a Cartan operator (a continuous generalization of the Cartan matrix) of a particularly special form. This coincidence is due to the definition of a Cartan subalgebra of the associative algebras given in [2, 3]. Further, it should be stressed that current algebras are also particular cases of the continuum algebras under consideration. Our approach, therefore, further develops the theory of Lie-algebra valued distributions which has been investigated in [4] and other papers.

In Sect. 1 we give an axiomatic formulation of the continuum Z-graded contragredient Lie algebras under investigation and establish some of their properties. Section 2 contains examples which, probably, exhaust the list of all these algebras with constant growth. It is remarkable that the Poisson bracket algebras generated by automorphisms of a "root" space as a compact are included here, as well as several examples isomorphic to $sl(2, E)$, such as current algebra and an algebra with a Cartan operator identical to the Hilbert operator. A theorem on some continuous limits of semisimple Lie algebras is formulated. The final (third) section is devoted to dynamical systems associated via a zero curvature type representation with the introduced Lie algebras. These include the continuous analogues of Toda lattices. The Conclusion contains a preliminary programme of our future investigations in this direction.

Although we begin this paper with a rather general axiomatic formulation, we discuss more specific examples in the later nonconceptual part of the paper, where the local algebra is parametrized not simply by a vector space but by a commutative algebra and the Cartan operator is a linear operator in it. The consideration of the most general situation is hitherto not motivated by applications in theoretical and mathematical physics.

As usual, generalization stimulates deeper understanding of the original ("classical") structure. An excellent illustration of this fact is, for example, the transition from finite-dimensional simple Lie algebras to Kac-Moody algebras. In their turn, the algebras we consider, include those with Cartan matrices of the finite type in a broader and rather unusual context. Actually we construct their continuous limits in which the root space is continuous and is a manifold or a more general space, for example with a measure. In the continuous case there exist analogues, of several types, of Lie algebras of exponential growth. In fact, one can have various types of "exponentialities." Moreover, rather natural Cartan operators, for example those having a δ'-function as a kernel, lead to quite "reasonable" Lie algebras, which nevertheless have an exponential growth in a literal sense (see Sect. 2). Evidently, these possibilities for continuum Lie algebras are very interesting. In any case the nonlinear differential and integro-differential systems generated by such algebras are worthy of attention. Note also that continuum analogues of contragredient Kac-Moody algebras discussed here admit a direct generalization to the case of Lie superalgebras.

1. Axiomatic Foundation of Continuum Z-Graded Contragredient Lie Algebras

Let E be a vector space over the field ϕ (R or C); K and S are two bilinear mappings $E\times E\to E$. Define a "local Lie algebra" $\hat{g}\equiv g_{-1}\oplus g_0\oplus g_{+1}$ as follows. Each of g_i, $i=0, \pm 1$, as a vector space is isomorphic to E, in other words the elements of g_i are parametrized by the vectors $\varphi\in E$ so that $g_i=\{X_i(\varphi), \varphi\in E, i=0, \pm 1\}$. Besides, there are relations

$$\begin{aligned}&[X_0(\varphi), X_0(\psi)]=0,\ [X_0(\varphi), X_{\pm 1}(\psi)]=\pm X_{\pm 1}(K(\varphi,\psi))\ ,\\ &[X_{+1}(\varphi), X_{-1}(\psi)]=X_0(S(\varphi,\psi))\end{aligned} \tag{1}$$

for all φ, $\psi\in E$.

Lemma. *The Jacobi identity for $\hat{g}$ is equivalent to the condition*

$$K(\varphi, K(\psi,\chi))=K(\psi, K(\varphi,\chi))\ ,\qquad S(\varphi, K(\psi,\chi))=S(K(\psi,\varphi),\chi)\ . \tag{2}$$

In what follows this condition is assumed to be satisfied.

Definition 1. Let $g'(E;K,S)$ be a Lie algebra freely generated by a local part $\hat{g}$ and J be the largest homogeneous ideal having a trivial intersection with g_0. Then $g(E;K,S)=g'(E;K,S)/J$ is called *a continuum contragredient Lie algebra* with the local part $\hat{g}$ and the defining relations (1).

Remark. For the case of the contragredient Lie algebras with the generalized Cartan matrix, which in what follows is called *the discrete case* for brevity, the quotient algebra can be defined by adding the Serre conditions to the defining relations for Chevalley generators of the local part (see for example [6]). In our continuum case we have managed to write down only part of the generalized Serre conditions. However, this does not hinder a constructive investigation of the algebras in question, just as it did not hinder the analogous investigation in the discrete case.

