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The Weil-Petersson Geometry of the Moduli Space of $SU(n \geqq 3)$ (Calabi-Yau) Manifolds I

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Abstract. The Weil-Petersson metric is defined on the moduli space of Calabi-Yau manifolds. The curvature of this Weil-Petersson metric is computed and its potential is explicitely defined. It is proved that the moduli space of Calabi-Yau manifolds is unobstructed (see Tian).

Dedicated to Lipman Bers on the occasion of his 75th birthday

0.1. Introduction

In this paper we are going to study some differential-geometric properties of the moduli space of compact complex manifolds of $\dim_{\mathbb{C}} \geqq 3$ which admit non-flat metrics g with holonomy groups $H(g) \neq \{0\}$ and $H(g) \subseteqq SU(n)$. Such manifolds we will call $SU(n)$ or Calabi-Yau manifolds. Before stating the main results, we will make several remarks.

Remark 0.1.1. It is not difficult to see that a metric on a compact complex manifold whose holonomy $H^0 \neq \{0\}$ and $H \subseteqq SU(n)$, will be Kähler and Ricci flat. We will call it the Calabi-Yau metric. (See [2]).

Remark 0.1.2. If M is a Calabi-Yau manifold, then from the theory of invariants of the group $SU(n)$ and the fact that the holonomy group $H^0 \neq \{0\}$ and $H \subseteqq SU(n)$, it follows $H^0(M, \Omega^i) = 0$ for $1 < i < n$ and $H^0(M, \Omega^n)$ is spanned by a holomorphic n-form w_0, which has no zeroes and no poles. This implies that $c_1(M) = 0$. Constructions of Calabi-Yau manifolds are based on the solution of the Calabi conjecture by Yau. See [15].

Recently $SU(3)$ manifolds have attracted the interest of physicists working on string theory and algebraic geometers working on the classification of threefolds and on algebraic cycles.

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Let me state the results that are contained in this paper. In Sect. 1 the following theorem is proved:

Theorem 1. *Let M be a Calabi-Yau ($SU(n \geqq 3)$) manifold, where $n=\dim_{\mathbb{C}} M$. Let $\pi: X \to S \ni 0$, $\pi^{-1}(0)=M$ be the Kuranishi family of M, then S is a non-singular complex analytic space such that*

$$\dim_{\mathbb{C}} S=\dim_{\mathbb{C}} H^1(M,\theta)=\dim_{\mathbb{C}} H^1(\Omega^{n-1}). \qquad \text{See also [13].}$$

More precisely we have proved Theorem 1′. From Theorem 1′ follows Theorem 1 and our curvature computations are based on Theorem 1′.

Theorem 1′. *Let M be a Calabi-Yau ($SU(n \geqq 3)$) manifold. Let $(g_{\alpha\bar{\beta}})$ be a Calabi-Yau metric on M. Let $\mathbb{H}^1(M,\theta)$ denote the harmonic elements of $H^1(M,\theta)$ with respect to $(g_{\alpha\bar{\beta}})$, let ϕ_1 be any element of $\mathbb{H}^1(M,\theta)$, then there exists a unique power series*

$$\phi(t)=\phi_1 t+\phi_2 t^2+\ldots+\phi_N t^N+\ldots$$

such that for $|t|<\varepsilon$,

a) $\phi(t) \in C^\infty(M, \Omega^{0,1} \otimes \theta_M)$.

b) $\bar{\partial}^* \phi(t)=0$, *where $\bar{\partial}^*$ is the adjoint operator of $\bar{\partial}$ with respect to $(g_{\alpha\bar{\beta}})$.*

c) $\bar{\partial}\phi(t)-\frac{1}{2}[\phi(t),\phi(t)]=0$.

d) *for each $K \geqq 2$ $\phi_K \lrcorner \omega_0=\partial \Psi_K$, where $\omega_0 \in H^0(M,\Omega^n)$ and ω_0 has no zeroes.*

Theorem 1 was first announced by F.A. Bogomolov in [18]. Later P. Candelas, G. Horowith, A. Strominger, and E. Witten proved Theorem 1 under the assumption that

$$H^2(M,\mathbb{Z}) \cong \mathbb{Z}. \qquad ([17])$$

Theorem 1 was also proved by Tian independently. Next we are going to describe the results in Sect. 2. So we need some definitions in order to formulate the results.

Definition. A pair (M,L) where M is a Calabi-Yau manifold and $L \in H^2(M,\mathbb{R})$ will be called a polarized $SU(n)$ manifold if $L=[\operatorname{Im} g^1_{\alpha\bar{\beta}}]$, where $(g^1_{\alpha\bar{\beta}})$ is a Kähler metric on M.

With $[\omega]$ we will denote the class of cohomology of a form ω. From now on we will suppose that L is fixed.

Suppose that $M \to S$ is the Kuranishi family of polarized Calabi-Yau manifold (M,L), so may be after shrinking S we may suppose that for each $s \in S$ on $M_s=\pi^{-1}(s)$ there exists a unique Ricci-flat Kähler metric $g_{\alpha\bar{\beta}}(s)$ such that $[\operatorname{Im}_{\alpha\bar{\beta}}(s)]=L$. The last fact follows from Yau's solution of Calabi's conjecture, Kodaira's stability theorem, which states that small deformations of Kähler manifold is Kähler and the fact that for $SU(n>3)$ manifolds $H^2(X,\Theta_X)=0$. From $h^{2,0}=0$ it follows that M is an algebraic manifold. Here we use the fact $n \geqq 3$, since if $n=2$ $h^{2,0}=1$. Now we can identify the tangent space at $s \in S$, $T_{s,S}$ with $\mathbb{H}^1(M_s,\Theta_s)$, where $\mathbb{H}^1(M_s,\Theta_s)$ is the harmonic part of $H^1(M_s,\Theta_s)$ with respect to $g_{\alpha\bar{\beta}}(s)$ and Θ_s is the sheaf of holomorphic vector fields. Now we are ready to define Weil-Petersson metric on S- the local moduli space of (M,L).

Definition. Let $\phi_1, \phi_2 \in T_{s,S}=\mathbb{H}^1(M_s,\Theta_s)$, then

$$\langle \phi_1, \phi_2 \rangle_{\text{w.p.}} := \int_{M_s} \phi^\mu_{1\bar{\alpha}} \overline{\phi^\nu_{2\bar{\beta}}} g_{\mu\bar{\nu}} g^{\beta\bar{\alpha}} \operatorname{vol}(g_{\alpha\bar{\beta}}(s)).$$

Here we are using the usual Einstein's conventions for summation.

In Sect. 2 we calculated the Weil-Petersson metric on the moduli space of polarized $SU(n\geqq 3)$ manifolds in terms of the standard cup product on $H^{n-1,1}$, i.e.

$$\langle u,v\rangle_{\text{w.p.}}=(-1)^{\frac{n(n-2)}{2}}(i)^{n-2}\int_M u\wedge\bar{v},\quad u,v\in H^{n-1,1}.$$

In order to simplify the computation of the curvature tensor $R_{\alpha\bar{\beta},\mu\bar{v}}$ of the Weil-Petersson metric $(h_{\mu\bar{v}})$ we need to find "good" local coordinates $(t^1,\dots,t^K)$ in S so that

$$h_{\mu\bar{v}}=\delta_{\mu\bar{v}}+\sum h_{\mu\bar{v},\alpha\bar{\beta}}t^\alpha\overline{t^\beta}+(\text{higher order terms}).$$

In Sect. 2 it is proved that such a coordinate system exists and so $\{h_{\mu\bar{v}}\}$, i.e. the Weil-Petersson metric is a Kähler metric. Let me describe how one fixes such a "good" coordinate system which we call "Kodaira-Spencer-Kuranishi" local in S. Let $\{n_v\}$ $v=1,\dots,K$ be a basis in $\mathbb{H}^1(M,\Theta)$ and let

$$\phi(t)=\sum\eta_\alpha t^\alpha+\sum_{(i_1,\dots,i_K)}^{\infty}\phi_{i_1,\dots,i_K}(t^1)^{i_1}\dots(t^K)^{i_K},\quad i_j>0,\quad \sum i_j\geqq 2$$

be the power series with the properties stated in Theorem 1′, then it is proved in Sect. 2, that $(t^1,\dots,t^K)$ will be a good local coordinate system, namely the following lemma is proved.

Lemma. *Let $(h_{\alpha\bar{\beta}})$ be the Weil-Petersson metric on S, then with respect to Kodaira-Spencer-Kuranishi local coordinates the following formula is true:*

$$\begin{aligned}h_{\alpha\bar{\beta}}(t,\bar{t})=(-1)^{\frac{n(n-1)}{2}}(i)^{n-2}\Bigg\{\Bigg[&\int_M(\eta_\alpha\lrcorner\omega_0)\wedge\overline{(\eta_\beta\lrcorner\omega_0)}\\&+4t^\alpha\overline{t^\beta}\int_M[\Lambda^2h_\alpha\lrcorner\omega_0]\wedge\overline{[\Lambda^2h_\beta\lrcorner\omega_0]}\\&+2t^\alpha\overline{t^\mu}\int_M[\Lambda^2\eta_\alpha\lrcorner\omega_0]\wedge[(\eta_\beta\wedge\bar{\eta}_\mu)\lrcorner\omega_0]\\&+t^\mu\overline{t^\nu}\int[\eta_\alpha\wedge\eta_\mu\lrcorner\omega]\wedge\overline{[\eta_\beta\wedge\eta_\nu\lrcorner\omega_0]}\\&+(\text{terms of order }\geqq 3)\Bigg]\Big[1-\sum\delta_{\alpha\bar{\beta}}t^\alpha\overline{t^\beta}\Big]\Bigg\},\end{aligned}\qquad (*)$$

where

$$\Lambda^2\eta_\alpha:\Lambda^2\Omega^{1,0}\to\Lambda^2\Omega^0(\Lambda^2\eta_\alpha(u\wedge v)$$
$$=\eta_\alpha(u)\wedge\eta_\alpha(v))\quad\text{and}\quad[\Lambda^2\eta_\alpha\lrcorner\omega_0],[\eta_\alpha\wedge\eta_\mu\lrcorner\omega_0]$$

denote the cohomology class of $\mathbb{H}(\Lambda^2\eta_\alpha\lrcorner\omega_0)$ and $\mathbb{H}[\eta_\alpha\wedge\eta_\mu\lrcorner\omega_0]$ in $H^{n-2,2}\subset H^n(M,\mathbb{C})$, where $\mathbb{H}$ is the harmonic projection. From this lemma we derive the following theorem:

