First Order Phase Transitions in Unbounded Spin Systems *I*: Construction of the Phase Diagram

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Abstract. The phase diagram and the corresponding infinite volume Gibbs states are constructed for a large class of continuous, unbounded spin models. Our construction relies on a partition of unity mapping our system onto an interacting contour system, a generalisation of Zahradnik's approach to Piragov Sinai theory to interacting contour systems, and a suitable mean field expansion around the minimas of the Hamiltonian.

Introduction

In this and a forthcoming paper, [1], we study continuous spin models with a single spin potential, V, whose minima are deep and widely separated. While such models have been extensively discussed in the literature if the minima of V are related by a symmetry, [2–5], results concerning the general case without any symmetry were only obtained in recent years.

The first work in this direction is that of Imbrie, [6], who treated two dimensional Euclidean field theories in which the potential, V, is a polynomial in the fields. His idea was to use the cluster expansion of [2] to obtain a hard core interacting contour system and to analyse this system using the techniques of Pirogov and Sinai, [7–9]. This turned out to be technically rather complicated because the resulting contour activities were not positive. Imbrie solved this problem by using a relatively involved resummation technique[1].

An alternative approach would be that of Bricmont et al., [12], who mapped certain lattice (and continuum) gases with three or more particle species to an interacting contour model (with positive activities) and then studied this model using the methods of [7–9]. Their method has been extended to bounded continuous spin systems on the lattice, but it is not clear whether their techniques extend to the unbounded case as well.

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[1] Recently, Borgs and Imbrie realised, [10], that this resummation can be avoided using the techniques of [11] rather than [7–9] since that approach may be generalised to contour models with complex activities. See [10] for details

Another approach is that of [13] (see also [14]) which starts from the very beginning with a generalised contour model and then analyses this model using the cluster expansion techniques of [15]. Their results are very similar to those described in the present work, with two differences. They only consider potentials, V, which have two minima, and they only consider the point in the phase diagram at which the two phases corresponding to these minima coexist. There is no doubt that their methods could be easily extended to include potentials with three or more minima, but extending their results to include points in the phase diagram away from coexistence would require substantial generalization.

Note also that none of the papers, [2–6], [12], or [13] treat the completeness question. That is, the question of whether all translation invariant infinite volume states are a convex combination of the particular states constructed, as expected.

The goal of the present work is to construct the phase diagram, and corresponding infinite volume states, of an unbounded spin model with single spin potential V and nearest neighbor interaction W (see Sect. 1 for precise definitions). We assume that V has N deep and widely separated minima of approximately the same depth, and that V depends on parameters $\mu \in R^{N-1}$ in such a way that by varying μ one shifts the differences between these minima. We will need assumptions concerning the behaviour of V and W near the minima of V. We choose assumptions (A.1 of Sect. 1) which allow us to control the Gibbs factor, e^{-H}, for fields lying near a minimum of V, through a cluster expansion around the Gaussian approximation to e^{-H}. This part of our analysis is quite flexible. We could use, instead, any other single phase type expansion or, more generally, any other method (for example that of [16]) which allows us to establish strong clustering properties for the models in which all spins R_x are constrained to lie near one of the minima of V.

Concerning the fluctuations between the various minima of V we proceed as follows. Let $\chi_q(R_x)$ denote a characteristic function which forces the spin, R_x, to lie in a certain neighborhood, U_q, of the qth minimum and let $\chi_0 = 1 - \sum_{q=1}^{N} \chi_q$. Given a volume, $\Lambda \subset Z^\nu$, we introduce a partition of unity

$$1 = \sum_{\omega: \Lambda \to \{0, 1, \dots, N\}} \chi_{\omega(x)}(R_x), \tag{0.1}$$

and consider ω as a spin configuration of a discrete spin system with ground states $1, \dots, N$. Defining contours as usual in the Pirogov–Sinai theory we obtain a representation of our model as a system of contours interacting via "fields" $R_x \in U_q$, for some $q \in \{1, 2, \dots, N\}$. (See Sect. 2 for details.)

At this point one is confronted with the problem lying at the core of the Pirogov–Sinai theory. One would like to obtain expansions by insisting that $\omega_x = q$ for x near the boundary of Λ and for some q. One then expects that if the qth minimum is stable (where the term stable must be defined) then the resulting contours have small activity. It is, however, not clear a priori which q are stable. In addition, one encounters such boundary conditions for all q at intermediate steps of the analysis. To deal with this problem we choose to use the technique developed in [11]. This technique is based on the introduction of auxiliary models,

called truncated models, for which all contours have small activities and which turn out to differ from the actual models that we are interested in only for rather large contours. The truncated models may then be analysed by Mayer expansion techniques. If h'_q are the free energies of the truncated models one shows that the truncated and actual models with q-boundary conditions are identical if

$$h'_q = h = \min_m h'_m. \tag{0.2}$$

Using standard methods, we then construct, for q satisfying (0.2), called the stable values for q, infinite volume states, $\langle \cdot \rangle_q$, which are small perturbations of the Gaussian approximation to e^{-H} in U_q (Theorem 1, Sect. 1). Making appropriate assumptions about the behaviour of V as μ varies (corresponding to the "degeneracy removing condition" needed in the Pirogov–Sinai theory) we then construct the phase diagram. That is, we show that there is a point, μ_0, for which all q are stable, that there are lines, $\mu_q(t)$ starting at μ_0, for which all q' except q are stable, that there are surfaces, $\mu_{q_1 q_2}(t, s)$, bounded by $\mu_{q_1}(t)$ and $\mu_{q_2}(t)$, for which all q except q_1 and q_2 are stable, and so on (Theorem 2, Sect. 1 and Sect. 6). In a second paper, [1], we then show that all translation invariant infinite volume states obeying certain regularity conditions on the growth of the boundary condition are convex linear combinations of the $\langle \cdot \rangle_q$. This then completes the analysis.

The paper is organized as follows. In Sect. 1 we define the model and state the assumptions we use to prove Theorems 1 and 2. In Sect. 2 we use the partition of unity, (0.1), to rewrite the model as an interacting contour model. In Sect. 3 the truncated models are introduced and we prove that the truncated and actual models are identical if $h'_q = h$. The cluster expansion we use to control the truncated models is introduced in Sect. 4, and its convergence is proven in Sect. 5. In Sect. 6 we discuss the construction of the phase diagram and in Sect. 7 we construct the infinite volume states, $\langle \cdot \rangle_q$.

1. The model, Assumptions, Results

We consider an unbounded spin model on the lattice $Z^\nu, \nu \geqq 2$, with spins $R_x \in R$,

$$H(R_A) = \sum_x V(R_x) + \frac{1}{2} \sum_{\substack{x, y \in A \\ |x - y| = 1}} W(R_x, R_y), \tag{1.1}$$

and partition function

$$Z(A) = \int e^{-H(R_A)} dR_A, \tag{1.2}$$

defined for arbitrary finite subsets $A \subset Z^\nu$. As usual we write $|x - y|$ for the lattice distance $\sum_{i=1}^{\nu} |x_i - y_i|$. V and W take values in $R \cup \{+\infty\}$ and depend on the inverse temperature β, as well as parameters $\mu \in \mathscr{V}^{(0)}$, where $\mathscr{V}^{(0)}$ is an open neighborhood of $0 \in R^{N-1}$, $N \geqq 2$. For $\mu = 0, V$ has N absolute minima[2], $R_1^{(0)}, \ldots, R_N^{(0)}$, which

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[2] In fact we allow for a slightly more general situation, see assumptions A.1–A.3 below

become deep and widely separated as β grows (see assumption A.2 below). As a typical example one may consider a potential, $V(R)$, obtained by rescaling from a β-independent potential, $v(r)$, with non-degenerate minima $r_1, \ldots, r_N$:

$$V(R) = \beta v(R/\sqrt{\beta}), \tag{1.3}$$

and similarly for W

$$W(R, R') = \beta w(R/\sqrt{\beta}, R'/\sqrt{\beta}) \tag{1.4}$$

with, for example, $w(r, r') = |r - r'|^2$.

We now will formulate several assumptions on V and W, which will allow us to prove the theorems mentioned in the introduction. We start with

Assumption A.0 (A Priori Bounds on V and W). *For all $\beta < \infty$, V and W are bounded below, and $\exp(-V)$ is integrable. We will assume that W is normalised in such a way that*

$$\min_{R, R'} W(R, R') = 0. \tag{1.5}$$

Note that $W(R_x, R_y)$ in (1.1) can be replaced by $\frac{1}{2}W(R_x, R_y) + \frac{1}{2}W(R_y, R_x)$ without changing $H(R_\Lambda)$. We therefore assume without loss of generality that W is symmetric, i.e.

$$W(R, R') = W(R', R). \tag{1.6}$$

Clearly the above assumption ensures that the partition function, $Z(\Lambda)$, exists. The next two assumptions state that in certain disjoint, open, intervals $U_q = U_q(\beta) \subset R$, $q = 1, \ldots, N$, the Gibbs factor, e^{-H}, is well approximated by appropriate Gaussians in U_q, whereas fluctuations away from U_q are suppressed. More precisely we assume the existence of a μ-independent, positive function $\tilde{\tau}(\beta) \to \infty$ as $\beta \to \infty$, such that for

$$U_q = \{R \in R \mid |R - R_q^{(0)}|^2 < \tilde{\tau}(\beta)\}, \tag{1.7a}$$

$$U_0 = R \bigg/ \bigcup_{q=1}^{N} U_q, \tag{1.7b}$$

the following assumptions are true (q and q' always denote integers between 1 and N).

Assumption A.1 (Gaussian Fluctuations). *There are constants η, η', η'' not depending on μ and β, as well as functions*

$$\begin{aligned}
m_q^2 &= m_q^2(\beta, \mu) \geqq \eta > 0, & m_q^2 &< \eta' < \infty \\
\kappa_q &= \kappa_q(\beta, \eta) \geqq 0, & \kappa_q &< \eta'' < \infty \\
R_q &= R_q(\beta, \mu) \in U_q, & e_q &= e_q(\beta, \mu) \in R
\end{aligned}$$

such that the following statements hold:

0) $U_q \cap U_{q'} = \phi$ *for* $q \neq q'$,
i) $V(R) \geqq (\eta/2)(R - R_q)^2 + e_q$ *for* $R \in U_q$,

ii) *there are functions* $\xi(\beta) \to \infty$ *as* $\beta \to \infty, \delta(\beta) \to 0$ *as* $\beta \to \infty$ *such that*

$$\left| V(R) - e_q - \frac{m_q^2}{2} |R - R_q|^2 \right| \leq \delta(\beta), \tag{1.8a}$$

$$\left| W(R, R') - \frac{\kappa_q}{2} |R - R'|^2 \right| \leq \delta(\beta), \tag{1.8b}$$

$$\text{if} \quad |R - R_q| \leq \xi(\beta), \quad |R' - R_q| \leq \xi(\beta),$$

iii) $[R_q - \xi(\beta), R_q + \xi(\beta)] \subset U_q(\beta)$ *and*

$$e^{-(\eta/2)\xi^2(\beta)} \leq \delta(\beta), \tag{1.9}$$

iv) $R_q(\beta, 0) = R_q^{(0)}.$

Assumption A.2 (Suppression of Non-Gaussian Fluctuations). *There are constants* $K_1 > 0, K_2 < \infty, 0 < \alpha < 1,$ *independent of* β *and* μ, *such that the following statements are true*:

i) $\min\limits_{R \in U_{q'}, R' \in U_{q'}} W(R, R') \geq K_1 \tilde{\tau}(\beta)$ *for all* $q \neq q',$

ii) $\inf\limits_{R \in R} V(R) \geq e_0 \equiv \min\limits_q e_q,$ $\tag{1.10}$

iii) $\int\limits_{U_0} dR e^{-\alpha(V(R) - e_0)} \leq K_2 e^{-K_1 \tilde{\tau}(\beta)}.$

Assumption A.3 (Regularity of e_q).

i) *For* $\mu = 0$ *all* e_q'*s are equal.*
ii) *There is a constant* $K_3 < \infty$, *independent of* μ *and* β, *such that for all* $\mu, \mu' \in \mathscr{V}^{(0)}$,

$$|e_q(\mu) - e_q(\mu')| \leq K_3 \tilde{\tau}(\beta) |\mu - \mu'|,$$

where $|\mu|$ *denotes the* l_2-*norm of* μ.

Note that assumption A.2 expresses the fact that fluctuations away from the absolute minimum of V are suppressed. Since we want to construct the phase diagram in a neighborhood of the coexistence point, we also need bounds on the differences between the values $V(R_q)$ of the various minima. These follow from assumption A.3.

Remarks.

i) Assumption A.1 will be used to prove the convergence of a convenient mean field expansion for the "restricted partition function," $Z_q^{\mathrm{res}}(\Lambda)$, obtained from $Z(\Lambda)$ by restricting all fields R_x to U_q. We therefore can replace this assumption by any other assumption which guarantees the convergence of a suitable expansion for $Z_q^{\mathrm{res}}(\Lambda)$.
ii) In the abstract context considered in this paper it might have been more natural to formulate assumptions A.2 and A.3 (as well as the following assumptions A.4 and A.5) using the free energy, h_q^{res}, of the restricted model instead of e_q. We choose e_q because the resulting assumptions are easier to check for the models we want to consider in the future (for example the model considered in [17]).

iii) In the case in which V and W are obtained by rescaling functions v and w, where $r_1, \ldots, r_N$ are the minima of v and $w(r, r') \geqq w(r, r) = 0$, we set

$$\tilde{\tau}(\beta) = O(\beta), \quad R_q = \sqrt{\beta} \, r_q,$$

$$e_q = \beta v(r_q), \quad m_q^2 = v''(r_q), \quad \kappa_q = \frac{\partial^2}{\partial r^2} w(r, r') \Bigg|_{r = r' = r_q},$$

$\xi(\beta)$ and $\delta(\beta)$ may, for example, be chosen as $O(\beta^\lambda), O(\beta^{3\lambda - 1/2})$, respectively, with $0 < \lambda < \frac{1}{6}$.

The goal of this paper is to construct infinite volume states, $\langle \cdot \rangle_q$, $q = 1, \ldots, N$, which are (for β large) small perturbations of the Gaussian with covariance $(m_q^2 - \kappa_q \Delta)^{-1}$ and mean R_q (see assumption A.1). The idea is to construct a cluster expansion for each choice of q, with "high temperature" polymers describing the fluctuations within $U_{q'}$ and Peierls countours describing the fluctuations away from U_q. Using the above assumptions it will be possible to establish the convergence of these expansions for a certain subset, $S = S(\beta, \mu) \subset \{1, \ldots, N\}$, of "stable phase" $q \in S$.

In fact we will show, following an idea of Zahradnik, that for β large enough and μ in a neighborhood of zero, S is always nonempty and is characterised by a certain free energy condition

$$S(\beta, \mu) = \{q \, | \, h_q'(\beta, \mu) = \min_{\tilde{q}} h_{\tilde{q}}'(\beta, \mu)\}, \tag{1.12}$$

where h_q' is the free energy of a certain auxiliary model (see Sect. 3).

We have assumed that for $\mu = 0$ the $e_q(\beta, 0)$ are all equal independent of q. If the e_q are well behaved as functions of μ then for each $q' \in \{1, 2, \ldots, N\}$ there will be a curve in $\mathcal{V}^{(0)}$, starting at $\mu = 0$, along which all of the e_q are equal for $q \neq q'$ while $e_{q'}$ differs from the rest. Similarly, for each distinct pair, $\{q_1, q_2\} \subset \{1, 2, \ldots, N\}$ there will be a surface in $\mathcal{V}^{(0)}$ along which e_q are equal only when $q \in \{1, 2, \ldots, N\} \setminus \{q_1, q_2\}$, and so on until one finds hypersurfaces of dimension $N - 1$ on which all e_q differ from each other. One may ask to what extent this structure reflects the behaviour of $S(\beta, \mu)$ in the sense that there is some $\mu_0 \in \mathcal{V}^{(0)}$ such that $S(\beta, \mu_0) = \{1, 2, \ldots, N\}$, that for each q there is a line starting at μ_0 along which $S(\beta, \mu) = \{1, 2, \ldots, N\} / \{q\}$, and so on. Phrased in the usual language of Piragov–Sinai theory we will show that, for β large enough, the phase diagram is a continuous deformation of the zero temperature phase diagram in a neighborhood of $\mu = 0$. To obtain this result, however, we will need some additional assumptions.

