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Resonance Theory in Atom-Surface Scattering

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Abstract. We study the problem of analytic extension of the resolvent for Hamiltonians arising in scattering of atoms by a quantum surface. We prove that the resolvent extends holomorphically to some regions of the lower half plane with isolated singularities called Landau resonances which are branch points of the resolvent. We study also the effect of impurities on the singularities of the resolvent and show that the presence of impurities adds poles to the Landau resonances.

Introduction

We study in this paper the theory of resonances for Hamiltonians arising in atom-surface scattering. The theory of atomic or molecular collisions with surfaces has been greatly developed by chemists in recent years since the measurement of scattering of atoms by a quantum surface is a way to study the surface structure of materials at atomic scales (see [Ge] for a review).

The typical form of the Hamiltonian is the following: $H = -\Delta + V(x,y)$ on $\mathbf{R}_x^{m-1} \times \mathbf{R}_y$, where y is the direction normal to the surface and $V(x,y)$ is an effective potential describing the interaction between an atom and a crystalline or non-crystalline material. Strictly speaking this Hamiltonian corresponds to a thin slab of material since it is possible for atoms to pass through the crystal. The Hamiltonian corresponding to an impenetrable material is the following: $H' = -\Delta + V(x,y)$ on $\mathbf{R}_x^{m-1} \times \mathbf{R}_y^+$ with Dirichlet boundary condition on $y=0$.

We will concentrate on H but all results proved for H hold for H' under corresponding hypotheses. (The proofs can be adapted almost verbatim.) When the surface is a perfect crystal, $V(x,y)$ tends to zero when y tends to infinity and is periodic in x with respect to some lattice T in $\mathbf{R}^{m-1}$. In this paper we will always assume that V is exponentially decreasing in y in a suitable sense.

For crystalline surfaces, H is usually studied using Bloch's theory to reduce the study of the resolvent $(H-\lambda)^{-1}$ to the study of $(H_p-\lambda)^{-1}$, where the Bloch number p belongs to a fundamental domain of the dual lattice T^* and $H_p = D_y^2 + (D_x+p)^2 + V(x,y)$, $D_{x_i} = (1/i)\partial/\partial x_i$, $D_y = (1/i)\partial/\partial y$ is a reduced

Hamiltonian on a cylinder $C_T = F_T \times \mathbf{R}$, where F_T is a fundamental domain of T.

It turns out that under classical hypotheses like exponential decay in y of $V(x, y)$ or dilation analyticity in y, it is possible to extend meromorphically $(H_p - \lambda)^{-1}$ to some region in the lower half plane with poles at the resonances of H_p. This kind of problem has been studied numerically. (See for example [Mo].)

Since H is obtained as a direct integral of H_p over p, one is naturally led to the guess that all resonances of H_p are singularities of $(H - \lambda)^{-1}$, which in the physical case $m = 3$ would lead to some closed set of positive measure as singularities of $(H - \lambda)^{-1}$.

The main result of this paper is that this guess is wrong.

We will consider two kinds of problems:

i) *Local Extension Problem*: given $\lambda_0 \in \sigma(H)$, extend analytically $(H - \lambda)^{-1}$ to a small neighborhood of λ_0 and describe its singularities.
ii) *Global Extension Problem*: extend analytically $(H - \lambda)^{-1}$ to some given open set $\mathscr{U}$ in $\mathbf{C}$ and describe its singularities.

For the *local extension problem*, we prove that for any $\lambda_0 \in \sigma(H)$, there exist some neighborhood $\mathscr{U}_{\lambda_0}$ of λ_0, and a finite set Σ of points called *Landau resonances* such that $(H - \lambda)^{-1}$ extends holomorphically to the universal covering of $\mathscr{U}_{\lambda_0} \setminus \Sigma$. The Landau resonances are usually branch points of $(H - \lambda)^{-1}$ instead of poles.

For the *global extension problem*, we have to add to Σ a closed set of measure zero Σ_∞ which corresponds to a complex essential spectrum (see Definition 4.5). Then $(H - \lambda)^{-1}$ extends holomorphically to the universal covering of $\mathscr{U} \setminus \Sigma \cup \Sigma_\infty$.

The points of Σ are generated by "condensation" of resonances of H_p, or more geometrically by pinching of some integration contours between some singularities of $(H_p - \lambda)^{-1}$. They are analogous to the Landau singularities of Feynman amplitudes. (See Theorem 4.7.)

Σ_∞ looks more like essential spectrum in the sense that Σ_∞ acts as a natural boundary for the extension of $(H - \lambda)^{-1}$ between suitable weighted L^2 spaces. Σ_∞ comes in part from the fact that we integrate operator-valued functions and that we have to take care of domain considerations.

To make this remark more clear let us compare H with two-body Schrödinger operators with exponentially decreasing potentials. In the last case the resolvent can be extended meromorphically to a strip $\{\lambda \in \mathbf{C} \mid \operatorname{Im} \lambda > -\alpha\}$ for α depending on the rate of decay of the potential. Σ_∞ plays a role similar to $\operatorname{Im} \lambda = -\alpha$ in this problem.

Let us remark that since V does not decay at infinity in all directions it is not surprising to obtain more complicated singularities for $(H - \lambda)^{-1}$ than in usual two-body Schrödinger operators.

Under a geometric hypothesis we study the growth and ramification properties of $(H - \lambda)^{-1}$ near Σ and prove that $(H - \lambda)^{-1}$ is of finite determination and of moderate growth near the points of Σ. (See Theorem 4.9.)

In generic cases, we can give an asymptotic expansion for $(H - \lambda)^{-1}$ near a Landau resonance $\lambda_0 \in \Sigma$ and show that the typical behavior is in $(\lambda - \lambda_0)^{\alpha/2}$ for $\alpha \in \mathbf{Z}$ or in $(\lambda - \lambda_0)^\alpha \log(\lambda - \lambda_0)$ with a leading singularity of finite rank. (See Theorem 4.10.)

In simple cases, we can associate "resonant functions" to the leading singularities of $(H-\lambda)^{-1}$ at λ_0.

In the last part of the paper we treat the case of a periodic crystal perturbed by impurities. One of the main issues in scattering by non-crystalline surfaces is to know what are the new effects in scattering quantities introduced by the presence of impurities or defects.

We model the impurities by adding to the potential V an additional potential W which is exponentially decreasing in (x, y).

Then we prove (see Theorems 5.1, 5.3) that the effects of impurities on $(H+W-\lambda)^{-1}$ is to add usual poles to the Landau resonances of $(H-\lambda)^{-1}$. This can be of practical significance since the Landau resonances should create logarithmic or square-root singularities on scattering cross-sections, which make resonance shapes very different from the usual Breit–Wigner resonance shape created by a pole.

The plan of the paper is the following:

In Sect. I, we introduce the Floquet–Bloch reduction which will be used in subsequent sections.
In Sect. II, we study the spectral theory of the reduced operators H_p using Mourre's commutator method.
In Sect. III, we prove the meromorphic extension in p and λ of $(H_p-\lambda)^{-1}$ and give formulas for $(H_p-\lambda)^{-1}$ using Fredholm determinants.
In Sect. IV, we prove the main results of the paper using the results of Sect. III and tools from complex analytic geometry.
In Sect. V, we study the resonances created by the presence of impurities.

To conclude let us mention that the methods used in this paper can be applied without much change to other similar problems: molecular scattering by surfaces, electromagnetic or acoustic scattering by periodic obstacles, resonances for periodic potentials.

We will come back to these problems in a subsequent publication.

I. Floquet Reduction

In this section, we recall the method of Floquet–Bloch to reduce the study of a periodic Schrödinger operator on $\mathbf{R}^m$ to the study of a family of Schrödinger operators on an m-torus. In our case, the Hamiltonian is periodic only in the directions tangent to the surface of the crystal, and the reduced Hamiltonian will live on a cylinder instead of a torus.

On $\mathbf{R}^m = \mathbf{R}_x^{m-1} \times \mathbf{R}_y$, we consider the following Hamiltonian: $H = (D_x)^2 + (D_y)^2 + V(x, y)$, where $D_{x_i} = (1/i)\partial_{x_i}, D_y = (1/i)\partial_y$ and $(D_x)^2 = \sum_{i-1}^{m-1} (D_{x_i})^2$.

We shall assume that V is a multiplicative potential which is T-periodic in the x variable for some lattice T in $\mathbf{R}^{m-1}$, i.e.: $V(x+\tau, y) = V(x, y)\ \forall \tau \in T$.

We will make in the next sections more precise hypotheses on the local singularities and the decay in y of $V(x, y)$.

We will follow the exposition of Skriganov [Sk] of the Floquet reduction.

We denote by T^* the dual lattice of T, which is defined as follows: if $(a_1,\ldots,a_{m-1})$ is a basis for T, a basis for T^* is given by the $(b_1,\ldots,b_{m-1})$ such that $\langle a_i, b_j\rangle = 2\pi\delta_{ij}$, where $\langle , \rangle$ is the Euclidean scalar product in $\mathbf{R}^{m-1}$.

We denote by F_T a fundamental domain of T, F_{T^*} a fundamental domain of T^*, which are chosen to be diffeomorphic to the $n-1$ torus Π^{n-1}. μ_T (respectively μ_{T^*}) will be the Lebesgue measure of F_T (respectively F_{T^*}).

For $\varphi\in\mathscr{S}(\mathbf{R}^m)$, the Schwartz space of rapidly decreasing C^∞ functions, and for $p\in F_{T^*}$, we set:

$$K_p\varphi(x,y) = \mu_T^{1/2}\sum_{\tau\in T}\varphi(x+\tau,y)e^{i\langle p,x+\tau\rangle}. \tag{1.1}$$

The sum in (1.1) is convergent because of the rapid decay of φ, and $K_p\varphi(x,y)$ is T-periodic in x, and satisfies the equations:

$$K_{p+p'}\varphi(x,y) = e^{i\langle p',x\rangle}K_p\varphi(x,y), \quad \text{for} \quad p'\in T^*. \tag{1.2}$$

If we consider $K_p\varphi(x,y)$ as a function of $(x,y,p)\in F_T\times\mathbf{R}\times F_T^*$, we see as in [Sk] that the family K_p extends as a unitary operator W_T:

$$L^2(\mathbf{R}^n)\to\mathscr{L} = \int_{F_{T^*}}^{\oplus} L^2(C_T)dp,$$

$$\varphi\mapsto K_p\varphi(x,y).$$

Here C_T is the cylinder $F_T\times\mathbf{R}$.

Assume now that V is T-periodic in x and is relatively bounded with respect to the Laplacian $\Delta = (D_x)^2 + (D_y)^2$ with relative bound strictly less than 1. Then (see [Sk]), we can decompose H as a direct integral of operators:

$$W_T H W_T^{-1} = \int_{F_{T^*}}^{\oplus} H_p dp \quad \text{where:}$$

—$H_p = (D_x+p)^2 + (D_y)^2 + V(x,y)$ with domain $H^2(C_T)$.
—W_T^{-1} is defined by the following formula:

$$W_T^{-1}\varphi(x,y,p) = (2\pi)^{-(m-1)/2}\mu_{T^*}^{-1/2}\int_{F_{T^*}} e^{-i\langle p,x\rangle}\varphi(x,y,p)dp. \tag{1.3}$$

We will denote by $K_p^{-1}: L^2(C_T)\to L^2_{\mathrm{loc}}(\mathbf{R}^m)$ the operator

$$K_p^{-1}: u\mapsto e^{-i\langle p,x\rangle}u(x,y), \tag{1.4}$$

where $u(x,y)$ is extended to $\mathbf{R}^m$ by T-periodicity in x.

Taking the usual Sobolev space on the cylinder $C_T, H^2(C_T)$ as domain for H_p means that we consider eigenfunctions of H_p ("Bloch waves") which are T-periodic in x, and in this version of the Floquet reduction, the p-dependence of the reduced operator is in the operator and not in his domain as in the usual reduction.

We will denote by $H_{0,p} = (D_x+p)^2 + (D_y^2)$ the free reduced operator with domain $H^2(C_T)$.

II. Spectral Theory of the Reduced Operators

In this section, we study the spectral theory of H_p and $H_{0,p}$. The spectral theory of $H_{0,p}$ is trivial by separation of variables. To get results on the spectrum of H_p, we will use the method of Mourre (see for example [M], [C.F.K.S.]) and construct a conjugate operator for H_p.

We will assume in this section that V satisfy the following hypotheses:

(H.1) $$V(H_{0,p}+i)^{-1} \text{ is compact.}$$

(H.2) If $A=\frac{1}{2}(y.D_y+D_y.y)$, the form $[V,iA]$ defined on $H^2(C_T)\cap D(A)$ extends as an operator bounded from $H^2(C_T)$ to $H^{-1}(C_T)$ and compact from $H^2(C_T)$ to $H^{-2}(C_T)$.

(H.3) The form $[[V,iA],iA]$ defined on $H^2(C_T)\cap D(A)$ extends as a bounded operator from $H^2(C_T)$ to $H^{-2}(C_T)$.

For example if $V\in C^2(C_T)$, V and $y.\nabla_y V$ tend to zero when y tends to infinity, and $y^2.\nabla_y^2 V$ is bounded, V satisfy $(H.i)i=1,2,3$.

II.A. Spectral Theory of $H_{0,p}$. Let $u\in H^2(C_T)$. Since u is T periodic in x, we can write:

$$u(x,y)=\sum_{n\in T^*} u_n(y)e^{i\langle n,x\rangle},$$

where

$$u_n(y)=\frac{1}{\mu_T^{1/2}}\int_{F_T} e^{-i\langle n,x\rangle}u(x,y)dx. \tag{2.1}$$

We have

$$H_{0,p}u(x,y)=\sum_{n\in T^*}(D_y^2+(n+p)^2)u_n(y)e^{i\langle n,x\rangle}.$$

This gives immediately the spectral decomposition of $H_{0,p}$: let us denote by F:

$$L^2(C_T)\to\mathscr{H}=\bigoplus_{T^*}L^2(\mathbf{R}),$$

$$u(x,y)\mapsto(\hat{u}_n(\xi))_{n\in T^*},$$

where $\hat{u}_n(\xi)=(2\pi)^{-1/2}\int e^{-iy.\xi}u_n(y)dy$.

