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Quantum Field-Theory Models on Fractal Spacetime

II. Hierarchical Propagators

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Abstract. Continuing the analysis of a previous paper, the present work applies rigorous renormalization group methods to the hierarchical models to establish the existence of field theories with non-Gaussian ultraviolet renormalization group fixed points in $4-\varepsilon$ dimensions.

1. Introduction

In the first paper [4] (hereafter denoted I), the subject of quantum field-theory models on fractal spacetimes was introduced and briefly motivated. Results on the scalar-field models with a realistic, proper-time propagator using available techniques are unfortunately scanty, but the analysis of the models with a hierarchical propagator can be carried substantially further. We adopt here the same definitions and notations as were introduced in the companion paper. As discussed there, a much considered type of hierarchical field-theory model [5–7], with free covariance

$$G_0(n,m)=\sum_{k=0}^{\infty} L^{-\alpha k}\delta_{[\![L^{-(k+1)}n]\!],[\![L^{-(k+1)}m]\!]}A(n_k)A(m_k). \tag{1.1}$$

on a fractal lattice version $*\mathbb{F}_M^N(d,L,\mathscr{C})$ of the Sierpinski carpet, [4, 9, 8], leads under a block-spin RG operation, to a recursion formula for the spin weight function $g(s)$ of the form:

$$g'(s)=\int g(L^{-\alpha/2}s+z)^{L^{\bar{d}}/2}g(L^{-\alpha/2}s-z)^{L^{\bar{d}}/2}d\mu(z)/\int g(z)^{L^{\bar{d}}}d\mu(z). \tag{1.2}$$

This is just the familiar Wilson approximate recursion formula, however, with the Euclidean dimension d replaced by the (generally non-integer) Hausdorff dimension $\bar{d}$ of the (hyper)carpet. The mathematical discussion of the fractal-spacetime, hierarchical models therefore reduces to the analysis of this recursion, which we carry out below. Section 2 gives the complete proof of a technical result, the existence of a stable domain for the RG recursion. In Sect. 3, the existence of a

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scale-invariant but non-Gaussian quantum field-theory is demonstrated on fractal spacetimes with Hausdorff dimension slightly less than four. In the final Sect. 4, it is proved that there are massive scalar theories on these same sets which have the non-Gaussian fixed point as their ultraviolet attractor. This last result, particularly, appears to be new.

2. Stable Domain of the Renormalization Group

This section is devoted to proof of the existence of a stable domain $\mathscr{D}$ of the hierarchical RG recursion (1.1). It is a somewhat refined version of Proposition I in the paper [5] of Gawedzki and Kupiainen. We define the domain of weight functions g to satisfy the following conditions (a), (b), (c):

(a) For $|\mathrm{Im}\, s|<|\log\varepsilon|$, $g(s)$ is an even analytic function, positive for real s and $g(0)=1$.

(b) For $|s|<|\log\varepsilon|$, $g(s)=e^{-w(s)}$ with $w(s)$ analytic and

$$w(s)=c+\frac{1}{2}r:s^2:+\frac{1}{4!}u:s^4:+\frac{1}{6!}t:s^6:+\frac{1}{8!}h:s^8:+\tilde{w}(s), \tag{2.1}$$

and $d^i\tilde{w}(s)/ds^i|_{s=0}=0$, $i=0,2,4,6,8$. Furthermore, with $\hat{u}$ the fixed-point coupling of 3^{rd} order perturbation theory,

$$u=\hat{u}+\tilde{u}, \quad |\tilde{u}|\leqq A\varepsilon^2L^{-\bar{d}}\log^2L, \tag{2.2}$$

$$r=r_{2c}u^2+\tilde{r}, \quad |\tilde{r}|\leqq B\varepsilon^3L^{-\bar{d}}\log^3L, \tag{2.3}$$

$$t=t_{2a}u^2+\tilde{t}, \quad |\tilde{t}|\leqq C\varepsilon^3L^{-\bar{d}}\log^3L, \tag{2.4}$$

$$|h|\leqq D\varepsilon^3L^{-\bar{d}}\log^3L, \tag{2.5}$$

$$\sup_{|s|\leqq|\log\varepsilon|}|\tilde{w}(s)|\leqq E\varepsilon^4|\log\varepsilon|^{24}L^{-\bar{d}}\log^4L. \tag{2.6}$$

(The constant c is fixed by the requirement that $w(0)=0$.)