Assumption A.4 (Degeneracy Removing Condition).

(i) $e_q(\mu)$ is C^1 in $\mathcal{V}^{(0)}$ and let

$$E_{qj} = \frac{d}{d\mu_j} (e_q - e_n). \tag{1.13}$$

Then E is invertible for all $\mu \in \mathcal{V}^{(0)}$.
(ii) $\|E^{-1}\| \leq 1/M(\beta)$ *for all* $\mu \in \mathcal{V}^{(0)}$, *and* $\mathcal{V}^{(0)}$ *contains the sphere of radius* $M(\beta)^{-\lambda}$ *around 0 for some* $\lambda < 1$ *and some constant* $M(\beta) \to \infty$ *as* $\beta \to \infty$.

Assumption A.5 (Regularity of $\partial V/\partial \mu$ and $\partial W/\partial \mu$). *There are constants K_5 and $K_6(\beta)$, where K_5 is independent of μ and β and K_6 satisfies*

$$\lim_{\beta \to \infty} \frac{K_6(\beta)}{M(\beta)} = 0,$$

and there is a constant $p > 0$ such that

$$\left| \frac{dV(R)}{d\mu} \right| \leqq K_5(\tilde{\tau}(\beta) + (V(R) - e_0)), \tag{1.14}$$

$$\left| \frac{dW(R, R')}{d\mu_j} \right| \leqq K_5(\tilde{\tau}(\beta) + W(R, R')), \tag{1.15}$$

and, for all $R \in U_q$, for each $q = 1, 2, \ldots, N$,

$$\left| \frac{d}{d\mu_j}(V(R) - e_q) \right| \leqq K_6(\beta)|R - R_q|^p, \tag{1.16}$$

$$\left| \frac{d}{d\mu_j} W(R, R') \right| \leqq K_6(\beta)(|R - R_q|^p + |R - R_q|^p). \tag{1.17}$$

Remark iv). For the rescaled model, a degeneracy removing condition on

$$\tilde{E}_{qj} = \frac{d}{d\mu_i}(v(r_q) - v(r_N)) \tag{1.13a}$$

implies assumption A.4 with $M(\beta) = O(\beta)$. Assuming, for example, that w does not depend on μ, together with the condition

$$\left| \frac{d}{d\mu_j} v(r) \right| \leqq \tilde{K}_5(1 + v(r)) \tag{1.14a}$$

and a Lipschitz condition on $dv/d\mu_j$ in an open neighborhood of r_q,

$$\left| \frac{d}{d\mu_j}(v(r) - v(r')) \right| \leqq \tilde{K}_6|r - r'|^p, \tag{1.16a}$$

gives assumptions A.5 with

$$K_6(\beta) = O(\beta^{1 - p/2}).$$

Using assumptions A.0 through A.5 we will prove the following two theorems. $A = A(R_{\text{supp}\,A})$ always denotes a local observable, i.e. a complex valued function of finitely many variables $R_x, x \in \text{supp}\,A$, $\| A \|^{(\alpha)}$ is the norm

$$\| A \|^{(\alpha)} = \sup_{R_{\text{supp}\,A}} \left| A(R_{\text{supp}\,A}) \prod_{x \in \text{supp}\,A} \exp\left(-(1 - \alpha)(V(R_x) - e_0)\right) \right|,$$

and $\langle \cdot \rangle_{q, \Lambda}$ denotes the expectation value

$$\langle \cdot \rangle_{q, \Lambda} = \frac{1}{Z_q(\Lambda)} \int \cdot \prod_{x \in \partial \Lambda} \chi_q(R_x) e^{-H(R_\Lambda)} dR_{\Lambda'}$$

with $Z_q(\Lambda)$ chosen so that $\langle 1 \rangle_{q,\Lambda} = 1$. $\chi_q(R)$ is the characteristic function of the event $R \in U_q$. The first theorem concerns the infinite volume limit of $\langle \cdot \rangle_{q,\Lambda}$.

Theorem 1. *Assume that assumptions A.0 through A.3 are valid. Then there is a finite constant β^* and an open neighborhood, $\mathscr{V}^{(1)} \subset \mathscr{V}^{(0)}$, of $\mu = 0$ such that if $\beta \geq \beta^*$ and $\mu \in \mathscr{V}^{(1)}$, then there is a non-empty subset, $S(\beta, \mu) \subset \{1, 2, \ldots, N\}$, of stable boundary conditions, $q \in S(\beta, \mu)$, such that the limit $\langle A \rangle_q = \lim\limits_{\Lambda \to \mathbb{Z}^d} \langle A \rangle_{q,\Lambda}$ exists for all A with $\| A \|^{(\alpha)} < \infty$. $\langle A \rangle_q$ is translation invariant and describes a small perturbation of the Gaussian with mean R_q and covariance $(m_q^2 - \kappa_q \Delta)^{-1}$ (see Theorems 7.3 and 7.4 for the precise bounds).*

The second theorem is the construction of the phase diagram. It states that there is an open subset, $\mathscr{V}^{(2)} \subset \mathscr{V}^{(0)}$, in which $\{\mu \in \mathscr{V}^{(2)} | S(\beta, \mu) = \{1, 2, \ldots, N\}\}$ is a point, $\mu_0 \in \mathscr{V}^{(2)}$; that, for each $q \in \{1, 2, \ldots, N\}$, $\{\mu \in \mathscr{V}^{(2)} | S(\beta, \mu) = \{1, 2, \ldots, N\} \setminus \{q\}\} = \tilde{\mathscr{V}}_q$ is an open curve one of whose end points is μ_0; that, for each pair of distinct $q_1, q_2 \in \{1, 2, \ldots, N\}$, $\{\mu \in \mathscr{V}^{(2)} | S(\beta, \mu) = \{1, 2, \ldots, N\} \setminus \{q_1, q_2\}\} = \tilde{\mathscr{V}}_{q_1, q_2}$ is an open surface whose boundary in $\mathscr{V}^{(2)}$ is $\tilde{\mathscr{V}}_{q_1} \cup \tilde{\mathscr{V}}_{q_2} \cup \{\mu_0\}$; and so on. More precisely

Theorem 2. *Assuming assumptions A.0 through A.5 to be valid there is a finite constant β^* and an open subset $\mathscr{V}^{(2)} \subset \mathscr{V}^{(1)} \subset \mathscr{V}^{(0)}$ such that for $\beta \geq \beta^*$ and for all non-empty subsets $M \subset \{1, 2, \ldots, N\}$ the set $\mathscr{V}_M = \{\mu \in \mathscr{V}^{(2)} | S(\beta, \mu) \subset M\}$ is a non-empty, continuous hypersurface of dimension $N - |M|$.*

For further discussion of the relation between Theorem 2 and the phase diagram see Sect. 6.

2. Dilute Partition Functions, Contours

In this section we begin the contour analysis of our model. We will introduce certain dilute partition functions, $Z_q^{\text{dil}}(\Lambda)$, for which the field ϕ_x is constrained to lie in U_q near $\partial \Lambda$, and then rewrite $Z_q^{\text{dil}}(\Lambda)$ as a sum over non-overlapping contours, interacting via a perturbed Gaussian.

We first fix some notation. For two points $x, y \in Z^\nu$ we usually use the maximum distance $\text{dist}(x, y) = \max\limits_{1 \leq i \leq \nu} |x_i - y_i|$, except for the notation of a nearest neighbor pair $\langle x, y \rangle$, for which

$$|x - y| = \sum_{i=1}^{\nu} |x_i - y_i| = 1.$$

For $V, W \subset Z^\nu$ we define $\text{dist}(V, W)$ as the minimum over all points $x \in V$, $y \in W$ of $\text{dist}(x, y)$, and $\text{diam}\, V = \max\limits_{x, y \in V} \text{dist}(x, y)$. We set $V^c = Z^\nu \setminus V$, $\partial V = \{x \in V | \text{dist}(x, V^c) \leq 1\}$ and say V is connected if $\text{dist}(W, V \setminus W) \leq 1$ for any non-trivial subset $W \subset V$. We say V is a volume *without holes* if V^c is connected and call two sets V, W *adjacent* or *touching* if $\text{dist}(V, W) \leq 1$.

Our partition functions will always be defined in sets Λ of the form

$$\Lambda = \bigcup_{x' \in \Lambda'} \square(x'), \tag{2.1}$$

where, for some odd integer $L > 1$ to be chosen later, Λ' is a finite subset of LZ^ν

and $\square(x')$ is an L-block centered at x',

$$\square(x') = \{x \in Z^\nu | \operatorname{dist}(x, y') \leq L/2\}.$$

We now introduce the partition of unity $1 = \sum_{i=0}^{N} \chi_i(R)$, with

$$\chi_i(R) = \begin{cases} 0 & \text{if} \quad R \notin U_i \\ 1 & \text{if} \quad R \in U_i \end{cases}, \tag{2.2}$$

and rewrite $Z(\Lambda)$ as

$$Z(\Lambda) = \sum_{\omega:\, \Lambda \to \{0,\ldots,N\}} \int \chi_\omega(R_\Lambda) e^{-H(R_\Lambda)} dR_\Lambda,$$

where we have introduced the notation $\chi_\omega(R_\Lambda) = \prod_{x \in \Lambda} \chi_{\omega(x)}(R_x)$.

The q-dilute partition function $Z_q^{\mathrm{dil}}(\Lambda)$ is obtained from $Z(\Lambda)$ by restricting the sum over ω to a certain set $\Omega_q^{\mathrm{dil}}(\Lambda)$. For volumes Λ without holes this is the set of configurations, ω, for which $\omega(x) = q$ for all $x \in \Lambda$ with $\operatorname{dist}(x, \partial \Lambda^c) \leq L + 1$. For volumes Λ with holes we will introduce additional constraints which forbid contours enclosing the holes of Λ (this turns out to be convenient at several places, in particular in Sect. 3 and Sect. 7). We now define $\Omega_q^{\mathrm{dil}}(\Lambda)$ for arbitrary volumes, Λ, of the form (2.1).

Given $\omega: \Lambda \to \{0, 1, \ldots, n\}$ a block $\square(x') \subset \Lambda$ is called q-correct with respect to $\omega, q = 1, \ldots, N$, if $\omega(x) = q \ \forall x \in \Lambda$ with $d(x, \square(x')) \leq 1$. $\square(x')$ is called *incorrect* with respect to ω, if there is no $q \in \{1, \ldots, N\}$ such that $\square(x')$ is q-correct. The union of all blocks incorrect with respect to ω is denoted $B(\omega)$, and $\Omega_q^{\mathrm{dil}}(\Lambda)$ is defined as the set of all $\omega: \Lambda \to \{0, 1, \ldots, N\}$, such that

 i) all cubes $\square(x') \subset \Lambda$ which touch $\partial \Lambda^c$ are q-correct with respect to ω and
 ii) all finite connect components of $(B(\omega))^c$ lie in Λ.

Letting

$$H(R_\Lambda | R_{\Lambda'}) = H(R_\Lambda) + \sum_{\substack{x \in \Lambda,\, y \in \Lambda' \\ |x - y| = 1}} W(R_x, R_y), \tag{2.3}$$

$\Lambda' \subset \Lambda^c$, we define, for all finite Λ of the form (2.1), $q \in \{1, \ldots, N\}$ and boundary conditions $R_{\Lambda'}$ in $(U_q)^{|\Lambda'|}$,

$$Z_q^{\mathrm{dil}}(\Lambda) = \sum_{\omega \in \Omega_q^{\mathrm{dil}}(\Lambda)} \int \chi_\omega(R_\Lambda) e^{-H(R_\Lambda)} dR_\Lambda,$$

$$Z_q^{\mathrm{dil}}(\Lambda | R_{\Lambda'}) = \sum_{\omega \in \Omega_q^{\mathrm{dil}}(\Lambda)} \int \chi_\omega(R_\Lambda) e^{-H(R_\Lambda | R_{\Lambda'})} dR_\Lambda. \tag{2.4}$$

We also define the restricted partition functions

$$Z_q^{\mathrm{res}}(\Lambda) = \int \chi_q(R_\Lambda) e^{-H(R_\Lambda)} dR_\Lambda,$$

$$Z_q^{\mathrm{res}}(\Lambda | R_{\Lambda'}) = \int \chi_q(R_\Lambda) e^{-H(R_\Lambda | R_{\Lambda'})} dR_\Lambda. \tag{2.5}$$

We now wish to rewrite $Z_q^{\mathrm{dil}}(\Lambda)$ as a sum over contours. We proceed as follows. Given a function $\omega \in \Omega_q^{\mathrm{dil}}(\Lambda)$ we decompose $B(\omega)$ into its connected components $V_1, \ldots, V_k$ and define ω_i to be the restrictions of ω to $V_i, i = 1, \ldots, k$. The pairs

$\gamma_1 = (V_1, \omega_1), \ldots, \gamma_k = (V_k, \omega_k)$ are called the *contours of* ω, and V_i is called the *support*, supp γ_i, of γ_i.

Fix a contour $\gamma = (\text{supp}\,\gamma, \omega_\gamma)$ of ω. Then, as may be verified from the definition of q-correct, ω is constant on the boundary, ∂K, of each connected component, K, of $(\text{supp}\,\gamma)^c$. We let $\text{Int}_m\,\gamma$ denote the union of all finite components, K, for which $\omega = m$ on ∂K and write

$$\text{Int}\,\gamma = \bigcup_{m=1}^{n} \text{Int}_m\,\gamma, \quad V(\gamma) = \text{Int}\,\gamma \cup \text{supp}\,\gamma, \quad \text{Ext}\,(\gamma) = Z^\nu \backslash V(\gamma). \tag{2.6}$$

We also introduce the notions

$$\partial\gamma = \partial\,\text{supp}\,\gamma, \quad \partial_m\gamma = \partial(\text{Int}_m\,\gamma)^c, \quad \partial_e\gamma = \partial V(\gamma)^c. \tag{2.7}$$

Note that ω is constant on $\partial V(\gamma)$ (we say γ is a *q-contour* if $\omega = q$ on $\partial V(\gamma)$) and that $\omega = m$ on $\partial_m\gamma$. A contour γ is called an *external contour* of ω if supp $\gamma \subset \text{Ext}\,(\gamma')$ for all contours $\gamma' \neq \gamma$ of ω.

We also introduce the following notation: A pair $\gamma = (\text{supp}\,\gamma, \omega_\gamma)$ is called a *contour in* Λ, if γ is a contour of ω for some $q \in \{1, \ldots, n\}$ and some $\omega \in \Omega_q^{\text{dil}}(\Lambda)$ (note that by our definition of $\Omega_q^{\text{dil}}(\Lambda)$ this forbids contours which surround the holes of Λ). Two contours, γ, γ', are *compatible* if dist $(\gamma, \gamma') \equiv \text{dist}\,(\text{supp}\,\gamma, \text{supp}') \geqq 2$ and *mutually external* if dist $(V(\gamma), V(\gamma')) \geqq 2$. Finally we let $\Omega_q^{\text{ext}}(\Lambda)$ denote the set of sets, $\{\gamma_1, \ldots, \gamma_k\}$, of mutually external q-contours in Λ.

Remark 2.1. Since supp γ is a union of blocks centered in LZ^ν, and any two such blocks, $\square(x'), \square(x'')$, which do not touch have at least the distance $L + 1$, any contour γ in Λ has at least distance $L + 1$ from Λ, and two compatible contours have at least distance $L + 1$ from one another.