F is obviously unitary from $L^2(C_T)$ into $\mathscr{H}$ and we have:

$$H_{0,p}F^*(\hat{u}_n)=F^*(((p+n)^2+\xi^2)\hat{u}_n).$$

For $p\in F_{T^*}$, we will call *thresholds of* H_p the numbers $(n+p)^2=(n_1+p_1)^2+\cdots(n_{m-1}+p_{m-1})^2$ for $n\in T^*$, and denote by $E_0(p)=\inf_{n\in T^*}\{(n+p)^2\}$.

From the explicit spectral decomposition of $H_{0,p}$, we get immediately:

Proposition 2.1.

$$\sigma(H_{0,p})=\sigma_{\mathrm{ac}}(H_{0,p})=[E_0(p),+\infty[.$$

Finally we will write for later use the resolvent kernel of $(H_{0,p}-\lambda)$: the kernel $K_{0,p}(x,y,x',y',\lambda)$ of $(H_{0,p}-\lambda)^{-1}$ has the following form:

$$K_{0,p}(x,y,x',y',\lambda)=\sum_{n\in T^*}e^{i\langle n,x-x'\rangle}\frac{1}{2(\lambda-(n+p)^2)^{1/2}}e^{i(\lambda-(n+p)^2)^{1/2}|y-y'|}. \tag{2.2}$$

$K_{0,p}$ is bounded from $L^2(C_T)$ into $H^2(C_T)$ if $\operatorname{Im}((\lambda-(n+p)^2)^{1/2})>0$ $\forall n\in T^*$. Equation (2.2) follows directly from the separation of variables used above.

II.B. Spectral Theory of H_p. We will prove in this subsection the following theorem:

Theorem 2.2. *Under the hypotheses* (H.1), (H.2), (H.3), *we have*

$$\sigma_{\mathrm{ess}}(H_p) = [E_0(p), +\infty[, \tag{2.3}$$

the eigenvalues of H_p can accumulate only at the thresholds $(n+p)^2$ for $n \in T^*$, (2.4)

$$\sigma_{\mathrm{sing}}(H_p) = \varnothing. \tag{2.5}$$

Proof. (2.3) Follows directly from (H.1), Proposition 2.1 and Weyl's theorem on essential spectrum. (cf. [Re–Si]). To prove (2.4), (2.5), we will apply Mourre's commutator method and we refer to the book [C.F.K.S.] for details.

As a conjugate operator we take $A = \frac{1}{2}(y.D_y + D_y.y)$. A is the generator of dilations in the y direction, which is the only direction in which classical particles can escape to infinity. Then we have

$$[H_{0,p}, iA] = 2(D_y)^2, \quad [H_p, iA] = 2(D_y)^2 - y.\nabla_y V. \tag{2.6}$$

We let the reader check that the technical hypotheses on $H, A, [H, iA]$ and $[[H, iA], iA]$ needed to apply Mourre's theorem are satisfied with this choice of A, under the hypotheses (H.1), (H.2), (H.3) (see [C.F.K.S.]).

Let now $\Delta \subset [E_0(p), +\infty[$ be a closed interval not containing any threshold. We will check that H_p satisfies a Mourre estimate on Δ.

Let $\chi \in C_0^\infty(\Delta)$ be a cutoff function with $\chi \equiv 1$ on $\Delta' \subset\subset \Delta$. Using that $(H_{p+i})^{-1} - (H_{0,p+i})^{-1}$ is compact by (H.1), we see that $\chi(H_p) - \chi(H_{0,p})$ is compact by a Stone–Weierstrass argument. (H.2) and (2.6) give that:

$$\chi(H_{0,p})[H_p, iA]\chi(H_{0,p}) = 2\chi(H_{0,p})(D_y)^2\chi(H_{0,p}) + K,$$

where K is a compact operator.

Using the spectral resolution of $H_{0,p}$, we have

$$F(\chi(H_{0,p})u)_n = \hat{u}_n(\xi) \times \chi(\xi^2 + (n+p)^2).$$

Since Δ does not contain any threshold, there exist some $\alpha > 0$ such that: $\forall n \in T^*$, $\xi \in \mathbf{R}$ such that $\chi(\xi^2 + (n+p)^2) \neq 0$, we have

$$\xi^2 \geqq \alpha.$$

This implies that

$$\chi(H_{0,p})[H_p, iA]\chi(H_{0,p}) \geqq 2\alpha\chi^2(H_{0,p}) + K. \tag{2.7}$$

If we replace $\chi(H_{0,p})$ by $\chi(H_p)$, we will introduce error terms of the type

$$(\chi(H_{0,p}) - \chi(H_p))[H_p, iA]\chi(H_{0,p})$$

and

$$\chi(H_{0,p})[H_p, iA](\chi(H_{0,p}) - \chi(H_p))$$

which are compact, using (H.2) and (2.6).

So we get

$$\chi(H_p)[H_p, iA]\chi(H_p) \geqq 2\alpha\chi^2(H_p) + \tilde{K}, \tag{2.8}$$

where $\tilde{K}$ is compact.

By composing (2.8) to the left and right by $E_{\Delta'}(H_p)$, we prove that H_p satisfies a Mourre estimate on Δ', and also on Δ if we enlarge a little Δ.

Using the theorem of Mourre (see [C.F.K.S.] Theorem 4.7 and Corollary 4.10) we get (2.4) and (2.5). □

We make now a few remarks on this theorem:

Remark 2.3. The clinder C_T is a special case of a Riemannian manifold with a finite number of cylindrical ends. The spectral theory of the Laplace–Beltrami operator on this kind of manifold (and also on more general ones) has been studied recently by Froese and Hislop (see [F.H], [H]), and by Guillopé ([G]). In particular Froese and Hislop use Mourre's commutator method to get results on the spectrum.

Remark 2.4. We cannot exclude the existence of embedded eigenvalues as the case of $V(x, y) = V(y)$ shows. In this case, denoting by $\lambda_k, k \in \mathbf{N}$ the eigenvalues of $(D_y)^2 + V(y)$, H_p has the eigenvalues $(n+p)^2 + \lambda_k, n \in T^*, k \in \mathbf{N}$, with an infinite number of embedded ones. However for a general potential, we expect that these eigenvalues will dissolve in the continuous spectrum due to coupling between different channels, and become resonances.

This mechanism is responsible for resonance creation in atom-surface scattering.

III. Analytic Extension of the Reduced Resolvent

In this section we will study the analytic extension in (p, λ) of the reduced resolvent $(H_p - \lambda)^{-1}$, when the potential V is exponentially decreasing in the y variable.

We will use a method originally used by Vainberg [Va] in the study of obstacle scattering, which has the advantage of eliminating the continuous spectrum. We will assume in general that $V \in L^\infty_\alpha(C_T) = \{V \mid e^{\alpha\langle y\rangle} V \in L^\infty(C_T)\}$ for some $\alpha > 0$ big enough. Here $\langle y\rangle = (1+y^2)^{1/2}$.

This is certainly not the optimal class since we can allow local singularities of V (see Remark 3.4).

We denote by $L^2_\alpha(C_T) = \{u \in L^2_{\mathrm{loc}}(C_T) \mid e^{\alpha\langle y\rangle} u \in L^2(C_T)\}$ and $H^1_a(C_T) = \{u \in H^1_{\mathrm{loc}}(C_T) \mid e^{\alpha\langle y\rangle} u \in H^1(C_T)\}$, for $a \in \mathbf{R}$. We put on these Hilbert spaces their natural norms.

In studying the analytic extension in (p, λ) of $(H_p - \lambda)^{-1}$, we must take into consideration the fact that for p fixed, $(H_{0,p} - \lambda)^{-1}$ (and hence $(H_p - \lambda)^{-1}$) has branch points in λ at the thresholds $(n+p)^2$ for $n \in T^*$ (see (2.2)). When λ varies continuously in some bounded open set $\mathscr{U}$ of $\mathbf{C}$, and p in some bounded complex neighborhood $\mathscr{W}$ of F_{T*} in $\mathbf{C}^{m-1}$, only a finite number of the functions $(\lambda - (n+p)^2)^{1/2}$ can change their determination. Let us denote by $\mathscr{I} \subset T^*$ the set of the n such that $(\lambda - (n+p)^2)^{1/2}$ can change of determination, and put $N = \mathrm{Card}(\mathscr{I}), \mathscr{I} = \{n_1, \ldots, n_N\}$.

To uniformize the functions $(\lambda - (n+p)^2)^{1/2}$ for $n \in \mathscr{I}$, we introduce the following complex analytic set:

$$\Gamma \subset \mathbf{C}^{N+m} = \{(p, \lambda, z_1, \ldots, z_N) \in \mathscr{W} \times \mathscr{U} \times \mathbf{C}^N \mid z_i^2 = \lambda - (n_i + p)^2, i = 1, \ldots, N\}.$$

On Γ we have a particular region denoted by Γ_∞, which corresponds to the "physical sheet." Γ_∞ is the subset of Γ, where $\operatorname{Im} z_i > 0, i = 1, \ldots, N$, and on Γ_∞, Γ is a smooth submanifold which can be parametrized by (p, λ). For $p \in F_{T^*}, \lambda \in \mathscr{U}$, $\operatorname{Im} \lambda > 0$, we can write the kernel of $(H_{0,p} - \lambda)^{-1}$ as the restriction to Γ_∞ of the function:

$$K_0(x, y, x', y', p, \lambda, z_1, \ldots, z_N) = \sum_{j=1}^{N} e^{i\langle n_j, x - x'\rangle} \frac{1}{2z_j} e^{iz_j|y-y'|} + \sum_{n \in T^* \setminus \mathscr{I}} e^{i\langle n, x - x'\rangle} \frac{1}{2(\lambda - (n+p)^2)^{1/2}} e^{i(\lambda - (n+p)^2)^{1/2}|y-y'|}. \tag{3.1}$$

We remark that Γ_∞ is the natural region where K_0 is the kernel of an operator $K_0(p, \lambda, z)$ which is bounded on $L^2(C_T)$. We denote for simplicity by z the N-uple $(z_1, \ldots, z_N)$, and let z vary in some set Z of the type: $Z = \{z \in \mathbf{C}^N | \operatorname{Im} z_i \geqq -\varepsilon\}$ for some $\varepsilon > 0$. We have the following Proposition:

Proposition 3.1. *For $a > \varepsilon$, $K_0(p, \lambda, z)$ extends meromorphically in $(p, \lambda, z) \in \mathscr{W} \times \mathscr{U} \times Z$ as a bounded operator from $L_a^2(C_T)$ into $H^1_{-a}(C_T)$ with singularities on $z_i = 0$, $i = 1, \ldots, N$.*

Proof. To study the boundedness of $K_0(p, \lambda, z)$ from $L_a^2(C_T)$ into $H^1_{-a}(C_T)$, we have to investigate the following reduced kernel:

$$K_{\text{red}}(x, y, x', y', p, \lambda, z) = \sum_{j=1}^{N} e^{i\langle n_j, x - x'\rangle} \frac{1}{2z_j} e^{iz_j|y-y'| - a\langle y\rangle - a\langle y'\rangle} + \sum_{n \in T^* \setminus \mathscr{I}} e^{i\langle n, x - x'\rangle} \frac{1}{2(\lambda - (n+p)^2)^{1/2}} e^{i(\lambda - (n+p)^2)^{1/2}|y-y'| - a\langle y\rangle - a\langle y'\rangle}.$$

For $(p, \lambda) \in \mathscr{W} \times \mathscr{U}, n \in T^* \setminus \mathscr{I}$, we have:

$$(\lambda - (n+p)^2)^{1/2} = i(n^2)^{1/2}\left(1 + \frac{2n.p}{n^2} + \frac{p^2}{n^2} - \frac{\lambda}{n^2}\right)^{1/2} = i|n|\left(1 + 0\left(\frac{1}{|n|}\right)\right),$$

uniformly for $(p, \lambda) \in \mathscr{W} \times \mathscr{U}$.

We use here the fact that $\operatorname{Im}((\lambda - (n+p)^2)^{1/2}) > 0$, for $(p, \lambda) \in \mathscr{W} \times \mathscr{U}, n \notin \mathscr{I}$.

Finally we have:

$$(\lambda - (n+p)^2)^{1/2} = i|n| + O(1) \quad \text{for} \quad (p, \lambda) \in \mathscr{W} \times \mathscr{U}, \quad n \notin \mathscr{I}. \tag{3.2}$$

We can now prove the Proposition: since K_0 splits on the direct sum decomposition $L^2(C_T) = \bigoplus_{n \in T^*} e^{i\langle n, x\rangle} \otimes L^2(\mathbf{R})$, we only have to estimate the following operators:

$$M_n : u(y) \mapsto e^{-i\langle n, x\rangle} M(e^{i\langle n, x\rangle} u(y))$$

for $M = K_{\text{red}}, \partial_y K_{\text{red}}, \partial_{x_i} K_{\text{red}}, i = 1, \ldots, m-1$, as operators on $L^2(\mathbf{R})$. For $M = K_{\text{red}}$, we get an operator M_n with kernel:

$$\frac{1}{2z_j} e^{iz_j|y-y'| - a\langle y\rangle - a\langle y'\rangle} \quad \text{if} \quad n = n_j,$$

$$\frac{1}{2(\lambda - (n+p)^2)^{1/2}} e^{i(\lambda - (n+p)^2)^{1/2}|y-y'| - a\langle y\rangle - a\langle y'\rangle} \quad \text{if} \quad n \notin \mathscr{I}.$$

Using (3.2) and the criterion of Schur, we get that M_n is bounded on $L^2(\mathbf{R})$ with norm $\|M_n\| \leqq C$ uniformly in n, provided we take $a > \varepsilon$, $z_i \neq 0$. For $M = \partial_y K_{\mathrm{red}}$ or $\partial_{x_i} K_{\mathrm{red}}$, we get an operator M_n with kernel equal respectively to (if $n = n_j$):

$$\frac{1}{2}\frac{y-y'}{|y-y'|}e^{iz_j|y-y'|-a\langle y\rangle - a\langle y'\rangle}$$

and

$$\frac{n_{ij}}{2z_j}e^{iz_j|y-y'|-a\langle y\rangle - a\langle y'\rangle}.$$

They satisfy the same estimates as above. The case $n \notin \mathscr{I}$ is similar, which proves the Proposition. □

We will need also the following lemma:

Lemma 3.2. *The residues of $K_0(p,\lambda,z)$ at $z_j = 0$ are finite rank operators.*

Proof. The singularity in z_j^{-1} at a threshold $(n_j + p)^2$ comes from the singularity of $(D_y^2 - z_j^2)^{-1}$, which has the kernel: $e^{iz_j|y-y'|}/2z_j$. We will show how to isolate this singularity for $(D_y^2 - z_j^2)^{-1}$ and then come back to $K_0(p,\lambda,z)$.