Theorem 2. a) *The following formulas are true for the curvature tensor $R_{\alpha\bar{\beta},\mu\bar{v}}$ of the Weil-Petersson metric on the moduli space of $SU(n\geqq 3)$ manifolds*

$$R_{\alpha\bar{\beta},\mu\bar{v}}=(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}2\int_M[\eta_\alpha\wedge\eta_\mu\lrcorner\omega_0]\wedge\overline{[\eta_\beta\wedge\eta_\nu\lrcorner\omega_0]}-\delta_{\mu\bar{v}}.$$

From observation 1 (**) it follows that

$$h_{\alpha\bar{\beta}}(t,\bar{t})=(-1)^{\frac{n(n-1)}{2}}(i)^{n-2}\left(\int_M \frac{d\omega_t}{dt^\alpha}\wedge\overline{\frac{d\omega_t}{dt^\beta}}\right)\left(\int_M \omega_t\wedge\bar{\omega}_t\right)^{-1} \qquad (***)$$

and $\phi(t,\bar{t})=\log\left[(-1)^{\frac{n(n-1)}{2}}(i)^{n-2}\left(\int_M \omega_t\wedge\bar{\omega}_t\right)\right]$ *is the potential for the Weil-Petersson metric.*

If $\alpha\neq\mu$ *and* $\beta\neq\nu$,

$$R_{\alpha\bar{\beta},\alpha\bar{\beta}}=+(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}8\int_M [\Lambda^2\eta_\alpha\lrcorner\omega_0]\wedge\overline{[\Lambda^2\eta_\beta\lrcorner\omega_0]},$$

$$R_{\alpha\bar{\beta},\alpha\bar{\nu}}=(-1)^{\frac{n(n+1)}{2}}(\sqrt{-1})^{n-2}4\int_M [\Lambda^2\eta_\alpha\lrcorner\omega_0]\wedge\overline{[\eta_\beta\wedge\eta_\nu\lrcorner\omega_0]}, \qquad \beta\neq\mu.$$

b) *The biholomorphic sectional curvature of the Weil-Petersson metric on the moduli space of* $SU(n\geqq 3)$ *manifolds is negative.*

c) *Curvature operator* $\leqq 0$.

The proof of the lemma is based on the following two observations.

Observation 1. $\langle\phi_1,\phi_2\rangle_{\text{w.p.}}=(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int(\phi_1\lrcorner\omega_0)\wedge(\overline{\phi_2\wedge\omega_0})$, where $\int\omega_0\wedge\bar{\omega}_0=\int\mathrm{vol}(g_{\alpha\bar{\beta}})$.

This formula says that in case of $SU(n)$ manifold we do not need the Calabi-Yau metric in order to define the Weil-Petersson metric. We only need the polarization class since if $\int\omega_t\wedge\bar{\omega}_t=\int L^n$, then we have a canonical isomorphism $\alpha: H^1(M,\Theta)\to H^1(\Omega^{n-1})\alpha(\phi)=\phi\lrcorner\omega_0$. On $H^{n-1,1}$ we have a canonical metric; $\langle a,b\rangle=(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_M a\wedge\bar{b}$. This is so since if $n\geqq 3$ all elements of $H^{n-1,1}$ are primitive and $H^{n-2,0}=+H^0(M,\Omega^{n-2})=0$.

Let w_t be the holomorphic n-form on X_t. Then we have the following formula for $w_t\in\Gamma(X,\Omega_{X/S})$:

$$\omega_t=\omega_0+\sum_{K=1}^{n}(-1)^{\frac{K(K-1)}{2}}\Lambda^K\phi(t)\lrcorner\omega_0,$$

where

$$\Lambda^K\phi(t)\in\Gamma(M,\mathrm{Hom}(\Lambda^1\Omega^{\Lambda,0},\Lambda^1\Omega^{0,1}))$$

and

$$\Lambda^K\phi(t)(u_1\Lambda\ldots\Lambda u_k)=\phi(t)(u_1)\wedge\ldots\wedge\phi(t)(u_k).$$

From (***) Theorem 2 follows almost directly.

Remark 1. It is a well known fact that the moduli space of marked polarized $K3$ surfaces is $SO(2,19)/SO(2)\times SO(19)$. From observation 1 it follows that the Weil-Petersson metric is the Bergmann metric on $SO(2,19)/SO(2)\times SO(19)$. (See also [11]).

Some Historical Notes. The purpose of introducing the invariant metric on the moduli space (in the case of Riemann surfaces on the Teichmüller space), is to provide information on the intrinsic properties of the space. The Weil-Petersson metric has successfully filled this role in the case of Riemann surfaces of genus $g\geqq 2$.

Ahlfors was the first to consider the curvature of the Weil-Petersson metric in the case of Riemann surfaces, i.e. on the Teichmüller space. See [1]. He obtained singular integral formulas for the Riemann curvature tensor. As an application he found that the Ricci, holomorphic sectional and scalar curvatures are all negative. Royden later showed that the holomorphic sectional curvature is bounded away from zero. Tromba gave a complete formula for the curvature of the Weil-Petersson metric on Teichmüller space and found that the general sectional curvature is negative. See [14].

Later Scott Wolpert gave other formulas for the curvature tensor of the Weil-Petersson metric on the Teichmüller space of Riemann surfaces of genus $g \geqq 2$. From his formulas S. Wolpert showed that the holomorphic sectional and Ricci curvatures are bounded above by $\frac{-1}{2\pi(g-1)}$ and the scalar curvature is bounded above by $\frac{-3(3g-1)}{4\pi}$. S. Wolpert showed that the curvatures are governed by the spectrum of the Laplacian. See [16]. J. Royden also obtained similar results. Later S. Wolpert used his calculations of the curvature tensor of the Weil-Petersson metric to get some information of the global structure of the moduli space of Riemann surfaces.

Siu generalized the formulas of S. Wolpert in the case of algebraic manifolds with Ricci <0 and complex dimension $\geqq 2$. See [12]. Nannacini obtained formulas similar to Siu's in the case of $SU(n \geqq 2)$ polarized manifolds. See [9]. Unfortunately Siu's and Nannacini's formulas did not say anything about the sign of the curvature. Royden also obtained some formulas for manifolds of $\dim \geqq 2$ and Ricci <0.

Koiso was the first to introduce the Weil-Petersson metric in $\dim \geqq 2$. See [7] and [4].

Review of Tian's results. See [13]. In his paper Tian proved that the Weil-Petersson metric is just the pullback of the Chern form of the tautological of $\mathbb{CP}^N$ restricted to the period domain, which is an open set of a quadric in $\mathbb{CP}^N$. From this description Tian obtained that the Weil-Petersson metric is a Kähler one and the holomorphic sectional curvature is bounded away from zero.

All the results in [13] that overlap with the results of this paper are obtained independently by Tian.

Wolf in his thesis obtained a similar results as Theorem 2.6 in the case of the Teichmüller theory of Riemann surfaces. See [16].

Recently the author found some applications of the results of the present paper. Namely we prove the analogue of the Global Torelli theorem for $SU(n \geqq 3)$ manifolds. The proof is based on the fact that the discs D_α, defined in *Observation 2* are totally geodesic submanifolds. We proved that the Weil-Petersson metric is complete on the Teichmüller space of the Calabi-Yau manifolds. From this result we obtained some interesting degenerations and simultaneous resolutions of singularrites of the one parameter family of Calabi-Yau manifolds. We also proved similar results to that of Beauville in the case of complete intersections, namely that the image of

$$\mathrm{Diff}_+(M) = \{\text{all diffeomorphisms that preserve the orientation of } M\}$$

has a finite index in $\mathrm{Aut}\, H^n(M, \mathbb{Z})$. See [3].

At last we should mention the following result, that will appear in a joint paper with D. Bao and T. Ratiu. Let M be the moduli space of polarized manifold (X_t, g_t), where g_t is a Kähler-Einstein-Calabi-Yau metric, $[\operatorname{Im} g_t]=L$, then as in the case of Calabi-Yau manifolds we can define the Weil-Petersson metric on M. Let $\det\bar\partial$ be the determinant line bundle on M that corresponds to different $\bar\partial_t$ operators on X_t and let $\|\ \|_Q$ be the Quillen metric on $\det\bar\partial$, then

Theorem $\partial\bar\partial\log\|\ \|_Q$ is the Weil-Petersson metric on M.

0.2. *Conventions on Some Relations.*

$(z^1, \dots, z^n)$ will denote a system of local coordinates on a compact complex manifold. (0.2.1.a)

$dz^1\wedge\dots\wedge\widehat{dz^{i_1}}\wedge\dots\wedge\widehat{dz^{i_K}}\wedge\dots\wedge dz^n$ means that if $i_1<\dots<i_K$ then $dz^{i_1}, \dots, dz^{i_K}$ are omitted. (0.2.1.b)

0.2.2 Given a Hermitian metric $ds^2=h_{\alpha\bar\beta}dt^\alpha \Lambda d\bar t^\beta$ on a complex manifold, we say that it is Kähler if

$$\frac{\partial h_{\alpha\bar\beta}}{\partial t^\gamma}=\frac{\partial h_{\gamma\bar\beta}}{\partial t^\alpha}. \tag{0.2.2.1}$$

A metric is Kähler if and only if we can find normal coordinates at each point, i.e. holomorphic coordinates such that at the point the metric tensor has the development $h_{\alpha\bar\beta}=\delta_{\alpha\bar\beta}+0(|t|^2)$. If the metric is Kähler and real analytic, one can introduce a set of canonical coordinates at a point which are characterized by the property that the power series for $h_{\alpha\bar\beta}$ contains no terms which are products only of unbarred (or only of barred variables). In terms of canonical coordinates

$$h_{\alpha\bar\beta}=\delta_{\alpha\beta}+\tfrac{1}{2}R_{\alpha\bar\beta,\gamma\bar\delta}t^\gamma t^{\bar\delta}+0(|t|^3), \tag{0.2.2.2}$$

where $h_{\alpha\bar\beta,\gamma\bar\delta}$ is the Riemann curvature tensor. If $(\xi^1, \dots, \xi^K)$ and $(\eta^1, \dots, \eta^K)$ are unit tangent vectors, the holomorphic bisectional curvature in direction ξ, η is given by:

$$K_{\xi\eta}=R_{\alpha\bar\beta,\gamma\bar\delta}\xi^\alpha\bar\xi^\beta\bar\xi^\gamma\bar\xi^\delta. \quad \text{See [10]} \tag{0.2.2.3}$$

and the holomorphic sectional curvature in direction ξ is given by

$$K_{\xi\xi}=R_{\alpha\bar\beta,\gamma\bar\delta}\xi^\alpha\bar\xi^\beta\bar\xi^\gamma\bar\xi^\delta. \quad \text{See [10]}. \tag{0.2.2.4}$$

So we have proved in Sect. 2 that Kodaira-Spencer-Kuranishi coordinates are normal coordinates. So we apply (0.2.2.3) and (0.2.2.4) in order to get Theorem 2.