The following procedure is standard in Pirogov–Sinai theory: Consider a configuration $\omega_\Lambda \in \Omega_q^{\text{dil}}(\Lambda)$ and its external contours $\gamma_1, \ldots, \gamma_m$. Resum over all ω_Λ with a given set, $C = \{\gamma_1, \ldots, \gamma_m\}$, of external contours. Introduce the notation

$$e^{-\phi(\gamma|R_\gamma)} = \chi_{\omega_\gamma}(R_\gamma) e^{-H(R_\gamma)} \tag{2.8}$$

with $R_\gamma = R_{\text{supp}\,\gamma}$. Then $Z_q^{\text{dil}}(\Lambda)$ becomes

$$Z_q^{\text{dil}}(\Lambda) = \sum_{C \in \Omega_q^{\text{ext}}(\Lambda)} \int dR_{\Lambda \backslash \text{Int}\,C}\, e^{-H(R_{\text{Ext}_\Lambda C}|R_{\partial V(C)})} \chi_q(R_{\text{Ext}_\Lambda C})$$

$$\cdot \prod_{\gamma \in C} e^{-\phi(\gamma|R_\gamma)} \prod_{m=1}^{N} Z_m^{\text{dil}}(\text{Int}_m\,\gamma | R_{\partial_m\gamma}), \tag{2.9}$$

where we have used the notation

$$\text{Int}\,C = \bigcup_{\gamma \in C} \text{Int}\,\gamma, \quad V(C) = \bigcup_{\gamma \in C} V(\gamma), \quad \text{and} \quad \text{Ext}_\Lambda C = \Lambda \backslash V(C).$$

We now define the relative free energy, F_q, of a q-contour γ by

$$e^{-F_q(\gamma|R_{\partial\gamma})} = \prod_{m=1}^{N} \frac{Z_m^{\text{dil}}(\text{Int}_m\,\gamma | R_{\partial_m\gamma})}{Z_q^{\text{dil}}(\text{Int}_m\,\gamma)}, \tag{2.10}$$

and the contour functional $\psi_q(\cdot|\cdot)$ by

$$\exp(-\psi_q(\gamma|R_{\partial_e\gamma})) \equiv \int dR_\gamma \chi_{\omega_\gamma}(R_\gamma) \exp(-H(R_\gamma|R_{\partial_e\gamma}) - F(\gamma|R_{\partial\gamma}))$$

$$= \int dR_\gamma \exp(-\phi(\gamma|R_\gamma) - F_q(\gamma|R_{\partial\gamma})) \prod_{x\in\partial\gamma, y\in\partial_e\gamma|x-y|=1} \exp(-W(R_x, R_y)). \tag{2.11}$$

Dividing each term in (2.9) by $\prod_{\gamma\in C} Z_q(\text{Int}_\gamma)$ and multiplying it back again we then obtain

$$Z_q^{dil}(\Lambda) = \sum_{C\in\Omega_q^{ext}(\Lambda)} \int dR_{\text{Ext}_\Lambda C} \chi_q(R_{\text{Ext}_\Lambda C})$$

$$\cdot \exp(-H(R_{\text{Ext}_\Lambda C})) \prod_{\gamma\in C} \exp(-\psi_q(\gamma|R_e\gamma)) Z_q^{dil}(\text{Int}_\gamma). \tag{2.12}$$

Equation (2.12) can be iterated and one obtains the following representation of $Z_q^{dil}(\Lambda)$ as a sum over sets, $C = \{\gamma_1, \ldots, \gamma_k\}$, of pair-wise compatible q-contours in Λ $\left(\text{we write this sum as } \sum_{C \text{ in } \Lambda}^q \right)$

$$\boxed{Z_q^{dil}(\Lambda) = \sum_{C \text{ in } \Lambda}^q \int dR_{\Lambda\setminus C} \chi_q(R_{\Lambda\setminus C}) \exp(-H(R_{\Lambda\setminus C})) \prod_{\gamma\in C} \exp(-\psi_q(\gamma|R_{\partial_e\gamma}))} \tag{2.13}$$

where we have used the notation $\Lambda\setminus C = \Lambda\setminus \bigcup_{\gamma\in C} \text{supp}\,\gamma$. We will sometimes also write (2.13) as

$$\sum_{C \text{ in } \Lambda}^q \int dR_{\Lambda\setminus C} \chi_q(R_{\Lambda\setminus C}) \exp(-H(R_{\Lambda\setminus C})) \exp(-\psi_q(C|R_{\partial_e C})), \tag{2.13}$$

where we have introduced the notation

$$\psi_q(C|R_{\partial_e C}) = \sum_{\gamma\in C} \psi_q(\gamma|R_{\partial_e\gamma}). \tag{2.14}$$

Equation (2.13) is the desired representation of $Z_q^{dil}(\Lambda)$ as a sum over non-overlapping contours interacting via a perturbed Gaussian. The convergence of the corresponding Mayer series for $\log Z_q^{dil}(\Lambda)$ will be studied in Sects. 3–5.

3. Stable and Unstable Contours, Truncated Contour Models

In this section we begin the analysis of the convergence of the Mayer expansion for $\log Z_q^{dil}(\Lambda)$. To motivate our strategy, assume for the moment that Z_q^{res}, defined in Eq. (2.5), can be written as the partition function of a dilute polymer model (this is in fact possible due to assumption A.1) and that ψ_q obeys a bound of the form

$$\exp(-\psi_q(\gamma|R_{\partial_e\gamma})) \leq \exp(-\tau + e_q)|\text{supp}\,\gamma|) \tag{3.1}$$

for all q-contours, γ, and all boundary conditions $R_{\partial_e\gamma} \in U_q^{|\partial_e\gamma|}$. Then, for τ large enough, we can combine the contours in the sum (2.13) with the polymers of Z_q^{res} to obtain a new polymer system which is again dilute. The resulting polymer expansion for $\log Z_q^{dil}(\Lambda)$ would be the desired convergent Mayer expansion.

While a bound of the form (3.1) can be established for the integral

$\int \exp(-\phi(\gamma|R_\gamma))dR_\gamma$, see Lemma 3.1 below, we cannot expect that (3.1) holds for $\psi_q(\cdot|\cdot)$, at least not in general (otherwise one could show that the Gibbs states corresponding to Z_q^{dil} are small perturbations of the corresponding Gaussian, for β large and for all $q\in\{1,\ldots,N\}$, which is clearly false in general).

To overcome this difficulty we follow the strategy of Zahradnik [11] and introduce certain truncated contour functionals, ψ'_q, for which a bound of the form (3.1) is true. In our context this is done by replacing $F_q(\gamma|R_{\partial\gamma})$ in the definition of ψ_q by $F'_q(\gamma|R_{\partial\gamma}) = \max\{F_q(\gamma|R_{\partial\gamma}),\ -(2\tau/3)|\mathrm{supp}\,\gamma|\}$. The corresponding partition function, Z'_q, can then be analysed by a convergent cluster expansion. In a final step one shows that ψ_q and ψ'_q agree for those q, for which $h'_q = h$, where h'_q is the free energy corresponding to Z'_q and $h = \min_m h'_m$. Thus, if $h'_q = \min_m h'_m$, the cluster expansion for the truncated partition function, Z'_q, is in fact a cluster expansion for Z_q^{dil}.

We begin by proving that the integral of $e^{-\phi}$ obeys a bound of the form (3.1). $K_1 > 0$ and $K_3 < \infty$ are the constants from assumptions A.2 and A.3.

Lemma 3.1. *There is a constant $c_0 < \infty$ not depending on $L, \beta,$ and μ such that for all*

$$\mu\in\mathscr{V}^{(1)} = \left\{\mu\in\mathscr{V}^{(0)}\,\middle|\,|\mu| < \frac{K_1}{4K_3}(2L^\nu)^{-1}\right\}$$

and all q-contours, γ, the following inequality holds:

$$\int \exp(-\phi(\gamma|R_\gamma))dR_\gamma \leqq \exp(-(\tau + e_q)|\mathrm{supp}\,\gamma|) \tag{3.2}$$

with

$$\tau = \tau(\beta, L) \equiv \frac{K_1}{2}\frac{\tilde{\tau}(\beta)}{(2L)^\nu} - \log c_0. \tag{3.3}$$

Proof. From assumption A.1 i),

$$\int e^{-V(R)}\chi_m(R)dR \leqq e^{-e_m}\int \chi_m(R)\exp(-\tfrac{1}{2}\eta(R-R_m)^2)dR$$

$$\leqq e^{-e_m}\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}\eta(R-R_m)^2)dR \leqq \left(\frac{2\pi}{\eta}\right)^{1/2}e^{-e_0}, \tag{3.4}$$

where $m \neq 0$ and we recall that e_0 was defined as $\min_q e_q$. Given a q-contour, γ, we let $N(\Gamma)$ be the number of points, $x\in\mathrm{supp}\,\gamma$, for which $\omega_\gamma(x) = 0$, and $\mathscr{L}(\gamma)$ be the number of links, $\langle x,y\rangle \subset \mathrm{supp}\,\gamma$, such that

$$0 \neq \omega_\gamma(x) \neq \omega_\gamma(y) \neq 0.$$

Letting $c_0 = \max\{K_2, (2\pi/\eta)^{1/2}\}$ we have from assumption A.2

$$\int \exp(-\phi(\gamma|R_\gamma))dR_\gamma = \int \exp(-H(R_\gamma))\chi_{\omega_\gamma}(R_\gamma)dR_\gamma$$

$$\leqq (c_0 e^{-e_0})^{|\mathrm{supp}\,\gamma|}\exp(-K_1\tilde{\tau}(N(\gamma) + \mathscr{L}(\gamma))). \tag{3.5}$$

Recall that by the definition of a contour each block $\square(x') \subset \mathrm{supp}\,\gamma$ must be incorrect. Therefore each block, $\square(x')$, must touch a link or point contributing to $\mathscr{L}(\gamma)$ or $N(\gamma)$ respectively. Since a link or point in Z^ν can touch at most 2^ν distinct

blocks we find that

$$N(\gamma) + \mathscr{L}(\gamma) \geq |\operatorname{supp} \gamma|(2L)^{-\nu}. \tag{3.6}$$

On the other hand, by assumption A.2,

$$|e_q - e_0| \leq 2K_3 \tilde{\tau}(\beta)|\mu| \leq \tfrac{1}{2}K_1 \frac{\tilde{\tau}(\beta)}{(2L)^\nu}, \tag{3.7}$$

provided $\mu \in \mathscr{V}^{(1)}$. Combined with (3.5) and (3.6) this gives the bound (3.3). ∎

We now define the truncated contour functional ψ'_q. We will assume from now on that τ is the constant from Lemma 3.1 and that β is chosen so large that $\tau > 0$.

Definition 3.1. We define the truncated contour functional $\psi'_q(\cdot|\cdot)$ by replacing $F_q(\gamma|R_{\partial\gamma})$ in (2.11) by

$$F'_q(\gamma|F_{\partial\gamma}) = \max\left\{F_q(\gamma|R_{\partial\gamma}), -\frac{2\tau}{3}|\operatorname{supp}\gamma|\right\}. \tag{3.8}$$

The truncated partition function $Z'_q(\Lambda)$ is defined by the relation (2.13) with ψ_q replaced by ψ'_q.

With this definition ψ'_q obeys a bound of the form (3.1) (with τ replaced by $\tau/3$). For β large enough, $Z'_q(\Lambda)$ can therefore be analysed by a convergent cluster expansion. This is done in Sects. 4 and 5, where we prove the following lemma (η, η', η'' are the constants from assumption A.1).

Lemma 3.2. *There are constants $L_0, c_1 < \infty$ not depending on L or β, as we all as β independent constants $\tau_1 = \tau_1(L)$, $\beta_1 = \beta_1(L) < \infty$ such that for $L \geq L_0, \mu \in \mathscr{V}^{(1)}$, $\beta \geq \beta_1$ and $\tau(\beta, L) \geq \tau_1$, and for each $q \in \{1, 2, \ldots, N\}$ there is a number h'_q which satisfies*

i) $|h'_q - e_q| \leq c_1$,
ii) $|\log Z'_q(\Lambda) + |\Lambda| h'_q| \leq c_1 |\partial \Lambda^c|$.

Definition 3.2. Let h'_q be the free energy for the truncated contour model. We then define

$$h = \min_m h'_m \quad \text{and} \quad a_q = h'_q - h. \tag{3.9}$$

We call a q-contour, γ, *stable* if, $\forall m \in \{1, \ldots, N\}$,

$$\frac{Z_m(\operatorname{Int}_m \gamma)}{Z_q(\operatorname{Int}_m \gamma)} \leq \exp(2\tau/3)|\partial_m \gamma|), \tag{3.10}$$

and *small* if

$$a_q \operatorname{diam} \gamma \leq \tau/3. \tag{3.11}$$

If γ is not small we say it is *large*. Finally we define q to be a *stable phase* if all q-contours are stable. We denote the set of stable phases by $S = S(\beta, \mu)$.

Remark 3.1. We have normalised W in such a way that $W \geq 0$. Therefore

$$\exp(-F(\gamma|R_{\partial\gamma})) \leq \prod_m \frac{Z_m(\operatorname{Int}_m \gamma)}{Z_q(\operatorname{Int}_m \gamma)}. \tag{3.12}$$

On the other hand

$$\sum_m |\partial_m \gamma| \leq |\partial \gamma| \leq |\text{supp}\, \gamma|, \tag{3.13}$$

which implies that $\psi_q(\gamma|\cdot) = \psi'_q(\gamma|\cdot)$ for stable contours γ. In particular $Z'_q(\Lambda) = Z_q^{\text{dil}}(\Lambda)$ if q is a stable phase. It is, however, not a priori clear which are the stable phases, and whether there is at least one stable phase.

The following theorem implies that the set S of stable phases is in fact not empty, and that a stable phase can be characterized by the condition $a_q = 0$.

Theorem 3.3. *Assume that L has been chosen according to Lemma 3.2. Then there is a constant* $\beta_2 = \beta_2(L) \geq \beta_1$, *such that for* $\beta \geq \beta_2$ *and* $\mu \in \mathscr{V}^{(1)}$ *the following statements hold:*

  i) $Z_q^{\text{dil}}(\Lambda) \geq \exp\{-h'_q|\Lambda| - c_1|\partial \Lambda^c|\}$,
 ii) $Z_q^{\text{dil}}(\Lambda) \leq \exp\{-h|\Lambda| + (c_1 + 1)|\partial \Lambda^c|\}$,
iii) *If* γ *is a small contour, then* γ *is stable.*

Here Λ is an arbitrary finite volume of the form (2.1) and β_1, c_1 are the constants of Lemma 3.2.

Remark 3.2. The constant β_2 is chosen in such a way that $\tau = \tau(\beta, L) \geq \tau_1$ for $\beta \geq \beta_2$. In addition we will need, in the course of the proof, the additional constraints

$$2c_1 + 1 \leq \tau/3, \tag{3.14a}$$

$$\tau - 3c_1 - 1 \geq \tau_2, \tag{3.14b}$$

where τ_2 is a constant depending only on an N and v.

Remark 3.3. Obviously Theorem 3.3 iii) implies that all phases q with $a_q = 0$ are stable. If, on the other hand, q is a stable phase, then all q-contours are stable and

$$Z'_q(\Lambda) = Z_q^{\text{dil}}(\Lambda) \geq Z_m^{\text{dil}}(\Lambda) \exp(-2\tau/3)|\partial \Lambda^c|)$$

for all Λ and all m by (3.10) (given Λ and m, one just has to choose a q-contour γ with $\text{Int}_m \gamma = \Lambda$, which is always possible). Combined with Lemma 3.2 and Theorem 3.3 i) the above bound shows that $h'_q \leq h'_m$ for all m, and hence $a_q = 0$. So Theorem 3.3 implies in fact that q is stable if and only if $a_q = 0$.

Proof of Theorem 3.3 i). The statement follows immediately from Lemma 3.2 and the fact that $\psi'(\gamma|R_{\partial_e \gamma}) \geq \psi(\gamma|R_{\partial_e \gamma})$.