We can write:

$$e^{iz_j|y-y'|} = 1 + \int_0^1 iz_j|y-y'|e^{iz_j|y-y'|t}dt. \tag{3.3}$$

Using (3.3), we can write:

$$(D_y^2 - z_j^2)^{-1}u = \frac{1}{2z_j}\int u(y')dy' + M(z_j)u,$$

where by the same arguments as in the proof of Proposition 3.1, we see that $M(z_j)$ is holomorphic in z_j as a bounded operator between $L_a^2(\mathbf{R})$ and $H_{-a}^1(\mathbf{R})$, for $a > -\operatorname{Inf}(\operatorname{Im} z_j)$.

We can now come back to $K_0(p,\lambda,z)$: for $u \in L_a^2(C_T)$, if we put

$$\pi_j u(x,y) = e^{i\langle n_j, x\rangle}\mu_T^{-1}\int_{F_T} e^{-i\langle n_j, x\rangle}u(x,y)dx,$$

π_j is bounded on $L_a^2(C_T)$.

We see directly by looking at expression (3.1) for $K_0(p,\lambda,z)$, that $K_0(p,\lambda,z)(1-\pi_j)$ has no singularities on $z_j = 0$, as a bounded operator from $L_a^2(C_T)$ into $H_{-a}^1(C_T)$.

On the other hand:

$$K_0(p,\lambda,z)\pi_j u = e^{i\langle n_j,x\rangle}(D_y^2 - z_j^2)^{-1}e^{-i\langle n_j,x\rangle}\pi_j u = \frac{\tilde{\pi}_j}{z_j}u + \tilde{M}(p,\lambda,z)u,$$

where $\tilde{M}(p,\lambda,z)$ is holomorphic near $z_j = 0$ and $\tilde{\pi}_j$ is a rank one operator

$$\tilde{\pi}_j u = e^{i\langle n_j,x\rangle}(2\mu_T)^{-1}\int_{C_T} e^{-i\langle n_j,x\rangle}u(x,y)dxdy.$$

This proves the lemma. □

We need one more result before proving Theorem 3.5. Let us denote by $C(Z)$ the constant $-\operatorname*{Inf}_{z\in Z}(\operatorname{Im} z_i)$. We have:

Proposition 3.3. *Assume that $V\in L^\infty_\alpha(C_T)$ for $\alpha > 2C(Z)$. Then $VK_0(p,\lambda,z)$ belongs to the Schatten class $\mathscr{F}_k$ on $L^2_a(C_T)$, for $a > C(Z)$, $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z, z_i\neq 0, k>m$.*

Proof. It clearly suffices to prove that $e^{-\alpha\langle y\rangle}K_0(p,\lambda,z)$ is in $\mathscr{F}_k$. Let us first remark that if $\alpha > 2C(Z)+\varepsilon_0$ for $\varepsilon_0>0$, we have that $e^{-(\alpha-\varepsilon_0)\langle y\rangle}K_0(p,\lambda,z)$ is bounded from $L^2_a(C_T)$ into $H^1_a(C_T)$, for $a > C(Z)$. This means that $e^{(a-\alpha+\varepsilon_0)\langle y\rangle}K_0(p,\lambda,z)e^{-a\langle y\rangle}$ is bounded from $L^2(C_T)$ into $H^1(C_T)$, or that

$$(1+|\Delta|)^{1/2}e^{(a-\alpha+\varepsilon_0)\langle y\rangle}K_0(p,\lambda,z)e^{-a\langle y\rangle}$$

is $L^2(C_T)$ bounded for $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z, z_i\neq 0$.

To prove the Proposition, it suffices to show that:

$$e^{-\varepsilon_0\langle y\rangle}(1+|\Delta|)^{-1/2} \quad \text{belongs to } \mathscr{F}_k \text{ for } k>m \text{ on } L^2(C_T). \tag{3.4}$$

Using Theorem 4.1 of Simon [Si], which can be easily adapted to our case, we see that (3.4) holds, which proves the Proposition. □

Remark 3.4. In order to include local singularities of V, we can introduce the space $L^k_\alpha(C_T)=\{V\in L^k_{\mathrm{loc}}(C_T)|e^{\alpha\langle y\rangle}V\in L^k(C_T)\}$. If $V\in L^k_\alpha(C_T)$ for $k>m$, V belongs to the Stummel class S_m (see [C.F.K.S.]) so H and H_p are self adjoint with domains $H^2(\mathbf{R}^m)$ and $H^2(C_T)$ respectively. Using Theorem 4.1 of [Si] we see that Proposition 3.3 still holds if $V\in L^k_\alpha(C_T), k>m, \alpha>2C(Z)$.

Theorem 3.5. *Assume that V belongs to $L^k_\alpha(C_T)$ for $k>m, \alpha>2C(Z)$. Then $(1+VK_0(p,\lambda,z))^{-1}$ can be written for $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z$ as*

$$(1-VK_0(p,\lambda,z))^{-1}=\frac{D(p,\lambda,z)}{f(p,\lambda,z)},$$

where D and f are holomorphic in $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z$ as a bounded operator on $L^2_a(C_T)$ and as a function respectively, for $a>C(Z)$.

Proof. From Lemma 3.2 and his proof we see that $K(p,\lambda,z)=VK_0(p,\lambda,z)$ can be written as:

$$K(p,\lambda,z)=M(p,\lambda,z)+\sum_{j=1}^{N}\frac{\pi_j}{z_j}=M(p,\lambda,z)+R(p,\lambda,z),$$

where

—$M(p,\lambda,z)$ is holomorphic in $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z$ and satisfies the properties of Proposition 3.3.

—$\pi_j\, j=1,\dots,N$ are rank one operators on $L^2_a(C_T)$.

We will write:

$$(1+K)^{-1}=(1+M)^{-1}(1+R(1+M)^{-1})^{-1}. \tag{3.6}$$

We first consider $(1+M(p,\lambda,z))^{-1}$. We will use the theory of regularized determinants for operators in $\mathscr{F}_k$. We follow the exposition of [Si] paragraph 9. We consider for $N_0 \geqq m+1$:

$$R(M) = (1+M)\exp\left(- \sum_1^{N_0-1} (-M)^k/k \right) - 1.$$

$R(M)$ is trace class with $\| R(M) \|_{T_r} \leqq C \| M \|_{N_0}$, where $\| \quad \|_{T_r}$ is the trace norm and $\| \quad \|_{N_0}$ is the norm in $\mathscr{F}_{N_0}$.

Then $1+M$ is invertible if and only if the usual Fredholm determinant $\det(\mathbf{1} + R(M))$ is non-zero, and:

$$(1+M)^{-1} = (1+R(M))^{-1} \exp\left(- \sum_1^{N_0-1} (-M)^k/k \right).$$

$R(M)$ is holomorphic in $(p, \lambda, z) \in \mathscr{W} \times \mathscr{U} \times Z$ as a bounded operator in $L_a^2(C_T)$ and belongs to $\mathscr{F}_1$ with $\| R(M) \|_{T_r} \leqq C$ uniformly in (p, λ, z). So we are reduced to the case when M is trace class. The Fredholm determinant of M is then: $\det(1+M) = \sum_{m=0}^{\infty} D_{0,m}(M)$, where $D_{0,m}(M)$ is a polynomial expression of the $\operatorname{tr}(M^k)$ for $1 \leqq k \leqq m$, satisfying:

$$|D_{0,m}(M)| \leqq \left(\frac{c}{m} \right)^{m/2} \| M \|_{T_r}^m.$$

(see [Si]). To show that $\det(1+M)$ is holomorphic in (p, λ, z), it suffices to show that $D_{0,m}(M)$ (i.e. $\operatorname{Tr}(M^k)$) is holomorphic in (p, λ, z). Then $\det(1+M)$ will be holomorphic as a uniformly convergent series of holomorphic functions.

It is easy to check that $\operatorname{Tr}(M^k)$ is holomorphic in (p, λ, z), which shows that $\det(1+M)$ is a holomorphic function.

If $\det(1+M) \neq 0$, $(1+M)^{-1}$ can be written as: $D_1(p, \lambda, z)/\det(1 + M(p, \lambda, z))$, where $D_1(p, \lambda, z)$ is of the form:

$$D_1(p, \lambda, z) = \sum_0^{+\infty} D_{1,m}(M),$$

and $D_{1,m}(M)$ is a polynomial expression of the $M^k, \operatorname{Tr}(M^k)$ for $1 \leqq k \leqq m$, satisfying:

$$\| D_{1,m}(M) \| \leqq C^m \frac{1}{m^{m/2}} \| M \|_{T_r}^m \quad \text{(see [Si])}.$$

As before, we prove that $D_{1,m}(M)$ and then $D_1(p, \lambda, z)$ are holomorphic for $(p, \lambda, z) \in \mathscr{W} \times \mathscr{U} \times Z$. So we can write $(1+M)^{-1}$ as: $D_1(p, \lambda, z)/f_1(p, \lambda, z)$, and using the estimates in the proof of Proposition 3.1, it is easy to check that $(1 + M(p, \lambda, z))$ is invertible for $p \in \mathscr{W}$, $\operatorname{Im} \lambda \gg 1$, $\operatorname{Im} z_i \gg 1$ which proves that $f_1(p, \lambda, z) \not\equiv 0$.

From (3.6), we see that to invert $(1+K)^{-1}$, it remains to invert $1 + \pi(z) D_1(p, \lambda, z)/g(z) f_1(p, \lambda, z)$, where $\pi(z) = \sum_{j=1}^{N} \pi_j z_1 \ldots z_{j-1} z_{j+1} \ldots z_N$, and $g(z) = z_1 \ldots z_N$. $\pi(z) D_1(p, \lambda, z)/g(z) f_1(p, \lambda, z)$ can be written as $\tilde{\pi}(p, \lambda, z)/h(p, \lambda, z)$, where $h = g \times f_1$, and $\tilde{\pi}$ has a fixed image F of finite dimension.

Now to solve

$$u + \frac{\tilde{\pi}}{h} u = v, \tag{3.7}$$

we try $u=(1-\pi_0)r+r$, where π_0 is the orthogonal projection in F and $r\in F$. We get:

$$r+\frac{\tilde{\pi}}{h}r=\pi_0 v-\frac{\tilde{\pi}}{h}(1-\pi_0)v=E(p,\lambda,z)v. \tag{3.8}$$

The equation in the finite dimensional space $F: r+(\tilde{\pi}/h)r=f$ can be solved by: $r=\tilde{D}(p,\lambda,z)/\tilde{h}(p,\lambda,z)$, where $\tilde{D}$ and $\tilde{h}$ are holomorphic in (p,λ,z) as an operator and as a function respectively.

This follows from finite dimensional matrix theory.

Finally we solve (3.7) by

$$u=(1-\pi_0)v+\frac{\tilde{D}(p,\lambda,z)}{\tilde{h}(p,\lambda,z)}E(p,\lambda,z)v. \tag{3.9}$$

Putting together (3.6), (3.9), we get that $(1+K(p,\lambda,z))^{-1}$ can be written as: $D(p,\lambda,z)/f(p,\lambda,z)$ where D and f are holomorphic in $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z$ as an operator on $L_a^2(C_T)$ and as a function respectively. This proves the theorem. □

IV. Analytic Extension of the Total Resolvent

In this section we will prove the existence of an analytic extension for the total resolvent $(H-\lambda)^{-1}$. As explained in the introduction, the singularities of $(H-\lambda)^{-1}$ are different when one considers the *local extension* of $(H-\lambda)^{-1}$ in a small neighborhood of $\lambda_0\in\mathbf{R}$, and when one considers the *global extension* of $(H-\lambda)^{-1}$ to $\mathscr{U}$, where $\mathscr{U}$ is a bounded set in $\mathbf{C}$ as in Sect. III.

In the local case, $(H-\lambda)^{-1}$ extends holomorphically as a bounded operator between some weighted L^2 spaces to the universal covering of $\mathscr{U}_{\lambda_0}\backslash\Sigma$, where $\mathscr{U}_{\lambda_0}$ is a neighborhood of λ_0 and Σ is a discrete set of points in $\mathscr{U}$, called *Landau resonances.*

In the global case in addition to Σ, $(H-\lambda)^{-1}$ can have singularities on a closed set of measure zero Σ_∞ which correspond to a kind of complex essential spectrum. We will study some properties of $(H-\lambda)^{-1}$ like finite determination and moderate growth under some additional hypotheses near the points of Σ. The points in Σ are analogous to Landau singularities in quantum field theory. Using results of Leray and Pham we will then study the behavior of $(H-\lambda)^{-1}$ near generic points of Σ and get asymptotic expansions which show that in general a Landau resonance is a branch point of $(H-\lambda)^{-1}$ rather than a pole.

In some cases it is even possible to associate to a Landau resonance a kind of "generalized resonant state."

We start by proving some formulas.

From the discussion in Sect. I, we see that for $\operatorname{Im}\lambda>0$, $(H-\lambda)^{-1}$ can be written as:

$$\text{for}\quad u\in C_0^\infty(\mathbf{R}^m): \tag{4.1}$$

$$(H-\lambda)^{-1}u=C(m,T)\int_{F_{T^*}} e^{-i\langle p,x\rangle}(H_p-\lambda)^{-1}\left(\sum_{\tau\in T}u(x+\tau,y)e^{i\langle p,x+\tau\rangle}\right)dp,$$

where $C(m,T)=(2\pi)^{-(m-1)/2}\mu_T^{1/2}\mu_{T^*}^{-1/2}$.