Statement. Lie algebra $g(E;K,S)$ is Z-graded, $g=\bigoplus_{n\in Z} g_n$. It is easy to convince oneself that $g_n=[g_{n-1}, g_1]$ for $n>0$, and $g_n=[g_{n+1}, g_{-1}]$ for $n<0$.

Definition 2. The Lie algebra $g(E;K,S)$ is called the *algebra of temperate growth* if for each n there exists a finite-dimensional subspace $L_n\subset g_{+1}$, $\dim L_n<\infty$, such that $g_n=[g_{n-1}, L_n]$.

Every graded Lie algebra in the discrete case obviously has a temperate growth in the sense of Definition 2. However, in our situation with the Cartan subalgebra being, in principle, infinite-dimensional this notion allows us to separate interesting possibilities from all "almost" free algebras. More restrictive is the notion of *polynomial growth* in terms of the Gel'fand-Kirillov dimension [7], however, in a functional sense in the spirit of Kolmogorov's ε-entropy. Finally, we speak of *constant growth* if $g_n\simeq g_1\simeq E$.

Further, in this paper, we will not consider the most general algebras $g(E;K,S)$, because all known applications hitherto occur within a more special and simple scheme. Let us assume that E is an associative commutative algebra (possibly, without unity) over field ϕ, while the mappings K and S have the linear form

$$K(\varphi,\psi)=K\varphi\cdot\psi,\ K:E\to E;\ S(\varphi,\psi)=S(\varphi\cdot\psi),\ S:E\to E\ . \tag{3}$$

Clearly, condition (2) is satisfied for them automatically: $S(\varphi K\psi\cdot\chi)\equiv S(K\psi\cdot\varphi\cdot\chi)$. Thus, the defining relations (1) take the form

$$\begin{aligned}&[X_0(\varphi),X_0(\psi)]=0,\ [X_0(\varphi),X_{\pm1}(\psi)]=\pm X_{\pm1}(K\varphi\cdot\psi)\ ,\\&[X_{+1}(\varphi),X_{-1}(\psi)]=X_0(S(\varphi\cdot\psi))\ .\end{aligned} \tag{4}$$

We will consider only these relations or those reducing to them.

The case with $S=I\equiv\mathrm{Id}$ is especially important. It will be called the *standard* one and the operator K for this case will be called the *Cartan operator*. Suppose that the operator S is invertible. Then the substitution $X_0(S\varphi)\to X_0(\varphi)$ reduces (4) to the standard relations

$$\begin{aligned}&[X_0(\varphi),X_0(\psi)]=0,\ [X_0(\varphi),X_{\pm1}(\psi)]=\pm X_{\pm1}(\tilde{K}\varphi\cdot\psi)\ ,\\&[X_{+1}(\varphi)),X_{-1}(\psi)]=X_0(\varphi\cdot\psi)\ ,\end{aligned} \tag{5}$$

with the Cartan operator $\tilde{K}\equiv KS$.

If operator S has a kernel[2] and $\mathrm{Ker}\,S\subset\mathrm{Ker}\,K$, then we have the central extension of the standard case (see below).

Now, our main problem is to describe, as was done for the discrete case (see for example [6]), the continuum contragredient Lie algebras of temperate or constant growth. A list of these algebras is given in Sect. 2.

2. Main Examples

Here we assume that E is one of the following spaces:

i) the space of tame functions with pointwise multiplication on a smooth manifold M;
ii) the algebra of polynomials;
iii) the space of jets of infinitely differentiable functions;
iv) the algebra of formal power series.

There is no essential difference between the algebras $g(E;K,S)$ for different choices of the space E from the above list.

2.1. Discrete Case (The Kac-Moody Algebras)

This case, in our approach, corresponds to the finite-dimensional algebra $E=C^n$ with coordinate multiplication in some basis. Here the Cartan operator K coincides with the generalized $n\times n$ Cartan matrix k, $S=I$. The local Lie algebra $\hat{g}$ is a linear hull of $3n$ elements: generators h_i of the Cartan subalgebra and Chevalley generators $X_{\pm i}$, $1\leqq i\leqq n$, with the defining relations [6]

$$[h_i,h_j]=0,\ [h_i,X_{\pm j}]=\pm k_{ji}X_{\pm j},\ [X_{+i},X_{-j}]=\delta_{ij}h_i\ ,$$

[2] Here, in distinction to the rest of the text, the term "kernel" should be understood not in the sense of a kernel of an integral operator but as a subset $E_0\subset E$ annihilated by S, that is $SE_0\to 0$

to which (5) reduce. The consideration of the quotient algebra $g(E;K,S) = g'(E;K,S)/J$ in Definition 1 is equivalent to imposing the Serre conditions $(\mathrm{ad}\, X_{\pm i})^{1-k_{ji}} X_{\pm j} = 0$, $i \neq j$.