Proof of Theorem 3.3 ii) and iii). The proof will be done by induction. We define the level of a contour γ to be the maximal n such that there are contours $\gamma_1, \ldots, \gamma_n = \gamma$, with $\text{supp}\, \gamma_i \subset \text{Int}\, \gamma_{i+1}$, $\text{dist}\,(V(\gamma_i), \gamma_{i+1}) > 2$ and the level of Λ as the maximal level of a contour in Λ.

Proof of ii) for level $\Lambda = 0$. Trivial, because in this case $Z_q^{\text{dil}}(\Lambda) = Z_q^{\text{res}}(\Lambda) = Z'_q(\Lambda)$.

Proof of iii) for level $\gamma = n$. $\text{Int}_m \gamma$ is a volume of level $\leq n - 1$. Therefore we can use the inductive assumption to bound

$$\frac{Z_m^{\text{dil}}(\text{Int}_m \gamma)}{Z_q^{\text{dil}}(\text{Int}_m \gamma)} \leq \exp(a_q|\text{Int}_m \gamma| + (2c_1 + 1)|\partial_m \gamma|).$$

Bounding $|\mathrm{Int}_m \gamma|$ by

$$|\partial_m \gamma| \,\mathrm{diam}\, \mathrm{Int}_m \gamma \leq |\partial_m \gamma| \,\mathrm{diam}\, \gamma,$$

and using the definition of a small contour we obtain the upper bound

$$\exp((2c_1 + 1 + \tau/3)|\partial_m \gamma|),$$

which shows that γ is stable if

$$2c_1 + 1 \leq \tau/3 \quad \text{(this is (3.14a))}.$$

Proof of ii) *for level* $\Lambda = n$. We use the relation (2.9). Fix, for a moment, the set C_1 of large external contours in C. To specify C completely, one has to specify, in addition to C_1, a set C_2 of mutually external, small contours in $\Lambda \backslash V(C_1) = \mathrm{Ext}_\Lambda(C_1)$. Using the fact that the volume $\mathrm{Ext}_\Lambda C$ appearing in (2.9) can be rewritten as $\mathrm{Ext}_{\Lambda \backslash V(C_1)} C_2$ we can resum the small contours in $\Lambda \backslash V(C_1)$ to obtain the relation

$$Z_q^{\mathrm{dil}}(\Lambda) = \sum_{C_1}' \int Z_q^{\mathrm{small}}(\Lambda \backslash V(C_1) | R_{\partial V(C_1)})$$

$$\cdot \prod_{\gamma \in C_1} \left[\exp(-\phi(\gamma | R_\gamma)) \prod_{m=1}^N Z_m^{\mathrm{dil}}(\mathrm{Int}_m \gamma | R_{\partial_m \gamma}) dR_\gamma \right],$$

where Z_q^{small} is obtained from Z_q^{dil} by restricting the sum over ω in (2.4) to a sum over configurations ω which contain only small external contours, and $\sum'$ is sum over sets of mutually external, large contours. We now bound

$$Z_q^{\mathrm{small}}(\Lambda \backslash V(C_1) | R_{\partial V(C_1)}) \leq Z_q^{\mathrm{small}}(\Lambda' V(C_1)),$$

$$Z_m^{\mathrm{dil}}(\mathrm{Int}_m \gamma | R_{\partial_m \gamma}) \leq Z_m^{\mathrm{dil}}(\mathrm{Int}_m \gamma),$$

and use the fact that all the contours appearing in the representation (2.13) for $Z_q^{\mathrm{small}}(\Lambda / V(C_1))$ are stable due to the inductive assumption iii). Therefore $Z_q^{\mathrm{small}}(\Lambda / V(C_1)) \leq Z_q'(\Lambda / V(C_1))$ which can be bounded using Lemma 3.2. Using the inductive assumption ii) for $Z_m^{\mathrm{dil}}(\mathrm{Int}_m \gamma)$ and Lemma 3.1 for $\int \exp(-\phi(\gamma | R_\gamma)) dR_\gamma$ we get the bound

$$Z_q^{\mathrm{dil}}(\Lambda) \leq \sum_C' \exp(-h_q' |\Lambda \backslash V(C)| + c_1 |\partial \Lambda^c| + c_1 |\partial V(C)|)$$

$$\prod_{\gamma \in C} \exp(-(\tau + e_q)|\mathrm{supp}\,\gamma|) \prod_m \exp(-h |\mathrm{Int}_m \gamma| + (1 + c_1)|\partial_m \gamma|).$$

We now extract a factor $\exp(-h|\Lambda|)$, bound

$$\exp(-e_q |\mathrm{supp}\,\gamma|) \leq \exp(-h_q' - c_1)|\mathrm{supp}\,\gamma|),$$

and use (3.13) to obtain the inequality

$$Z_q^{\mathrm{dil}}(\Lambda) \leq \exp(-h|\Lambda| + c_1 |\partial \Lambda^c|) \sum_C' \exp(-a_q |\Lambda \backslash \mathrm{Int}\, C|) \prod_{\gamma \in C} \exp(-(\tau - 3c_1 - 1)|\mathrm{supp}\,\gamma|).$$

$$(3.15)$$

To complete the proof, we need the following technical lemma, proven in [11]. For the convenience of the reader we give the proof of this lemma at the end of this section.

Lemma 3.4 ([11]). *Consider a contour functional $\tilde{K}:\gamma\mapsto\tilde{K}(\gamma)\geqq 0$ and let $\tilde{Z}$ be the partition function*

$$\tilde{Z}(\Lambda)=\sum_{C\ \text{in}\ \Lambda}^{q}\prod_{\gamma\in C}\tilde{K}(\gamma)e^{|\gamma|}.$$

Assume that $K(\gamma)\leqq\tilde{\varepsilon}^{|\gamma|}$. Then there is a constant τ_2 depending only on v and N, such that for $\tilde{\varepsilon}\leqq e^{-\tau_2}$ the following statement is true. Let $\tilde{s}$ be the free energy corresponding to $\tilde{Z}$. Then, for all $\tilde{a}\geqq-\tilde{s}$,

$$\sum_{C}' \exp\left(-\tilde{a}|\Lambda\backslash\text{Int}\,C|\right)\prod_{\gamma\in C}\tilde{K}(\gamma)\leqq e^{|\partial\Lambda^{c}|},$$

where the sum goes over sets of mutually external q-contours in Λ.
To apply this lemma we put $\tilde{a}=a_q$,

$$\tilde{\varepsilon}=\exp\left(-\gamma-3c_1-1)\right)$$

and

$$\tilde{K}(\gamma)=\begin{cases}\tilde{\varepsilon}^{|\gamma|} & \text{if}\quad\gamma\text{ is large}\\ 0 & \text{if}\quad\gamma\text{ is small}\end{cases}$$

Using the fact that $\tilde{Z}$ contains only contours γ for which

$$|\sup\gamma|\geqq(\tau/3a_q)$$

(since $\text{diam}\,\gamma\leqq|\text{supp}\,\gamma|$, because $\text{supp}\,\gamma$ is connected) one easily finds that

$$-\tilde{s}\leqq O(\tilde{\varepsilon}^{(\tau/3a_q)})\leqq a_q$$

if τ is chosen large enough. We therefore can apply the lemma provided

$$\tau-3c_1-1\geqq\tau_2$$

(this is the restriction 3.14b). ∎

We close this section with a proof of Lemma 3.4.

Proof of Lemma 3.4. $\tilde{Z}$ is the partition function of a polymer with activities $K^*(\gamma)=\tilde{K}(\gamma)e^{|\gamma|}$. For $\tilde{\varepsilon}$ small enough (depending only on N and v) $\tilde{Z}$ can be controlled by a convergent expansion and

$$|\log\tilde{Z}(W)+\tilde{s}|W||\leqq O(\tilde{\varepsilon})|\partial W^{c}|\leqq|\partial W^{c}|.$$

Putting $W=\text{Int}\,C$ and using (3.13) together with the assumption $\tilde{a}\geqq-\tilde{s}$ we get

$$\sum_{C}' \exp\left(-\tilde{a}|\Lambda\backslash\text{Int}\,C|\right)\prod_{\gamma\in C}\tilde{K}(\gamma)\leqq e^{\tilde{s}|\Lambda|}\sum_{C}'\prod_{\gamma\in C}\tilde{K}(\gamma)e^{|\gamma|}\tilde{Z}(\text{Int}\,\gamma)$$

$$=e^{\tilde{s}|\Lambda|}\tilde{Z}(\Lambda)\leqq e^{|\partial\Lambda^{c}|}. \quad\blacksquare$$

4. Polymer Expansion for the Truncated Model

In Sects. 4 and 5 we prove Lemma 3.2 by a convenient cluster expansion. In principle we could use the Glimm–Jaffe cluster expansion in the form of [18] to obtain a representation of Z_q^{res} as a partition function of a hard core interacting

polymer system and combine it with the contour expansion (2.13). This approach has however the disadvantage that it needs C^∞-differentiability of V and W near R_q, which is not natural for the lattice models considered here.

Instead we use a slightly different approach, which, after expanding in the perturbing potentials

$$V_q(\phi) \equiv V(R_q + \phi) - \frac{m_q^2}{2}\phi^2 - e_q,$$

$$W_q(\phi, \phi') \equiv W(R_q + \phi, R_q + \phi') - \frac{\kappa_q}{2}(\phi - \phi')^2,$$

(4.1)

uses interpolation parameters (s-parameters) only in regions where no factors of V_q, W_q or $e^{-\psi_q}$ arise. Since we decouple these regions using non-zero Dirichlet data on their boundaries, we do not generate field derivatives of V_q or W_q, and therefore don't need differentiability assumptions on V_q or W_q. The convergence of this expression is established in Sect. 5.

We recall the definition of the truncated partition function ($\Lambda\backslash C$ is defined as $\Lambda\backslash \bigcup_{\gamma\in C} \mathrm{supp}\,\gamma$)

$$Z_q'(\Lambda) = \sum_C \int dR_{\Lambda\backslash C}\chi_q(R_{\Lambda\backslash C})\exp\left(-H(R_{\Lambda\backslash C})\right) \prod_{\gamma\in C} \exp\left(-\psi_q'(\gamma\,|\,R_{\partial_e\gamma})\right),$$

and rewrite, with $\phi_x = R_x - R_q$, where R_q is the minimum of the quadratic approximation of H in U_q,

$$H(R_{\Lambda\backslash C}) = e_q|\Lambda\backslash\mathrm{supp}\,C| + H_q^{(0)}(\phi_{\Lambda\backslash C}) + \frac{1}{2}\sum_{x,y\in\Lambda\backslash C} W_q(\phi_x, \phi_y) + \sum_{x\in\Lambda\backslash C} V_q(\phi_x), \quad (4.2)$$

where V_q, W_q are defined in (4.1) and

$$H_q^{(0)}(\phi_\Omega) = \frac{\kappa_q}{4}\sum_{\substack{x,y\in\Omega \\ |x-y|=1}}(\phi_x - \phi_y)^2 + \frac{m_q^2}{2}\sum_{x\in\Omega}\phi_x^2. \quad (4.3)$$

We now expand in V_q and W_q. More precisely, we define

$$F_{xy}(\phi_x, \phi_y) = \exp\left(-W_q(\phi_x, \phi_y)\right) - 1, \quad (4.4a)$$

$$F_x(\phi_x) = \chi_q(\phi_x + R_q)\exp\left(-V_q(\phi_x)\right) - 1 \quad (4.4b)$$

and, with the convention $F_\varnothing = 1$,

$$F_X(\phi_X) = \sum_{\substack{X_1, B\,\mathrm{s.t.} \\ X = X_1\cup X(B)}} \prod_{x\in X_1} F_x(\phi_x) \prod_{\langle x,y\rangle\in B} F_{xy}(\phi_x, \phi_y), \quad (4.4c)$$

where B denotes a set of nearest neighbor pairs and $X(B)$ is the corresponding set of points. Then

$$Z_q'(\Lambda) = e^{-e_q|\Lambda|}\sum_C \sum_{X\subset\Lambda\backslash C} \int d\phi_{\Lambda\backslash C}\exp\left(-H^{(0)}(\phi_{\Lambda\backslash C})\right)$$

$$\cdot F_X(\phi_X)\prod_{\gamma\in C}\exp\left(e_q|\mathrm{supp}\,\gamma| - \psi_q'(\gamma\,|\,R_q + \phi_{\partial_e\gamma})\right). \quad (4.5)$$

With the notation

$$Z_q^{(0)}(\Omega|\phi_{\Omega'}) = \int d\phi_\Omega \exp(-H_q^{(0)}(\phi_\Omega|\phi_{\Omega'})),$$

$$H_q^{(0)}(\phi_\Omega|\phi_{\Omega'}) = H_q^{(0)}(\phi_\Omega) + \frac{\kappa_q}{2}\sum_{\substack{x\in\Omega,\,y\in\Omega'\\|x-y|=1}}|\phi_x-\phi_y|^2,$$
(4.6)

this can be rewritten as

$$Z_q'(\Lambda) = e^{-e_q|\Lambda|}Z_q^{(0)}(\Lambda)\sum_C\sum_{X\subset\Lambda\backslash C}\int d\phi_{X\cup\partial_e C}\exp(-H_q^{(0)}(\phi_{X\cup\partial_e C}))$$

$$\cdot\prod_{\gamma\in C}\exp(-\psi_q'(\gamma|R_q+\phi_{\partial_e\gamma}) + e_q|\mathrm{supp}\,\gamma|)F_X(\phi_X)\frac{Z_q^{(0)}(\Lambda_1|\phi_{\partial_e C\cup\partial X})}{Z_q^{(0)}(\Lambda)},\quad (4.7)$$

where $\Lambda_1 = \Lambda\backslash(X\cup\mathrm{supp}\,C\cup\partial_e C)$. In order to decouple different connected components of $X\cup\mathrm{supp}\,C\cup\partial_e C$ we now choose a convenient expansion for the ratio of the Gaussian partition functions in Eq. (4.7). The expansion we choose is essentially that used in [18].