Using the second resolvent formula, we have:

$$(H_p-\lambda)^{-1}=(H_{0,p}-\lambda)^{-1}(1+V(H_{0,p}-\lambda)^{-1})^{-1}, \quad \text{for} \quad \operatorname{Im}\lambda>0, \quad p\in F_{T^*}.$$

On the other hand, $(H_{0,p}-\lambda)^{-1}$ is the restriction to Γ_∞ of the operator $K_0(p,\lambda,z)$: if we denote for $p\in F_{T^*}$, $\operatorname{Im}\lambda>0$, by $z_i(p,\lambda)$ the determination of $(\lambda-(n_i+p)^2)^{1/2}$ with positive imaginary part, we have: $(H_{0,p}-\lambda)^{-1}=K_0(p,\lambda,z(p,\lambda))$.

From Theorem 3.5, we can rewrite (4.1) as:

$$\begin{aligned}(H-\lambda)^{-1}u&=C(m,T)\int_{F_{T^*}} e^{-i\langle p,x\rangle}\frac{\tilde{D}(p,\lambda,z(p,\lambda))}{f(p,\lambda,z(p,\lambda))}\left(\sum_{\tau\in T}u(x+\tau,y)e^{i\langle p,x+\tau\rangle}\right)dp\\&=C(m,T)\int_{F_{T^*}}\frac{M(p,\lambda,z,(p,\lambda))}{f(p,\lambda,z(p,\lambda))}u\,dp.\end{aligned}$$

For $a\in\mathbf{R}$, we denote by $L^2_a(\mathbf{R}^m)$ the space

$$\{u\in L^2_{\text{loc}}(\mathbf{R}^m)\,|\,e^{a(\langle x\rangle+\langle y\rangle)}u\in L^2(\mathbf{R}^m)\}$$

and by $H^1_a(\mathbf{R}^m)$ the space

$$\{u\in H^1_{\text{loc}}(\mathbf{R}^m)\,|\,e^{a(\langle x\rangle+\langle y\rangle)}u\in H^1(\mathbf{R}^m)\}.$$

$L^2_a(\mathbf{R}^m)$ and $H^1_a(\mathbf{R}^m)$ are Hilbert spaces with their natural norms. We have the following Proposition:

Proposition 4.1. *If* $V\in L^k_\alpha(C_T)$ *for* $k>m$, $\alpha>2(C(Z)+\sup|\operatorname{Im}p|_{p\in\mathscr{W}})$, $M(p,\lambda,z)$ *extends holomorphically to* $(p,\lambda,z)\in\mathscr{W}\times\mathscr{U}\times Z$ *as a bounded operator from* $L^2_a(\mathbf{R}^m)$ *into* $H^1_{-a}(\mathbf{R}^m)$ *for* $a>(C(Z)+\sup|\operatorname{Im}p|_{p\in\mathscr{W}})$.

Proof. Using Theorem 3.5, we see that it suffices to prove that K_p (defined in (1.1)) extends holomorphically in p as a bounded operator from $L^2_a(\mathbf{R}^m)$ into $L^2_a(C_T)$ and that $(K_p)^{-1}$ (defined in (1.4)) extends holomorphically in p as a bounded operator from $H^1_{-a}(C_T)$ into $H^1_{-a}(\mathbf{R}^m)$. If $a>C\sup|\operatorname{Im}p|_{p\in\mathscr{W}}$ such that $|e^{-a\langle x\rangle}e^{i\langle p,x\rangle}|\leqq e^{-\varepsilon\langle x\rangle}$ for $p\in\mathscr{W}$, it is clear that K_p is uniformly bounded from $L^2_a(\mathbf{R}^m)$ into $L^2_a(C_T)$. Consider now the function $e^{-i\langle p,x\rangle}u(x,y)$ for $u\in H^1_{-a}(C_T)$ extended by periodicity in x as a function in $H^1_{\text{loc}}(\mathbf{R}^m)$.

We have to estimate:

$$\|e^{-a(\langle x\rangle+\langle y\rangle)}e^{-i\langle p,x\rangle}u(x,y)\|^2_{L^2(\mathbf{R}^m)}=\sum_{\tau\in T^*}\|e^{-a(\langle x+\tau\rangle+\langle y\rangle)}e^{-i\langle p,x+\tau\rangle}u(x,y)\|^2_{L^2(F_T)}. \tag{4.3}$$

If $a>C\sup|\operatorname{Im}p|_{p\in\mathscr{W}}$ we have for $p\in\mathscr{W}$, $x\in F_T$: $|e^{-a\langle x+\tau\rangle}e^{-i\langle p,x+\tau\rangle}|\leqq Ce^{-\varepsilon|\tau|}$ for some $\varepsilon>0$.

So we get a convergent sequence in (4.3) and we have:

$$\|e^{-i\langle p,x\rangle}u\|_{L^2_{-a}}(\mathbf{R}^m)\leqq C\|u\|_{L^2_{-a}(C_T)}.$$

We can estimate the derivatives of $e^{-i\langle p,x\rangle}u$ in the same way, which proves the Proposition. $\square$

We now use Cauchy's formula in order to eliminate the multivalued functions $z_i(p,\lambda)$. By making a linear change of coordinates in p, we may assume that $F_{T^*}=[-1,1]^{m-1}$.

We will consider the operator $(H-\lambda)^{-1}$ for λ near some point $\lambda_0 \in \mathscr{U} \cap \{\operatorname{Im} \lambda > 0\}$. Since $f(p,\lambda,z(p,\lambda)) \neq 0$ for $p \in F_{T^*}$, $\operatorname{Im}\lambda > 0$, we can find some $\varepsilon > 0$ small enough such that if p is in F_{T^*}, λ in a small neighborhood of λ_0, and $|z - z(p,\lambda)| \leqq \varepsilon$, we have $f(p,\lambda,z) \neq 0$, and such that $z \in Z$ if $|z - z(p,\lambda)| \leqq \varepsilon$. Let $\gamma_i(p)$ be the circle of center $(\lambda_0 - (n_i + p)^2)^{1/2}$ and of radius $\varepsilon/2$. By Cauchy formula, we have, since $f(p,\lambda,z) \neq 0$ if $|z - z(p,\lambda)| \leqq \varepsilon$:

$$\frac{M(p,\lambda,z(p,\lambda))}{f(p,\lambda,z(p,\lambda))} = \frac{1}{i\pi} \int_{\gamma_1(p)} \frac{M(p,\lambda,z_1,z_2(p,\lambda),\dots,z_N(p,\lambda))}{f(p,\lambda,z_1,z_2(p,\lambda),\dots,z_N(p,\lambda))} \frac{(\lambda-(n_1+p)^2)^{1/2}}{z_1^2-\lambda+(n_1+p)^2} dz_1 .$$

By iterating this process, we can write $(H-\lambda)^{-1}$ as follows: Let Δ be the $m+N-1$ dimensional cycle which is the image of $[-1,1]^{m-1} \times T^N$ (here T^N is the N-torus) under the map: $(p,\omega) \to (p, z(p, z(p,\lambda_0) + (\varepsilon/2)\omega) \in \mathbf{C}^{m-1} \times \mathbf{C}^N$. Then we have for λ near λ_0:

$$(H-\lambda)^{-1} = C(m,T)(i\pi)^{-N} \int_\Delta \frac{M(p,\lambda,z)}{f(p,\lambda,z)} \prod_{i=1}^{N} (\lambda-(n_i+p)^2)^{1/2} (z_i^2 - \lambda + (n_i+p)^2)^{-1} dz dp. \tag{4.2}$$

So we have written $(H-\lambda)^{-1}$ for λ near λ_0 as an integral on Δ of an operator valued function $G(p,\lambda,z)$ which is holomorphic in (p,λ,z) on the universal covering $(\mathscr{W} \times \mathscr{U} \times Z \backslash S)^*$ of $\mathscr{W} \times \mathscr{U} \times Z \backslash S$, where S is the analytic set defined by the equations:

$$S = \{(p,\lambda,z) \in \mathscr{W} \times \mathscr{U} \times Z \,|\, f(p,\lambda,z) = 0$$
$$\text{or } \lambda - (n_i+p)^2 = 0 \text{ or } z_i - \lambda + (n_i+p)^2 = 0 \text{ for } 1 \leqq i \leqq N\}.$$

It is well known that if $G(p,\lambda,z)$ is a usual holomorphic function on $(\mathscr{W} \times \mathscr{U} \times Z \backslash S)^*$ the singularities of $\int_\Delta G(p,\lambda,z) dz_1 \wedge \dots \wedge dz_N \wedge dp_1 \wedge \dots \wedge dp_{m-1}$ are much smaller than the crude guess

$$\S_{\text{sing}} = \{\lambda \,|\, \exists (p,z) \in \Delta \text{ such that } (p,\lambda,z) \in S\}.$$

(Note that this guess would produce "lines" of singularities if $m=2$ and "balls" of singularities if $m=3$.)

Indeed this integral is holomorphic as long as the integration cycle can be deformed continuously avoiding the set S. This is the basis of the analysis of Landau singularities in quantum field theory (see for example [F.F.L.P], [P], [B.P]), of ramification of holomorphic integrals (see [V], [L], [K]) and of functions of the Nilsson class (see [Me]).

The singularities of $(H-\lambda)^{-1}$ will come from the following three types of obstructions:

a) *end point singularities*: a boundary of the integration cycle is tangent to S.
b) *pinching singularities*: the cycle is pinched between some strata of S.
c) *pinching singularities at infinity*: the cycle is pinched between some stratum of S and the infinity or the boundary of the holomorphy domain of $G(p,\lambda,z)$.

A problem very similar to ours (with usual holomorphic functions) has been studied by Kobayashi [K] in relation to singularities of solutions of Cauchy problems in the complex domain. We will follow the exposition of Kobayashi. We introduce now some notation.

We put: $Y = \mathscr{W} \times \mathscr{U} \times Z \backslash S, F_i^{\pm} = \{(p,z) \in \mathscr{W} \times Z \,|\, p_i = \pm 1\}$ for $i = 1, \ldots, m-1$,

$$F = \bigcup_{+,-i} F_i^{\pm}, \tilde{F}_i^{\pm} = F_i^{\pm} \cap \pi_{p,z}(Y), \quad \tilde{F} = F \cap \pi_{p,z}(Y),$$

where $\pi_{p,z}$ is the projection $(p, \lambda, z) \mapsto (p, z)$, and finally $\tilde{S}$ the analytic set $S \cup (F \times \mathscr{U})$.

If $\pi:(p, \lambda, z) \mapsto \lambda$ is the other projection, we will denote by A_λ the fiber $\pi^{-1}(\lambda) \cap A$ for $\lambda \in \mathscr{U}$, A some set in $\mathscr{W} \times \mathscr{U} \times Z$. To describe obstruction c), we will have to replace the domain $\mathscr{W} \times Z$ of the fiber variables (p, z) by a smaller domain having a real analytic boundary. To this end, we consider the function

$$\delta(p,z) = C_0 \sum_{i=1}^{m-1} (\operatorname{Im} p_i)^2 + C_1 \left(\sum_{i=1}^{N} (\operatorname{Im} z_i + C(Z))^2 \right)^{-1} + C_2 \sum_{i=1}^{N} (\operatorname{Im} z_i)^2,$$

where the constants C_0, C_2 have to be chosen large enough and C_1 small enough.

We will replace $\mathscr{W} \times Z$ by one of the following sets:

$$\mathscr{B}_r = \{(p,z) \in \mathscr{W} \times Z \,|\, \delta(p,z) < r\} \quad \text{for} \quad 0 < r_1 < r < r_0.$$

We will now describe the singular set denoted by Σ corresponding to the obstructions a), b). (We will see that obstruction c) occurs only for the global extension problem.) The set $\tilde{S}$ is a complex analytic set and has a stratification with strata consisting of smooth submanifolds. Since the basis is one dimensional, it is well known that there exist a stratification of $(\mathscr{W} \times \mathscr{U}, \tilde{S})$ such that the rank of $d\pi$ is constant on each stratum and satisfying Thom A_π condition. (See [Hi1].) (In the sequel we will always consider such a stratification.) We are indebted to C. Sabbah for numerous discussions and indications concerning the Propositions 4.3–4.6 and Lemma 4.4.

Definition 4.2. *$\Sigma \subset \mathscr{U}$ is the projection on $\mathscr{U}$ of $\bar{M}$, for each stratum M of $\tilde{S}$ in $\mathscr{W} \times \mathscr{U} \times Z$, such that $d\pi|_{TM} = 0$.*

Σ is analogous to the Landau variety in the study of Feynman integrals. We will call the points of Σ *Landau resonances*. It is important to notice that the set Σ in $\mathscr{U}$ *depends* on the set $\mathscr{W} \times Z$ used to make the contour deformations.

Proposition 4.3. *Σ is a finite set of points.*

Proof. Since $\tilde{S}$ is an analytic set in $\mathscr{W} \times \mathscr{U} \times Z$ which extends outside $\mathscr{W} \times \mathscr{U} \times Z$, the number of strata of $\tilde{S}$ in $\bar{\mathscr{W}} \times \bar{\mathscr{U}} \times \bar{Z}$ is finite. Let M be a stratum of $\tilde{S}$ such that $d\pi|_{TM} = 0$. By the curve selection lemma (see [Mi]), if $(p_0, \lambda_0, z_0) \in \bar{M}$, there exist a real analytic curve $\rho:[0, \varepsilon_0[\to \bar{M}$ such that $\rho(0) = (p_0, \lambda_0, z_0)$ and $\rho(t) \in M$, $\forall t > 0$. Consider now the curve $\pi \circ \rho(t)$ in $\mathscr{U}$. One has $d(\pi \circ \rho(t)) \equiv 0$ since $\rho(t) \in M$, $\forall t > 0$, so $\pi \circ \rho(t) = \pi(p_0, \lambda_0, z_0) = \lambda_0$. By the connectedness of $\bar{M}$, $\pi(\bar{M}) = \lambda_0$, which proves the Proposition. □

Let us now describe the set Σ_∞ corresponding to obstruction c).