2.2. *Current Algebra*

Let M be a manifold or a topological space and E be a space of C^∞-functions on M, $K=2I$, $S=I$. Then $g(E;K,S)$ is identical to the current algebra $sl(2, C^\infty(M))$.

If E is a space of vector-functions on M and one chooses $I \otimes k$ as the Cartan operator with k being the Cartan matrix of a simple Lie algebra g, then we obtain the Lie algebra of currents taking values in g, $g(E;K,S) = C^\infty(M;g)$. Of course, the current algebra is not simple, it is a (continual) sum of algebras g.

2.3. *Poisson Brackets Algebra*

Let E be the algebra of trigonometrical polynomials on a circle, $K=S=-id/dt$.

Theorem 1. *The continuum contragredient algebra $g(E; -id/dt, -id/dt)$ is isomorphic to the Lie algebra of functions on a two-dimensional torus T^2 with Poisson bracket $\{f,g\} = \partial f/\partial t \partial g/\partial s - \partial g/\partial t \partial f/\partial s$.*

Proof. In this case, relations (4) take the form

$$[X_0(\varphi), X_0(\psi)] = 0, \; [X_0(\varphi), X_{\pm 1}(\psi)] = \mp i X_{\pm 1}(\varphi'\psi) \;,$$
$$[X_{+1}(\varphi), X_{-1}(\psi)] = -iX_0((\varphi\psi)') \;,$$

where $\varphi' = d\varphi/dt$. Then $g_n \simeq E$ and

$$[X_n(\varphi), X_m(\psi)] = iX_{n+m}(m\varphi'\psi - n\psi'\varphi) \;.$$

Comparing the functions $\varphi_n(t)e^{ins}$ on T^2 to the elements $X_n(\varphi) \in g_n$, we see that the Poisson bracket relations

$$\{\varphi_n(t)e^{ins}, \psi_m(t)e^{ims}\} = i(m\varphi'_n\psi_m - n\varphi_n\psi'_m)e^{i(m+n)s}$$

correspond to the algebra $g(E; -id/dt, -id/dt) = \bigoplus_{n \in Z} g_n$. □

Note that the notion of roots for our algebras depends on the operators K and S. In particular, if $K=S=-id/dt$, then the roots of this algebra are $n\delta'(t-t')$, whereas in the standard form, for the algebra with $K=d^2/dt^2$, $S=I$, one must consider $n\delta''(t-t')$ as the roots. This algebra is unlike the current algebra, but is nevertheless contained in our list. It arose in connection with integrable (in the sense of Liouville) systems of a special type in [8], see Sect. 3. However, the fact that the Poisson algebra on the torus is a graded algebra, as well as the fact that this algebra is a natural limit of the series A_n, has probably not been noted before.

2.4. *The Simplest Continuous Limit of the Series A_n; (kernel: δ'')*

It is easy to convince oneself using the shift operators $e^{\pm \partial/\partial j}$ that in the appropriate continuous limit the Cartan matrix for the series A_n takes the form of the Cartan operator $K=d^2/dt^2$ (with the symmetrical kernel $\delta''(t-t')$); here $S=I$. This case, being the standard one in accordance with our terminology, gives the algebra $g(E; d^2/dt^2, I)$, which is isomorphic to those considered in Example 3 up to a quotient over constants.

2.5. The Nonsymmetrizable Case ; (kernel: δ')

Let $K=d/dt$, $S=I$. In this the kernel of the Cartan operator is $\delta'(t-t')$ and, is clearly not symmetrizable. This yields an algebra of temperate growth, in the sense of Definition 2, but not of constant growth. Moreover, in this case $\dim L_n=2$. This example is a continuum analogue of the contragredient Lie algebra $g(k)$, however, with a matrix k [in the defining relations (6)] which is not a generalized Cartan matrix. Nevertheless, the Cartan operator here is a continuous limit of the Cartan matrix of the superalgebra $sl(n/n+1)$ for the choice of the simple (odd) root system corresponding to the Dynkin scheme $\otimes - \ldots - \otimes$.