We note that there is a characteristic length, l_q, the correlation length of the gaussian, Eq. (4.3), over which points in the lattice are strongly correlated. It is this fact which motivated our introduction of the coarse lattice, LZ^ν. Choosing $L > l_q$ we treat the couplings across faces of L-blocks as perturbations, but only for those L-blocks far enough from $\partial X\cup\partial_e C$ to sufficiently moderate the effects of the conditioning. Thus, given X and C, define $\bar{X}$ to be the smallest union of L-blocks such that $X\cup\partial_e C\subset\bar{X}\subset\Lambda\backslash\mathrm{supp}\,C$ and $\mathrm{dist}(\Lambda\backslash(\bar{X}\cup\mathrm{supp}\,C),$ $\partial X\cup\partial_e C) > r_0$, where $r_0\gg l_q$ will be fixed, along with L, in the next section. Given two nearest neighbor points, x', y', in LZ^ν define the face separating the adjacent L-blocks x' and y' to be the set of the $L^{\nu-1}$ nearest neighbor pairs $\{x,y\}\subset Z^\nu$ such that $\{x,y\}\cap\square(x')\neq\varnothing$ and $\{x,y\}\cap\square(y')\neq\varnothing$. Given a union of L-blocks, $\Omega\subset Z^\nu$, we say that a face, b, intersects Ω (written $b\cap\Omega$) if at least one of the L-blocks separated by b is a subset of Ω. Introduce decoupling parameters s_b for all faces $b\cap\Lambda\backslash(\bar{X}\cup\mathrm{supp}\,C)$ by defining, for each set Γ of faces intersecting $\Lambda\backslash(\bar{X}\cup\mathrm{supp}\,C)$,

$$s_{xy}(\Gamma) = \begin{cases} s_b & \text{if } \{x,y\}\in b\in\Gamma \\ 0 & \text{if } \{x,y\}\in b\notin\Gamma, \quad b\cap\Lambda\backslash(\bar{X}\cup\mathrm{supp}\,C). \\ 1 & \text{otherwise} \end{cases}$$

Denote the Gaussian partition functions obtained from $Z_q^{(0)}(\Lambda)$ and $Z_q^{(0)}(\Lambda_1|\phi_{\partial X\cup\partial_e C})$ upon substituting $s_{xy}(\Gamma)(\phi_x-\phi_y)^2$ for $(\phi_x-\phi_y)^2$ by $Z_q^{(0)}(\Lambda, s_\Gamma)$ and $Z_q^{(0)}(\Lambda_1, s_\Gamma|\phi_{\partial X\cup\partial_e C})$. Then by the fundamental theorem of calculus,

$$\frac{Z_q^{(0)}(\Lambda_1|\phi_{\partial X\cup\partial_e C})}{Z_q^{(0)}(\Lambda)} = \frac{Z_q^{(0)}(\bar{X}\backslash(X\cup\partial_e C)|\phi_{\partial X\cup\partial_e C})}{Z_q^{(0)}(\bar{X}\cup\mathrm{supp}\,C)}\exp\left(\sum_{\Gamma\neq\phi}W(\Gamma|\phi_{\partial X\cup\partial_e C})\right),\quad (4.8)$$

where

$$W(\Gamma|\phi_{\partial X\cup\partial_e C}) = \int_{[0,1]^{|\Gamma|}}ds_\Gamma\,\partial^\Gamma\{\log Z_q^{(0)}(\Lambda_1, s_\Gamma|\phi_{\partial X\cup\partial_e C}) - \log Z_q^{(0)}(\Lambda, s_\Gamma)\},\quad (4.9)$$

with $ds_\Gamma = \prod_{b\in\Gamma}ds_b$, $\partial^\Gamma = \prod_{b\in\Gamma}\partial/\partial s_b$; the sum over Γ is over all non-empty Γ

intersecting $\Lambda \backslash (\bar{X} \cup \text{supp}\, C)$. We note, see also [18], that $W(\Gamma | \cdot)$ is zero unless Γ satisfies the connectivity conditions:

i) $X(\Gamma) \cap (\bar{X} \cup \text{supp}\, C) \neq \varnothing$,
ii) $X(\Gamma)$ is contained in one and only one connected component of $X(\Gamma) \cup (\bar{X} \cup \text{supp}\, C)$.

Here, and in the following, $X(\Gamma)$ denotes the union of all L-blocks which intersect a face in Γ. If i) is violated, then the terms from the two partition functions in (4.8b) are equal, and thus cancel each other. If ii) is violated, then the derivative ∂^{Γ} vanishes because it acts on a sum of terms, one for each connected component of $X(\Gamma) \cup \bar{X} \cup \text{supp}\, C$, none of which depend on all of the s_b. Expanding the exponential in (4.8) and inserting the result into Eq. (4.6) we obtain

$$Z'_q(\Lambda) = e^{-e_q |\Lambda|} Z_q^{(0)}(\Lambda) \sum_C \sum_{X \subset \Lambda \backslash C} \sum_{\varGamma} \frac{1}{|\varGamma|!}$$

$$\cdot \frac{1}{Z_q^{(0)}(\bar{X} \cup \text{supp}\, C)} \int d\phi_{\bar{X}} \exp(-H_q^{(0)}(\phi_{\bar{X}})) F_X(\phi_X)$$

$$\cdot \prod_{\gamma \in C} \exp(-\psi'_q(\gamma | R_q + \phi_{\partial_e \gamma}) + e_q |\text{supp}\, \gamma|) \prod_{\Gamma \in \varGamma} W(\Gamma | \phi_{\partial X \cup \partial_e C}), \quad (4.10)$$

where the last sum is over all sequences $\varGamma$ of sets Γ satisfying the connectivity conditions i) and ii).

Equation (4.10) is essentially the expansion we are looking for. Defining a polymer in Λ as a connected union of L-blocks, $P \subset \Lambda$, and its activity

$$\rho(P) = \sum_{\substack{C, X, \varGamma \text{ s.t.} \\ \text{supp}\, C \cup \bar{X} \cup X(\varGamma) = P}} \frac{1}{|\varGamma|!} \frac{1}{Z_q^{(0)}(\bar{X} \cup \text{supp}\, C)} \int d\phi_{\bar{X}} \exp(-H_q^{(0)}(\phi_{\bar{X}}))$$

$$\cdot \prod_{\gamma \in C} \exp(-\psi'_q(\gamma, R_q + \phi_{\partial_e \gamma})) F_X(\phi_X) \prod_{\Gamma \in \varGamma} W(\Gamma | \phi_{\partial X \cup \partial_e \Gamma}), \quad (4.11)$$

where $X(\varGamma) = \bigcup_{\Gamma \in \varGamma} X(\Gamma)$, we obtain, after rearranging with the help of the multi-nomial theorem,

$$\boxed{Z'_q(\Lambda) = e^{-e_q |\Lambda|} Z_q^{(0)}(\Lambda) \sum_{\mathbb{P}} \sum_{P \in \mathbb{P}} \rho(P)} \quad (4.12)$$

where the sum is over sets of unions of L-blocks, $\mathbb{P} = \{P_1, P_2 \ldots\}$, such that each P_j is connected and if $j \neq k$ then P_j and P_k have no face in common.

In the next section we prove the following theorem, from which the convergence of the Mayer expansion for $\log Z'_q(\Lambda)$ immediately follows.

Theorem 4.1. *For all $b < \infty$ one can choose $L_0 < \infty$ independent of β and L, and $\gamma_0 = \gamma_0(L)$, $\beta_1 = \beta_1(L)$, $\tau_1 = \tau_1(L)$ not depending on β, such that*

$$|\rho(P)| \leq e^{-b|P'|} = e^{-bL^{-\nu}|P|}$$

for all polymers P, provided $L \geq L_0$, $\mu \in \mathscr{V}^{(1)}$, $\tau = \tau(\beta, L) \geq \tau_1$ and $\beta \geq \beta_1$.

Lemma 3.2 is an immediate consequence of Theorem 4.1: Let $\theta_q(\Lambda)$ be the

polymer partition function

$$\theta_q(\Lambda) = \sum_{\mathbb{P}} \prod_{P \in \mathbb{P}} \rho(P). \tag{4.13}$$

Choose b large enough, for example so large that the sum $\sum'$ over all polymers touching a given L-cube $\square(x')$

$$\sum_{P}' e^{-(b-1)|P'|} < 1. \tag{4.14}$$

Then [19] the Mayer expansion for $\log\theta_q(\Lambda)$ is absolutely convergent. It follows that the limit

$$s_q = -\lim_{\Lambda\uparrow\mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log\theta_q(\Lambda)$$

exists, and that

$$|s_q| \leqq \varepsilon(b),$$

and

$$|\log\theta_q(\Lambda) + s_q|\Lambda| | \leqq \varepsilon(b)|\partial\Lambda^c|$$

for a constant $\varepsilon(b) \to 0$ as $b \to \infty$. This, together with the corresponding bounds on the Gaussian partition function $Z_q^{(0)}(\Lambda)$—proven for example by random walk methods—immediately gives Lemma 3.2.

5. Convergence of the Mayer Series for the Truncated Models

In this section we prove Theorem 4.1. The proof will follow from two lemmas. We first state the lemmas. Then using them we prove Theorem 4.1. Finally, we prove the two lemmas themselves.

Lemma 5.1. $\forall c < \infty$ there is a constant L_0 (depending on c, η, η'' and ν) such that, $\forall L \geqq L_0$, $\forall \varepsilon > 0$ and a suitable choice of r_0 (depending on $\varepsilon, L, \eta, \eta''$ and ν),

$$\sum_{\substack{\Gamma \text{ s.t.} \\ \operatorname{supp} C \cup \bar{X} \cup X(\Gamma) = P}} \frac{1}{|\Gamma|!} \prod_{\bar{\Gamma} \in \Gamma} |W(\bar{\Gamma}|\phi_{\partial X \cup \partial_e C})|$$

$$\leqq \exp\left\{ -cL^{-\nu}|P\backslash(\bar{X} \cup \operatorname{supp} C)| + f_1|\bar{X} \cup \operatorname{supp} C| + \frac{\varepsilon}{2}\sum_{x \in \partial X \cup \partial_e C} \phi_x^2 \right\},$$

f_1 is a constant depending only on L, ν, η and η''.

Lemma 5.2. $\forall \tilde{c} < \infty$, $\forall \varepsilon < \eta$ there is a choice of β_1, (depending on $\tilde{c}, \varepsilon, \nu, \eta, \eta', \eta''$) such that

$$\frac{1}{Z_q^{(0)}(\bar{X} \cup \operatorname{supp} C)} \int |F_X(\phi_X)| \exp\left(-H_q^{(0)}(\phi_{\bar{X}}) + \frac{\varepsilon}{2}\sum_{x \in \partial X \cup \partial_e C} \phi_x^2 \right) d\phi_{\bar{X}}$$

$$\leqq \exp\{ -\tilde{c}|x| + f_2|X \cup \operatorname{supp} C \cup \partial_e C|,$$

provided $\beta \geqq \beta_1$. The constant f_2 depends only on $\varepsilon, \nu, \eta, \eta'$ and η''.

5.1 Proof of Theorem 4.1. We choose $\varepsilon = \eta/2$, $c = b + \log 2$, and $\tau_1 = 3\tilde{c}_1$, and we fix $L \geqq L_0$ and r_0 according to Lemma 5.1. $\tilde{c} > b$ will be chosen later. Using the bound

$$\exp\left(-\psi_q'(\gamma | R_{\partial_e C}) + e_q |\operatorname{supp} \gamma|\right) \leqq \exp\left(-\frac{\tau}{3} |\operatorname{supp} C|\right)$$

together with Lemma 5.1, Lemma 5.2 and the fact that $|\partial_e C| \leqq (3^\nu - 1)|\operatorname{supp} C|$ we obtain

$$|\rho(P)| \leqq \sum_{\substack{C, X \text{ s.t.} \\ \bar{X} \cup \operatorname{supp} C \subset P}} \exp\left(-cL^{-\nu}|P| + (f_1 + cL^{-\nu})|\bar{X} \cup \operatorname{supp} C|\right)$$

$$\cdot \exp\left(-(c - 3^\nu f_2)|X \cup \operatorname{supp} C|\right),$$

provided $\tau \geqq \tau_1 = 3\tilde{c}$ and $\beta \geqq \beta_1 = \beta_1(\tilde{c})$. Next we use the definition of $\bar{X}$ to conclude that

$$|\bar{X}| \leqq (2r_0 + 1)^\nu |X \cup \partial_e C| \leqq (2r_0 + 1)^\nu (|X| + (3^\nu - 1)|\operatorname{supp} C|),$$

which shows that

$$|X \cup \operatorname{supp} C| \geqq 3^{-\nu}(2r_0 + 1)|\bar{X} \cup \operatorname{supp} C|.$$

Bounding the sum over C and X with fixed "support," $Y = \bar{X} \cup \operatorname{supp} C$, by $2^{|Y|}(N+1)^{|Y|}$, choosing

$$\tilde{c} = 3^\nu f_2 + [f_1 + cL^{-\nu} + \log(2N+2)](6r_0 + 3)^\nu$$

and bounding the remaining sum by $2^{|P|L^{-\nu}}$ we obtain Theorem 4.1. ∎

5.2 Proof of Lemma 5.1. Using the fact that the contributions from $\Lambda \backslash P$ cancel in the definition of $W(\Gamma | \phi_{\partial X \cup \partial_e C})$ we rewrite

$$W(\Gamma | \phi_{\partial X \cup \partial_e C}) = \int ds_\Gamma \partial^\Gamma \{\log Z_q^{(0)}(P_1, s_\Gamma | \phi_{\partial X \cup \partial_e C}) - \log Z_q^{(0)}(P, s_\Gamma)\} \qquad (5.1)$$

with $P_1 = P \backslash (X \cup \operatorname{supp} C \cup \partial_e C)$. Let $C^{(2)}(s_\Gamma)$ be the covariance matrix corresponding to $Z_q^{(0)}(P, s_\Gamma)$, i.e.

$$C^{(2)}(s_\Gamma) = \langle \phi_x \phi_y \rangle_{P, s_\Gamma}^0 - \langle \phi_x \rangle_{P, s_\Gamma}^{(0)} \langle \phi_y \rangle_{P, s_\Gamma}^{(0)},$$

where $\langle \ \rangle_{P, s_\Gamma}^{(0)}$ denotes the expectation with respect to the Gibbs factor $\exp(-H_q^{(0)}(\phi_P, s_\Gamma))$ obtained from $c^{-H_q^{(0)}(\phi_P)}$ by substituting $s_{xy}(\Gamma)(\phi_x - \phi_y)^2$ for $(\phi_x - \phi_y)^2$, and let $C^{(1)}(s_\Gamma)$ be the covariance matrix corresponding similarly to $Z_q^{(0)}(P_1, s_\Gamma | \phi_{\partial X \cup \partial_e C})$. Evaluating the Gaussian partition function in (5.1) we rewrite $W(\Gamma | \cdot)$ as

$$W(\Gamma | \phi_{\partial X \cup \partial_e C}) = W(\Gamma) + \frac{1}{2} \sum_{x, y \in \partial X \cup \partial_e C} \phi_x B_{xy}(\Gamma) \phi_y \qquad (5.2)$$

with

$$W(\Gamma) = \tfrac{1}{2} \int ds_\Gamma \partial^\Gamma \{\operatorname{tr} \log C^{(1)}(s_\Gamma) - \operatorname{tr} \log C^{(2)}(s_\Gamma)\}, \qquad (5.3a)$$

$$B_{xy}(\Gamma) = \sum_{\substack{u, v \in P \\ |x - v| = 1 \\ |y - u| = 1}} \int ds_\Gamma \partial^\Gamma C_{uv}^{(1)}(s_\Gamma). \qquad (5.3b)$$

Our starting point is the following basic estimate (proven e.g. in [18])

$$|\partial^{\Gamma} C_{xy}^{(1)}| \leq d_0^{|\Gamma|+1} \exp(-m_0 d(x, \Gamma, y)), \tag{5.4}$$

where $d_0 < \infty, m_0 > 0$ are constants depending only on η, η'' and v (recall that $m_q^2 \geq \eta$, $\kappa_q \leq \eta''$) and $d(x, \Gamma, y)$ is the length of the shortest path from x to y that visits each bond $b \in \Gamma$ at least once.

Proposition 5.3. *Given η, η'' and v there is a constant $d_1 < \infty$ such that*

$$\sum_{\Gamma \text{ s.t.} |\Gamma|=n} |W(\Gamma, \phi_{\partial X \cup \partial_e C})| \leq (d_1 e^{-(m_0 L/4v)})^n \left[f_1 |\bar{X} \cup \operatorname{supp} C| + \frac{\varepsilon_1}{2} \sum_{x \in \partial X \cup \partial_e C} \phi_x^2 \right],$$

where

$$f_1 = (d_1/L) e^{Lm_0/2}, \quad \varepsilon_1 = d_1 e^{-m_0(r_0 - L/2)}. \tag{5.5}$$

Proof. We first bound $|W(\Gamma)|$. Fixing a bond $b \in \Gamma$ and using the formula

$$\frac{\partial}{\partial s_b} C^{(i)} = -C^{(i)} \frac{\partial [C^{(i)}]^{-1}}{\partial s_b} C^{(i)},$$

we obtain, noting that

$$I_b = -\frac{\partial}{\partial s_b} (C^{(1)})^{-1} = -\frac{\partial}{\partial s_b} (C^{(2)})^{-1}$$

does not depend on s_{Γ},

$$W(\Gamma) = \tfrac{1}{2} \int ds_{\Gamma} [\operatorname{tr} I_b \partial^{\Gamma \setminus \{b\}} C^{(2)} - \operatorname{tr} I_b \partial^{\Gamma \setminus \{b\}} C^{(1)}].$$

Using (5.4) together with the explicit representation

$$(I_b)_{xy} = \kappa_q \sum_{\{u,v\} \in b} \{ \delta_{ux} \delta_{vy} + \delta_{vx} \delta_{uy} - \delta_{ux} \delta_{uy} - \delta_{vx} \delta_{vy} \}$$

and the bound $\kappa_q \leq \eta''$ we obtain

$$|W(\Gamma)| \leq 4\eta'' L^{v-1} d_0 |\Gamma| e^{-m_0 d(\Gamma)},$$

where $d(\Gamma)$ is the length of the shortest closed path that visits each face in Γ at least once. Using the fact that $W(\Gamma) = 0$ if $X(\Gamma) \cap (\bar{X} \cup \operatorname{supp} C) = \emptyset$ we can always label the bonds $b_1, \ldots, b_n \in \Gamma$ in such a way that

i) $b_1 \cap \square(x') \neq \emptyset$ for some $\square(x') \subset (\bar{X} \cup \operatorname{supp} C)$,

ii) $d(\Gamma) \geq \sum_{i=1}^{n-1} \operatorname{dist}(b_i, b_{i+1})$.