(c) For $|s|\geqq|\log\varepsilon|$, $|\mathrm{Im}\, s|<L^{-\alpha/2}|\log\varepsilon|$ with $\bar{u}=\beta_2^{-1}(1-L^{-\varepsilon})$:

$$|g(s)|\leqq\exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\, s)^4+(\mathrm{Re}\, s)^2)\right]. \tag{2.7}$$

Then, we have the following:

Theorem 1. *There are an integer L_0 and real number $\varepsilon_0(L)>0$ such that for $L>L_0$, $\varepsilon<\varepsilon_0(L)$ the* RG *transform g' of g defined by* (1.1) *exists and also obeys the conditions* (a), (b), (c) *except possibly* (4) *of* (b). *However, for each $g\in\mathscr{D}$ and each $k\leqq 0$, there exists a compact $\mathscr{B}\subset\mathbb{C}$ (dependent upon g) such that $\mathscr{B}_k\subseteqq\mathscr{B}_{k+1}$ and if $\tilde{r}\in\mathscr{B}_k$, then each of the* RG *iterates g_l, $k\leqq l\leqq 0$, exists and belongs to $\mathscr{D}$. If $\tilde{r}\in\mathscr{B}\equiv\bigcap_{k\leqq 0}\mathscr{B}_k$, then all* RG *iterates g_k, $k\leqq 0$ exist and belong to $\mathscr{D}$.*

It is also demonstrated that r', u', t', h' obey the third-order RG iteration equations with rigorous bounds:

$$|r'-L^2[r-(\alpha_2u^2+\alpha_1ru+\alpha_4tu)+(\alpha_3'u^3)]|\leqq B'\varepsilon^4|\log\varepsilon|^{24}\log^4L, \tag{2.8}$$

$$|u'-L^{\varepsilon}[u-(\beta_2u^2+\beta_1ru+\beta_4tu)+(\beta_3'u^3)]|\leqq A'\varepsilon^4\log\varepsilon|^{24}\log^4L, \tag{2.9}$$

$$|t'-L^{-2(1-\varepsilon)}[t-(\delta_2u^2+\delta_4tu)+(\delta_3'u^3)]|\leqq C'\varepsilon^4|\log\varepsilon|^{24}\log^4L, \tag{2.10}$$

$$|h'-L^{-4+3\varepsilon}[h-(\gamma_4tu)+(\gamma_3'u^3)]|\leqq D'\varepsilon^4|\log\varepsilon|^{24}\log^4L. \tag{2.11}$$

Moreover, the domain $\mathscr{D}$ is shown to be nonempty: if $g(s) \equiv e^{-w(s)}$ for

$$w(s) \equiv \frac{1}{2} r : s^2 : + \frac{1}{4!} u : s^4 : + c \tag{2.12}$$

with r and u obeying (3) and (4) of (b), then g obeys all of the conditions (a), (b), (c). Define the set $\mathscr{K} \subseteqq \mathscr{D}$ to be those $g \in \mathscr{D}$ for which $\tilde{r} \in \mathscr{B}(g)$. Then, $\mathscr{K}$ is also nonempty: if $g \in \mathscr{D}$, there exists an $R = R(g)$ such that

$$g^c(s) \equiv g(s) e^{-\frac{1}{2} R s^2} \in \mathscr{K} . \tag{2.13}$$

Proof. Define

$$w_0(s) = c + \frac{1}{2} r : s^2 : + \frac{1}{4!} u : s^4 : + \frac{1}{6!} t : s^6 : + \frac{1}{8!} h : s^8 : \tag{2.14}$$

so that

$$w(s) = w_0(s) + \tilde{w}(s) . \tag{2.15}$$

Following the notation in (4.10), define also

$$q_0(s, z) = \frac{1}{2} [w_0(L^{-\alpha/2} s + z) + w_0(L^{-\alpha/2} s - z)] , \tag{2.16}$$

$$\tilde{q}(s, z) = \frac{1}{2} [\tilde{w}(L^{-\alpha/2} s + z) + \tilde{w}(L^{-\alpha/2} s - z)] . \tag{2.17}$$