Theorem 2. *The dimensions of subspaces g_m of the graded algebra $g(E; d/dt, I)$ are $d_m = \dim_{g_0} g_m = 2^{|m|-2}$ for $|m|>2$ ($d_{0,\pm 1}=1$ by definition); the element $X_0(1)$ is the centre of the algebra.*

Proof. The last statement is verified directly. The rest is based on an explicit construction of a basis for this algebra using the Jacobi identities and is carried out by induction.

Put $X_{\pm 2}(\varphi_2, \varphi_1) \stackrel{\text{def}}{=} [X_{\pm 1}(\varphi_2),\ X_{\pm 1}(\varphi_1)]$ and consider the structure of the subspace g_m with $m \in Z_+$. Similarly for $m \in Z_-$. Then it is easy to convince oneself that $g_{+2}=X_{+2}(\phi)$, where $\phi=\varphi_1\varphi_2'-\varphi_1'\varphi_2$ and also

$$[X_{-1}(\chi), X_{+2}(\phi)]=X_{+1}(\phi\chi),\ [X_0(\chi), X_{+2}(\phi)]=-2X_{+2}(\phi\chi')\ .$$

Analogously it can be shown that $g_{+3}=\{X_{+3}^{(1)}(\psi), X_{+3}^{(2)}(\psi)\}$, where

$$X_{+3}^{(i)}(\psi)=2^{i-3}\{[X_{+1}(1), X_{+2}(\psi)]+(-1)^i[X_{+1}(\psi), X_{+2}(1)]\},\ i=1,2\ .$$

Here,

$$[X_{-1}(\chi), X_{+3}^{(i)}(\psi)]=(-1)^i X_{+2}(d^{i-1}(\psi\chi)/dt^{i-1})\ ,$$

$$[X_0(\chi), X_{+3}^{(i)}(\psi)]=-3X_{+3}^{(i)}(\psi\chi')\ ,$$

from which there follows a linear independence of the elements $X_{+3}^{(1)}$ and $X_{+3}^{(2)}$.

Continuing the induction we see that if the elements $X_{+m}^{(1)}(\psi),\ldots,X_{+m}^{(d_m)}(\psi)$ generate a basis of g_{+m}, $m \geqq 3$, then the subspace $g_{+(m+1)}$ has the basis

$$X_{+(m+1)}^{(s)}(\psi)=\lambda_s\{[X_{+1}(1), X_{+m}^{(s)}(\psi)]-[X_{+1}(\psi), X_{+m}^{(s)}(1)]\}\ ,$$

$$X_{+(m+1)}^{(s+d_m)}(\psi)=\mu_s\{[X_{+1}(1), X_{+m}^{(s)}(\psi)]+[X_{+1}(\psi), X_{+m}^{(s)}(1)]\}\ , \qquad 1 \leqq s \leqq d_m\ ,$$

where λ_s and μ_s are some nonzero constants. □

Note that a partial analogue of the Serre conditions, some of which are already contained in the initial scheme, is provided by the relations $[g_{-2}, g_{+2}]=0$, $[g_{\mp 2}, X_{\pm(m+1)}^{(s)}]=0$, $m \geqq 2$, $1 \leqq s \leqq d_m$. In this connection recall that in the discrete case with the generalized Cartan matrix, the corresponding conditions can be expressed as $[g_{\mp 1}, (\mathrm{ad}\, X_{\pm i})^{1-k_{ji}} X_{\pm j}] = 0, i \neq j$. The qualitative difference consists of the fact that here the growth begins with the subspaces $g_{\pm 3}$. More details on the analogues of the Serre relations will be given in another paper.

2.6. *Cases with δ-Type Kernels*

Let M be a manifold of C-class and T be its C-diffeomorphism; $E = C^\infty(M)$, $T\varphi(t) = \varphi(Tt)$. Define the operator K in the form $K\varphi(t) = \varphi(t) - \varphi(Tt)$, that is $K = I - T$, $S = I - T^{-1}$.

Theorem 3. *The algebra $g(E; I-T, I-T^{-1}) \equiv g(E; T)$ is isomorphic to the algebra of finite sequences $\{X_n(\varphi)\}$, $n \in Z$, with the defining relations*

$$[X_0(\varphi), X_0(\psi)] = 0,\ [X_0(\varphi), X_{\pm 1}(\psi)] = \pm X_{\pm 1}(\psi(\varphi - T\varphi)) \ ,$$

$$[X_{+1}(\varphi), X_{-1}(\psi)] = X_0(\varphi\psi - T^{-1}(\varphi\psi)) \ ,$$

or in standard form (5), *where* $\tilde{K} = KS = 2I - T - T^{-1}$.