Bounding the sum over sets Γ by a sum over sequences $b_1, \ldots, b_n$ we get

$$\sum_{\substack{\Gamma \text{ s.t.} \\ |\Gamma|=n}} |W(\Gamma)| \leq L^{-v} |\bar{X} \cup \operatorname{supp} C| 4\eta'' L^{v-1} d_0^n \cdot e^{Lm_0/2} e^{-n(Lm_0/4v)}$$

$$\cdot \sup_{x'} \sum_{\substack{b_1, \ldots, b_n \text{ s.t.} \\ b_1 \cap \square(x') \neq \phi}} \exp\left(-\frac{m_0}{2} \sum_{i=1}^{n-1} \operatorname{dist}(b_i, b_{i+1}) \right),$$

where we have used the bound $d(\Gamma) \geq (|\Gamma|/2v - 1)L$ (for a proof, see e.g.

[18]) to extract the factor $e^{Lm_0/2}e^{-|\Gamma|Lm_0/4v}$. We now use the fact that there are not more than $2v$ bonds, b_1, with $\square(x')\cap b_1 \neq \varnothing$, together with the bound $\sum_b \exp((-m_0/2)\mathrm{dist}(b,b')) \leq K$, where $K < \infty$ is a constant depending only on v and m_0, to get the estimate

$$\sum_{\substack{\Gamma \text{ s.t.} \\ |\Gamma|=n}} |W(\Gamma)| \leq |\mathrm{supp}\, C \cup \bar{X}|(8v\eta''/L)e^{Lm_0/2}(Kd_0 e^{-Lm_0/4v})^n. \tag{5.6}$$

On the other hand, by (5.4) and the definition of $B_{xy}(\Gamma)$,

$$\sum_{|\Gamma|=n} \sum_{y \in \partial X \cup \partial_e C} |B_{xy}(\Gamma)| \leq 4v^2 d_0^{n+1} e^{Lm_0/2} e^{-(Lm_0/4v)n}$$

$$\cdot e^{-m_0 r_0} \sup_u \sum_{|\Gamma|=n} \sum_v \exp\left(\frac{-m_0}{2}d(u,\Gamma,v)\right),$$

where we have used the fact that

$$\mathrm{dist}(x,\Gamma) \geq r_0, \quad \mathrm{dist}(y,\Gamma) \geq r_0$$

to extract the additional factor $e^{-m_0 r_0}$. Bounding the sum over Γ with $|\Gamma| = n$ as before, and noticing that

$$|\phi_x \phi_y| \leq \tfrac{1}{2}(\phi_x^2 + \phi_y^2),$$

which implies that

$$\sum_{x,y \in \partial X \cup \partial_e C} |\phi_x B_{xy}\phi_y| \leq \sum_{x \in \partial X \cup \partial_e C} \phi_x^2 \sum_{y \in \partial X \cup \partial_e C} |B_{xy}|,$$

we find

$$\sum_{x,y \in \partial X \cup \partial_e C} \sum_{\substack{\Gamma \text{ s.t.} \\ |\Gamma|=n}} |\phi_x B_{xy}(\Gamma)\phi_y| \leq (kd_0 e^{-Lm_0/4v})^n \cdot 4v^2 d_0 e^{Lm_0/2 - m_0 r} K \sum_{x \in \partial X \cup \partial_e C} \phi_x^2. \tag{5.7}$$

Choosing

$$d_1 = \max\{Kd_0, 8v\eta'', 8v^2 Kd_0\},$$

we obtain the proposition. $\blacksquare$

We now prove Lemma 5.1. Given c we choose L_0 so large that

$$d_1 e^{-m_0 L_0/4v} \leq \tfrac{1}{2}e^{-2c}.$$

Using the fact that $X(\Gamma) \cup \bar{X} \cup \mathrm{supp}\, C = P$, together with Proposition 5.3 and the bound $|\Gamma| \geq \tfrac{1}{2}L^{-v}|X(\Gamma)|$ we may extract a factor

$$\prod_{\Gamma \in \bar{\Gamma}} e^{-2c|\Gamma|} \leq \exp(-cL^{-v}|P\backslash(\bar{X} \cup \mathrm{supp}\, C)|)$$

from the left-hand side of Lemma 5.1 and bound the remaining sum by

$$\sum_{k=0}^{\infty} \frac{1}{k!}\left\{\left[f_1 L^{-v}|\bar{X} \cup \mathrm{supp}\, C| + \frac{\varepsilon_1}{2}\sum_{x \in \partial X \cup \partial_e C} \phi_x^2\right]\sum_{n=1}^{\infty}(\tfrac{1}{2})^n\right\}^k$$

$$= \exp\left[f_1 L^{-v}|\bar{X} \cup \mathrm{supp}\, C| + \frac{\varepsilon_1}{2}\sum_{x \in \partial X \cup \partial_e C} \phi_x^2\right],$$

which proves Lemma 5.1 (choose r_0 so large that $\varepsilon_1 = \varepsilon_1(r_0, L) \leq \varepsilon$). $\blacksquare$

5.3 Proof of Lemma 5.2. We first bound, using the fact that $W \geq 0$,

$$\frac{1}{Z_q^{(0)}(\bar{X} \cup \operatorname{supp} C)} \int d\phi_{\bar{X}} F_X(\phi_X) \exp\left(-H_q^{(0)}(\phi_{\bar{X}}) + \frac{\varepsilon}{2} \sum_{x \in \partial X \cup \partial_e C} \phi_x^2\right)$$

$$\leqq \frac{Z_q^{(0)}(\bar{X} \setminus (X \cup \partial_e C))}{Z_q^{(0)}(\bar{X} \cup \operatorname{supp} C)} \int d\phi_{X \cup \partial_e C} F_X(\phi_X) \exp\left(-H_q^{(0)}(\phi_{X \cup \partial_e C}) + \frac{\varepsilon}{2} \sum_{x \in \partial X \cup \partial_e C} \phi_x^2\right)$$

$$\leqq \exp(\tilde{f}_2 |X \cup \operatorname{supp} C \cup \partial_e C|) \int d\phi_{X \cup \partial_e C} F_X(\phi_X)$$

$$\cdot \exp\left(-H_q^{(0)}(\phi_{X \cup \partial_e C}) + \frac{\varepsilon}{2} \sum_{x \in \partial X \cup \partial_e C} \phi_x^2\right),$$

where $\tilde{f}_2$ is a constant depending only on v, η, η' and η''. Using the fact that the sum in the definition (4.3c) of $F_X(\phi_X)$ contains at most $2^{|X|} 2^{2v|X|}$ terms, we are left with the proof of the following proposition (choose $Y = X \cup \partial_e C$ and $f_2 = \tilde{f}_2 + (2v + 1)\log 2 + \frac{1}{2}|\log(2\pi/\tilde{\eta})|$).

Proposition 5.4. $\forall \tilde{c} < \infty$ $\forall \varepsilon < \eta$ *there is a choice of* β_1 *(depending on* $\tilde{c}, \varepsilon, \eta, \eta'$ *and* η''*) such that*

$$\int d\phi_y \left| \prod_{x \in X_1} F_x(\phi_x) \prod_{\langle xy \rangle \in B} F_{xy}(\phi_x, \phi_y) \right| \exp\left(-H_q^{(0)}(\phi_Y) + \frac{\varepsilon}{2} \sum_{x \in Y} \phi_x^2\right)$$

$$\leqq \left(\frac{2\pi}{\tilde{\eta}}\right)^{(1/2)|Y|} \exp(-\tilde{c}|X_1 \cup X(B)|), \tag{5.8}$$

provided $X_1 \cup X(B) \subset Y$. $\tilde{\eta}$ *is the constant* $\eta - \varepsilon$.

Proof. We introduce the notation

$$G_{xy} = |F_{xy}(\phi_x, \phi_y)| \exp\left(-\frac{\kappa_q}{2} |\phi_x - \phi_y|^2\right),$$

$$G_x = \exp\left(-\frac{m_q^2 - \varepsilon}{2} \phi_x^2\right) \begin{cases} 1 & \text{if } x \notin X_1 \\ |F_x(\phi_x)| & \text{if } x \notin X_1 \end{cases}$$

and $\tilde{\eta} = \eta - \varepsilon$. The left-hand side of (5.8) can be bounded by

$$\int \prod_{\langle xy \rangle \in B} G_{xy}(\phi_x, \phi_y) \prod_{x \in Y} G_x(\phi_x) d\phi_x$$

$$\leqq \prod_{x \in Y} [\int d\phi_x G_x]^{1 - n_x/2v} \prod_{\langle xy \rangle \in B} [\int d\phi_x d\phi_y G_x G_{xy}^2 G_y]^{1/2v}, \tag{5.9}$$

where we used Hölder's inequality and the fact that each point in Z^v can be shared by at most $2v$ nearest neighbor pairs. n_x is the number of nearest neighbor pairs in B that contain the point x. The integrals in (5.9) are estimated using the following bounds (they all follow from assumption A.1 and the definition of G_x, G_{xy}):

$$|G_x(\phi_x)| \leqq (e^{\delta(\beta)} - 1) e^{-(\tilde{\eta}/2)\phi_x^2}$$

if $x \in X_1$ and $|\phi_x| \leqq \xi(\beta)$,

$$|G_x(\phi_x)| \leqq e^{(\varepsilon/2)\phi_x^2} \left[\exp(-(V(\phi_x + R_q) - e_q)) + \exp\left(-\frac{m_q^2}{2} \phi_x^2\right) \right] \leqq 2e^{-(\tilde{\eta}/2)\phi_x^2}$$

if $x \in X_1$ and $|\phi_x| \geq \xi(\beta)$,

$$|G_x(\phi_x)| \leq 2e^{-(\tilde{\eta}/2)\phi_x^2}$$

if $x \in Y \setminus X_1$,

$$|G_{xy}(\phi_x \phi_y)| \leq e^{\delta(\beta)} - 1,$$

if $|\phi_x| \leq \xi(\beta)$ and $|\phi_y| \leq \xi(\beta)$, and

$$|G_{xy}(\phi_x, \phi_y)| \leq 1,$$

if $|\phi_x| > \xi(\beta)$ or $|\phi_y| > \xi(\beta)$.

Combined with (5.9) and the estimates

$$\int_{-\xi(\beta)}^{\xi(\beta)} d\phi_x e^{-(\tilde{\eta}/2)\phi_x^2} \leq \int_{-\infty}^{\infty} d\phi_x e^{-(\tilde{\eta}/2)\phi_x^2} = \frac{\sqrt{2\pi}}{\tilde{\eta}}$$

$$\int_{|\phi_x| \geq \xi(\beta)} e^{-(\tilde{\eta}/2)\phi_x^2} \leq \frac{\sqrt{2\pi}}{\tilde{\eta}} e^{-(\tilde{\eta}/2)\xi(\beta)^2},$$

we obtain the bound

$$\left[2\frac{\sqrt{2\pi}}{\tilde{\eta}}\right]^{|Y|} [\tfrac{1}{2}(e^{\delta(\beta)} - 1) + e^{-(\tilde{\eta}/2)\xi(\beta)^2}]^{|X_1|} \times [(e^{\delta(\beta)} - 1)^{2\nu} + 2e^{-(\tilde{\eta}/2)\xi(\beta)^2}]^{|B|/2\nu}$$

for the left-hand side of (5.8). Using the fact that $|X(B)| \leq 2|B|$ and that $e^{-(\eta/2)\xi^2(\beta)} \leq \delta(\beta)$, we obtain the proposition with $e^{-c} = O(\delta(\beta)^\lambda)$, for some $\lambda > 0$ depending on $(\eta - \varepsilon)/\eta$ and ν (for $\varepsilon = \eta/2$, $\lambda = 1/8\nu$). ∎

6. The Phase Diagram

In Sect. 3 it was shown that the free energies of the truncated contour models,

$$h'_q = \lim_{\Lambda \uparrow Z^\nu} \frac{1}{|\Lambda|} \log Z'_q(\Lambda),$$

are enough to determine which phases, q, are stable in the sense that $\psi'_q = \psi_q$ (which implies, as we show in the next section, that infinite volume Gibbs states, $\langle \cdot \rangle_q$, can be constructed for such q). The stable q's are those for which h'_q is minimal. Thus, recalling Eq. (3.8), the a_q indicates which phases are stable: q is stable iff $a_q = 0$. If q is not stable then $a_q > 0$ so that a_q determines a function from $\mathscr{V}^{(1)} \subset R^{N-1}$ into $R_0^N = \{x \in R^N | \forall k, x_k \geq 0 \text{ and } \exists j \text{ with } x_j = 0\}$. In the Pirogov–Sinai theory the inverse of this function is called the phase diagram [9]. The standard physicist's notion of the phase diagram corresponds to the images, in $\mathscr{V}^{(1)}$, of the various hyperplanes given by $x_{j_1} = x_{j_2} = \cdots = x_{j_m} = 0$ for all possible non-empty $\{j_1, \ldots, j_m\} \subset \{1, 2, \ldots, N\}$. These consist of all $\mu \in \mathscr{V}^{(1)}$ for which the phases $j_1, \ldots, j_m$ are stable.

In order to simplify the presentation we prefer to do away with R_0^N by considering, rather than a_q, the function

$$b = \begin{pmatrix} h_1 - h_N \\ h_2 - h_N \\ \vdots \\ h_{N-1} - h_N \end{pmatrix} \tag{6.1}$$

which maps $\mathscr{V}^{(1)}$ into R^{N-1}. (Note that there is nothing special about N in Eq. (6.1). We could have chosen any $j \in \{1, \ldots, N\}$ and defined b by $b_k = h_k - h_j$.) The phase diagram is here given by the images, under b^{-1}, of the sets $\{y \in R^{N-1} | y_j \geq 0$ for all $j, y_{j_1}, \ldots, y_{j_m} = 0\}$ for all possible $\{j_1, \ldots, j_m\} \subset \{1, \ldots, N-1\}$ (including the empty set as a possibility) and the sets $\{y \in R^{N-1} | y_{j_1} = y_{j_2} = \cdots = y_{j_m} < 0\}$ for all possible non-empty $\{j_1, \ldots, j_m\} \subset \{1, \ldots, N-1\}$. The advantage of using b is that it simplifies the following

Theorem 6.1. *If β is large enough then there exist open sets, $\mathscr{V}, \mathscr{W} \subset R^{N-1}$ such that*

i) $b: \mathscr{V} \to \mathscr{W}$ *is a homeomorphism,*
ii) $0 \in \mathscr{W}$.

Comments. The point of (i) is that b is in fact continuously invertible. (ii) shows that the point of maximal phase co-existence, when all phases, q, are stable, is in the phase diagram. This insures that all possible combinations of co-existing phases can be obtained by varying μ in $\mathscr{V}$. It is also possible to demonstrate that, for β large enough, $0 \in \mathscr{V}$. $b(0)$ is not, however, in general 0.