If M is a stratum of $\tilde{S}$ in $\mathscr{W} \times \mathscr{U} \times Z, M_\lambda$ is a union of smooth submanifolds for $\lambda \in \mathscr{U}$. This is obvious if $d\pi|_{TM} \neq 0$, and follows from the proof of Proposition 4.3, and Thom A_π condition if $d\pi|_{TM} = 0$. For $\lambda \in \mathscr{U}$, we denote by $D(\lambda)$ the set of $r \in [r_1, r_0]$ such that for some stratum M of $\tilde{S}, M_\lambda$ is tangent to ∂B_r.

We have the following lemma:

Lemma 4.4.
i) *$D(\lambda)$ is a finite subset of* $[r_1, r_0]$.
ii) $K = \bigcup_{\lambda \in \mathscr{U}} \{\lambda\} \times D(\lambda)$ *is a closed set.*

Proof.
i) can be proved as Proposition 4.4, using the critical variety cM_λ of M_λ with respect to the real analytic map δ. $\overline{c\mathscr{M}_\lambda}$ is a real analytic set and we can apply the real analytic version of the curve selection lemma to each connected component of $\overline{c\mathscr{M}_\lambda}$ to prove that $D(\lambda)$ is a finite set.

Let us now prove ii).

Let $\lambda_0 \in \mathscr{U}, r_0 \in [r_1, r_0]$, and assume that there exist a sequence $(\lambda_n, r_n) \to (\lambda_0, r_0)$, $\lambda_n \in \mathscr{U}, r_n \in D(\lambda_n)$. Then we can find a stratum M of $\tilde{S}$, a sequence $(p_n, z_n) \in M_{\lambda_n}$, such that $T_{(p_n, z_n)} M_{\lambda_n} \subset T_{(p_n, z_n)} \partial B_{r_n}$. By compactness we can assume that (p_n, z_n) tends to some $(p_0, z_0) \in \partial B_{r_0}$, such that $(p_0, \lambda_0, z_0) \in \bar{M}$. If $(p_0, \lambda_0, z_0) \in M$, $T_{(p_0, z_0)} M_{\lambda_0} \subset T_{(p_0, z_0)} \partial B_{r_0}$, and $r_0 \in D(\lambda_0)$.

If $(p_0, \lambda_0, z_0) \in N$, for some stratum N of $\tilde{S}$ adjacent to M, then using Thom A_π condition, $T_{(p_0, z_0)} N_{\lambda_0} \subset T_{(p_0, z_0)} \partial B_{r_0}$, and $r_0 \in D(\lambda_0)$, which proves that K is closed. □

From now on, we replace $\mathscr{W} \times Z$ by B_{r_0}, and consider ∂B_{r_0} as the boundary of the holomorphy domain of $G(p, \lambda, z)$ in the variables (p, z).

Definition 4.5. *Σ_∞ is the set of $\lambda \in \mathscr{U}$ such that $r_0 \in D(\lambda)$*

Proposition 4.6. *It is possible to choose r_0 such that Σ_∞ is disjoint from Σ and is included in a closed subanalytic set of measure zero.*

We refer to [Hi1] for the definition of subanalytic sets.

Proof. We first check that it is possible to choose r_0 such that Σ_∞ is disjoint from Σ. From Proposition 4.5 ii), we can find some subinterval $[r'_1, r'_0]$ of $[r_1, r_0]$, and a small neighborhood $\mathscr{V}$ of Σ in $\mathscr{U}$, such that $[r'_1, r'_0] \cap D(\lambda) = \emptyset$, $\forall \lambda \in \mathscr{V}$.

In the sequel the new r_0 will be chosen in $[r'_1, r'_0]$. We first take a semianalytic stratification (M) of the pair $(\mathscr{U} \times \bar{B}_{r_0}, \tilde{S})$ (see [Hi1]) such that the strata of (M) in $\mathscr{U} \times B_{r_0}$ are the strata of the stratification of $(\mathscr{U} \times B_{r_0}, \tilde{S})$. Let us denote by $\hat{\Sigma}$ the union of the sets of critical values of the mappings $(p, \lambda, z) \in M \to (\lambda, \delta(p, z)) \in \mathscr{U} \times [r_1, r_0]$, for all strata M of (M). It is easy to see that if $r \in D(\lambda), (\lambda, r) \in \hat{\Sigma}$. Moreover, $\hat{\Sigma}$ is a subanalytic set as a proper image of a semianalytic set in $\mathscr{U} \times \bar{B}_{r_0}$, and is of measure zero by Sard's theorem. By Fubini's theorem, we can find some $r_3 \in [r'_1, r'_0]$ such that the fiber $\hat{\Sigma}_{r_3}$ in $\mathscr{U}$ of $\hat{\Sigma}$ over r_3 is of measure zero. This fiber is again a subanalytic set and contains the set Σ_∞ if we take $r_0 = r_3$. This proves the Proposition. □

We have the following theorem, which is the main result of this section:

Theorem 4.7. *Assume that V belongs to $L^k_\alpha(C_T)$, for $k > m$, $\alpha > 0$.*
Then the following results hold:

i) *Local Extension Problem: for any $\lambda_0 \in \mathscr{U} \cap \mathbf{R}$, there exist a neighborhood $\mathscr{U}_{\lambda_0}$ of*

λ_0 in $\mathcal{U}$, such that $(H-\lambda)^{-1}$ extends holomorphically from $\{\operatorname{Im}\lambda>0\}\cap\mathcal{U}_{\lambda_0}$ to the universal covering $(\mathcal{U}_{\lambda_0}\backslash\Sigma)^*$, as a bounded operator from $L^2_a(\mathbf{R}^m)$ into $H^1_{-a}(\mathbf{R}^m)$ for $a>\alpha/2$.

ii) *Global Extension Problem: $(H-\lambda)^{-1}$ extends holomorphically from $\{\operatorname{Im}\lambda>0\}\cap\mathcal{U}$ to the universal covering $(\mathcal{U}\backslash\Sigma\cup\Sigma_\infty)^*$, as a bounded operator from $L^2_a(\mathbf{R}^m)$ into $H^1_{-a}(\mathbf{R}^m)$ for $a>\alpha/2$.*

Proof. Since $\alpha>0$, we can find $\mathcal{W}$ and Z such that $\alpha>2(C(Z)+\sup|\operatorname{Im}p|_{p\in\mathcal{W}})$, so that Proposition 4.1 applies to $\mathcal{W}\times\mathcal{U}\times Z$. We will follow the proof of Kobayashi [K]. Let us first prove ii). Let $\Delta_i^\pm=\{(p,z)\in\Delta\,|\,p_i=\pm1\}$ be the components of the boundary of Δ, and let us denote by ω the operator valued form $G(p,\lambda,z)dp_1\wedge\cdots\wedge dp_{m-1}\wedge dz_1\wedge\cdots\wedge dz_N$.

Since ω is holomorphic of maximal degree, we have: $d\omega|_{Y_\lambda}=0$, $\omega|_{\tilde F_\lambda}=0$. This implies that if γ is some chain in Y_λ, $\int_\gamma\omega$ depends only on the homology class $[\gamma]$ of γ in the relative homology group $H_{m-1}(Y_\lambda,\tilde F_\lambda)$.

We consider now the problem of extending $(H-\lambda)^{-1}=\int_\Delta\omega$ from $\operatorname{Im}\lambda>0$, to some point λ_1 along a path $\ell:[0,1]\to\mathcal{U}$, with $\operatorname{Im}\ell(0)>0$, $\ell(1)=\lambda_1$. Applying the result of Kobayashi ([K] Theorem 1.3), we get that $(H-\lambda)^{-1}$ can be extended holomorphically along ℓ if there exist a deformation $\gamma_t:(\Delta,\Delta_i^\pm)\to(Y_{\ell(t)},\tilde F^\pm_{i,\ell(t)})$ of Δ such that $\gamma_0=\mathrm{Id}$ and $\gamma:(t,s)\in[0,1]\times\Delta\to\gamma_t(s)$ is continuous. This reduces the problem of holomorphic continuation to a problem of finding a continuous deformation of the relative cycle Δ in $H_{m-1}(Y_\lambda,\tilde F_\lambda)$ along the path ℓ.

Let now $\ell:[0,1]\to\mathcal{U}\backslash\Sigma\cup\Sigma_\infty$ be a path with $\ell(0)=\lambda_0$, $\operatorname{Im}\lambda_0>0$. Let J be the set of $t\in[0,1]$ such that Δ can be deformed along ℓ between 0 and t satisfying the above conditions. J is obviously open and since $[0,1]$ is connected it suffices to prove that J is closed to prove that $I=J$.

Let $t_n\in J$ a sequence with $t_n\to t_0$ when $n\to+\infty$. $\lambda_1=\ell(t_0)$ belongs to $\mathcal{U}\backslash\Sigma\cup\Sigma_\infty$, and if V is a small neighborhood of λ_1, M_λ is transversal to ∂B_{r_0} for all $\lambda\in V$, all strata M of $\tilde S$, if r_0 is defined in Proposition 4.6. Then we can apply to $\bar B_{r_0}\times V$ the local isotropy lemma of [F.K.]. This implies that $\pi:\bar B_{r_0}\times V\to V$ is a locally trivial fibration with respect to $S\cap(\bar B_{r_0}\times V)$, $F_i^\pm\cap(\bar B_{r_0}\times V)$.

This implies that one can find a deformation of $\gamma_{\ell(t_n)}$ along the part of ℓ which stays in V, such that $\gamma_{\ell(t)}\subset\bar B_r\backslash S_{\ell(t)}$ and the faces of $\gamma_{\ell(t)}$ are in the submanifolds $F^\pm_{i,\ell(t)}$. This implies that $I=J$, hence that $(H-\lambda)^{-1}$ can be extended along any path in $\mathcal{U}\backslash\Sigma\cup\Sigma_\infty$.

Then it follows from the monodromy Theorem that $(H-\lambda)^{-1}$ extends as a function on $(\mathcal{U}\backslash\Sigma\cup\Sigma_\infty)^*$, which proves ii). Let us now prove i). From Lemma 4.5, we can find some $r_0>0$, some neighborhood $\mathcal{U}_{\lambda_0}$ of λ_0 in $\mathcal{U}$, such that $r_0\notin D(\lambda)$, $\forall\lambda\in\mathcal{U}_{\lambda_0}$. Then the result follows by applying the arguments above to $\bar B_{r_0}\times\mathcal{U}_{\lambda_0}$. This proves the Theorem. □

We will now study the behavior of $(H-\lambda)^{-1}$ near a Landau resonance.

We first recall some definitions.

Definition 4.8. *An operator-valued function $M(\lambda)$ is of finite determination near some point λ_0 if there exist a neighborhood $\mathcal{V}$ of λ_0 such that the branches of $M(\lambda)$ over any simply connected subset of $\mathcal{V}\backslash\{\lambda_0\}$ span a vector space of finite dimension in $\mathcal{L}(L^2_a(\mathbf{R}^m),H^1_{-a}(\mathbf{R}^m))$.*

$M(\lambda)$ has a moderate growth at a point $\lambda_0 \in \Sigma$, if there exist a neighborhood V of λ_0 in $\mathscr{U}$ such that for any simply connected subset $\tilde{V}$ of $V \backslash \{\lambda_0\}$, for any branch of $M(\lambda)$ on $\tilde{V}$, denoted by $\tilde{M}(\lambda)$, there exist $C_0, N_0 > 0$ such that:

$$\|\tilde{M}(\lambda)\| \leqq C_0 |\lambda - \lambda_0|^{-N_0}.$$

Here $\|\tilde{M}(\lambda)\|$ is the operator norm in $\mathscr{L}(L^2_a(\mathbf{R}^m), H^1_{-a}(\mathbf{R}^m))$.

—an analytic set $\tilde{S}$ in $\mathscr{W} \times \mathscr{U} \times Z$ is a *divisor with normal crossings* if near any point $(p_0, \lambda_0, z_0) \in \tilde{S}$, there exist holomorphic coordinates $(z_1, \dots, z_{m+N})$ such that $\tilde{S}$ is given by the equation $z_1 \dots z_k = 0$ near (p_0, λ_0, z_0).

To study the growth of $(H-\lambda)^{-1}$ near a point of Σ, we need a geometric hypothesis on $\tilde{S}$.

We first add to $\tilde{S}$ the fiber $\pi^{-1}(\lambda_0)$ for each $\lambda_0 \in \Sigma$, which does not change the set Σ.

Hironaka desingularisation theorem says that there exist an analytic space X and a proper morphism $\beta: X \to \mathscr{W} \times \mathscr{U} \times Z$ such that:

—$\beta: X \backslash \beta^{-1}(\tilde{S}) \to \mathscr{W} \times \mathscr{U} \times Z \backslash \tilde{S}$ is an isomorphism.

—$\tilde{S}' = \beta^{-1}(\tilde{S})$ is a divisor with normal crossings. (See [Hi2]).

We make the same hypothesis as in Mercier [Me]:

(T) There exist a stratification $(\tilde{M})$ of the pair $(X, \tilde{S}')$ such that for each $\tilde{M}$ there exist a stratum M of $(\mathscr{W} \times \mathscr{U} \times Z, \tilde{S})$ such that $\beta|_{\tilde{M}}: \tilde{M} \to M$ is a submersion.

The hypothesis is made to ensure that the strata of $\tilde{S}'$ intersecting $\beta^{-1}(\partial B_{r_0} \times V)$ intersect it transversally.