This algebra is written more simply as

$$[X_n(\varphi), X_m(\psi)] = X_{n+m}(\varphi T^n \psi - \psi T^m \varphi) \ .$$

Here, in terms of the general problem with relations (1), $K(\varphi, \psi) = K_m(\varphi, \psi) \equiv \psi(\varphi - T^m \varphi)$, $m = \pm 1$, $n = 0$, $S(\varphi, \psi) = \varphi T\psi - \psi T^{-1}\varphi$.

Note that here the substitution $X_{-1}(\varphi) \to X_{-1}(T^{-1}\varphi)$ leads to the preceding relations with $K_{+1} = K_{-1}$.

Theorem 4. *The algebra g is simple if the diffeomorphism T is minimal, that is all its orbits are everywhere dense.*

The proof is based on the Zeller-Meier theorem (see [2]) on the simplicity of cross-products.

These algebras generate a wide class of examples which are studied (in the sense of cross-products) as associative algebras [2, 3]. Their completions lead to the construction of C^*-algebras and von Neumann factors. As already noted in the Introduction, a special subclass of these algebras was used in [5] in connection with the construction of some dynamical systems. Here, in particular Example 4 and its generalizations to arbitrary dimensions are also contained.

2.7. *Vector Field on a Manifold*

Let V be a vector field on a manifold M, E be the algebra $C^\infty(M)$ and $K = S = V$ in relations (4). Then the Z-graded contragredient algebra $g(E; V, V)$ is defined by the monomial brackets

$$[X_n(\varphi), X_m(\psi)] = X_{n+m}(m\varphi V\psi - n\psi V\varphi) \ .$$

This algebra can be equivalently represented as an algebra of functions on the direct product $S^1 \times M$ with the brackets $\{\varphi, \psi\} = \partial\varphi/\partial t\, V\psi - \partial\psi/\partial t\, V\varphi$.

2.8. *The Hilbert-Cartan Operator*

In all previous examples the Cartan operator was a differential or difference-differential operator, while the space E was defined with the usual (pointwise) multiplication. Now we consider the simplest case when the Cartan operator is of an integral type in E with a pointwise product, or, equivalently, $K=S=I$ and the multiplication in the space E is of another type.

Let E be a space of functions φ on C^1 which satisfy the Hölder condition and are expanded into a sum of holomorphic φ_+ and antiholomorphic φ_- parts with respect to some domains V. Define the multiplication in E as $\varphi \circ \psi = \varphi_+ \psi_+ + \varphi_- \psi_-$. It is useful to stress that

$$\varphi \circ \psi = 1/2(H\varphi \cdot \psi + \varphi \cdot H\psi) , \tag{6}$$

where H is the Hilbert transformation with the integral over $L=\partial V$. Following our general scheme consider the Lie algebra $g(E;H)$ with defining relations (5) and $\tilde{K}=I$, that is

$$[X_0(\varphi), X_0(\psi)]=0, \ [X_0(\varphi), X_{\pm 1}(\psi)] = \pm X_{\pm 1}(\psi \circ \varphi) ,$$

$$[X_{+1}(\varphi), X_{-1}(\psi)] = X_0(\varphi \circ \psi) .$$

Theorem 5. *The algebra $g(E;H)$ is isomorphic to the algebra $sl(2,E)$ with product* (6) *in E.*

This theorem is essential for an investigation of the nonlinear equations associated with the Hilbert-Cartan operator (see Sect. 3).

Let us formulate now (without proof) a theorem which partially formulates the continuum limits of the semi-simple Lie algebras as the graded algebras. The formal details are left to our next publication. Note that a limit of a graded Lie algebra gives a graded Lie algebra in any sense of the word "limit." The notion of the limit needs to be elucidated only for the Cartan subalgebra and the Cartan matrix (operator) because the algebra is generated by its local part in accordance with the adopted axioms. Once this has been done, it suffices to note that the finite-dimensional Cartan subalgebra in the formulation of the theorem is a finite-dimensional commutative semi-simple algebra as above, while its ("weak") limit is an infinite-dimensional algebra of functions on the limit space. So, the Cartan operator is a weak limit of the Cartan matrix as an operator in these algebras. In other words, the discrete objects play the role of a net-approximation of their continuum limits. Note that precisely this interpretation follows also from the theory of integrable systems, whereas it has not appeared earlier in the theory of Lie algebras.