1. The Kuranishi Space of a $SU(n)$ Manifold M is Unobstructed

1.1. Remark. a) From now on we will suppose that M is an $SU(n)$ manifold with a fixed Calabi-Yau metric $(g_{\alpha\bar\beta})$, i.e. $g_{\alpha\bar\beta}$ is a Kähler, Ricci-flat metric on M.

b) If ϕ is any element of $H^j(M, \Lambda^K\Theta)$, then by $\mathbb{H}\phi$ we will denote the harmonic part of ϕ and by $\mathbb{H}^j(M, \Lambda^K\Theta)$ all harmonic tensors on M which are elements of $H^j(M, \Lambda^K\Theta)$ with respect to the Calabi-Yau metric.

c) For any point $x \in M$ from now on we will chose the local coordinates $(z^1, \ldots, z^n)$ in $U \ni x$ in such a way that

$$\omega_0|_U = dz^1 \wedge \ldots \wedge dz^n,$$

where ω_0 is the holomorphic form without zeroes on M.

Theorem 1.2. *Let M be a $SU(n)$ manifold and let*

$$\begin{array}{ccc} M & \subset & X \\ \downarrow & & \downarrow \pi \\ 0 & \in & S \end{array}$$

be the Kuranishi family of M, then

a) *S is a non-singular complex manifold,*

b) $\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} H^1(M, \Theta)$.

Proof. Let us first remember how the Kuranishi family is defined. We define $\bar{\partial}^*$ to be the adjoint of $\bar{\partial}$ with respect to the Calabi-Yau metric, $\square$ to be the Laplace operator, and G to be the Green operator. Let $\{\eta_\nu | \nu = 1, \ldots, m\}$ be a base for $\mathbb{H}^1(M, \Theta)$. Kuranishi proved that the power series solution of the equation

$$\phi(t) = \eta(t) + \tfrac{1}{2}\bar{\partial}^* G[\phi(t), \phi(t)],$$

where $\eta(t) = \sum_{\nu=1}^{m} t_\nu \eta_\nu$ has a unique convergent power series solution. And this $\phi(t)$ satisfies

$$\bar{\partial}\phi(t) - \tfrac{1}{2}[\phi(t), \phi(t)] = 0$$

if and only if

$$\mathbb{H}[\phi(t), \phi(t)] = 0.$$

Let $\{\beta_\lambda | \lambda = 1, \ldots, Z\}$ be an orthonormal base of $\mathbb{H}^2(M, \Theta)$ and let $\langle\ ,\ \rangle$ be the inner product in

$$A^2 = \Gamma(\Omega^{0,2} \otimes \Theta).$$

Then

$$\mathbb{H}[\phi(t), \phi(t)] = \sum_{\lambda=1}^{\tau} \langle[\phi(t), \phi(t)], \beta_\nu\rangle \beta_\nu.$$

Hence $\mathbb{H}[\phi(t), \phi(t)] = 0$ iff $\langle[\phi(t), \phi(t)], \beta_\nu\rangle = 0$ for $\lambda = 1, \ldots, \tau$. Since $\lambda = 1, \ldots, \tau$. Since $\phi(t)$ is a power series in t so is $\langle[\phi(t), \phi(t)], \beta_\nu\rangle = b_\nu(t)$. Thus $b_\nu(t)$ is holomorphic in t for $\lambda = 1, \ldots, \tau$ and $|t| < \varepsilon$. Then Kuranishi proved that S is defined as follows:

$$S = \{t \,|\, |t| < \varepsilon, b_\lambda(t) = 0, \lambda = 1, \ldots, \tau\}.$$

We have a family $X \xrightarrow{\pi} S$ such that it is locally complete and $\pi^{-1}(0) = M$. From all this it follows that if we prove that for each η_ν, $\nu = 1, \ldots, r$, there exists a power series (convergent)

$$\phi_\nu(t) = \eta_\nu t + \phi_2^\nu t^2 + \ldots + \phi_K^\nu t^K \ldots$$

such that:

a) $$\bar{\partial}\phi_\nu(t)=\tfrac{1}{2}[\phi_\nu(t),\phi_\nu(t)]$$

b) $\phi_\nu(t)$ fulfills the following equation:

$$\phi_\nu(t)=\eta_\nu t+\tfrac{1}{2}\bar{\partial}^*G[\phi_\nu(t),\phi_\nu(t)],$$

then $b_\nu(t)=0$ and so S is an open subset in $\mathbb{H}^1(M,0)$, i.e. S is an non-singular manifold of dimension equal to the $\dim H^1(M,\Theta)$.

So we need to prove that for each $\eta_\nu\in\mathbb{H}^1(M,\Theta)$ $\nu=1,\ldots,\gamma$ we can find a power series

$$\phi_\nu(t)=\eta_\nu t+\phi_2^\nu t^2+\ldots+\phi_K^\nu t^K+\ldots$$

such that

a) $$\bar{\partial}\phi_\nu(t)=\tfrac{1}{2}[\phi_\nu(t),\phi_\nu(t)],$$

b) $$\phi_\nu(t)=\eta_\nu t+\tfrac{1}{2}\bar{\partial}^*G[\phi_\nu(t),\phi_\nu(t)].$$

Lemma 1.2.1. *Let $\phi_\nu(t)=\eta_\nu t+\phi_2^\nu t^2+\ldots+\phi_K^\nu t^K+\ldots$ be convergent power series such that*

a) $$\bar{\partial}^*\phi_\nu(t)=0$$

b) $$\bar{\partial}\phi_\nu(t)=\tfrac{1}{2}[\phi_\nu(t),\phi_\nu(t)],$$

then

$$\phi_\nu(t)=\eta_\nu t+\tfrac{1}{2}\bar{\partial}^*G[\phi_\nu(t),\phi_\nu(t)].$$

Proof. $\phi_\nu(t)-\mathbb{H}\phi_\nu(t)=G\Box\phi_\nu(t)=G(\bar{\partial}^*\bar{\partial}+\bar{\partial}\bar{\partial}^*)\phi_\nu(t)=G\bar{\partial}^*\bar{\partial}\phi_\nu(t)$. This is so since $\bar{\partial}^*\phi_\nu(t)=0$. From the equality

$$\phi_\nu(t)-\mathbb{H}\phi_\nu(t)=G\bar{\partial}^*(\bar{\partial}\phi_\nu(t)),$$

and from $\bar{\partial}\phi_\nu(t)=\frac{1}{2}[\phi_\nu(t),\phi_\nu(t)]$ we get

$$\phi_\nu(t)=\mathbb{H}\phi_\nu(t)+\tfrac{1}{2}\bar{\partial}^*G[\phi_\nu(t),\phi_\nu(t)]=\eta_\nu t+\tfrac{1}{2}\bar{\partial}^*G[\phi_\nu(t),\phi_\nu(t)].\quad \text{Q.E.D.}$$

From all these facts it follows that we need to solve by induction the following equations:

$$\begin{aligned}&\bar{\partial}\phi_2^\nu=\tfrac{1}{2}[\eta_\nu,\eta_\nu],\quad \text{where}\quad \bar{\partial}^*\phi_2^\nu=0\\ &\vdots\\ &\bar{\partial}\phi_{N+1}^\nu=\tfrac{1}{2}([\phi_N^\nu,\eta_\nu]+[\phi_{N-1}^\nu,\phi_2^\nu]+\ldots+[\eta_\nu,\phi_N^\nu])\end{aligned}\qquad (*)$$

where $\bar{\partial}^*\phi_{N+1}^\nu=0$.

The solutions of (*) is based on the following lemmas:

Lemma 1.2.2. *For each $\eta\in\mathbb{H}^1(M,\Theta)$ $\eta\lrcorner\omega_0$ is a harmonic form of type $(n-1,1)$.*

Proof. Let $\eta|_U=\sum\eta_{\bar{\alpha}}^\mu d\bar{z}^\alpha\otimes\dfrac{\partial}{\partial z^0}$ and $\omega_0|_U=dz^1\wedge\ldots\wedge dz^n$, then

$$\eta\lrcorner\omega_0|_U=\sum(-1)^{\mu-1}\eta_{\bar{\alpha}}^\mu d\bar{z}^\alpha\wedge\ldots\wedge d\hat{z}^\mu\wedge\ldots\wedge dz^n.$$

Now clearly

$$\bar{\partial}\eta=0 \Rightarrow \bar{\partial}(\eta\lrcorner\omega_0)\equiv 0.$$

Next we need to prove that

$$\bar{\partial}^*(\eta\lrcorner\omega_0)=0.$$

The proof of this fact is based on the following fact:

$$(\bar{\partial}^*\phi)_{A_p,\bar{B}_q}=(-1)^{p+1}\sum_{\beta} g^{\bar{\beta}\alpha}\nabla_\alpha\phi_{A_p,\bar{\beta}\bar{B}_q}, \qquad \text{see [8]}. \tag{**}$$

From the formula

$$\nabla(\eta\lrcorner\omega_0)=\nabla\eta\lrcorner\omega_0\pm\eta\lrcorner\nabla\omega_0$$

and from the Bochner principle, that on any Ricci flat compact complex manifolds any holomorphic tensor is parallel, we get that $\nabla\omega_0=0$. See [8]. So

$$\nabla_\alpha(\eta\lrcorner\omega_0)=\nabla_\alpha\eta\lrcorner\omega_0.$$

From this formula we get that

$$\bar{\partial}^*(\eta\lrcorner\omega_0)=(\bar{\partial}^*\eta)\lrcorner\omega_0=0. \quad \text{Q.E.D.}$$

Lemma 1.2.3. *For each* $\eta\in\mathbb{H}^1(M,\Theta)$ *we have that if* $\eta|_U=\sum\eta^\mu_{\bar{\alpha}}d\bar{z}^\alpha\otimes\frac{\partial}{\partial z^\mu}$, *then*

$$\sum_{\mu=1}^{\mu}\partial_\mu\phi^\mu_{\bar{\alpha}}=0 \quad \forall\alpha=1,\dots,n.$$

Proof. We know that $\eta\lrcorner\omega_0$ is a harmonic form on a Kähler manifold so

$$\partial(\eta\lrcorner\omega_0)=0.$$

On the other hand

$$\eta\lrcorner\omega_0|_U=\sum(-1)^{\mu-1}\eta^\mu_{\bar{\alpha}}d\bar{z}^\alpha\wedge dz^1\wedge\dots\wedge d\hat{z}^\mu\wedge\dots\wedge dz^\mu.$$

So

$$\partial(\eta\lrcorner\omega_0|_U)=\sum_\alpha\left(\sum_\mu\partial_\mu\phi^\mu_{\bar{\alpha}}\right)d\bar{z}^\alpha\wedge dz^1\wedge\dots\wedge dz^\mu\wedge\dots\wedge dz^\mu\equiv 0.$$

From here $\Rightarrow\sum\partial_\mu\phi^\mu_{\bar{\alpha}}=0.$ Q.E.D.