We have the following theorem:

Theorem 4.9.

i) *$(H-\lambda)^{-1}$ is of finite determination near any point of Σ.*

ii) *if condition (T) holds, $(H-\lambda)^{-1}$ has a moderate growth near any point of Σ.*

Proof. Since the properties of finite determination and moderate growth are local, we can consider $(H-\lambda)^{-1}$ near a point $\lambda_1 \in \Sigma$. Let us consider a branch of $(H-\lambda)^{-1}$ near λ_1, obtained by analytic continuation along a homotopy class $[\gamma]$ in $\mathscr{U} \backslash \Sigma \cup \Sigma_\infty$. The continuation of $(H-\lambda)^{-1}$ in $(V | \{\lambda_1\})^*$, where V is a small neighborhood of λ_1, is obtained by deforming the integration cycle Δ in (4.2) inside the set B_{r_0}. We claim that it is possible to choose r_0 in Proposition 4.6 such that $r_0 \notin D(\lambda)$, $\forall \lambda \in V$ and the following condition holds:

(T') any stratum of $\tilde{S}$ intersecting $\partial B_{r_0} \times \mathscr{U}$ intersect it transversally. Indeed it suffices to remark as in Lemma 4.5 that the set of critical values of δ on M is finite, for all strata M of $\tilde{S}$.

Let λ_0 a point in $V \backslash \{\lambda_1\}$. We will adapt the arguments of Mercier [Me] to our problem. Let us assume that one branch of $(H-\lambda)^{-1}$ at λ_0 is obtained by integrating in (4.2) on some relative cycle γ_0.

We introduce the locally finite family of analytic sets given by $S_{\lambda_0}, F^{\pm}_{i,\lambda_0}$ and ∂B_{r_0}. By Lojasiewicz Theorem (see [Me]), we can find a semianalytic triangulation of $\bar{B}_{r_0}$ which is finite and compatible with this family. This induces a triangulation of the pair $(\bar{B}_{r_0} \backslash S_{\lambda_0}, \tilde{F}_{\lambda_0})$ by (K, L), where K is the simplicial complex made of the simplexes of the previous triangulation which do not intersect S_{λ_0} and L is the simplicial complex made of the simplexes of the previous triangulation which intersect $\tilde{F}_{\lambda_0}$ and not S_{λ_0}.

Then $H_{m-1}(\bar{B}_{r_0}\backslash S_{\lambda_0}, \tilde{F}_{\lambda_0})$ is isomorphic to $H_{m-1}(|K|,|L|)$ and we can write $[\gamma_0]$ using the simplexes of the triangulation of $(|K|,|L|)$: $[\gamma_0] = \sum_{j\in J_0} b_j\sigma_j$, $b_j\in\mathbf{Z}$, J_0 a finite set.

For λ near λ_0, we can also take $[\gamma_\lambda] = \sum_{j\in J_0} b_j\sigma_j$.

Then $(H-\lambda)^{-1} = \sum_{j\in J_0} b_j \int_{\sigma_j} \omega$.

It is clear that $\omega = G(p,\lambda,z)dp_1 \wedge \cdots \wedge dp_{m-1} \wedge dz_1 \wedge \cdots \wedge dz_N$ is of finite determination on any of the simply connected subsets σ_j, since the multivaluedness of ω comes only from the $(\lambda-(n_i+p)^2)^{1/2}$, which are of finite determination.

So $(H-\lambda)^{-1}$ is a finite sum of the functions $\int_{\sigma_j} \omega$, each of finite determination, which proves i).

We prove now ii): we consider a point $\lambda_0\in\Sigma$.

Let us denote by $\beta: X \to \mathscr{W} \times \mathscr{U} \times Z$ the desingularisation of $\tilde{S}$ in $\mathscr{W}\times\mathscr{U}\times Z$. We can write $(H-\lambda)^{-1} = \int_{\gamma_\lambda}\omega = \int_{\beta_*\gamma'_\lambda}\omega = \int_{\gamma'_\lambda}\beta^*\omega$, where $\gamma'_\lambda = (\beta^{-1})_*\gamma_\lambda$. (Here γ'_λ exist because β is an isomorphism outside $\tilde{S}$.) Using condition (T), we see that $\beta^{-1}(\partial B_{r_0}\times\mathscr{U})$ is transversal to all strata of $\tilde{S}'$. So denoting again $\tilde{S}'$ by $\tilde{S}$, we are reduced to the case where $\pi^{-1}(\lambda_0)\subset\tilde{S}$, $\tilde{S}$ is a divisor, and $\partial B_{r_0}\times\mathscr{U}$ is transversal to all strata of $\tilde{S}$.

We fix a small neighborhood V of λ_0, in which we will estimate the growth of $(H-\lambda)^{-1}$.

We can now finish the proof as in [Me].

We will only indicate the principal steps of the proof. Modulo a change of coordinates, we can assume that $\lambda_0 = 0$. The idea of the proof is to lift the radial vector field on $\mathbf{C}$: $\zeta = -(\lambda(\partial/\partial\lambda) + \bar{\lambda}(\partial/\partial\bar{\lambda})) = -r(\partial/\partial r)$, where $r = |\lambda|$, to a vector field ξ in $\mathscr{W}\times V$ which is tangent to all strata of $\tilde{S}$ and to $\partial B_{r_0}\times V$. This vector field is then used to construct the deformations of γ_λ as λ tends to 0 along an integral curve of ζ, which is of the form: $\alpha_s(\lambda_1) = \lambda_1 e^{-s}$ for $s\in\mathbf{R}^+$.

Step 1: Construction of ξ.

We will construct ξ locally near any point of $B_{r_0}\times V$ and patch together the local vector fields with a partition of unity.

1. *near* $(p_0,\lambda_0,z_0)\in\bar{B}_{r_0}\times V\backslash(\tilde{S}\cup\partial B_{r_0}\times V)$:

$$\text{we take } \xi = \left(\lambda\frac{\partial}{\partial\lambda} + \bar{\lambda}\frac{\partial}{\partial\bar{\lambda}}\right).$$

2. *near* $(p_0,\lambda_0,z_0)\in\bar{B}_{r_0}\times V\cap(\tilde{S}\backslash\partial B_{r_0}\times V)$:

Since $\tilde{S}$ is a divisor, we can take a coordinate chart near (p_0,λ_0,z_0), $(z_1,\ldots,z_{m+N})$ such that $\tilde{S}$ is given near (p_0,λ_0,z_0) by the equation $z_1\ldots z_p = 0$, and $(p_0,\lambda_0,z_0) = (0,\ldots,0)$. Since $\pi^{-1}(0)\subset\tilde{S}$, by the Nullstellensatz (see [Me] p. 82) we see that $\pi(z) = r_0(z)z_1^{\alpha_1}\cdots z_k^{\alpha_k}$, $\alpha_1,\ldots,\alpha_k\in\mathbf{N}$, $r_0(0)\neq 0$, $k\leqq p$. Changing for example z_1, we can assume that $\pi(z) = z_1^{\alpha_1}\cdots z_k^{\alpha_k}$. We can take

$$\xi = -\frac{1}{\alpha_1}\left(z_1\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial z_1}\right),$$

and we have $\pi_*\xi = \zeta$.

3. *near* $(p_0, \lambda_0, z_0) \in \bar{B}_{r_0} \times V \cap (\tilde{S} \cap \partial B_{r_0} \times V)$:

As before we can find a coordinate chart such that $\tilde{S}$ has the equation $z_1 \cdots z_p = 0$, and $\pi(z) = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$.

Since ∂B_{r_0} is transversal to each stratum of $\tilde{S}$, we can extend the set of local coordinates $(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_p, \operatorname{Im} z_p)$ by $u_{p+1}, v_{p+1}, \ldots, u_{m+N}, v_{m+N}$ such that $u_{p+1} = 0$ is a C^∞ equation of ∂B_r near (p_0, λ_0, z_0). We take

$$\xi = -\frac{1}{\alpha_1}\left(z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}\right).$$

ξ is tangent to S and to ∂B_{r_0}, and $\pi_* \xi = \zeta$.

4. *near* $(p_0, \lambda_0, z_0) \in \bar{B}_{r_0} \times V \cap (\partial B_{r_0} \times U \backslash \tilde{S})$:

We can take

$$\xi = -\left(\lambda \frac{\partial}{\partial \lambda} + \bar{\lambda} \frac{\partial}{\bar{\partial}}\right).$$

ξ is tangent to ∂B_{r_0}. We now patch together ξ with a C^∞ partition of unity in $\bar{B}_{r_0+\varepsilon} \times V_\varepsilon$, where V_ε is a small neighborhood of V. We obtain a vector field supported in $\bar{B}_{r_0+\varepsilon} \times V_\varepsilon$.

Step 2: Estimates of $(H-\lambda)^{-1}$.

We want now to control the growth of $(H-\lambda)^{-1} = \int_{\gamma_\lambda} \omega$, when λ tends to 0 along a ray $\alpha_s(\lambda_1) = \lambda_1 e^{-s}$.

Here for $\lambda \in V \backslash \{\lambda_0\}, \gamma_\lambda$ is a deformation of the cycle γ_{λ_1} which stays in the ball $\bar{B}_{r_0}$. The integral curves of the vector field ξ induce a 1-parameter family of diffeomorphisms $j_s = \exp(s\xi)$, from $(\bar{B}_{r_0} \backslash \tilde{S}_{\lambda_1}) \times \{\lambda_1\}$ into $(\bar{B}_{r_0} \backslash \tilde{S}_{\alpha_s(\lambda_1)}) \times \{\alpha_s(\lambda_1)\}$. We use here the fact that ξ is tangent to ∂B_{r_0} and $\tilde{S}$, and that $\pi_* \xi$ is the radial vector field pointing inwards. Moreover j_s is a homeomorphism from the pair

$$(\bar{B}_{r_0} \backslash S_{\lambda_1}, \tilde{F}_{\lambda_1}) \quad \text{to the pair} \quad (\bar{B}_{r_0} \backslash \tilde{S}_{\alpha_s(\lambda_1)}, \tilde{F}_{\alpha_s(\lambda_1)}).$$

Since $H_{m-1}(\bar{B}_{r_0} \backslash S_\lambda, \tilde{F}_\lambda)$ is a locally constant sheaf over V, (using for example the triangulation in the proof of i)), we see that $[\gamma_{\alpha_s(\lambda_1)}] = (j_s)_* [\gamma_{\lambda_1}]$.

So we can write:

$$(H - \alpha_s(\lambda_1))^{-1} = \int_{(js)_* \gamma_{\lambda_1}} \omega = \int_{\gamma_{\lambda_1}} j_s^* \omega.$$

To prove ii) it remains to show that $\| j_s^* \omega \| \leqq C e^{cs}$, where $\| \quad \|$ is the norm induced by the Riemannian structure on $\mathscr{W} \times Z$, uniformly on $\bar{B}_{r_0}$ and when λ_1 is on some arc of circle, $\{\theta_0 \leqq \operatorname{Arg} \lambda_1 \leqq \theta_1, |\lambda_1| = \varepsilon_0\}$. This can be done as in the proof of [Me] Theorem 3.1, with the modifications of [Me] Theorem 3.2. □

Behavior of the Resolvent Near a Landau Resonance in Generic Cases. We will now study more precisely the behavior of $(H-\lambda)^{-1}$ near some points of Σ. We consider the case of a Landau resonance $\lambda_0 \in \Sigma$ generated by a pinch of the integration cycle at a point $(p_0, z_0) \in \bar{B}_{r_0}$. For simplicity, we will only consider Landau resonances created by polar singularities, i.e. by strata of $\tilde{S}$ away from $\lambda = (n_i + p)^2 i = 1, \ldots, N$. However, monodromy formulas for $(H-\lambda)^{-1}$ near a Landau resonance created by ramified singularities can be obtained using the results of Pham ([P] Chapter VII).

We make the following hypotheses:

(4.5) there exist neighborhoods V of λ_0, $\tilde{W}$ of (p_0, z_0) such that if $\lambda \in V$, $(p, z) \in \bar{B}_{r_0} \setminus \tilde{W}$ then $(p, \lambda, z) \notin S$, and (p_0, λ_0, z_0) is the only critical point of π in $\tilde{W} \times V$.

(4.6) $$S_{\lambda_0} \cap F_{\lambda_0} = \varnothing.$$

(4.7) near (p_0, λ_0, z_0) S is the union of the complex hypersurfaces $S_1, \ldots, S_k$ intersecting in general position at (p_0, λ_0, z_0). This means that near (p_0, λ_0, z_0) S_i has a irreducible equation $s_i(p, \lambda, z) = 0$ with $s_i(p_0, \lambda_0, z_0) = 0$, $ds_1, \ldots, ds_k$ linearly independent at (p_0, λ_0, z_0).

(4.8) (p_0, λ_0, z_0) is a non-degenerate critical point of π for the stratum $A = \bigcap_{i=1}^{k} S_i$ and is not a critical point of any other stratum of S.

(4.9) S_i are polar manifolds of

$$G(p, \lambda, z) = \frac{M(p, \lambda, z)}{f(p, \lambda, z)} \prod_1^N (\lambda - (n_i + p)^2)^{1/2} (z_1^2 - \lambda + (n_i + p)^2)^{-1}.$$

We note that it is possible to choose $M(p, \lambda, z)$ and $f(p, \lambda, z)$ near (p_0, λ_0, z_0) such that $M(p_0, \lambda_0, z_0)$ is a finite rank operator. This follows easily from the holomorphic Fredholm theorem.

We consider the behavior of one branch of $(H - \lambda)^{-1}$, denoted by $(H - \lambda)_\gamma^{-1}$ obtained by integrating $G(p, \lambda, z)$ on a cycle γ, for λ near $\lambda_0, \lambda \neq \lambda_0$.

Using (4.5), we can write $\gamma = \tilde{\gamma} + \sigma$, where σ is an absolute cycle in $\tilde{W}$ and $\tilde{\gamma}$ avoids S_{λ_0}.

So $\int_{\tilde{\gamma}} G(p, \lambda, z) dp_1 \wedge \cdots \wedge dp_{m-1} \wedge dz_1 \cdots \wedge dz_N$ is holomorphic for λ near λ_0, and modulo a holomorphic term, we can assume that $\gamma = \sigma$ is an absolute cycle. We will denote by N the intersection index of σ with the vanishing cell e defined by the manifolds $S_1, \ldots, S_k$. (See [P] Chap. V for the definitions of N and e.) Practically N is zero when the path σ is not pinched between the $S_1, \ldots, S_k$.