Expanding q_0 as $q_0(s, z) = \sum_{i=0}^{4} \breve{g}_{2i}(s) : z^{2i} :_1$ with

$$\breve{g}_{2i} = c L^{i\alpha} + \frac{1}{2} r \binom{2}{2i} L^{-(1-i)\alpha} : s^{2-2i} : + \frac{1}{4!} u \binom{4}{2i} L^{-(2-i)\alpha} : s^{4-2i} :$$
$$+ \frac{1}{6!} t \binom{6}{2i} L^{-(3-i)\alpha} : s^{6-2i} : + \frac{1}{8!} h \binom{8}{2i} L^{-(4-i)\alpha} : s^{8-2i} : , \tag{2.18}$$

note that for $i > 0$

$$\sup_{|s| < |\log \varepsilon|} |\breve{g}_{2i}(s)| = O(\varepsilon |\log \varepsilon|^2 L^{-\bar{d}} \log L) . \tag{2.19}$$

Set

$$q_0(s, z) \equiv \breve{g}(s) + \bar{q}_0(s, z) \tag{2.20}$$

and

$$\bar{q}(s, z) \equiv \bar{q}_0(s, z) + \tilde{q}(s, z) , \tag{2.21}$$

so that

$$q(s, z) = \breve{g}_0(s) + \bar{q}(s, z) . \tag{2.22}$$

We now show (a′), (b′), (c′) follow from (a), (b), (c).

(a′) This is essentially direct since convergence of the integral in (5.1) is uniform in the strip $|\mathrm{Im}\, s| < |\log \varepsilon|$ by (b), (c).

(b′) Denote $g(L^{-\alpha/2} s \pm z)$ by $g_\pm$. Observe that

$$g'(s) \equiv g'_\chi(s) g'_\perp(s) \tag{2.23}$$

with

$$g'_\chi(s) \equiv \int g_+^{L^{\bar{d}}/2} g_-^{L^{\bar{d}}/2} \chi d\mu / \int g^{L^{\bar{d}}} \chi d\mu \tag{2.24}$$

and

$$g'_\perp = [1 + (\int g_+^{L^{\bar{d}}/2} g_-^{L^{\bar{d}}/2} \chi^\perp d\mu / \int g_+^{L^{\bar{d}}/2} g_-^{L^{\bar{d}}/2} \chi d\mu)] \cdot [1 + (\int g^{L^{\bar{d}}} \chi^\perp d\mu / \int g^{L^{\bar{d}}} \chi d\mu)], \tag{2.25}$$

where χ is the characteristic function for the set $\{s : |s| < \delta|\log\varepsilon|\}$, $0 < \delta < 1$ chosen small enough that $L^{\alpha/2}(1-\delta) > 1$. We see that for $|s| < |\log\varepsilon|$, $g'_\chi(s) = 1 + O(\varepsilon|\log\varepsilon|^4 \log L)$ and $g'_\perp(s) = 1 + O(e^{-C|\log\varepsilon|^2})$ so that $g'(s) = e^{-w(s)}$ with $w'(s)$ analytic.

Consider the contribution $w'_\chi = -\log g'_\chi$. Clearly, $w'_\chi(s) = y'_\chi(s) - y'_\chi(0)$ with

$$y'_\chi(s) = -\log \int e^{-L^{\bar{d}} q(s,z)} \chi(z) d\mu(z) = L^{\bar{d}} \breve{g}_0(s) - \log \int e^{-L^{\bar{d}} \bar{q}(s,z)} \chi(z) d\mu(z). \tag{2.26}$$

We write the second piece as

$$-\log \int d\mu \chi e^{-L^{\bar{d}}(\bar{q}_0 + \tilde{q})} = -\log \int d\mu \chi e^{-L^{\bar{d}} \bar{q}_0} + L^{\bar{d}} \int_0^1 dt \langle \tilde{q} \rangle_{0,t}, \tag{2.27}$$

where $\langle \cdot \rangle_{0,t} = \int (\cdot) e^{-L^{\bar{d}}(\bar{q}_0 + t\tilde{q})} \chi d\mu / \int e^{-L^{\bar{d}}(\bar{q}_0 + t\tilde{q})} \chi d\mu$. Set $y'_\chi = y'_{\chi 0} + \tilde{y}'_\chi$ with $y'_{\chi 0} \equiv L^{\bar{d}} \breve{g}_0(s) - \log \int d\mu \chi e^{-L^{\bar{d}} \bar{q}_0}$ and $\tilde{y}'_\chi = L^{\bar{d}} \int_0^1 dt \langle \tilde{q} \rangle_{0,t}$. The main contribution comes from $y'_{\chi 0}$:

$$\begin{aligned} y'_{\chi 0} &= L^{\bar{d}} \langle q_0 \rangle_\chi - \frac{1}{2} L^{2\bar{d}} \langle \bar{q}_0^2 \rangle_\chi^T + \frac{1}{3!} L^{3\bar{d}} \langle \bar{q}_0^3 \rangle_\chi^T \\ &\quad + \frac{L^{4\bar{d}}}{3!} \int_0^1 dt (t-1)^3 \langle \bar{q}_0^4 \rangle_{\chi,t}^T \end{aligned} \tag{2.28}$$

with self-explanatory notations. Denote the quartic remainder by $R'_{\chi 0}$ and separate it into terms contributing to y'_0 and to $\tilde{w}'$:

$$R'_{\chi 0}(s) = \sum_{m=0}^{4} \frac{\breve{f}_m}{(2m)!} : s^{2m} : + \tilde{R}'_{\chi 0}(s). \tag{2.29}$$

An easy error analysis employing Cauchy estimates gives

$$\tilde{R}'_{\chi 0}(s) \leqq L^{(\bar{d} - 5\alpha)} \cdot \text{const}\, \varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}} \log^4 L, \tag{2.30}$$

and for the couplings $\breve{f}_m$ in (2.29):

$$\breve{f}_m = O(\varepsilon^4 L^{\bar{d} - \alpha m} L^{\bar{d}} \log^4 L). \tag{2.31}$$

The remainder term $\tilde{y}'_\chi$ is handled similarly. Clearly, by (b) and the χ in the expectation $\langle \cdot \rangle_{0,t}$ we have

$$\langle \tilde{q} \rangle_{0,t} = O(\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}} \log^4 L). \tag{2.32}$$

Write

$$\begin{aligned} \tilde{y}'_\chi(s) &= L^{\bar{d}} \int_0^1 dt \langle \tilde{q} \rangle_{0,t} (L^{-\alpha/2} s) \\ &= \text{const} + \frac{\breve{b}_2}{2!} : (L^{-\alpha/2} s)^2 : + \frac{\breve{b}_4}{4!} : (L^{-\alpha/2} s)^4 : + \frac{\breve{b}_6}{6!} : (L^{-\alpha/2} s)^6 : \\ &\quad + \frac{\breve{b}_8}{8!} : (L^{-\alpha/2} s)^8 : + \tilde{\tilde{w}}'_\chi(s). \end{aligned} \tag{2.33}$$

The constants $\tilde{b}_2, \ldots, \tilde{b}_8$ are all $O(\varepsilon^4|\log\varepsilon|^{24}\log^4 L)$ by a Cauchy estimate. Thus $\tilde{w}'_\chi$ makes contributions $\tilde{r}'_\chi, \tilde{u}'_\chi, \tilde{t}'_\chi, \tilde{h}'_\chi$ to r', u', t', h' of orders $O(\varepsilon^4|\log\varepsilon|^{24} L^{-p\alpha}\log^4 L)$, $p=1,2,3,4$ respectively. Its contribution $\tilde{\tilde{w}}'_\chi$ to $\tilde{w}'$ is

$$\begin{aligned}\tilde{\tilde{w}}'_\chi &= L^{\bar{d}} \int_0^1 dt \frac{1}{10!} D^{10}\langle\tilde{q}\rangle_{v_0,t}(0) O(|L^{-\alpha/2}s|^{10}) \\ &= L^{\bar{d}-5\alpha} O(\varepsilon^4|\log\varepsilon|^{24} L^{-\bar{d}}\log^4 L)\end{aligned} \tag{2.34}$$

by (b) and a Cauchy estimate. This is the same order as in (2.30).

All that is left to examine of the contributions from y'_χ are the first three terms of (2.28), which give the main contribution y'_M:

$$\begin{aligned} y'_M &= L^{\bar{d}}\langle q_0\rangle_\chi - \frac{1}{2} L^{2\bar{d}}\langle \bar{q}_0^2\rangle_\chi^T + \frac{1}{3!}\langle \bar{q}_0^3\rangle_\chi^T \\ &= L^{\bar{d}}\langle q_0\rangle - \frac{1}{2!} L^{2\bar{d}}\langle \bar{q}_0^2\rangle^T + \frac{1}{3!} L^{3\bar{d}}\langle \bar{q}_0^3\rangle^T + O(e^{-C|\log\varepsilon|^2}), \end{aligned} \tag{2.35}$$

the formal third-order perturbative contribution plus terms of $O(e^{-C|\log\varepsilon|^2})$ which may be absorbed to fourth order. The perturbative contributions to the couplings r', u', t', h' are of the form