Lemma 1.2.4. *Let* $\phi,\psi\in\Gamma(M,\Omega^{0,1}\otimes\Theta)=\Gamma(M,\mathrm{Hom}(\Omega^{1,0},\ \Omega^{0,1}))$ *and* $\partial(\phi\lrcorner\omega_0)=\partial(\psi\lrcorner\omega_0)=0$, *then*

$$2\partial(\phi\wedge\psi\lrcorner\omega_0)=([\phi,\psi]\lrcorner\omega_0),$$

where $\phi\wedge\Psi\in\Gamma(M,\mathrm{Hom}(\Lambda^2\Omega^{1,0},\Lambda^2\Omega^{0,1}))$ *and*

$$(\phi\wedge\psi)(u\wedge v):=\phi(u)\wedge\psi(v).$$

Proof. We have

$$2\phi\wedge\psi|_U=\left(\sum_{i<j}\phi^i\wedge\psi^j-\psi^i\wedge\phi^j\right)\otimes\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j}.$$

Here

$$\phi|_U = \sum \phi^i_{\bar\mu} d\bar z^\mu \otimes \frac{\partial}{\partial z^i}, \quad \psi|_U = \sum \psi^j_{\bar\nu} d\bar z^\nu \otimes \frac{\partial}{\partial z^j}$$

and

$$\phi^i = \sum_\mu \phi^i_{\bar\mu} d\bar z^\mu, \quad \psi^j = \sum_\nu \psi^j_{\bar\nu} d\bar z^\nu .$$

From these formulas we get:

$$\begin{aligned} &2(\phi \wedge \psi \perp \omega_0)|_U \\ &= \sum_{i<j} (-1)^{i+j-2} (\phi^i \wedge \psi^j - \psi^i \wedge \phi^j) dz^1 \wedge \ldots \wedge d\hat z^i \wedge \ldots \wedge d\hat z^j \wedge \ldots \wedge dz^n . \end{aligned}$$

Let us compute the coefficient of $2\partial(\phi \wedge \psi \perp \omega_0)$ in front of $dz^1 \wedge \ldots \wedge dz^i \wedge \ldots d\hat z^j \wedge \ldots \wedge dz^\mu$. So we have

$$\begin{aligned} 2\partial(\phi \wedge \psi \perp \omega_0)|_U = \sum_j \Big[\sum_i (-1)^{j-1} (\partial_i \phi^i \wedge \psi^j - \partial_i \psi^i \wedge \phi^j + \phi^i \partial_i \psi^j - \psi^i \wedge \partial_i \phi^j) \\ \wedge dz^1 \wedge \ldots \wedge dz^i \wedge \ldots \wedge d\hat z^j \wedge \ldots \wedge dz^\mu \Big], \end{aligned}$$

where

$$\partial_i \phi^i = \sum_{\mu=1}^{n} (\partial_i \phi^i_{\bar\mu}) d\bar z, \qquad \partial_i \psi^i = \sum_{\nu=1}^{\mu} \partial_i \psi^i_{\bar\nu} d\bar z^\nu . \tag{1.2.4.1}$$

From

$$\partial(\phi \perp \omega_0) = \partial(\psi \perp \omega_0) = 0 \Rightarrow \sum_{i=1}^{\mu} \partial_i \phi^i = \sum_{i=1}^{\mu} \partial_i \psi^i = 0 .$$

So from these formulas and (1.2.4.1) it follows that:

$$\begin{aligned} &2\partial(\phi \wedge \psi \perp \omega_0)|_U \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} (-1)^{1+j} (\phi^i \partial_i \psi^j - \psi^i \partial_i \phi^j) \wedge dz^1 \wedge \ldots \wedge d\hat z^j \wedge \ldots \wedge dz^n \right). \end{aligned} \tag{1.2.4.2}$$

From the definition of $[\phi, \psi]$, i.e.

$$[\phi, \psi]|_U = \sum_j \left(\sum_i (\phi^i \partial_i \psi^j - \psi^i \partial_i \phi^j) \right) \otimes \frac{\partial}{\partial z^j}, \tag{1.2.4.3}$$

and (1.2.4.2) we get that

$$2\partial[(\phi \wedge \psi) \perp \omega_0] = [\phi, \psi] \perp \omega_0 . \quad \text{Q.E.D.}$$

Lemma 1.2.5. *Let* $\phi_i \in \Gamma(M, \mathrm{Hom}(\Omega^{1,0}, \Omega^{0,1}))$ *for* $2 \leqq i \leqq N$ *and*

a) $\bar\partial \phi_i = \frac{1}{2}([\phi_{i-1}, \phi_1] + [\phi_{i-2}, \phi_2] + \ldots + [\phi_2, \phi_{i-1}] + [\phi_1, \phi_{i-1}],$

b) $\bar\partial \phi_1 = 0 .$

Then

$$\bar{\partial}(\tfrac{1}{2}([\phi_N, \phi_1]+[\phi_{N-1}, \phi_2]+\ldots+[\phi_2, \phi_{N-1}]+[\phi_1, \phi_N])=0\,.$$

Proof. Clearly we have:

$$\frac{1}{2}\bar{\partial}\left(\sum_{K=0}^{N-1}[\phi_{N-K}, \phi_{K+1}]\right)=\frac{1}{2}\left(\sum_{K=0}^{N-1}([\bar{\partial}\phi_{N-K}, \phi_{K+1}]-[\phi_{N-K}, \bar{\partial}\phi_{K+1}]\right). \tag{1.2.5.1}$$

From $-[\phi_j, \bar{\partial}\phi_i]=[\bar{\partial}\phi_i, \phi_j]$ $[\phi_i, \phi_j]=[\phi_j, \phi_i]$ (see [8]) and

$$\bar{\partial}\phi_{K+1}=\frac{1}{2}([\phi_K, \phi_1]+\ldots+[\phi_1, \phi_K])=\frac{1}{2}\left(\sum_{i=1}^{K}[\phi_i, \phi_{K-i+1}]\right),$$

we get

$$\frac{1}{2}\bar{\partial}\left(\sum_{K=0}^{N-1}[\phi_{N-K}, \phi_{K+1}]\right)=(2[[\phi_{N-1}, \phi_1], \phi_1]+[[\phi_1, \phi_2], \phi_{N-1}])$$

$$+\sum_{\substack{i>1, j>1, K>1\\ i\neq j\neq K\\ i+j+K=N+1}}[[\phi_i, \phi_j], \phi_K]+[[\phi_K, \phi_i], \phi_j]+[[\phi_j, \phi_K], \phi_i])$$

$$+\sum_{\mu\neq N-2\mu+1}([[\phi_{N-2\mu+1}, \phi_\mu, \phi_\mu]+\tfrac{1}{2}[[\phi_\mu, \phi_\mu], \phi_{N-2\mu+1}])+(\tfrac{1}{2}[[\phi_K, \phi_K], \phi_K]]) \tag{1.2.5.2}$$

if $N+1=3K$). From (1.2.5.2) and Jacobi identity we get that

$$\frac{1}{2}\bar{\partial}\left(\sum_{K=0}^{N-1}[\phi_{N-K}, \phi_{K+1}]\right)=0\,. \quad \text{Q.E.D.}$$

1.2.6. Now we are ready to solve the equations (*) from Sect. 0, 1. We will solve them inductively.

Induction Hypothesis. Suppose that for any $2\leqq K\leqq N$, we have

a) $$\bar{\partial}\phi_K=\frac{1}{2}\left(\sum_{i=1}^{K-1}[\phi_{K-i}, \phi_i]\right),$$

b) $$\bar{\partial}^*\phi_K=0,$$

c) $$\phi_K\perp\omega_0=\partial\psi_K \quad \text{and so} \quad \partial(\phi_K\perp\omega_0)=0\,.$$

We must find ϕ_{N+1} such that

a) $$\bar{\partial}\phi_{N+1}=\frac{1}{2}\left(\sum_{i=1}^{N}[\phi_{N-i+1}, \phi_i]\right)$$

b) $$\bar{\partial}^*\phi_{N+1}=0$$

c) $$\partial(\phi_{N+1}\perp\omega_0)=0 \quad \text{and moreover} \quad \phi_{N+1}\perp\omega_0=\partial\psi_{N+1}\,.$$

From Lemma 1.2.4 it follows that

$$\frac{1}{2}\left(\sum_{i=1}^{N}[\phi_{N-i+1}, \phi_i]\right)\perp\omega_0=\partial\left(\sum_{i+k=N+1}(\phi_i\wedge\phi_k)\perp\omega_0\right). \tag{1.2.6.1}$$

From Lemma 1.2.5 it follows that

$$\frac{1}{2}\bar{\partial}\left(\sum_{i=1}^{N}[\phi_{N-i+1},\phi_i]\right)\perp\omega_0=\bar{\partial}\partial\left(\sum_{i+K=N+1}(\phi_i\wedge\phi_K)\perp\omega_0\right)=0. \quad (1.2.6.2)$$

From the Hodge theorem, the fact that M is a Kähler manifold we get

$$\partial\left[\left(\sum_{i+K=N+1}\phi_i\wedge\phi_K\right)\perp\omega_0\right]=\partial\bar{\partial}(-\Psi^1_{N+1})=\bar{\partial}\partial\Psi^1_{N+1}. \quad (1.2.6.3)$$

From the Hodge theorem and the fact that M is a Kähler manifold we get

$$\partial\Psi^1_{N+1}=\bar{\partial}\Psi''_{N+1}+\bar{\partial}^*\Psi'''_{N+1}, \quad (1.2.6.4)$$

where $\bar{\partial}^*\Psi'''_{N+1}=\partial\Psi_{N+1}$ and so $\bar{\partial}^*\partial\Psi_{N+1}=0$ since $\bar{\partial}^*\circ\bar{\partial}^*=0$. Define

$$\phi_{N+1}=\partial\Psi_{N+1}\perp\omega_0^*,$$

where

$$\omega_0^*\in\Gamma(M,\Lambda^n\Theta) \quad \text{and} \quad \langle\omega_0^*,\omega_0\rangle=1 \quad \text{pointwise, i.e.}$$

$$\omega_0^*|_U=\frac{\partial}{\partial z^1}\wedge\ldots\wedge\frac{\partial}{\partial z^n}. \quad (1.2.6.5)$$

Clearly from the fact that

$$\nabla_\alpha\omega_0^*=0 \quad \text{(Bochner principle)}$$

we get immediately that

$$\bar{\partial}^*(\partial\psi_{N+1}\perp\omega_0^*)=(\bar{\partial}^*\partial\psi_{N+1}\perp\omega_0)=0.$$

(For more details see Lemma 1.2.2). So

$$\bar{\partial}^*\phi_{N+1}=0 \quad \text{and} \quad \bar{\partial}\phi_{N+1}=\frac{1}{2}\left(\sum_{i=1}^{N}[\phi_{N+1-1},\phi_i]\right).$$

The theorem is proved. Q.E.D.