An interesting question is whether the set of potentials in $L_\alpha^\infty(C_T)$ such that S has the above structure near any critical point in $W \times \mathscr{U} \times Z$ is an open dense set of $L_\alpha^\infty(C_T)$.

Under the hypotheses above, we can write near (p_0, λ_0, z_0), $G(p, \lambda, z)$ as:

$$\frac{\tilde{M}(p, \lambda, z)}{s_1^{\alpha_1}(p, \lambda, z) \cdots s_k^{\alpha_k}(p, \lambda, z)} \quad \text{for} \quad \alpha_1, \ldots, \alpha_k \in \mathbb{N}, \tilde{M} \text{ as } M.$$

The case when some α_i are bigger than 1 means that there is a constant degeneracy in the singularities of $(1 + VK_0(p, \lambda, z))^{-1}$. This happens for example if the crystalline material represented by the potential V has a symmetry group. We put then $\alpha = \alpha_1 + \cdots + \alpha_k$.

We have the following theorem:

Theorem 4.10. *Under the hypotheses (4.i) $i = 5, \ldots, 9$, the branch $(H - \lambda)_\gamma^{-1}$ can be written for λ near λ_0 as:*

i) *if $m + N + k$ is odd:*

$$(H - \lambda)_\gamma^{-1} = E_0(\lambda) + C_0 N (\lambda - \lambda_0)^{(m+N+k)/2 - 1 - \alpha} (M(p_0, \lambda_0, z_0) + (\lambda - \lambda_0) E_1(\lambda)),$$

ii) *if* $m+N+k$ *is even,* $m+N+k \geqq 2\alpha+2$:

$$(H-\lambda)_\gamma^{-1} = E_0(\lambda) + C_0 N(\lambda-\lambda_0)^{(m+N+k)/2-1-\alpha} \mathrm{Log}(\lambda-\lambda_0)(M(p_0,\lambda_0,z_0) + (\lambda-\lambda_0)E_1(\lambda)),$$

iii) *if* $m+N+k$ *is even,* $m+N+k < 2\alpha+2$:

$$(H-\lambda)_\gamma^{-1} = E_0(\lambda) + C_0 N(\lambda-\lambda_0)^{(m+N+k)/2-1-\alpha}(M(p_0,\lambda_0,z_0) + (\lambda-\lambda_0)E_1(\lambda)) + N\,\mathrm{Log}(\lambda-\lambda_0) \times E_2(\lambda),$$

iv) *if* $m+N=k$:

$$(H-\lambda)_\gamma^{-1} = E_0(\lambda) + C_0 N(\lambda-\lambda_0)^{m+N-\alpha-1}(M(p_0,\lambda_0,z_0) + (\lambda-\lambda_0)E_1(\lambda)),$$

where: E_0, E_1, E_2 *are holomorphic functions in* $\mathscr{L}(L_a^2(\mathbf{R}^m), H_{-a}^1(\mathbf{R}^m))$, *and* C_0 *is a non-vanishing constant.*

Proof. We use the results of Leray [L] and Pham [P] as stated in the book of Pham (see [P] Chap. VI).

We first reduce ourselves to a case when $\tilde{M}(p,\lambda,z)$ is independent of λ. To do this, we use the fact (see [P] Sect. V.2) that under the hypotheses (4.7), (4.8), there exist a neighborhood of (p_0,λ_0,z_0) still denoted by $\tilde{W} \times V$ and a holomorphic change of coordinates defined on $\tilde{W} \times V: (p,\lambda,z) \to (\tilde{p}(p,z,\lambda),\lambda)$ such that in the new coordinates $(\tilde{p},\lambda)$ the functions $s_1,\dots,s_k$ take the simple form:

$$\begin{aligned} s_1 &= \lambda - (\tilde{p}_1 + \dots + \tilde{p}_{k-1} + \tilde{p}_k^2 + \dots + \tilde{p}_{m+N-1}^2) = \lambda - f(\tilde{p}) \\ s_2 &= \tilde{p}_1 \\ &\vdots \\ s_k &= \tilde{p}_{k-1}. \end{aligned}$$

For simplicity, we will still denote by (p,λ,z) the new coordinates. We can also assume that the cycle σ is contained in $\tilde{W}$, by subtracting to $(H-\lambda)^{-1}$ some operator holomorphic in λ. This can be done as in the proof of VI.2.1. in [P].

Using Taylor's formula, we can write:

$$\begin{aligned} \frac{\tilde{M}(p,\lambda,z)}{s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_k^{\alpha_k}} &= \sum_{0 \leqq \beta \leqq \alpha_1 - 1} \frac{1}{\beta!} \partial_\lambda^\beta \tilde{M}(p,\lambda,z)|_{\lambda=f(p,z)} s_1^{-\alpha_1+\beta} s_2^{-\alpha_2} \cdots s_k^{-\alpha_k} \\ &\quad + \tilde{R}(p,\lambda,z) s_2^{-\alpha_2} \cdots s_k^{-\alpha_k} = \sum_{0 \leqq \beta \leqq \alpha_1 - 1} \tilde{M}_\beta(p,z) s_1^{-\alpha_1+\beta} s_2^{-\alpha_2} \cdots s_k^{-\alpha_k} \\ &\quad + \tilde{R}(p,\lambda,z) s_2^{-\alpha_2} \cdots s_k^{-\alpha_k}. \end{aligned} \tag{4.10}$$

From (4.8), it follows that (p_0,λ_0,z_0) is not a critical point of π for the stratum $S_2 \cap \dots \cap S_k$, hence $\int_\sigma \tilde{R}(p,\lambda,z) s_2^{-\alpha_2} \cdots s_k^{-\alpha_k} dp_1 \wedge \dots \wedge dp_{m-1} \wedge dz_1 \wedge \dots \wedge dz_N$ is holomorphic near λ_0.

So we are reduced to the study of $\int_\sigma \tilde{M}_\beta(p,z) s_1^{-\alpha_1+\beta} s_2^{-\alpha_2} \cdots s_k^{-\alpha_k} dp_1 \wedge \dots \wedge dp_{m-1} \wedge dz_1 \wedge \dots \wedge dz_N$ for $\beta \leqq \alpha_1 - 1$.

Then the theorem follows directly by applying VI.2.1 in [P] to each of the terms in (4.10). In (4.10) only the first term corresponding to $\tilde{M}(p, f(p,z)z)$ contributes to the leading singularity at $\lambda = \lambda_0$. The only thing that we have to

check is that some constant appearing in the formulas VI.2.1. of [P] is non-zero:
More precisely we can write:

$$d\lambda = \sum_{i=1}^{k} a_i ds_i \quad \text{at} \quad (p_0, \lambda_0, z_0).$$

Let us check that $a_i \neq 0$ for $i = 1, \ldots, k$. If for example $a_1 = 0$, then (p_0, λ_0, z_0) would be an element of the critical manifold of the stratum $S_2 \cap \cdots \cap S_k$, which is excluded by (4.8). This concludes the proof of the theorem. □

Remark 4.11. The interest of Theorem 4.10 is that $M(p_0, \lambda_0, z_0)$ is always a finite rank operator, but $M(p_0, \lambda_0, z_0)$ can be equal to zero in some cases. When $M(p_0, \lambda_0, z_0) \neq 0$, Theorem 4.10 shows that the leading singularity of $(H - \lambda)^{-1}$ at a Landau resonance λ_0 is of finite rank. Note however that $E_1(\lambda), E_2(\lambda)$ are in general not of finite rank.

We will now apply Theorem 4.10 in some simple cases where it is possible to compute explicitly $M(p_0, \lambda_0, z_0)$.

Landau Resonance Generated by a Simple Resonance Curve. We assume that $\{s_1(p, \lambda, z) = 0\}$ is a component of $\{f(p, \lambda, z) = 0\}$ and that (modulo a change of indices) $s_i(p, \lambda, z) = z_i^2 - \lambda + (n_i + p)^2$ for $i = 2, \ldots, k$. We denote by z'' the rest of the z variables. Since we have assumed that (z_0, p_0, λ_0) is away from the hypersurfaces $\lambda = (n_i + p)^2$, we can find determinations of the functions $(\lambda - (n_i + p)^2)^{1/2}$ near (λ_0, p_0) such that $z_{i0} = (\lambda_0 - (n_i + p_0)^2)^{1/2}$, for $i = 2, \ldots, k$. Then it is easy to check that (p_0, λ_0, z_0) is a critical point of π for A if and only if (p_0, λ_0, z_0'') is a critical point of π for the hypersurface given by $s_1(p, \lambda, (\lambda - (n_2 - p)^2)^{1/2}, \ldots, (\lambda - (n_k + p)^2)^{1/2}, z'') = 0$. If we denote by $s_1(p, \lambda, z'')$ this new function, we have $(\partial s_i/\partial \lambda)(p_0, \lambda_0, z_0'') \neq 0$, i.e. $\lambda = \lambda_0$ is a simple root of $s_1(p_0, \lambda, z_0'')$.

For a function $u \in L^2_{\mathrm{loc}}(\mathbf{R}^m)$, which is p_0-Floquet periodic (i.e. $e^{i\langle p_0, x\rangle} u$ is T-periodic), we denote by $R_0(\lambda)$ the following operator:

$$R_0(\lambda)u = \sum_{n_j \in J} (\lambda - z_{jo}^2 - (n_j + p_0)^2) e^{-i\langle n_j + p_0, x\rangle} u_{n_j}(y),$$

where $u_{n_j}(y)$ is the Fourier coefficient of order n_j (in the x variable) of $e^{i\langle p_0, x\rangle} u$. (In particular if $k = N + 1, R_0(\lambda_0) = 0$.) We will assume for simplicity that $m \geqq 3$. Then we have the following corollary:

Corollary 4.12. *Under the hypotheses above, $M(p_0, \lambda_0, z_0)$ is a finite rank operator π_0: satisfying: $(H - \lambda_0 + R_0(\lambda_0))\pi_0 = \pi_0(H - \lambda_0 + R_0(\lambda_0)) = 0$.*

Proof. We can write for λ near λ_0: $\tilde{D}(p_0, \lambda, z_0)/f(p_0, \lambda, z_0) = \tilde{E}_0(\lambda) + \tilde{\pi}_0/\lambda - \lambda_0$, where $\tilde{E}_0(\lambda)$ is a holomorphic operator.

In the Appendix Proposition A.1, we prove that: $(H_{p_0} - \lambda_0 + \tilde{R}_0(\lambda_0))\tilde{\pi}_0 = \tilde{\pi}_0(H_{p_0} - \lambda_0 + \tilde{R}_0(\lambda_0)) = 0$, where $\tilde{R}_0(\lambda) = e^{i\langle p_0, x\rangle} R_0(\lambda) e^{-i\langle p_0, x\rangle}$.

We have $G(p_0, \lambda, z_0) = E_0(\lambda) + \pi_0/\lambda - \lambda_0$, where $E_0(\lambda)$ is holomorphic and $\pi_0 u = c(m, T) e^{-i\langle p_0, x\rangle} \tilde{\pi}$ for $u \in L^2_a(\mathbf{R}^m)$. From this we get easily that π_0 satisfies the corollary. □

In paricular if $k = N + 1$, the singular part of $(H - \lambda)^{-1}$ at λ_0 is associated with p_0-Floquet periodic solutions of $Hu - \lambda_0 u = 0$.

V. Resonances Created by Impurities

As discussed in the introduction, one of the main problems in atom-surface scattering is to investigate what are the effects introduced in scattering quantities by the presence of impurities. (See [Ge] Sect. II.C). We will consider the periodic potentials as a background potential and treat the impurities as a relatively compact perturbation.

Let us assume that the effect of impurities can be described with an additional real potential $W(x, y)$ with $W \in L^\infty_\alpha(\mathbf{R}^m) = \{V \in L^\infty(\mathbf{R}^m) | e^{\alpha(\langle x \rangle + \langle y \rangle)} V \in L^\infty(\mathbf{R}^m)\}$. As in Sects. III, IV, we can allow local singularities of W. (See Remark 5.2.)

We denote by $\tilde{H}$ the Hamiltonian $\tilde{H} = H + W$ with domain $H^2(\mathbf{R}^m)$. Then we have the following theorem.

Theorem 5.1. *Assume that V belongs to $L^k_\alpha(C_T)$ for $k > m, \alpha > 0$ and $W \in L^\infty_{\alpha'}(\mathbf{R}^m)$ for $\alpha' > 0$. Then the following results hold:*

i) *Local Extension Problem: for any $\lambda_0 \in \mathscr{U} \cap \mathbf{R}$, there exist a neighborhood $\mathscr{U}_{\lambda_0}$ of λ_0 in $\mathscr{U}$ such that $(\tilde{H} - \lambda)^{-1}$ extends meromorphically from $\{\operatorname{Im} \lambda > 0\} \cap \mathscr{U}_{\lambda_0}$ to $(\mathscr{U}_{\lambda_0} \backslash \Sigma)^*$ as a bounded operator from $L^2_a(\mathbf{R}^m)$ into $H^1_{-a}(\mathbf{R}^m)$ for $a > \frac{1}{2}(\alpha + \alpha')$, with poles in $(\mathscr{U}_{\lambda_0} \backslash \Sigma)^*$ having finite rank residues.*

ii) *Global Extension Problem: $(\tilde{H} - \lambda)^{-1}$ extends meromorphically from $\{\operatorname{Im} \lambda > 0\}$ to $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$ as a bounded operator from $L^2_a(\mathbf{R}^m)$ into $H^1_{-a}(\mathbf{R}^m)$ for $a > \frac{1}{2}(\alpha + \alpha')$, with poles in $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$ having finite rank residues.*

We will call *resonances* of $\tilde{H}$ the poles of $(\tilde{H} - \lambda)^{-1}$ in $(\mathscr{U}_{\lambda_0} \backslash \Sigma)^*$ or $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$.