$$r'_M = L^2[r - (\alpha_2 u^2 + \alpha_1 ru + \alpha_4 tu) + (\alpha'_3 u^3) + O(\varepsilon^4 L^{-\bar{d}}\log^4 L)], \tag{2.36}$$

$$u'_M = L^\varepsilon[u - (\beta_2 u^2 + \beta_1 ru + \beta_4 tu) + (\beta'_3 u^3) + O(\varepsilon^4 L^{-\bar{d}}\log^4 L)], \tag{2.37}$$

$$t'_M = L^{-2(1-\varepsilon)}[t - (\delta_2 u^2 + \delta_4 tu) + (\delta'_3 u^3) + O(\varepsilon^4 L^{-\bar{d}}\log^4 L)], \tag{2.38}$$

$$h'_M = L^{-4+3\varepsilon}[h - (\gamma_4 tu) + (\gamma'_3 u^3) + O(\varepsilon^4 L^{-\bar{d}}\log^4 L)], \tag{2.39}$$

where the coefficients of the quadratic terms are $O(L^{\bar{d}})$ and the coefficients of the cubic terms are $O(L^{2\bar{d}})$. The contribution $\tilde{w}'_M$ to w' is of the form

$$\tilde{w}'_M(s) = \sum_{i=5}^{10} \frac{g'_{2i}}{(2i)!} :s^{2i}:. \tag{2.40}$$

The largest of the induced couplings is

$$g'_{10} = L^{-6+4\varepsilon}[\eta_5 t^2 + \eta_6 uh + \eta'_4 tu^2 + \ldots] = O(\varepsilon^4 L^{-5\alpha}\log^4 L), \tag{2.41}$$

while the smallest arises from a single contribution: $g'_{20} = L^{-16+9\varepsilon}\xi h^3 = O(\varepsilon^9 L^{-10\alpha}\log^9 L)$. Thus, at least $g'_{2i} = O(\varepsilon^4 L^{-5\alpha}\log^4 L)$ for all i, so that

$$\sup_{|s|<\log\varepsilon} |\tilde{w}'_M(s)| = O(\varepsilon^4|\log\varepsilon|^{20} L^{-5\alpha}\log^4 L). \tag{2.42}$$

If we now assemble all of the contributions from y'_χ, we find

$$\begin{aligned} r'_\chi &= r'_M + \tilde{r}'_\chi + \tilde{f}_2 = L^2[r - (\alpha_2 u^2 + \alpha_1 ru + \alpha_4 tu) + (\alpha'_3 u^3)] \\ &\quad + O(\varepsilon^4|\log\varepsilon|^{24} L^{-\alpha}\log^4 L), \end{aligned} \tag{2.43}$$

$$\begin{aligned} u'_\chi &= u'_M + \tilde{u}'_\chi + \tilde{f}_4 = L^\varepsilon[u - (\beta_2 u^2 + \beta_1 ru + \beta_4 tu) + (\beta'_3 u^3)] \\ &\quad + O(\varepsilon^4|\log\varepsilon|^{24} L^{-2\alpha}\log^4 L), \end{aligned} \tag{2.44}$$

$$\begin{aligned} t'_\chi &= t'_M + \tilde{t}'_\chi + \tilde{f}_6 = L^{-2(1-\varepsilon)}[t - (\delta_2 u^2 + \delta_4 tu) + (\delta_3 u^3)] \\ &\quad + O(\varepsilon^4|\log\varepsilon|^{24} L^{-2\alpha}\log^4 L), \end{aligned} \tag{2.45}$$

$$\begin{aligned} h'_\chi &= h'_M + \tilde{h}_\chi + \tilde{f}_8 = L^{-4+3\varepsilon}[h - (\gamma_4 tu) + (\gamma'_3 u^3)] \\ &\quad + O(\varepsilon^4|\log\varepsilon|^{24} L^{-4\alpha}\log^4 L), \end{aligned} \tag{2.46}$$

and

$$\tilde{w}'_\chi = \tilde{w}'_M + \tilde{\tilde{w}}'_\chi + \tilde{R}'_\chi = L^{-6+4\varepsilon} O(\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}} \log^4 L) \quad \text{for} \quad |s| < |\log\varepsilon| . \tag{2.47}$$

We lastly consider the contribution $w'_\perp(s) \equiv -\log g'_\perp(s)$. However, since $g'_\perp(s) = 1 + O(e^{-C|\log\varepsilon|^2})$, $w'_\perp(s) = O(e^{-C|\log\varepsilon|^2})$, which may be absorbed to fourth order. Thus, we conclude that Eqs. (2.8–11), (2.6) give the correct results for the total contributions, using the negative powers of L to select constants E, B', A', C', D'.