Let $f_j(\mu) = e_j(\mu) - e_N(\mu)$ for $j \in \{1, 2, \ldots, N-1\}$ and let $s_j(\mu) = h_j(\mu) - h_N(\mu) - f_j(\mu)$. Then Eq. (6.1) reads

$$b = f(\mu) + s(\mu). \tag{6.2}$$

By Lemma 3.2 $|s_j(\mu)| \leq 2c_1$ for all $\mu \in \mathscr{V}^{(1)}$, where $\mathscr{V}^{(1)}$ is the set defined in Lemma 3.1. By assumption (A.4) and the inverse function theorem there is an open set $\mathscr{W}^{(1)} \subset R^{N-1}$ containing a sphere of radius $M(\beta)^{1-\lambda}$ around 0 such that $\mathscr{V}^{(1)}$ and $\mathscr{W}^{(1)}$ are diffeomorphic under the action of f. Let

$$\mathscr{W} = \{b \in \mathscr{W}^{(1)} | b' \in \mathscr{W}^{(1)} \forall b' \text{ s.t. } |b' - b| \leq 2Nc_1\}.$$

Then, $\forall b \in \mathscr{W}$ and all $k \geq 1, \mu^{(0)} = f^{-1}(b)$ and $\mu^{(k)} = f^{-1}(b - s(\mu^{(k-1)}))$ are well defined, with $\mu^{(k)} \in \mathscr{V}^{(1)}$.

Again, assumption A.4 and Lemma 3.2 allow us to conclude that, for $b \in \mathscr{W}$ and $\mu, \mu' \in \mathscr{V}^{(1)}$,

$$|\mu^{(0)}| = |f^{-1}(b)| \leq \frac{1}{M(\beta)} |b|,$$

$$|\mu^{(0)} - \mu^{(1)}| \leq \frac{1}{M(\beta)} |s(\mu^{(0)})| \leq \frac{2Nc_1}{M(\beta)}$$

and

$$|f^{-1}(b - s(\mu)) - f^{-1}(b - s(\mu'))| \leq \frac{1}{M(\beta)} |s(\mu) - s(\mu')|. \tag{6.3}$$

Therefore, the existence of the limit $\mu = \lim_{k \to \infty} \mu^{(k)}$ solving Eq. (6.2), as well as the continuity of the resulting b^{-1} follows from the following estimate.

Lemma 6.2. *There is a constant $c(\beta)$ such that*

$$\lim_{\beta \to \infty} \frac{c(\beta)}{M(\beta)} = 0$$

and, for all $\mu, \mu' \in \mathscr{V}^{(1)}$

$$|s(\mu) - s(\mu')| \leq c(\beta)|\mu - \mu'|.$$

Proof. Letting $h'_{q,\Lambda}(\mu) = (1/|\Lambda|)\log Z'_q(\Lambda)$ it is enough to bound the one sided derivatives $(d/d\mu_{\pm})(h'_{q,\Lambda}(\mu) - e_q(\mu))$ uniformly in Λ for all q. Note that

$$\frac{d}{d\mu_{\pm}}h'_{q,\Lambda}(\mu) = \frac{1}{|\Lambda|}(g_q^{(1)}(\mu) + g_q^{(2)}(\mu)),$$

where

$$g_q^{(1)}(\mu) = \frac{-1}{Z'_q(\Lambda)}\sum_C \int dR_{\Lambda\backslash C}\exp(-\psi'_q(C|R_{\partial_e C}))\frac{d}{d\mu_{\pm}}\exp(-H(R_{\Lambda\backslash C}))\chi_q(R_{\Lambda\backslash C})$$

and

$$g_q^{(2)}(\mu) = \frac{-1}{Z'_q(\Lambda)}\sum_C \int dR_{\Lambda\backslash C}\exp(-H(R_{\Lambda\backslash C}))\chi_q(R_{\Lambda\backslash C})\frac{d}{d\mu_{\pm}}\exp(-\psi'_q(C|R_{\partial_e C})).$$

We find

$$g_q^{(1)}(\mu) = \frac{1}{Z'_q(\Lambda)}\sum_{x\in\Lambda}\sum_{C\text{ s.t. }x\notin\text{supp}C}\int dR_{\Lambda\backslash C}\chi_q(R_{\Lambda\backslash C})$$

$$\cdot\exp(-\psi'_q(C|R_{\partial_e C}) - H(R_{\Lambda\backslash C}))\frac{d}{d\mu}\left(V(R_x) + \frac{1}{2}\sum_{y\in\Lambda\backslash C\text{ s.t. }|x-y|=1}W(R_x, R_y)\right).$$

Using assumption A.5 to estimate derivatives, the bound (3.1) (with $\tau/3$ instead of τ) for all contours γ with $x\in\text{Int}\,\gamma$ and $W\geq 0$ to eliminate couplings between partition functions we can re-sum the remaining contours and obtain

$$\left|g_q^{(1)}(\mu) - |\Lambda|\frac{de_q(\mu)}{d\mu}\right| \leq (2v+1)(N-1)K_6(\beta)$$

$$\cdot\sum_{x'\in\Lambda'}\int dR_{\square(x')}\chi_q(R_{\square(x')})\left(\sum_{x\in\square(x')}|R_x - R_q|^p\right)\exp\left(-\sum_{x\in\square(x')}\left(e_q + \frac{\eta}{2}(R_x - R_q)^2\right)\right)$$

$$\cdot\sum_{\substack{C\text{ s.t.}\\ \square(x')\subset\text{Int}\gamma\\ \forall\gamma\in C}}\frac{Z'_q(\Lambda)(\text{supp}\,C\cup\square(x')))}{Z'_q(\Lambda)}\prod_{\gamma\in C}\exp\left(-\left(\frac{\tau}{3} + e_q\right)|\text{supp}\,\gamma|\right),$$

where η is the constant from assumption A.1. The existence of a convergent cluster expansion for $Z'_q(\cdot)$ allows us to estimate

$$\frac{Z'_q(\Lambda\backslash(\square(x')\cup\text{supp}\,C))}{Z'_q(\Lambda)} \leq \exp((e_q + c_1)|\square(x')\cup\text{supp}\,C|),$$

where c_1 is the constant from Lemma (3.2). Thus, for τ large enough, we may bound the sum over C by $1 + O(e^{-(1/3)\tau}) \leq 2$, obtaining

$$\left|g_q^{(1)}(\mu) - |\Lambda|\frac{de_q(\mu)}{d\mu}\right| \leq |\Lambda|(2v+1)(N-1)\left(\frac{2\pi}{\eta}\right)^{L^v-1}K_6(\beta)\cdot e^{c_1 L^v}\cdot 2\int dR|R|^p e^{-(1/2)\eta R^2}$$

$$(6.4)$$

To estimate $g_q^{(2)}(\mu)$ we again use $W\geq 0$ to decouple, and bound ratios of partition

functions as in the step preceding Eq. (6.4) to obtain, where $\sum'$ is the sum over sets C of pairwise compatible contours, γ', all compatible with γ, such that $\operatorname{supp}\gamma \subset \operatorname{int}\gamma'$,

$$|g_q^{(2)}(\mu)| \leqq \sum_\gamma \sum_C{}' \left| \frac{d}{d\mu_\pm} \exp\left(-\psi_q'(\gamma|R_{\partial_e c})\right) \right| \cdot \exp\left((e_q+c_1)|\operatorname{supp}\gamma| - \left(\frac{\tau}{3}-c_1\right)|\operatorname{supp}C| \right)$$

$$\leqq 2\sum_\gamma \left| \frac{d}{d\mu_\pm} \exp\left(-\psi_q'(\gamma|R_{\partial_e\gamma})\right) \right| \exp\left((e_q+c_1)|\operatorname{supp}\gamma|\right). \tag{6.5}$$

More that either

$$\frac{d}{d\mu_\pm} \exp\left(-\psi_q'(\gamma|R_{\partial_e\gamma})\right) = 0$$

or

$$\frac{d}{d\mu_\pm} \exp\left(-\psi_q'(\gamma|R_{\partial_e\gamma})\right) = \frac{d}{d\mu} \exp\left(-\psi_q(\gamma|R_{\partial_e\gamma})\right).$$

Thus, we need only estimate

$$\left| \frac{d}{d\mu} \exp\left(-\psi_q(\gamma|R_{\partial_e\gamma})\right) \right|$$

$$\leqq \frac{1}{Z_q^{\text{dil}}(\operatorname{Int}\gamma)} \int dR_{V(\gamma)} \chi_{\omega_\gamma}(R_{\operatorname{supp}\gamma\cup\partial'\operatorname{Int}\gamma})$$

$$\cdot \exp\left(-H(R_{V(\gamma)}|R_{\partial_e\gamma})\right) \left| \frac{dH(R_{V(\gamma)}|R_{\partial_e\gamma})}{d\mu} \right|$$

$$+ \exp\left(-\psi_q(\gamma|R_{\partial_e\gamma})\right) \frac{1}{Z_q^{\text{dil}}(\operatorname{Int}\gamma)} \int dR_{\operatorname{Int}\gamma} \chi_q(R_{\partial'\operatorname{Int}\gamma})$$

$$\cdot \exp\left(-H(R_{\operatorname{Int}\gamma})\right) \left| \frac{dH(R_{\operatorname{Int}\gamma})}{d\mu} \right|, \tag{6.6}$$

where $\partial'\operatorname{Int}\gamma$ is the union of all L-blocks in $\operatorname{Int}\gamma$ which touch $\partial(\operatorname{Int}\gamma)$.

Using assumptions (A.5) we have, for any finite $\Omega \subset Z^\nu$,

$$\left| \frac{dH(R_\Omega)}{d\mu} \right| \leqq K_5(N-1)\{(1+2\nu)\tilde{\tau}(\beta)|\Omega| + H(R_\Omega) - e_0|\Omega|\}$$

and similarly for $H(R_\Omega|R_{\Omega'})$. Using the notation $\chi_\Omega(R_\Omega) = \prod_{x\in\Omega} \chi_x(R_x)$, where $\chi_x = 1$ or χ_j for some $j = 0, 1, \ldots, N$, we have integrals of the following form to estimate:

$$\int dR_\Omega \chi_\Omega(R_\Omega) e^{-H(R_\Omega)} \left| \frac{dH(R_\Omega)}{d\mu} \right|$$

$$\leqq \int dR_\Omega \chi_\Omega(R_\Omega) e^{-H(R_\Omega)} K_5(N-1)\{(1+2\nu)\tilde{\tau}(\beta)|\Omega| + H(R_\Omega) - e_0|\Omega|\}$$

$$\leqq K_5(2\nu+1+\rho)\tilde{\tau}(\beta)|\Omega|(N-1) \int dR_\Omega \chi_\Omega(R_\Omega) e^{-H(R_\Omega)}$$

$$+ e^{-e_0|\Omega|} K_5(N-1) \int\limits_{R_\Omega\in D} dR_\Omega \exp\left(-(H(R_\Omega) - e_0|\Omega|)\right)$$

$$\cdot (H(R_\Omega) - e_0|\Omega|)\chi_\Omega(R_\Omega), \tag{6.7}$$

where $D = \{R_\Omega | H(R_\Omega) - e_0|\Omega| \geqq \rho\tilde\tau(\beta)|\Omega|\}$ and where $\rho > 0$ is a constant independent of β to be chosen in a moment. Given $\alpha > 0$ as in assumption A.2 we now bound the second term on the right-hand side of 6.7) by

$$(N-1)K_5 e^{-e_0|\Omega|} \sup_{x \geqq \rho\tilde\tau|\Omega|} xe^{-(1-\alpha)x} \int dR_\Omega \exp(-\alpha(H(R_\Omega) - e_0|\Omega|))\chi_\Omega$$

$$\leqq (N-1)K_5 e^{-e_q|\Omega|} \exp\left(\frac{K_1}{2}\frac{\tilde\tau(\beta)}{(2L)^\nu}|\Omega|\right) \sup_{x \geqq \rho\tilde\tau|\Omega|} xe^{-(1-\alpha)x} \cdot \left\{K_2 + N\frac{\sqrt{2\pi}}{\eta\alpha}\right\}^{|\Omega|},$$

where we used assumption A.1 and A.2 to bound the integral over R_Ω and (3.7) to bound $e_0 - e_q$. We now choose

$$\rho = \max\left\{\frac{1}{1-\alpha}, \frac{1}{1-\alpha}\frac{K_1}{(2L)^\nu}\right\}$$

to conclude that

$$\int dR_\Omega(R_\Omega)\chi_\Omega(R_\Omega)e^{-H(R_\Omega)}\left|\frac{dH(R_\Omega)}{d\mu}\right|$$

$$\leqq K_5(N-1)(2\nu+1+\rho)\tilde\tau(\beta)|\Omega|\int dR_\Omega\chi_\Omega(R_\Omega)e^{-H(R_\Omega)}$$

$$+ K_5(N-1)\left(\frac{N}{\sqrt\alpha}e^{-e_q-\tau}\right)^{|\Omega|}\rho\tilde\tau(\beta)|\Omega|, \tag{6.8}$$

with τ as in Lemma 3.1, Eq. (3.4). Combined with (6.5), (6.6) and the bound analogous to (6.8) for $\int dR_\Omega\chi_\Omega\exp(-H(R_\Omega|R_{\Omega'})|dH(R_\Omega|R_{\Omega'})/d\mu|$ we obtain the estimate

$$\frac{1}{|\Lambda|}|g_q^{(2)}(\mu)| \leqq 4(N-1)K_5(2\nu+1+\rho)\tilde\tau(\beta)\sum_\gamma \exp\left(-\left(\frac{\tau}{3}-c_1\right)|\operatorname{supp}\gamma|\right)$$

$$+ 2(N-1)K_5\rho\tilde\tau(\beta)\sum_\gamma \frac{\exp(e_q+c_1)|\operatorname{supp}\gamma|)}{Z_q^{\mathrm{dil}}(\operatorname{Int}\gamma)}\left(\frac{N}{\sqrt\alpha}e^{-e_q-\tau}\right)^{|V(\gamma)|}$$

$$+ 2(N-1)K_5\rho\tilde\tau(\beta)\sum_\gamma \frac{\exp\left(-\left(\frac{\tau}{3}-c_1\right)|\operatorname{supp}\gamma|\right)}{Z_q^{\mathrm{dil}}(\operatorname{Int}_\gamma)}\left(\frac{N}{\sqrt\alpha}e^{-e_q-\tau}\right)^{|\operatorname{Int}\gamma|}.$$

Using Theorem 3.3 i to bound $Z_q^{\mathrm{dil}}(\operatorname{Int}\gamma)$ and Lemma 3.2 to bound $h_q' - e_q$ we can sum over γ and obtain

$$\frac{1}{|\Lambda|}(g_q^{(2)}(\mu)) \leqq O(\tilde\tau(\beta)e^{-\tau/3}). \tag{6.9}$$

Combined with (6.4) this proves Lemma 6.2 and hence Theorem 6.1. ∎

7. Expectation Values

In this section we construct the infinite volume states $\langle\cdot\rangle_q$ corresponding to the dilute partition functions for stable *q*'s.

We consider local observables A, that is complex valued functions $A = A(R_{\text{supp}\,A})$ depending only on a finite number of variables $R_x, x \in \mathscr{A} = \text{supp}\,A$, for which the norm

$$\| A \|^{(\alpha)} = \sup_{R_{\mathscr{A}}} | A(R_{\mathscr{A}}) | \exp\left(-(1-\alpha) \sum_{x \notin \mathscr{A}} [V(R_x) - e_0] \right) \tag{7.1}$$

is finite ($\alpha < 1$ is the constant from assumption A.2). We assume without loss of generality that $\mathscr{A}$ is a connected union of cubes.