Proof. As in Sect. III, we use the second resolvent formula: For $\operatorname{Im} \lambda > 0$, we have:

$$(\tilde{H} - \lambda)^{-1} = (H - \lambda)^{-1}(\mathbf{1} + W(H - \lambda)^{-1})^{-1} = (H - \lambda)^{-1}(\mathbf{1} + \tilde{K}(\lambda))^{-1}. \tag{5.1}$$

From Theorem 4.7 and the fact that $W \in L^\infty_\alpha(\mathbf{R}^m)$ for $\alpha > 0$, we see that $\tilde{K}(\lambda)$ extends holomorphically from $\operatorname{Im} \lambda > 0$ to $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$ as a compact operator on $L^2_a(\mathbf{R}^m)$. Since $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$ is connected and $1 + \tilde{K}(\lambda)$ is invertible for $\operatorname{Im} \lambda \gg 1$, the theorem follows from the analytic Fredholm theorem. □

Remark 5.2. As in Sect. III, Theorem 5.1 still holds if

$$e^{\alpha(\langle x \rangle + \langle y \rangle)} W(\Delta + 1)^{-1/2} \quad \text{is compact for} \quad \alpha > 0.$$

Let us make some comments on this result:

The resonances of $\tilde{H}$ live on the universal covering $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$ which means that one can discover new resonances by turning around the Landau resonances of H.

The physical interpretation of resonances in "upper sheets" of $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^*$ is not clear to us.

We have the following corollary:

Corollary 5.2. *Let us denote by $\Gamma_\mathbf{R}$ the set of resonances of $\tilde{H}$ in $(\mathscr{U} \backslash \Sigma \cup \Sigma_\infty)^* \cap \{\operatorname{Im} \lambda \geqq 0\}$, and by $\Sigma_\mathbf{R}$ the set $\Sigma \cap \mathbf{R}$. Then:*

i) $\sigma_{\mathrm{pp}}(\tilde{H}) \subset \Gamma_\mathbf{R} \cup \Sigma_\mathbf{R}, \Gamma_\mathbf{R} \subset \sigma_{\mathrm{pp}}(\tilde{H})$,

ii) *$\sigma_{\mathrm{sc}}(\tilde{H})$ is empty and the eigenvalues of $\tilde{H}$ can accumulate only at $\Sigma_\mathbf{R}$.*

Proof. i) Can be proved easily by using an idea of Balslev–Combes [B.C]. We have the following formula, if $d\tilde{E}_\lambda$ denotes the spectral measure of $\tilde{H}$:

$$\tilde{E}_{]-\infty,\lambda]} - \tilde{E}_{]-\infty,\lambda[} = s \lim_{z\to\lambda,\operatorname{Im} z>0} (z-\lambda)(\tilde{H}-z)^{-1}.$$

Let $\lambda_0 \in \sigma_{pp}(\tilde{H})$. Then $\tilde{E}_{]-\infty,\lambda_0]} - \tilde{E}_{]-\infty,\lambda_0[} = \tilde{E}_{\{\lambda_0\}} \neq 0$. Since $L_a^2(\mathbf{R}^m)$ is dense in $L^2(\mathbf{R}^m)$, we can find some $\varphi \in L_a^2(\mathbf{R}^m)$, such that $(\tilde{E}_{\{\lambda_0\}}\varphi, \varphi) = \lim_{z\to\lambda_0,\operatorname{Im} z>0} (z-\lambda_0)((\tilde{H}-z)^{-1}\varphi,\varphi) \neq 0$. Then $(\tilde{H}-z)^{-1}$ must have a singularity at $z=\lambda_0$, so $\lambda_0 \in \Gamma_{\mathbf{R}} \cup \Sigma_{\mathbf{R}}$. Suppose now that $\lambda_0 \in \Gamma_{\mathbf{R}}$, and that $\lambda_0 \notin \sigma_{pp}(\tilde{H})$. Then for any $\varphi_1, \varphi_2 \in L_a^2(\mathbf{R}^m)$, we have $\lim_{z\to\lambda_0,\operatorname{Im} z>0} (z-\lambda_0)((\tilde{H}-z)^{-1}\varphi_1,\varphi_2) = 0$, which is impossible if we choose φ_1 and φ_2 correctly with respect to the residue in the Laurent expansion of $(\tilde{H}-z)^{-1}$ at λ_0. This proves i). To prove ii), we notice that Theorem 5.1 implies that $\sigma_{sc}(\tilde{H}) \subset \Gamma_{\mathbf{R}} \cup \Sigma_{\mathbf{R}}$ (see [Re–Si]). $\Gamma_{\mathbf{R}} \cup \Sigma_{\mathbf{R}}$ is a set of points having only a locally finite set of accumulation points, so it cannot support a continuous measure, which proves that $\sigma_{sc}(\tilde{H}) = \emptyset$. The properties of the eigenvalues of $(\tilde{H})$ follows directly from i). □

Concerning the problem mentioned at the beginning of the section, we see that the effect of impurities is to add poles to the Landau resonances already present for a perfect crystalline surface. An obvious example where an impurity can create a pole of $(\tilde{H}-\lambda)^{-1}$ is by taking $W \leqq 0$ sufficiently negative to create a bound state of $\tilde{H}$, i.e. a real pole of $(\tilde{H}-\lambda)^{-1}$. According to Theorem 4.10, the Landau resonances are typically branch points of $(H-\lambda)^{-1}$ (the case of a polar singularity is very unusual), and should be still singularities of $(\tilde{H}-\lambda)^{-1}$. (See the discussion below.) So the presence of impurities should be seen on scattering cross-sections or spectral functions since polar singularities give rise to the well-known Breit–Wigner shapes and singularities like $(\lambda-\lambda_0)^{-1/2}$ or $\log(\lambda-\lambda_0)$ produce very different resonance shapes.

We will now briefly discuss the singularity of $(\tilde{H}-\lambda)^{-1}$ near a Landau resonance λ_0, assuming that the hypotheses (4.5), ..., (4.9) of Theorem 4.10 hold. Recall that it follows from this theorem that the Riemann surface on which $(H-\lambda)^{-1}$ is holomorphic and univalued for λ near λ_0 is the Riemann surface T of $(\lambda-\lambda_0)^{1/2}$ if $m+N+k$ is odd and of $\operatorname{Log}(\lambda-\lambda_0)$ if $m+N+k$ is even. Then we have the following result:

Theorem 5.3. *Under the hypotheses of Theorem 4.10, if $m+N+k>2\alpha$ and $m+N+k \geqq 2\alpha+2$ if $m+N+k$ is even, the number of resonances of $\tilde{H}$ on a sheet of T in a small neighborhood of λ_0 is finite.*

Proof. From (5.1) and Theorem 4.10, we have:

$$\mathbf{1} + W(H-\lambda)^{-1} = \mathbf{1} + E_0(\lambda) + d(\lambda)(K_1 + (\lambda-\lambda_0)E_1(\lambda)),$$

where $E_0(\lambda), E_1(\lambda)$ are holomorphic and compact operators on $L_a^2(\mathbf{R}^m)$, K_1 is a finite rank operator and:

$$d(\lambda) = (\lambda-\lambda_0)^{(m+N+k)/2-\alpha-1} \quad \text{if} \quad m+N+k \text{ is odd}$$

$$d(\lambda) = (\lambda-\lambda_0)^{(m+N+k)/2-\alpha-1}\operatorname{Log}(\lambda-\lambda_0) \quad \text{is} \quad m+N+k \text{ is even.}$$

Since $m+N+k>2\alpha$, $d(\lambda)(\lambda-\lambda_0) = 0((\lambda-\lambda_0)^\varepsilon)$ for some $\varepsilon>0$.

Assume first that $m+N+k$ is odd. The if we introduce the variable $(\lambda-\lambda_0)^{1/2}$, i.e. we uniformize the Riemann surface T, the theorem follows directly from the meromorphic Fredholm theorem.

Assume now that $m+N+k$ is even. Without loss of generality we can put $\lambda_0=0$. Using the fact that $E_0(\lambda)$ is compact and holomorphic in λ, we can write:

$$E_0(\lambda)=E_0(0)+R(\lambda)=K_0+R_0+R(\lambda), \quad \text{with} \quad \|R(\lambda)\|=0(|\lambda-\lambda_0|), \quad |R_0|\leqq 1/2,$$

K_0 of finite rank. If we introduce the dummy variable $t=\lambda d(\lambda)$, we can invert $\mathbf{1}+R_0+R(\lambda)+tE_1(\lambda)$ by a Neumann series for $|\lambda|\leqq\varepsilon_0, |t|\leqq\varepsilon_0$. $(\mathbf{1}+R_0+R(\lambda)+tE_1(\lambda))^{-1}=\mathbf{1}+F(t,\lambda)$, where $F(t,\lambda)$ is holomorphic for $|\lambda|\leqq\varepsilon_0$, $|t|\leqq\varepsilon_0$.

Using again the second resolvent formula, we see that we are reduced to the inversion of $\mathbf{1}+K_0(t,\lambda)+d(\lambda)K_1(t,\lambda)$ for $t=\lambda d(\lambda)$, where $K_i(t,\lambda)=K_i(1+F(t,\lambda))$.

As in the proof of Theorem 3.5, we are reduced to the inversion of a finite dimensional matrix $\mathbf{1}+K_0(t,\lambda)+d(\lambda)K_1(t,\lambda)=1/\lambda(\lambda\mathbf{1}+\lambda K_0(t,\lambda)+tK_1(t,\lambda))=(1/\lambda)M(t,\lambda)$.

So λ with $0<|\lambda|\leqq\varepsilon_0$ is a resonance of $\tilde{H}$ if and only if $f(\lambda d(\lambda),\lambda)=0$, where $f(t,\lambda)=\det(M(t,\lambda))$.

Using Taylor's formula in λ, we can write:

$$f(t,\lambda)=\sum_{n=N}^{\infty} a_n(t)\lambda^n=\lambda^N g(t,\lambda), \quad \text{with} \quad a_N(t)\not\equiv 0.$$

(We use here the fact that $g(t,\lambda)\not\equiv 0$, since the resonances of $\tilde{H}$ are discrete in $\{\lambda|0<|\lambda|\leqq\varepsilon_0\}$. We can replace $f(t,\lambda)$ by $g(t,\lambda)$, and since $a_N(t)\not\equiv 0$, there exist $N_0\in\mathbf{N}$ such that $\partial_t^{N_0}g(0,0)\neq 0$. Using Weierstrass preparation theorem, we can hence replace $g(t,\lambda)$ by a polynomial in t of degree N_0:

$$g(t,\lambda)=t^{N_0}+b_1(\lambda)t^{N_0-1}+\cdots+b_{N_0}(\lambda).$$

By Puiseux theorem, the solutions of $g(t,\lambda)=0$ are branches of analytic functions $t_i(\lambda)$, $i=1,\ldots,M_0$, with:

$$t_i(\lambda)=\lambda^{\alpha_i}(c_i+O(\lambda^{\varepsilon})), \quad \alpha_i\in\mathbf{Q}, \quad \varepsilon>0, \quad \text{if} \quad t_i(0)=0.$$

It is then easy to check that the number of solutions of $\lambda d(\lambda)=t_i(\lambda)$ on a sheet of T is bounded, which proves the theorem.

Remark 5.4. If the hypotheses of Theorem 5.3 are not satisfied, it may happen that $(\tilde{H}-\lambda)^{-1}$ has an essential singularity at $\lambda_0\in\Sigma$. This is due to the fact that the singular part of $(H-\lambda)^{-1}$ at λ_0 is in general not of finite rank.

Appendix

We prove here the result used in the proof of Corollary 4.12.

Proposition A.1. *The residue $\tilde{\pi}_0$ of $\tilde{D}(p_0,\lambda,z_0)/f(p_0,\lambda,z_0)$ at $\lambda=\lambda_0$ satisfies the following identities:*

$$(H_{p_0}-\lambda_0+\tilde{R}_0(\lambda_0))\tilde{\pi}_0=\tilde{\pi}_0(H_{p_0}-\lambda_0+\tilde{R}_0(\lambda_0))=0.$$

Proof. Since $V\in L_\alpha^k(C_T)$, $k>m, m\geqq 3$, using Sobolev inequalities, we see that H_{p_0}

is bounded between $H^1_{-a}(C_T)$ and $H^{-1}_{-a}(C_T)$, and between $H^2_a(C_T)$ and $L^2_a(C_T)$.

Let us put $\tilde{G}(p_0, \lambda, z_0) = \tilde{D}(p_0, \lambda, z_0)/f(p_0, \lambda, z_0)$. It is easy to see by analytic continuation that one has:

$$(H_{p_0} - \lambda + \tilde{R}_0(\lambda))\tilde{G}(p_0, \lambda, z_0) \equiv \mathbf{1}_1, \tilde{G}(p_0, \lambda, z_0)(H_{p_0} - \lambda + \tilde{R}_0(\lambda)) = \mathbf{1}_2,$$

where $\mathbf{1}_1$ is the injection of $L^2_a(C_T)$ into $H^{-1}_{-a}(C_T)$ and $\mathbf{1}_2$ is the injection of $H^2_a(C_T)$ into $H^1_{-a}(C_T)$.

Then we have, if γ is a small circle around λ_0:

$$\begin{aligned}(H_{p_0} - \lambda_0 + \tilde{R}_0(\lambda_0))\tilde{\pi}_0 &= \frac{1}{2i\pi} \int_\gamma (H_{p_0} - \lambda_0 + \tilde{R}_0(\lambda_0))\tilde{G}(p_0, \lambda, z_0)d\lambda \\ &= \frac{1}{2i\pi} \int_\gamma \mathbf{1} + O(\lambda - \lambda_0)\tilde{G}(p_0, \lambda, z_0)d\lambda = 0.\end{aligned}$$

On the other hand, let $\varphi \in H^2_a(C_T)$, and $u = (H_{p_0} - \lambda_0 + \tilde{R}_0(\lambda_0))\varphi \in L^2_a(C_T)$. We have: $\tilde{G}(p_0, \lambda, z_0)u = \varphi + \tilde{G}(p_0, \lambda, z_0)O(\lambda - \lambda_0)\varphi$, so $\tilde{G}(p_0, \lambda, z_0)u$ is holomorphic near $\lambda = \lambda_0$, which implies that $\tilde{\pi}_0 u = 0$. This proves the proposition. $\square$

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