We have proved the following theorem:

Theorem 1.2′. *Let M be a $SU(N)$ manifold and let $\eta\in\mathbb{H}^1(M,\Theta)$, then there exists a convergent power series in norms defined in* [8]

$$\phi(t)=\eta t+\phi_2t^2+\ldots+\phi_Kt^K+\ldots$$

such that

1. $\phi_i\in\Gamma(M,\Omega^{0,1}\otimes\Theta)$,
2. $\bar{\partial}^*\phi_i=0$,
3. $\phi_i\perp\omega_0=\partial\psi_i$,
4. $\bar{\partial}\varphi(t)=\frac{1}{2}[\phi(t),\phi(t)]$.

Remark. It is proved in [8] that if $\phi(t)$ fulfills 1), 2), and 4), then $\phi(t)\in C^\infty(M,\Omega^{0,1}\otimes\Theta)$.

2. Computation of the Curvature Tensor of the Weil-Petersson Metric

2.1. Let S be the Kuranishi space of a Calabi-Yan manifold M_0 and let $\pi: X \to S$ be the Kuranishi family of M_0. The tangent space $T_{s,S}$ at the point $s \in S$ can be identified with $\mathbb{H}^1(M_s, \Theta_S)$, where $\mathbb{H}^1(M_s, \Theta_S)$ is the harmonic part of $H^1(M_s, \Theta_S)$ with respect to the Calabi-Yau metric $g_{\alpha\bar{\beta}}(s)$ on M_s and we suppose that for all $s \in S$

$$[\operatorname{Im}(g_{\alpha\bar{\beta}}(s))] = [\operatorname{Im} g_{\alpha\bar{\beta}}(0)] = L.$$

We know from [15] that $g_{\alpha\bar{\beta}}(s)$ is a unique Kähler-Einstein metric on M_s.

Definition. Let $\phi_1, \phi_2 \in T_{s,S} \equiv \mathbb{H}^1(M_s, \Theta_S)$, then we can define the Weil-Petersson metric as follows:

$$\langle \phi_1, \phi_2 \rangle = \int_{M_s} (\phi_1)^{\mu}_{\bar{\alpha}} \overline{(\phi_2)^{\nu}_{\bar{\beta}}} g_{\mu\bar{\nu}} g^{\beta\bar{\alpha}} \operatorname{vol}(g_{\tau\bar{\mu}}(s)). \tag{2.1.1}$$

From Lemma 2.2, it will follow that the Weil-Petersson metric is topologically defined, i.e., in the case of Calabi-Yan manifolds M $\mathbb{H}^1(M, \Theta_S)$ can be identified with $H^1(M, \Omega^{n-1})$. If $\dim M \geqq 3$ it is easy to see that $\mathbb{H}^1(M, \Omega^{n-1})$ consists of primitive classes of forms of type $(n-1, 1)$. On $H^1(M, \Omega^{n-1})$ we can define in a natural way a scalar product, i.e., if $\omega_1, \omega_2 \in H^1(M, \Omega^{n-1})$, then

$$\langle \omega_1, \omega_2 \rangle = (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^{n-1} \int \omega_1 \wedge \bar{\omega}_2. \tag{2.1.2}$$

Remark (2.1.3). Notice that the identification

$$H^1(M, \Theta) \cong H^1(M, \Omega^{n-1})$$

is given by

$$\alpha \to \alpha \lrcorner\, \omega_M(n, 0).$$

Since L is fixed we may suppose that all $\omega_{M_s}(n, 0)$ are fixed since we may assume that

$$\int_{M_s} \omega_{M_s}(n, 0) \wedge \overline{\omega_{M_s}(n, 0)} = \int_{M_s} L^n.$$

So fixing L we have fixed the identification

$$H^1(M_s, \Theta_S) \cong H^1(M_s, \Omega^{n-1}).$$

Lemma 2.2.

$$\begin{aligned}\langle \phi_1, \phi_2 \rangle &= \int_{M_0} \overline{(\phi_1)^{\mu}_{\bar{\alpha}} (\phi_2)^{\nu}_{\bar{\beta}}} g_{\mu\bar{\nu}} g^{\beta\bar{\alpha}} \operatorname{vol}(g_{\tau\bar{\mu}}) \\ &= (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^{n-1} \int (\phi_1 \lrcorner\, \omega_0(n, 0)) \wedge \overline{(\phi_2 \lrcorner\, \omega_0(n, 0))},\end{aligned}$$

where $\omega_0(n, 0)$ is a holomorphic n-form on M_0 and

$$\int_{M_0} \omega_0(n, 0) \wedge \overline{\omega_0(n, 0)} = \int_{M_0} \operatorname{vol}(g_{\tau\bar{\mu}}).$$

Proof. From $\nabla \omega_0(n, 0) = 0 \Rightarrow \nabla(\omega_0(n, 0) \wedge \overline{\omega_0(n, 0)}) = 0$. Since $\nabla(\operatorname{vol}(g_{\tau\bar{\mu}})) = 0$ we may assume that

$$\omega_0(n, 0) \wedge \overline{\omega_0(n, 0)} \equiv \operatorname{vol}(g_{\tau\bar{\mu}}). \tag{2.2.1}$$

From $\phi \in \mathbb{H}^1(M_0, \Theta)$ it follows that $\phi_{\bar{\alpha}\bar{\beta}} = g_{\mu\bar{\beta}}\phi^{\mu}_{\bar{\alpha}} = \phi_{\bar{\beta}\bar{\alpha}}$. (See [9].) Using this fact we get that:

$$(\phi_1)^{\mu}_{\bar{\alpha}}\overline{(\phi_2)^{\nu}_{\bar{\beta}}}g_{\mu\bar{\nu}}g^{\beta\bar{\alpha}}\operatorname{vol}(g_{i\bar{j}}) \equiv (\phi_1 \lrcorner\, \omega_0(n,0)) \wedge (\phi_2 \lrcorner\, \omega_0(n,0)). \quad \text{Q.E.D.}$$

2.3. a) Let $\{\phi_i\}_{i=1}^{k}$ be an orthonormal basis in $\mathbb{H}^1(M_0, \Theta)$. Let

$$\phi(t_1, \ldots, t_k) = \sum_{i=1}^{k} \phi_i t_i + \ldots + \sum_{\substack{i_1 + \ldots + i_k = n \\ i_j \geqq 0}} \phi_{i_1, \ldots, i_k} t_1^{i_1} \ldots t_k^{i_k} + \ldots,$$

where $\forall (i_1, \ldots, i_k)\phi_{i_1}, \ldots, i_k)\phi_{i_1, \ldots, i_k} \in \Gamma(M, \Theta \otimes \Omega^{0,1})$ and for $\phi(t_1, \ldots, t_k)$ the following conditions are fulfilled:

1. $\bar{\partial}^*(\phi(t_1, \ldots, t_k) = 0$.
2. $\bar{\partial}\phi(t_1, \ldots, t_k = 1/2[\phi(t_1, \ldots, t_k), \phi(t_1, \ldots, t_k)]$.
3. For $i_i, \ldots, i_k$ such that $i_1 + \ldots + i_k \geqq 2$ we have $\phi_{i_1, \ldots, i_k} \lrcorner\, \omega_0(n,0) = \partial\psi_{i_1, \ldots, i_k}$ then $t = (t_1, \ldots, t_k)$ will be a local coordinate system in S. From Theorem 1.2 it follows that $\phi(t_1, \ldots, t_k) = \phi(t)$ exists and $\phi(t) \in \Gamma(M_0, \Omega^{0,1} \otimes \Theta)$, $t = (t_1, \ldots, t_k)$ we will call Kodaira-Spencer-Kuranishi coordinates. From now on we will fix these local coordinates.

b) Let $\{U_\alpha\}$ be a covering of M_0 and $(z_\alpha^1, \ldots, z_\alpha^n)$ be local coordinates in U_α such that

$$dz_\alpha^1 \wedge \ldots \wedge dz_\alpha^n = \omega_0(n,0)|_{U_\alpha}.$$

c) Let $\Theta_t^i = dz_\alpha^i + \sum \phi_j^i(t)\overline{dz_\alpha^j}$. From the definition of $\phi(t) = \sum \phi_j^i(t) d\bar{z}^j \otimes \dfrac{\partial}{\partial z^i}$ it follows that for each $t = (t^1, \ldots, t^n) \in S\{\Theta_t^j\}$ $(j = 1, \ldots, n)$ is a basis for $\Omega_t^{1,0}|_{U_\alpha}$.

Lemma 2.4. $d(\Theta_t^1 \wedge \ldots \wedge \Theta_t^n) = 0$. (See also [19] and Weil, Collected work, vol. 2).

Proof. Since $\phi(t) \in \Gamma(M, \operatorname{Hom}(\Omega_0^{1,0}, \bar{\Omega}_0^{1,0}))$ then for each $k \leqq n$, $k > 0$ we can define

$$\Lambda^K\phi(t) \in \Gamma(M, \operatorname{Hom}(\Lambda^K\Omega_0^{1,0}, \Lambda^K\Omega_0^{0,1})),$$

where

$$(\Lambda^K\phi(t))(u_1 \wedge \ldots \wedge u_K) = \phi(t)(u_1) \wedge \ldots \wedge \phi(t)(u_K).$$

Next we have the following formula

$$\begin{aligned} \theta_t^1 \wedge \ldots \wedge \theta_1^n &= dz^1 \wedge \ldots \wedge dz^n + \sum_{K=1}^{n} (-1)^{\frac{K(K-1)}{2}} \Lambda^K\phi \perp dz^1 \wedge \ldots \wedge dz^n \\ &= \omega_0|_U + \sum (-1)^{\frac{K(K-1)}{2}} (\Lambda^K\phi \perp \omega_0)|_0. \end{aligned} \tag{2.4.1}$$

Formula (2.4.1) follows from the definitions of $\Lambda^K\phi(t)$ and $\theta_t^1 \wedge \ldots \wedge \theta_t^n$.