Theorem 6. *(On some continuum limits of semi-simple Lie algebras). Let M be a compact manifold and g a simple contragredient graded Lie algebra of constant growth having $g_0 \simeq E = C^\infty(M)$ as the Cartan subalgebra. Then, if the Cartan operator has one of the following two forms:*

i) $K\varphi(t) = \varphi(Tt) - 2\varphi(t) + \varphi(T^{-1}t)$, *where T is a diffeomorphism of M,*
ii) $K\varphi(t) = V\varphi(t)$, *where T is a vector field on M, the algebra g is a continuum limit of a discrete case.*

These algebras themselves are described in Examples 4, 6 and 7.

Remark. 1. It is likely that each of the contragredient algebras of constant growth is a limit (in an appropriate sense) of a discrete one. However, and this is crucial, not every simple algebra is a limit of a simple one.
2. The case (ii) is, of course, a limiting case of (i), however we find it convenient to consider it separately.

If it is possible to introduce on a manifold, a measure which is invariant with respect to a diffeomorphism, or in case (ii), such a measure that the field determines a self-adjoint operator, then on the corresponding algebra g there exists an invariant Killing form. This fact allows us to construct a root system and to develop a structural theory.

3. Nonlinear Systems Associated with Continuum Contragredient Lie Algebras

In [1] we suggested continuum analogue of the two-dimensional generalized Toda lattice generated via the Maurer-Cartan equation for two-dimensional 1-form $A=A_+dz_+ + A_-dz_-$, $dA+A\wedge A=0$, that is

$$[\partial/\partial z_+ + A_+, \partial/\partial z_- + A_-]=0 \; . \tag{7}$$

Here the functions $A_\pm$ take values in subspaces $g_0 \oplus g_{\pm 1}$ of the local part of the algebra (with the Cartan operator rather than matrix). However, these algebras have been defined in less generality in comparison with those given in Sect. 2. It is more essential in view of the investigation of these equations that the realization of the components $A_\pm$ of the 1-form used in [1] does not contain the necessary number of arbitrary functions. For this reason representation (7) had a formal meaning there. Now we give an explicit construction of the nonlinear equations, which are differential or integro-differential depending on the properties of the Cartan operator. The equations are associated with the algebras $g(E;K,S)$ given by the defining relations in form (4).

In accordance with the general algebraic approach for two-dimensional nonlinear dynamical systems [9] the equations generated by representation (7) with the functions $A_\pm$ taking values in the subspaces $\bigoplus_{0\leqq m\leqq m_\pm} g_{\pm m}$ of an arbitrary Z-graded Lie algebra $g=\bigoplus_{m\in Z} g_m$,

$$A_\pm = \sum_{0\leqq m\leqq m_\pm} E_{\pm m} \; , \qquad E_{\pm m} = \sum_{1\leqq \alpha\leqq d_{\pm m}} X^{(\alpha)}_{\pm m}(f^\alpha_{\pm m}) \; , \tag{8}$$

have the form

$$\begin{gathered} \partial E_{-0}/\partial z_+ - \partial E_{+0}/\partial z_- + [E_{+0}, E_{-0}] + \sum_{1\leqq m\leqq \min(m_\pm)} [E_{+m}, E_{-m}] = 0 \; , \\ \sum_{1\leqq m\leqq m_\pm} (\partial E_{\pm m}/\partial z_\mp + [E_{\mp 0}, E_{\pm m}]) + \sum_{1\leqq m<n\leqq m_\pm} [E_{\mp m}, E_{\pm n}] = 0 \; . \end{gathered} \tag{9}$$

Here $X^{(\alpha)}_{\pm m}$ are the basis elements of $g_{\pm m}$, $f^\alpha_{\pm m}(z_+, z_-)$ are differentiable functions of variables z_+ and z_-, $m_\pm$ are some integers. These equations coincide only formally with the corresponding system associated in [9] with the Lie algebras of the discrete case. However, they are actually much more general, because, preserving the same differential properties over $z_\pm$, our equations for the case of the algebra $g(E;K,S)$ have a continuous (differential or integral) but not only matrix structure as the equations for the functions f. It is caused by that fact the the structural "constants" for the algebra $g(E; K, S)$ really are the kernels of the operators generated by K and S.