In order to construct the expectation value, $\langle A \rangle_q$, of A in the infinite volume limit,

$$\langle A \rangle_q = \lim_{\Lambda \uparrow Z^\nu} \langle A \rangle_{q,\Lambda}^{\text{dil}}, \tag{7.2}$$

$$\langle A \rangle_{q,\Lambda}^{\text{dil}} = Z_q^{\text{dil}}(\Lambda)^{-1} \sum_{\omega \in \Omega_q(\Lambda)} \int dR_\Lambda \chi_\omega(R_\Lambda) e^{-H(R_\Lambda)} A, \tag{7.3}$$

we consider the modification function

$$Z_q^{\text{dil}}(\Lambda, A) = \sum_{\omega \in \Omega_q(\Lambda)} \int dR_\Lambda \chi_\omega(R_\Lambda) e^{-H(R_\Lambda)} A. \tag{7.4}$$

Our goal is to derive a polymer representation for $Z_q^{\text{dil}}(\Lambda, A)$. We always assume that Λ is chosen so large that $\text{dist}(\mathscr{A}, \Lambda^c) \geq 2$.

We proceed as in the derivation of (2.9), the only difference being that we fix not only the external contours of a configuration $\omega \in \Omega_q(\Lambda)$, but also all contours γ of ω which are *incompatible with* A, that is all γ for which $\text{dist}(V(\gamma), \mathscr{A}) \leq 1$. We now introduce the notion of an A-contour.

Consider a configuration $\omega \in \Omega_q(\Lambda)$ and the contours $\gamma_1, \ldots, \gamma_k$ of ω which are incompatible with A. Let γ_A be the pair $(\text{supp}\,\gamma_A, \omega_{\gamma_A})$, where

$$\text{supp}\,\gamma_A = \mathscr{A} \cup \bigcup_{i=1}^{k} \text{supp}\,\gamma_i \tag{7.5}$$

and ω_{γ_A} is the restriction of ω to $\text{supp}\,\gamma_A$. We call γ_A the A-contour of ω. Consider a connected component K of $(\text{supp}\,\gamma_A)^C$. Then ω is constant on ∂K and equals q if K is the infinite component of $(\text{supp}\,\gamma_A)^c$. As before we define $\text{Int}_m \gamma_A$ to be the union of all finite components, K, for which $\omega = m$ on ∂K,

$$\text{Int}\,\gamma_A = \bigcup_{m=1}^{N} \text{Int}_m \gamma_A \quad \text{and} \quad V(\gamma_A) = \text{Int}\,\gamma_A \cup \text{supp}\,\gamma_A.$$

Defining

$$\exp(-\phi(\gamma_A | R_{\gamma_A})) = A \exp(-H(R_{\gamma_A})) \chi_{\omega_{\gamma_A}}(R_{\gamma_A})$$

and

$$\exp(-\psi_q(\gamma_A | R_{\partial_e \gamma_A})) = \int dR_{\gamma_A} \exp(-\phi(\gamma_A | R_{\gamma_A}))$$

$$\prod_{\substack{x \in \partial \gamma_A, y \in \partial_e \gamma_A \\ |x-y|=1}} \exp(-W(R_x, R_y)) \prod_{m=1}^{N} \frac{Z_m(\text{Int}_m \gamma_A, R_{\partial_m \gamma_A})}{Z_q(\text{Int}_m \gamma_A)}, \tag{7.7}$$

one immediately obtains the analog of (2.12), namely

$$Z_q^{\text{dil}}(\Lambda, A) = \sum_{\gamma_A} \sum_{\substack{C \in \Omega_q^{\text{ext}}(\Lambda) \\ C \text{ in Ext}_\Lambda \gamma_A}} \int dR_{\Lambda \setminus C} \chi_q(R_{\Lambda \setminus C_A})$$

$$\cdot \exp(-H(R_{\Lambda \setminus C_A})) \prod_{\gamma \in C_A} \exp(-\psi_q(\gamma | R_{\partial_e \gamma})) Z_q(\text{Int } \gamma), \qquad (7.8)$$

where the first sum goes over A-contours in Λ and the second over sets $C = \{\gamma_1, \ldots, \gamma_m\}$ of mutually external contours for which each γ_i lies in $\text{Ext}_\Lambda \gamma_A = \Lambda \setminus V(\gamma_A)$. C_A is the union of C and $\gamma_A : C_A = \{\gamma_A, \gamma_1, \ldots, \gamma_m\}$, and $\Lambda \setminus C_A \equiv \Lambda \setminus \text{supp } C_A$. Again, (7.8) can be literated giving

$$Z_q^{\text{dil}}(\Lambda, A) = \sum_{C_A \text{ in } \Lambda}^q \int \exp(-\psi_q(C_A | R_{\partial_e C_A}))$$

$$\cdot \exp(-H(R_{\Lambda \setminus C_A})) \chi_q(R_{\Lambda \setminus C_A}) dR_{\Lambda \setminus C_A}, \qquad (7.9)$$

where the sum goes over sets, C, of pairwise compatible q-contours in Λ which contain exactly one A-contour. Here compatibility between an ordinary contour γ and an A-contour, γ_A, is defined by the requirement $\text{dist}(V(\gamma), \text{supp } \gamma_A) \geq 2$.

We now combine the expansion (7.9) with the high temperature expansion for Z_q^{res}. Up to Eq. (4.10) the procedure is exactly the same as in Sect. 4. The only difference arises when we decompose the set $\text{supp } C \cup \bar{X} \cup X(\Gamma)$ into polymers where we have to introduce the notion of A-polymers. We introduce, for a connected set, P, of L-blocks, the volume $\text{Int } P$ as the union of all finite connected components of P^c and the volume $V(P) = P \cup \text{Int } P$. We define: a union P_A of L-blocks is an A-polymer if P_A contains the set $\mathcal{A}$ and $\text{dist}(V(P), \mathcal{A}) \leq 1$ for all connected components C of P_A. Compatibility between an ordinary polymer P (i.e. a connected union of L-blocks) and an A-polymer P_A is again defined by the requirement

$$\text{dist}(P_A, V(P)) \geq 2. \qquad (7.10)$$

We finally introduce the activity $\rho_A(P_A)$ by replacing the sum over C in (4.11) by a sum over sets C_A containing exactly one A-contour, γ_A (note that $\rho_A(P_A)$ is linear in A by our definition of $e^{-\psi_q(\gamma_A, \cdot)}$; we also use ψ_q instead of ψ_q' because q is assumed to be stable).

With these definitions we obtain the following analog of (4.72)

$$Z_q^{\text{dil}}(\Lambda, A) = e^{-e_q |\Lambda|} Z_q^{(0)}(\Lambda) \sum_{\mathbb{P}_A} \rho_A(P_A) \prod_i \rho(P_i), \qquad (7.11)$$

where the sum goes over sets

$$\mathbb{P}_A = (P_A, P_1, \ldots, P_k)$$

of mutually compatible polymers containing exactly one A-polymer. We now divide (7.11) by $Z_q^{\text{dil}}(\Lambda)$ by the standard algebraic procedure for polymer systems, see e.g. [20]. One obtains

$$\boxed{\langle A \rangle_{\Lambda, q}^{\text{dil}} = \sum_{n=0}^{\infty} \sum_{P_A, P_1, \ldots, P_n} \frac{\phi_c(P_A, \ldots, P_n)}{n!} \rho_A(P_A) \prod_{i=1}^n \rho(P_i)} \qquad (7.12)$$

where ϕ_c is the usual combinatoric factor (it is called $a(X)$ in [20]).

To prove the convergence of this expansion we need the following Lemma, which is the appropriate generalization of Lemma 3.1. $\mathscr{V}^{(1)}$ is the region from Lemma 3.1.

Lemma 7.1. *There is a constant $\tilde{c}_0$ not depending on L, β or μ such that*

$$\int |\exp(-\phi_A|R_{\gamma_A}))|dR_{\gamma_A} \leq \tilde{K}_A \exp(-e_q|\operatorname{supp}\gamma_A| - \tau|\operatorname{supp}\gamma_A \setminus \mathscr{K}|),$$

provided $\mu \in \mathscr{V}^{(1)}$, $\tau = \tau(L, \beta)$ is the constant from Lemma 3.1 and

$$\tilde{K}_A = (\tilde{c}_0 e^{e_q - e_0})^{|\mathscr{A}|} \|A\|^{(\alpha)}.$$

Proof. We first extract a factor $\|A\|^{(\alpha)}$ and then proceed as in the proof of Lemma 3.1 to bound

$$\int |\exp(-\phi(\gamma_A|R_{\gamma_A}))|dR_{\gamma_A} \leq \|A\|^{(\alpha)} \exp(-e_0|\operatorname{supp}\gamma_A|) \int \prod_{x \in \mathscr{A}} \exp(-\alpha(V(R_x) - e_0))$$

$$\cdot \prod_{x \in \operatorname{supp}\gamma_A \setminus \mathscr{A}} \exp(-(V(R^x) - e_0)) \prod_{\langle xy \rangle} \exp(-W(R_x, R_y)) \prod_{x \in \operatorname{supp}\gamma_A} \chi_{\omega(x)}(R_x)dR_x$$

$$\leq \|A\|^{(\alpha)} \exp(-\tilde{\tau}(\mathscr{L} + N) - e_0|\operatorname{supp}\gamma_A|)c_0^{|\operatorname{supp}\gamma_A \setminus \mathscr{A}|}(c_0/\sqrt{\alpha})^{|\mathscr{A}|},$$

where N and $\mathscr{L}$ are defined as in the proof of Lemma 3.1. Using the fact that all cubes in $\operatorname{supp}\gamma_A \setminus \mathscr{A}$ are incorrect we obtain $\mathscr{L} + N \geq (2L)^{-\nu}|\operatorname{supp}\gamma_A \setminus \mathscr{A}|$ and hence, again proceeding as in the proof of Lemma 3.1,

$$\int |\exp(-\phi(\gamma_A|R_{\gamma_A}))|dR_{\gamma_A}$$

$$\leq \|A\|^{(\alpha)} \left(\frac{c_0}{\sqrt{\alpha}} e^{e_q - e_0} \right)^{|\mathscr{A}|} \exp(-e_q|\operatorname{supp}\gamma_A|) \exp(-\tau|\operatorname{supp}\gamma_A \setminus \mathscr{A}|)$$

provided $\mu \in \mathscr{V}^{(1)}$ (we have used $c_1 \geq 1$ without loss of generality). This proves the lemma with $\tilde{c}_0 = c_0/\sqrt{\alpha}$. ∎

Corollary 7.2. *For all $b < \infty$ one can choose L_0 independent of β and L, and $r_0 = r_0(L)$, $c_3 = c_3(L)$, $\beta_1 = \beta_1(L)$ and $\tau_1 = \tau_1(L)$ not depending on β, such that*

$$|\rho_A(P_A)| \leq c_3^{|\mathscr{A}|} \|A\|^{(\alpha)} e^{-b|P_A|},$$

provided $L \geq L_0$, $\mu \in \mathscr{V}^{(1)}$, $\tau = \tau(\beta, L) \geq \tau_1$ and $\beta \geq \beta_1$.

Proof. We fix $\varepsilon, c, \tilde{c}, \tau_1 = 3\tilde{c}, L \geq L_0$ and r_0 as in the proof of Lemma 4.1. Assuming without loss of generality that Theorem 3.3 is valid for $\beta \geq \beta_1$ we bound

$$|\exp(-\psi(\gamma_A|R_{\partial_e \gamma_A}))|\exp(e_q|\operatorname{supp}\gamma_A|)$$

$$\leq \tilde{K}_A \exp(-\tau|\operatorname{supp}\gamma_A \setminus \mathscr{A}| + (2c_1 + 1)|\partial\operatorname{supp}\gamma_A|)$$

$$\leq \tilde{K}_A \exp((2c_1 + 1|\mathscr{A}| - (\tau/3)|\operatorname{supp}\gamma_A \setminus \mathscr{A}|).$$

If q is stable, $h'_q \leq h'_m$ for all m and, by Lemma 3.2, $e_q \leq e_m + 2c_1$, which shows that

$$\tilde{K}_A \exp((2c_1 + 1)|\mathscr{A}|) \leq \|A\|^{(\alpha)} (\tilde{c}_0 e^{4c_1 + 1})^{|\mathscr{A}|}.$$

Combining these bounds with Lemma 5.1 and Lemma 5.2 and proceeding as in

the proof of Theorem 4.1, one obtains

$$|\rho_A(P_A)| \leq \|A\|^{(\alpha)} (\tilde{c}_0 e^{4c_1 + 1})^{|\mathscr{A}|}$$

$$\sum_{\substack{C_A, X, \text{s.t.} \\ \bar{X} \cup \text{supp} C_A \subset P_A}} \exp\left(- cL^{-\nu}|P_A| + (f_1 + cL^{-\nu})|\tilde{X} \cup \text{supp} C_A|\right)$$

$$\cdot \exp\left(- \tilde{c}|X \cup \text{supp} C_A \setminus \mathscr{A}| + 3^\nu f_2 |X \cup \text{supp} C_A|\right).$$

Extracting a factor $e^{\tilde{c}|\mathscr{A}|}$ we can bound the sum as before and obtain the corollary with

$$c_3 = \tilde{c}_0 e^{4c_1 + 1} e^{\tilde{c}}. \quad \blacksquare$$

Corollary 7.2, together with Theorem 4.1, immediately imply convergence of the cluster expansion (7.12), and hence the following

Theorem 7.3. *There are constants $\varepsilon > 0$, $K < \infty$ and $\beta^* < \infty$, such that the following statements are true for $\beta \geq \beta^*$ and $\mu \in \mathscr{V}^{(1)}$.*

i) *For stable boundary conditions q, and any observable A with $\|A\|^{(\alpha)} < \infty$, the thermodynamic limit*

$$\langle A \rangle_q = \lim_{\Lambda \to Z^\nu} \langle A \rangle_{q, \Lambda}^{\text{dil}}$$

exists, is translation invariant and shows exponential clustering.
ii) *Let $\langle \cdot \rangle_q^{\text{res}}$ be the infinite volume state corresponding to Z_q^{res}. Then, for stable boundary conditions q,*

$$|\langle A \rangle_q - \langle A \rangle_q^{\text{res}}| \leq K^{|\text{supp} A|} \|A\|^{(\alpha)} e^{-\varepsilon \tilde{\tau}(\beta)},$$

where $\tilde{\tau}(\beta)$ is the constant from assumption A.2.

Proof. i) is standard and ii) is proven by comparing the cluster expansion for $\langle A \rangle_q$ and $\langle A \rangle_q^{\text{res}}$. Since the activities of these two expansions differ only by terms containing at least one contour, we may extract a factor $e^{-\tilde{c}\tau} \leq e^{-\varepsilon \tilde{\tau}(\beta)}$; the remaining sums are bounded as before giving the factor $K^{|\text{supp} A|} \|A\|^{(\alpha)}$. $\quad \blacksquare$

The following theorem is proven in the same way as Theorem 7.3 ii). $\langle \cdot \rangle_q^{(0)}$ denotes the infinite volume state of the Gaussian with covariance $C_q = (m_q^2 - \kappa_q \Delta)^{-1}$ and mean R_q. $\delta(\beta)$ is the constant from assumption A.1.

Theorem 7.4. *For a polynominally bounded function $A(\cdot)$ denote by A_q the shifted function, $A_q(R) = A_q(R - R_q)$. Then, for $\beta > \beta^*$,*

$$|\langle A_q \rangle_q^{\text{res}} - \langle A_q \rangle_q^{(0)}| \leq K_A \delta(\beta)^\varepsilon,$$

where K_A depends on A, but not on β, and $\varepsilon > 0$ is independent of β and A.

Remark. One may have observed that the definition of $\langle \cdot \rangle_q$ in Sect. 1 differs from that given here. Since $Z_q(\Lambda)$ and $Z_q^{\text{dil}}(\Lambda)$ and hence also $\langle \cdot \rangle_{q, \Lambda}$ and $\langle \cdot \rangle_{q, \Lambda}^{\text{dil}}$ differ only by contours touching $\partial \Lambda$, one may easily check that

$$\lim_{\Lambda \to Z^\nu} \langle A \rangle_{q, \Lambda} = \lim_{\Lambda \to Z^\nu} \langle A \rangle_{q, \Lambda}^{\text{dil}}$$

if q is stable, $\beta \geq \beta^*$ and $\|A\|^{(\alpha)} < \infty$.

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