Proposition 2.4.2. $(-1)^{\frac{K(K-1)}{2}} \bar{\partial}(\Lambda^K\phi \perp \omega_0) + (-1)^{\frac{K(K+1)}{2}} \partial(\Lambda^{K+1}\phi \perp \omega_0) = 0$.

Proof. So it is enough to prove

$$\bar{\partial}(\Lambda^K\phi \perp \omega_0) + (-1)^K \partial(\Lambda^{K+1}\phi \perp \omega_0) = 0.$$

From $\bar{\partial}\phi(t) = 1/2[\phi(t), \phi(t)]$ it follows that

$$\bar{\partial}\phi^i(t) = \sum_{j=1}^{n} \phi^j \partial_j \phi^i. \tag{2.4.2.1}$$

Since

$$\Lambda^K\phi = \sum_{i_\Lambda<\ldots<i_K} \phi^{i_1}\wedge\ldots\wedge\phi^{i_K}\otimes\frac{\partial}{\partial z^{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial z^{i_K}}, \tag{2.4.2.2}$$

we get

$$(\Lambda^K\phi\perp\omega_0)|_U = \sum(-1)^{(i_1-1)+\ldots+(i_K-1)}\phi^{i_1}\wedge\ldots\wedge\phi_{i_K}\wedge dz^1\wedge\ldots\wedge\hat{dz}^{i_1}\wedge\ldots\wedge\hat{dz}^{i_K}\wedge\ldots\wedge dz^n, \tag{2.4.2.3}$$

$$\bar\partial(\Lambda^K\phi\perp\omega_0)|_v = \sum(-1)^{(i_1-1)+\ldots+(i_K-1)+(\mu-1)}\phi^{i_1}\wedge\ldots\wedge\bar\partial\phi^{i_\mu}\wedge\ldots\phi_{i_K}\wedge dz^1\wedge\ldots\wedge\ldots\hat{dz}^{i_1}\wedge\ldots\wedge\hat{dz}^{i_K}\wedge\ldots\wedge dz^n. \tag{2.4.2.4}$$

From (2.4.2.1) and (2.4.2.3) we get

$$\begin{aligned}\bar\partial(\Lambda^K\phi\perp\omega_0)|_v &= \sum(-1)^{(i_1-1)+\ldots+(i_K-1)+(\mu-1)}\phi^{i_1}\wedge\ldots\wedge\sum_j\phi^j\wedge\partial_j\phi^{i_\mu}\wedge\ldots\wedge\phi^{i_K}\\ &\quad\wedge dz^1\wedge\ldots\wedge dz^{i_1}\wedge\ldots\wedge dz^{i_K}\wedge\ldots\wedge dz^n\\ &= \sum(-1)^{(i_1-1)+\ldots+(i_K-1)}\left(\sum_j\phi^j\wedge\phi^{i_1}\wedge\ldots\wedge\partial_j\phi^i\wedge\ldots\wedge\phi^{i_K}\right)\\ &\quad\wedge dz^1\wedge\ldots\hat{dz}^{i_1}\wedge\ldots\wedge d\hat{z}^{i_K}\wedge\ldots\wedge dz^n.\end{aligned} \tag{2.4.2.5}$$

Next we must compute $(-1)^K\partial(\Lambda^{K+1}\phi\perp\omega_0)=?$

Suppose that $i_1<i_2<\ldots<i_{l-1}\leqq j<i_1<\ldots<i_K$,

$$\begin{aligned}\partial((-1)^K\Lambda^{K+1}\phi\perp\omega_0) &= (-1)^K\partial(\sum(-1)^{(i_1-1)+\ldots+(j-1)+\ldots+(i_K-1)}\phi^{i_1}\wedge\ldots\phi\\ &\quad\wedge\ldots\wedge\phi^{i_K}\wedge dz^1\wedge\ldots\wedge\hat{dz}^{i_1}\wedge\ldots\wedge\hat{dz}^{j}\wedge\ldots\wedge\hat{dz}^{i_K}\wedge\ldots\wedge dz^n\\ &= (-1)^K\sum_j\sum(-1)^{(i_1-1)+\ldots+(j-1)+\ldots+(i_K-1)+(l-1)+(j-1)+(K+1)}\\ &\quad\times\phi_{i_1}\wedge\ldots\wedge\partial_j\phi_{i_\mu}\wedge\ldots\\ &\quad\wedge\ldots\wedge\phi_{i_K}\wedge dz^1\wedge\hat{dz}^{i_1}\wedge\ldots\wedge dz^j\wedge\ldots\wedge\hat{dz}^{i_K}\wedge\ldots\wedge dz^n.\end{aligned} \tag{2.4.2.6}$$

From $\sum\partial_j\phi^j=0$ and (2.4.2.6) we get that

$$\begin{aligned}&\partial((-1)^K\Lambda^{K+1}\phi\perp\omega_0)\\ &= -\left(\sum_{i_1<\ldots<i_K}(-1)^{(i_1-1)+\ldots+(i_K-1)}\left(\sum_{j=1}^n\phi^j\right)\wedge\phi^{i_1}\wedge\ldots\wedge\partial_j\phi^{i_\mu}\right.\\ &\quad\wedge\ldots\wedge\phi^{i_K}\wedge dz^1\wedge\ldots\wedge d\hat{z}^{i_1}\wedge\ldots\wedge d\hat{z}^{i_K}\wedge\ldots\wedge dz^n.\end{aligned} \tag{2.4.2.7}$$

From (2.4.2.5) and (2.4.2.7) we get that

$$\bar\partial(\Lambda^K\phi\perp\omega_0)+(-1)^K\partial(\Lambda^{K+1}\phi\perp\omega_0)=0.\quad\text{Q.E.D.}$$

From 2.4.2 it follows that

$$d(\theta_t^1\wedge\ldots\wedge\theta_t^n)=0.\quad\text{Q.E.D.}$$

Remark 2.4.9. Since ω_0 and for all K $\Lambda^K\phi$ are globally defined tensors it follows that

$$\omega_t=\omega_0+\sum_{K=1}^n(-1)^{\frac{K(K-1)}{2}}(\Lambda^K\phi\perp\omega_0) \tag{2.4.9.1}$$

is also globally defined. From (2.4.1) it follows that ω_t is a holomorphic n-form on M_t, since $d\omega_t=0$ and ω_t is of type $(n,0)$ on M_t.

Definition 2.5.

$$\phi(t_1,\dots,t_k,\bar{t}_1,\dots,\bar{t}_k)=\log\left((-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\left(\int_{M_0}\omega_t\wedge\bar{\omega}_t\right)\right).$$

Theorem 2.6. a) $(h_{i\bar{j}})=\left(\frac{\partial^2\phi}{\partial t_i\partial\bar{t}_j}\right)$ *is the Weil-Petersson metric.*

b) $$R_{i\bar{j},k\bar{l}}=\begin{cases} 2(-1)^{\frac{n(n+1)}{2}}(\sqrt{-1})^{n-2}\int_{M_0}[\phi_i\wedge\phi_k\lrcorner\,\omega_0]\wedge[\overline{(\phi_j\wedge\phi_l\lrcorner\,\omega_0)}] \\ \text{if } i\neq k \text{ and } j\neq l \\ 4(-1)^{\frac{n(n+1)}{2}}(\sqrt{-1})^{n-1}\int_{M_0}[(\Lambda^2\phi_i\lrcorner\,\omega_0)]\wedge[(\bar{\phi}_j\wedge\bar{\phi}_l\lrcorner\,\bar{\omega}_0)] \\ \text{if } j\neq l \\ 8(-1)^{\frac{n(n+1)}{2}}\sqrt{-1}\int_{M_0}[\Lambda^2\phi_i\lrcorner\,\omega_0]\wedge[\Lambda^2\phi_j\lrcorner\,\bar{\omega}_0]-\delta_{kl} \\ \text{if } i=k,\ j=l, \end{cases}$$

where $[\phi_i\wedge\phi_j\lrcorner\,\omega_0]$ *is the cohomology class of* $\phi_i\wedge\phi_j\lrcorner\,\omega_0$ *in* $H^{n-2,2}\hookrightarrow H^n(M_0,\mathbb{C})$.

c) *For all* μ *and* ν $[\phi_\mu\wedge\phi_\nu\lrcorner\,\omega_0]$ *is a primitive class of cohomology in* $H^{n-2,2}(M_0)$.

Proof of 2.6a. Since $\phi: S\to\mathbb{R}$ it follows that $(h_{i\bar{j}})=\left(\frac{\partial^2\phi}{\partial t_i\partial\bar{t}_j}\right)$ is a Hermitian matrix. From the definition of ϕ it follows that

$$h_{i\bar{j}}=\left((-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-2}\int_{M_0}\frac{\partial\omega_t}{\partial t_i}\wedge\overline{\frac{\partial\omega_t}{\partial t_j}}\right)\left(\int_{M_0}\omega_t\wedge\bar{\omega}_t\right)^{-1} \qquad (2.6a.1)$$

Griffiths proved in [6] that

a) $$\frac{\partial\omega_t}{\partial t_i}=\frac{\partial\phi(t)}{\partial t_i}\lrcorner\,\omega_t$$

b) $$\frac{\partial\omega_t}{\partial t_i}$$

defines a non-zero class of cohomology of type $(n-1,1)$ on M_t. (For the proof of b) see the appendix.)

Remark. b) is the so-called local Torelli theorem. Since for the Calabi-Yan manifold of $\dim\geqq 3$ each class of cohomology of type $(n-1,1)$ is primitive we get that $(h_{i\bar{j}})>0$. This follows from the following well-known fact from Kähler geometry: If η is a primitive form on M of type (a, b) then

$$*\eta=(-1)^{\frac{(a+b)(a+b+1)}{2}}(\sqrt{-1})^{a-b}\frac{1}{(n-a-b)!}L^{n-a-b}\eta,$$

where $n=\dim_{\mathbb{C}}M$, $L=$ the class of the cohomology of the imaginary part of the Kähler metric and $*$ is the Hodge operator.