Note that Eq. (9) admit a gauge arbitrariness which can be eliminated, for example, by equating E_{-0} to zero with an appropriate transformation $A_\pm \to g^{-1}(\partial/\partial z_\pm + A_\pm)g$ with a function $g = \exp X_0(\varphi(z_+, z_-))$.

In the simplest case when the functions $A_\pm$ take values in the local part of g, that is $m_\pm = 1$,

$$A_\pm = X_0(u_\pm) + X_{\pm 1}(f_\pm) \ , \tag{10}$$

$u_\pm \equiv f^1_{\pm 0}, f_\pm \equiv f^1_{\pm 1}$, and system (9) reduces with account of commutation relations (4) to the unknown equations

$$\partial u_-/\partial z_+ - \partial u_+/\partial z_- + S(f_+ f_-) = 0 \ , \qquad K u_\pm = \pm \partial/\partial z_\pm \ln f_\mp \ . \tag{11}$$

They lead to the following nonlinear equation of the second order for the gauge invariant function $\varrho \equiv \ln f_+ f_-$

$$\Delta\varrho \equiv \partial^2 \varrho/\partial z_+ \partial z_- = \tilde{K} \exp \varrho \ , \tag{12}$$

where $\tilde{K} \equiv KS$. This equation was given in [1] as a continuous generalization of the Toda lattices together with a formal solution of the Goursát problem to it for the case of invertible operator $\tilde{K}$. Evidently, the algebras g and g' with $KS = K'S'$ give the same Eq. (12).

In gauge $E_{-0} = 0$ the functions $A_\pm$ from (10) are rewritten in the form

$$A_+ = X_0(u) + X_{+1}(1) \ , \qquad A_- = X_{-1}(\exp \varrho) \ .$$

Equation (11) reduce to

$$\partial u/\partial z_- = S \exp \varrho \ , \qquad Ku = \partial \varrho/\partial z_+ \ , \tag{13}$$

whereas Eq. (12) remains unchanged.

For the case of invertible operators K and/or S Eqs. (12) and (13) admit various equivalent forms. For example, for the invertible operator $\tilde{K}$ we have

$$\partial u/\partial z_- = S \exp \tilde{K}x \ , \qquad u = S \partial x/\partial z_+ \ , \qquad \Delta x = \exp \tilde{K}x \ , \tag{14i}$$

where $x = \tilde{K}^{-1}\varrho$; for invertible K:

$$\partial u/\partial z_- = S \exp Ky \ , \qquad u = \partial y/\partial z_+ \ ; \qquad \Delta y = S \exp Ky \ , \tag{14ii}$$

where $y = K^{-1}\varrho$;

$$\partial u/\partial z_- = S \exp K^2 \omega \ , \qquad u = K \partial \omega/\partial z_+ \ , \qquad \Delta \omega = K^{-1} S \exp K^2 \omega \ , \tag{14iii}$$

where $\omega = K^{-2}\varrho$.

Note that for the invertible and symmetrizable operator $\tilde{K}$ Eq. (12) is a variational Lagrange-Euler equation with the Lagrangian density

$$L = \int dt v(t) \, [1/2 \partial x(t)/\partial z_+ \int dt' \tilde{K}(t, t') \partial x(t')/\partial z_- + \exp \tilde{K}x(t)] \ .$$

Here we omit for brevity the dependence of the function x on the variables $z_\pm$; $v(t)\tilde{K}(t, t') = v(t')\tilde{K}(t', t)$.

Let us discuss some particular cases of these equations. Firstly, consider $\tilde{K} \equiv KS = 0$, for example, when $K = 1 + iH \equiv H_+$, $S = 1 - iH$, with H being the

Hilbert operator with the Cauchy kernel ($H^2=-1$), that is, the kernels $H_\pm$ are $\delta_\pm$-functions. Then Eq. (12) reduces to the Laplace equation, $\Delta\varrho=0$. If $\tilde{K}=1$ (the kernel is δ-function) the equation coincides with the Liouville equation, $\Delta\varrho=e^\varrho$. A nontrivial example of Eq. (12) of integro-differential type, considered in [1], is given by the operator $\tilde{K}=H_+$ or $\tilde{K}=H_-$, $\Delta\varrho(t)=\int_L dt' \exp\varrho(t')/(t'-t\pm i0)$. Example 8 in Sect. 2 gives an explanation why this equation is exactly integrable and why its general solution is related with those of the Liouville equation (see [1]).