Now we are ready to prove that $(h_{i\bar{j}})$ is the Weil-Petersson metric on S. Since $\frac{\partial \omega_t}{\partial t_i}$ is a class of cohomology of type $(n-1,1)$ it follows from Lemma 2.2 that $\exp(f)(h_{i\bar{j}})=$ Weil-Petersson metric, where $f: S \to \mathbb{R}$. In [9] it is proved that Weil-Petersson is a Kähler metric, so

$$d(\exp(f)\Sigma h_{i\bar{j}} dt_i \wedge dt_j)=0.$$

But

$$d(\exp(F)\Sigma h_{i\bar{j}} dt_i \wedge \overline{dt_j})=d(\exp(f)) \wedge \Sigma h_{i\bar{j}} dt_i \wedge \overline{dt_j}=0. \qquad (2.6.a.2)$$

This is so since $(h_{i\bar{j}})$ is a Kähler metric. Next we will prove that from (2.6.a.2) $\to d(\exp(f))=0$ and so $f=\text{const}$. Indeed at a point $s \in S$ we may suppose that $(h_{i\bar{j}}) = (I)$ – the identity matrix. So from here it follows that

$$(df) \wedge \Sigma dt_i \wedge \overline{dt_i} = \Sigma \frac{\partial f}{\partial t_j} dt_j \wedge dt_i \wedge \overline{dt_i} + \Sigma \frac{\overline{\partial f}}{\partial \overline{t_k}} \overline{dt_k} \wedge dt_i \wedge \overline{dt_j} = 0$$

so $d(\exp f)=0 \Rightarrow f=\text{const}$. So since these two metrics are different by a constant and coincide at $s_0 \in S$ we have proved 2.6a. Q.E.D.

Proof of 2.6b. Everything follows from the following two formulas:

$$\exp(\phi(t))=\sum_{i=1}^{k} \phi_i t_i + \sum_{i=1}^{k} \phi_{i_2} t_i^2 + \sum_{i \neq j} \phi_{i,j} t_i t_j + \Sigma \phi_{i_3} t_i^3 + \sum_{i<j<k} \phi_{i,j,k} t_i t_j t_k + \dots\dots \qquad (2.6.b.1)$$

where $\phi_{i_2} \lrcorner \omega_0 = \partial \psi_{i_2}$, $\phi_{i,j} \lrcorner \omega_0 = \partial \psi_{i,j}$ and $\phi_{i_3} \lrcorner \omega_0 = \partial \psi_{i_3}$,

$$\omega_t = \omega_0 + \Sigma(-1)^{\frac{k(k-1)}{2}} (\wedge^k \phi(t)) \lrcorner \omega_0. \qquad (2.6.b.2)$$

From (2.6.b.1) and (2.6.b.2) it follows that

$$\begin{aligned}\frac{\partial \omega_t}{\partial t_i} &= (\phi_i \lrcorner \omega_0) + 2t_i(\phi_{i_2} \lrcorner \omega_0) + \sum_{j \neq i} (\phi_{i,j} \lrcorner \omega_0) t_j + 3t_i^2(\phi_{i_3} \lrcorner \omega_0) \\ &+ \sum_{\mu_1} \phi_{i,\mu,\nu} t_\mu t_\nu - 2t_i(\Lambda^2 \phi_i \lrcorner \omega_0) - \sum_{\mu} (\phi_i \wedge \phi_\mu \lrcorner \omega_0) t_\mu + \dots . \end{aligned} \qquad (2.6.b.3)$$

From (2.6.b.3) we get that

$$\begin{aligned}(h_{i\bar{j}}) = \left(\frac{\partial^2 \phi}{\partial t_i \partial \bar{t}_j}\right) &= (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^{n-2} \Big(\int_{M_0} (\phi_i \lrcorner \omega_0) \wedge \overline{(\phi_j \lrcorner \omega_0)} \\ &+ 2t_i \int_{M_0} (\phi_{i_2} \lrcorner \omega_0) \wedge \overline{(\phi_j \lrcorner \omega_0)} + 2\bar{t}_j \int_{M_0} (\phi_i \lrcorner \omega_0) \wedge \overline{(\phi_{j_2} \lrcorner \omega_0)} \\ &+ \sum_{k \neq l \neq i \neq j} t_k \bar{t}_l \int_{M_0} [(\phi_k \wedge \phi_i \lrcorner \omega_0) + (\phi_{i,k} \lrcorner \omega_0)] \\ &\wedge \overline{[-(\phi_l \wedge \phi_j \lrcorner \omega_0) + (\phi_{j,l} \lrcorner \omega_0)]} \\ &+ 2 \sum_{\substack{l=1 \\ l=!j+1i}} t_i \bar{t}_l \int_{M_0} (-(\Lambda^2 \phi_i \lrcorner \omega_0) + t_i(\phi_{i_2} \lrcorner \omega_0)] \\ &\wedge \overline{[-(\phi_l \wedge \phi_j \lrcorner \omega_0) + (\phi_{j,l} \lrcorner \omega_0)]} \\ &+ 4t_i \bar{t}_j \int_{M_0} (-\Lambda^2 \phi_i \lrcorner \omega_0 + \phi_{i_2} \lrcorner \omega_0) \wedge \overline{(-\Lambda^2 \phi_j \lrcorner \omega_0 + \phi_{i_2} \lrcorner \omega_0)} \\ &+ 3t_i^2 \int_{M_0} (\phi_{i_3} \lrcorner \omega_0) \wedge \overline{(\phi_j \lrcorner \omega_0)} + 3\bar{t}_j^2 \int (\phi_i \lrcorner \omega_0) \wedge \overline{(\phi_{j_3} \lrcorner \omega_0)} \\ &+ (\text{terms of order higher than } 3)] \ [\|\omega_t\|^{-2} = 1 - \Sigma \delta_{i\bar{j}} t_i \bar{t}_j + \dots]. \end{aligned}$$

Proposition (2.6.b.4). *The coefficients in* (2.6.b.3) *in front of* $2t_i$, $2\bar{t}_j$, $3t_i^2$, *and* $3\bar{t}_j^2$ *are equal to zero.*

Proof. We know from Sect. 1 that

$$(\phi_{i_2} \lrcorner \omega_0) = \partial\psi_{i_2}, \quad (\phi_{i_3} \lrcorner \omega_0) = \partial\psi_{i_3},$$
$$(\phi_{j_2} \lrcorner \omega_0) = \partial\psi_{j_2}, \quad (\phi_{j_3} \lrcorner \omega_0) = \partial\psi_{j_3},$$

and for each i $d(\phi_i \lrcorner \omega_0) = 0$. Let us prove that the coefficient in front of t_i is zero in (2.6.b.3),

$$\int_{M_0} (\phi_i \lrcorner \omega) \wedge \overline{(\phi_{j_2} \lrcorner \omega_0)} = \int_{M_0} (\phi_i \lrcorner \omega_0) \wedge \overline{\partial\psi_{j_2}} = \int_{M_0} d((\phi_i \lrcorner \omega_0) \wedge \bar{\psi}_{j_2})$$
$$= (\text{Stokes's theorem}) = 0.$$

Exactly in the same manner we prove that the other coefficients are zero. Q.E.D.

From (2.6.b.3) and (2.6.b.4) it follows that we have

$$\begin{aligned} h_{i\bar{j}} = \Bigg[& (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^{n-2} \Bigg(\int_{M_0} (\phi_i \lrcorner \omega_0) \wedge \overline{(\phi_j \lrcorner \omega_0)} \\ & + \sum_{\substack{k \neq i \\ l \neq j}} t_k \bar{t}_l \int_{M_0} ((\phi_i \wedge \phi_k \lrcorner \omega_0) - (\phi_{i,k} \lrcorner \omega_0)) \wedge \overline{(\phi_j \wedge \phi_l \lrcorner \omega_0) - (\phi_{j,l} \lrcorner \omega_0))} \\ & + 2 \sum_{l \neq j} t_i \bar{t}_l \int_{M_0} (\Lambda^2 \phi_i \lrcorner \omega_0 - \phi_{i_2} \lrcorner \omega_0) \wedge \overline{(\phi_j \wedge \phi_l \lrcorner \omega_0 - \phi_{j,l} \lrcorner \omega_0)} \\ & + 2 \sum_{k \neq i} t_k \bar{t}_j \int_{M_0} (\phi_i \wedge \phi_k \lrcorner \omega_0 - \phi_{i,k} \lrcorner \omega_0) \wedge \overline{(\Lambda^2 \phi_j \lrcorner \omega_0 - \phi_{j_2} \lrcorner \omega_0)} \\ & + 4 t_i \bar{t}_j \int_{M_0} (\Lambda^2 \phi_i \lrcorner \omega_0 - \phi_{i_2} \lrcorner \omega_0) \wedge \overline{(\Lambda^2 \phi_j \lrcorner \omega_0 - \phi_{j_2} \lrcorner \omega_0)} \\ & + (\text{terms of order} \geqq 3) \Bigg] [\|\omega_t\|^{-2} = 1 - \Sigma \delta_{i\bar{j}} t_i \bar{t}_j + \dots]. \end{aligned} \tag{2.6.b.5}$$

From $d\omega_t = 0$ we get that

$$(d(\phi_i \wedge \phi_\mu \lrcorner \omega_0 - \phi_{i,\mu} \wedge \omega_0) = 0 \quad \text{for all } i, \mu. \tag{2.6.b.6}$$

Since $\phi_{i,\mu} \lrcorner \omega_0 = \partial\psi_{i,\mu}$ we get that

$$\mathbb{H}(\phi_i \wedge \phi_\mu \lrcorner \omega_0 - \phi_{i,\mu} \lrcorner \omega_0) = \mathbb{H}(\phi_i \wedge \phi_\mu \lrcorner \omega_0), \tag{2.6.b.7}$$

where $\mathbb{H}$ is the harmonic projection. From (2.6.b.5) and (2.6.b.7), (2.6.b) follows directly. Q.E.D.

Proof of (2.6.6). From the definition of ω_i we get that the following formula is true:

$$[\omega_t] = [\omega_0] + \Sigma t_i [\phi_i \lrcorner \omega_0] - \sum_{i,j} t_i t_j [\phi_i \wedge \phi_j \lrcorner \omega_0] + \dots, \tag{2.6.c.1}$$

where [ ] means the class of cohomology. If $[\phi_i \wedge \phi_j \lrcorner \omega_0]$ is not a primitive class of cohomology, then

$$[\phi_i \wedge \phi_j \lrcorner \omega_0] = [L] \wedge [\psi_{ij}]. \tag{2.6.c.2}$$

Since $[\omega_t]$ is an $(n,0)$ form it follows that $[\omega_t]$ is a primitive for all $t\in S$ with respect to $L=[\operatorname{Im} g_{\alpha\bar\beta}(t)]$ fixed. So (2.6.c.2) will contradict the fact that ω_t is a primitive class of cohomology. Q.E.D.