The purely differential (with respect to variables z_+, z_- and t) subclass of Eq. (12) is obtained, for example, using the operators $K=\partial^2/\partial t^2$ and $S=I$ (Example 4 from Sect. 2, the kernel is $\delta_{,tt}$), or, equivalently, $K=S=\partial/\partial t$ (the kernels of K and S are $\delta_{,t}$). Equations (12) and (14) then have the forms

$$\Delta\varrho=(\exp\varrho)_{,tt}\ ,\quad \Delta x=\exp(x_{,tt})\ ,\quad \Delta y=y_{,tt}\exp y_{,t}\ ,\quad \Delta\omega=\exp\omega_{,tt}\ , \tag{15}$$

respectively. This example for functions depending only on z_++z_- and t, was considered in paper [8] in the context of integrable Hamiltonian systems associated with the Poisson algebra (see Example 3 in Sect. 2). Equations (15) are a direct continuous analogue of the two-dimensional Toda lattice related with the series A_n. This was discussed in [5].

A different (and also purely differential) example is given by the operator $K=\partial/\partial t$ (with the kernel $\delta_{,t}$, see example 5 in Sect. 2) and $S=I$.

Note that the involutive integrals of motion I_k for the system described by Eq. (12) with an invertible operator $\tilde{K}$ in which the function ϱ depends on two variables $z_++z_-\equiv\tau$ and t, that is, $\varrho_{,\tau\tau}=\tilde{K}\exp\varrho$, have the form $I_k=SpL^k$, $\partial I_k/\partial\tau=0$. Here $L=X_0(\tilde{K}^{-1}\varrho_{,\tau})+X_{+1}(1)+X_{-1}(\exp\varrho)$ – is the Lax-operator, $L_{,\tau}=[L,A]$, and $2A=X_0(\tilde{K}^{-1}\varrho_{,\tau})+X_{+1}(1)-X_{-1}(\exp\varrho)$.

For this case, the continuous analogue of the (proper) Bäcklund transformation (i.e. difference KdV equation) is the equation

$$\varrho_{,\tau}=d/dt(e^\varrho)\ . \tag{16}$$

This equation has the following Lax-pair

$$L=X_{+1}(e^\varrho)+X_{-1}(\varphi)\ ,\qquad A=X_{+2}(\phi)\ .$$

It immediately follows from the Lax representation (up to trivial transformations) that $\phi=\exp 2\varrho$ yields Eq. (16). The differentiation of (16) with respect to τ gives $2\varrho_{,\tau\tau}=d^2/dt^2(e^{2\varrho})$.

Conclusion

To conclude, we would like to stress that this paper represents a prelude rather than the complete story on a (in our opinion) very promising new class of infinite-dimensional continuum contragredient Lie algebras. These algebras, generated by their local part whose elements satisfy the defining commutation relations, can be considered as a continuous limit of the ordinary contragredient Lie algebras. In particular, the analogues of Kac-Moody algebras of finite growth arise when the generalized Cartan matrix is replaced by some distribution. In some special cases our algebras coincide with known algebras, which are considered here from a new standpoint.

It remains to develop a structural theory for the continuum contragredient Lie algebras with an appropriate bilinear form, system of roots, their "reflections," and the theory of their representations, in particular, with the highest weight modules, formulas for the characters, etc. Then, it will be possible to develop effective methods for the solution of the integrability problem for nonlinear systems (differential or integro-differential) associated with these algebras, which are the continuous analogues (in the sense of the Volterra method) of the discrete case.

We believe that the construction of the theory of the algebras in question will be useful for many branches of mathematics and physics and will lead to unexpected possibilities for them.

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Note added in proof. The axiomatic formulation of continuum Lie algebras $g(E; K, S)$ given in this paper can be generalized for the case when E is an arbitrary associative *noncommutative* algebra and/or the local algebra does not, in general, generate $g(E; K, S)$ as a whole. These generalizations will be published in our new paper as well as several new examples of continuum Lie algebras, in particular, an algebra of polynomial differential operators, different versions of $gl(\infty)$, and a wide class of cross-product Lie algebras. Moreover, there we will also describe Kac-Moody algebras, the algebra $S_0 \operatorname{Diff} T^2$ of infinitesimal area-preserving diffeomorphisms of the torus T^2, Fairlie-Fletcher-Zachos sine-algebras and their generalizations, etc., as special cases of cross-product Lie algebras. Note that already at the end of April (when the present paper appeared as a preprint), D. B. Fairlie illuminated a connection between Example 2.3 and sine-algebra and made us acquainted with this result. We are grateful to him for this information.