Corollary (2.6.1). *The biholomorphic sectional curvature of* $(h_{i\bar j})$ *is negative.*

Proof. Let $E\subseteqq \mathbb{H}^n$ be the subspace spanned by

$$i[\varphi_i \lrcorner\, \omega_0]\ \&\ [\phi_i\wedge\phi_j\lrcorner\,\omega_0] \quad \text{for all } i,j.$$

So from 2.6.c it follows that $E\subseteqq \mathbb{H}^n_0$, where $\mathbb{H}^n_0$ are primitive classes of cohomology. If $\omega\in E$, then we have

$$(-1)^{\frac{n(n+1)}{2}}(\sqrt{-1})^{n-2}\int_{M_0}\omega\wedge\bar\omega<0. \tag{2.6.1.1}$$

Let $\omega=\sum_\alpha\left\{\left[\sum_\beta \xi^\alpha\eta^\beta[\varphi_\alpha\wedge\varphi_\beta\lrcorner\,\omega_0]+i\xi^\alpha\eta^\beta[\varphi_\beta\lrcorner\,\omega_0]\right]+2\sum_\beta\xi^\alpha\eta^\alpha[\Lambda^2\varphi_\alpha\lrcorner\,\omega_0]\right\}$. It is easy to check that

$$(-1)^{\frac{n(n+1)}{2}}(\sqrt{-1})^{n-1}\int_{M_0}\omega\wedge\bar\omega=\Sigma R_{\alpha\bar\beta,\mu\bar\nu}\xi^\alpha\bar\xi^\beta\eta^\mu\bar\eta^\nu. \tag{2.6.1.2}$$

Since $\omega\in E$, it follows that

$$\Sigma R_{\alpha\bar\beta,\mu\bar\nu}\xi^\alpha\bar\xi^\beta\eta^\mu\bar\eta^\nu<0. \tag{2.6.1.3}$$

Expression (2.6.1.3) is exactly the biholomorphic sectional curvature. Q.E.D.

Corollary 2.6.2. *The sectional curvature of the Weil-Petersson metric is* $\leqq 0$.

Proof. The proof is based on the following observation:

Observation. Let $\{\phi_i\}$ be an orthonormal basis of harmonic forms in $\mathbb{H}^i(M,\Theta_M)$ $(\dim_{\mathbb{C}}\mathbb{H}^1(M,\Theta_M)=N)$, then $\{\phi_j\}$ can be viewed as a global section of

$$\operatorname{Hom}(\Omega^{1,0},\Omega^{0,1}).$$

Since

$$\Omega^{1,0}_M+\overline{\Omega^{1,0}_M}=T^*_M(\mathbb{R})\otimes\mathbb{C},$$

so we can view ϕ_j for every j as a global section of

$$\operatorname{Hom}(T^*_M(\mathbb{R})\otimes\mathbb{C},T^*_M(\mathbb{R})\otimes\mathbb{C})).$$

Now we can define

$$\phi_i\wedge\bar\phi_j\in\Gamma(M,\operatorname{Hom}(\Lambda^2(T^*_M(\mathbb{R})\otimes\mathbb{C}),\Lambda^2(T^*_M(\mathbb{R})\otimes\mathbb{C})),$$

where

$$\phi_i\wedge\bar\phi_j(u\wedge v):=\phi_i(u)\wedge\bar\phi_j(v) \text{ (this is defined pointwise for } x\in M).$$

Definition. Let $(\lambda^1,\ldots,\lambda^N)$ and $(\zeta^1,\ldots,\zeta^N)$ be any two linearly independent vectors in $\mathbb{C}^N$. Then we will define

$$\omega\in\Gamma(M,\operatorname{Hom}(\Lambda^2(T^*_M(\mathbb{R})\otimes\mathbb{C}),\Lambda^2(T^*_M(\mathbb{R})\otimes\mathbb{C}))$$

in the following way:

$$\omega:=\sum_{i,j}(\lambda^i\bar\zeta^j-\zeta^i\bar\lambda^j)(\phi_i\wedge\bar\phi_j).$$

Proposition 2.6.2.1. $(\omega \perp \bar{\omega})(m) = \|\omega(m)\|^2 \geqq 0$, *where* $(\omega \perp \bar{\omega})(m)$ *is the construction of the tensors* ω & $\bar{\omega}$ *at a point* $m \in M$ *and* $\|\omega(m)\|^2$ *is the norm of the tensor* ω *at the point* $m \in M$ *with respect to the induced metric on* $\mathrm{Hom}(T^*(\mathbb{R}) \otimes \mathbb{C}, T^*(\mathbb{R}) \otimes \mathbb{C})(m)$ *from the restriction of the Calabi-Yau metric on* $T^*(\mathbb{R})(m)$.

Proof. Let

$$\{dz^i\} \quad \text{and} \quad \left\{\frac{\partial}{\partial z^i}\right\}$$

be the orthonormal basis of $T^*(\mathbb{C})^{1,0}(m)$ and $T(\mathbb{C})^{1,0}(m)$. If we write down ω in these coordinates and compare the definition of the construction $(\omega \perp \bar{\omega})(m)$ and the norm $\|\omega(m)\|^2$ we will see that they coincide. So Proposition 2.6.2.1 is proved. Q.E.D.

Proposition 2.6.2.2.a.

$$(-1)^{\frac{n(n+1)}{2}} (i)^n \int_M (\omega \perp \bar{\omega}) \omega(n,0) \wedge \omega(0,n) \geqq 0 (\omega_M(n,0)$$

is the holomorphic n-form on M and we suppose that $\omega(n,0) \wedge \omega(0,n) = \mathrm{vol}(g_{\alpha,\bar{\beta}})$, $(g_{\alpha,\bar{\beta}})$ *is the Calabi-Yau metric on M*).

Proposition 2.6.2.2.b. *Let* $\omega' = \left(\sum_{i,j} (\lambda^i \bar{\zeta}^j - \zeta^i \bar{\lambda}^j)(\phi_i \perp \bar{\phi}_j)\right) \omega_M(n,0)$, *then*

$$(-1)^{\frac{n(n+1)}{2}} (i)^{n-2} \int_M \omega' \wedge \bar{\omega}' \leqq 0 .$$

Proposition 2.6.2.2.c.

$$(-1)^{\frac{n(n+1)}{2}} (i)^{n-2} \left(\int_M (\omega \perp \bar{\omega}) \omega_M(n,0) \wedge \omega_M(0,n) + \int_M \omega' \wedge \bar{\omega}' \right)$$
$$= \sum (\lambda^i \bar{\zeta}^j - \zeta^i \bar{\lambda}^j)(\lambda^k \bar{\zeta}^l - \zeta^k \bar{\lambda}^l) R_{i\bar{j},k\bar{l}} \leqq 0 .$$

Proof of 2.6.2.2.a. We know from [5] that if $\omega(n,0)$ is a form of type $(n,0)$ on an n-dimensional complex manifold, then ω is primitive from M so

$$(-1)^{\frac{n(n+1)}{2}} (i)^{n-2} \int_M \omega_M(n,0) \wedge \omega_M(0,n) \geqq 0 .$$

Now (2.6.2.2.a) follows immediately from (2.6.2.1). Q.E.D.

Proof of 2.6.2.2.b and 2.6.2.2.c. This follows immediately from the definition of the construction of tensors, Theorem 2.6 and the fact that $(i)^2 = -i$. Q.E.D.

The End of the Proof of 2.6.2. Notice that

$$\sum (\lambda^i \bar{\zeta}^j - \zeta^i \bar{\lambda}^j)(\lambda^k \bar{\zeta}^l - \zeta^k \bar{\lambda}^l) R_{i\bar{j},k\bar{l}}$$

is exactly the sectional curvature of the Weil-Petersson metric in the plane spanned by

$$(\mathrm{Re}\lambda^1, \ldots, \mathrm{Re}\lambda^N) \quad \text{and} \quad (\mathrm{Re}\zeta^1, \ldots, \mathrm{Re}\zeta^N) .$$

(For this fact see [10].)

So 2.6.2 is proved. Q.E.D.

Remark. In the same manner we can prove that the curvature operator is negative in Nakano's sense.

Appendix

Proposition 3. $\frac{d\omega_t}{dt_i}$ *defines a non-zero class of cohomology of type* $(n-1,1)$ *on* X_t.

Proof. From 2.3 it follows that locally

$$\omega_t=(A_t dz^1)\wedge\ldots\wedge(A_t dz^n)=\Theta_t^1\wedge\ldots\wedge\Theta_t^n,$$

where $A_t=\mathrm{id}+\Sigma\phi_i t_i+$(terms of order $\geqq 2$) and $\Theta_t^j=A_t dz^j$. From the fact that $dz^i=A_{t_0}^{-1}\Theta_{t_0}^i$ we get that

$$\omega_t=(A_t(A_{t_0}^{-1}\Theta_{t_0}^1))\wedge\ldots\wedge(A_t(A_{t_0}^{-1}\Theta_{t_0}^n))=((A_t\circ A_{t_0}^{-1})\Theta_{t_0}^1)\wedge\ldots\wedge((A_tA_{t_0}^{-1})\Theta_{t_0}^n). \tag{3.1}$$

From the expression $A_{t_0}=\mathrm{id}+\sum_i\phi_i(t_i)_0+\ldots$ we get that $A_{t_0}^{-1}=\mathrm{id}-\Sigma\phi_i(t_i)_0+\ldots$ and so

$$A_tA_{t_0}^{-1}=\mathrm{id}+\Sigma\phi_i(t_i-(t_i)_0)+\text{terms of order }\geqq 2. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\omega_t=\Theta_{t_0}^1\wedge\ldots\wedge\Theta_{t_0}^n+\sum_{i,\mu}(t_i-(t_i)_0)(\phi_i)_{\bar\alpha}^{\mu}(-1)^{\mu-1}\bar\Theta_{t_0}^{\alpha}\wedge\Theta_{t_0}^1\wedge\ldots\wedge\hat\Theta_{t_0}^{\mu}\wedge\ldots\wedge\Theta_{t_0}^n$$
$$+\text{terms of order }\geqq 2. \tag{3.3}$$

Since $\Theta_{t_0}^1\wedge\ldots\wedge\Theta_{t_0}^n=\omega_{t_0}$ and $d\omega_t=0$ we obtain from (3.3) that

$$\left.\frac{d\omega_t}{dt_i}\right|_{t_i=(t_i)_0}=-\sum_{\mu,\alpha}(\phi_i)_{\bar\alpha}^{\mu}\bar\Theta_t^{\alpha}\wedge\Theta_{t_0}^1\wedge\ldots\wedge\hat\Theta_{t_0}^{\mu}\wedge\ldots\wedge\Theta_{t_0}^n. \tag{3.4}$$

So $\frac{d\omega_t}{dt_i}$ is a closed form of type $(n-1,1)$ on X_{t_0} for $t=t_0$. That $\frac{d\omega_t}{dt_i}$ is a non-zero class of cohomology follows immediately from the so-called local Torelli theorem. See [6].

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