It remains only to verify the bounds (2.2, 4–5). This is easy for $\tilde{t}$, h using the irrelevancy of these variables, as appears from (2.10), (2.11). Note here that t_{2a} is defined by the requirement that t be close to its asymptotic form:

$$t_{as}(u) = t_{2a} u^2 + \tilde{t} , \tag{2.48}$$

where

$$t_{2a} = -(L^2 - 1)^{-1} \delta = O(L^{\bar{d}-2}) . \tag{2.49}$$

From (2.9–10) one sees that

$$\tilde{t}' = L^{-2(1-\varepsilon)} [\tilde{t} + \delta_3 u^3 + O(\varepsilon^4 |\log\varepsilon|^{24} \log^4 L)] \tag{2.50}$$

with

$$\delta_3 = 0(L^{2\bar{d}}) . \tag{2.51}$$

With Eq. (2.50) the bound (2.4) clearly iterates by choosing $L > L_0$ so that $L^{-2(1-\varepsilon)}$ is sufficiently small. The same argument applies to the bound (2.5) for h, using directly Eq. (2.11).

To iterate the bound (2.4), we now note from (2.9), (2.3–4) that

$$u' = L^\varepsilon [u - \beta_2 u^2 + \beta_3 u^3 + O(\varepsilon^4 |\log\varepsilon|^{24} \log^4 L)] \tag{2.52}$$

with

$$\beta_3 = O(L^{2\bar{d}}) . \tag{2.53}$$

Defining $\hat{u}$ to be the fixed point solution of the 3rd order equation,

$$\hat{u} = L^\varepsilon [\hat{u} - \beta_2 \hat{u}^2 + \beta_3 \hat{u}^3] , \tag{2.54}$$

the iteration equation for $\tilde{u} = u' - \hat{u}$ is found to be

$$\tilde{u}' = L^{-\varepsilon} \tilde{u} + O(\varepsilon^4 |\log\varepsilon|^{24} \log^4 L) . \tag{2.55}$$

If C' is the constant in the O-bound of the remainder, we can require that $C' \varepsilon^2 |\log\varepsilon|^{24} L^{\bar{d}} \log^2 L < (1 - L^{-\varepsilon})$, so that the inequality in (2.2) iterates.

(c′). We consider $|s| \geqq |\log\varepsilon|$, $|\mathrm{Im}\, s| < L^{-\alpha/2} |\log\varepsilon|$. The analysis here is so similar to that in the last step of the proof Gawedzki and Kupiainen gave of their Proposition 1 in [5] that it would harldy merit any discussion, except that there is a small technical error in that step of their proof. We point out here how to remedy that flaw. One observes first, as those authors do, that for $|z| < \frac{1}{2} L^{-\alpha/2} |\mathrm{Re}\, s|$,

$$|g_\pm| \leqq \exp\left[-\frac{\bar{u}}{96} ((\mathrm{Re}\, S_\pm)^4 + (\mathrm{Re}\, S_\pm)^2) \right] . \tag{2.56}$$

Therefore,

$$\left|\int_{|z|<\frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,s|} g_+^{L^{\bar{d}}/2} g_-^{L^{\bar{d}}/2} d\mu\right|$$
$$\leqq \exp\left[-\frac{\bar{u}}{96}(L^{\bar{d}-2\alpha}(\mathrm{Re}\,s)^4+L^{\bar{d}-\alpha}(\mathrm{Re}\,s)^2)\right]$$
$$\times \int_{|z|<\frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,s|} \exp\left[-\frac{\bar{u}}{96}\cdot 6L^{\bar{d}-\alpha}(\mathrm{Re}\,s)^2 z^2\right] d\mu(z)$$
$$\leqq \exp\left[-\frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\,s)^2\right] \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\,s)^4+(\mathrm{Re}\,s)^2)\right]$$
$$\times \int \exp\left[-\frac{\bar{u}}{96}\cdot 24L^{\bar{d}}z^4\right] d\mu(z)$$
$$\leqq \exp\left[-\frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\,s)^2\right] \int \exp\left[-\frac{\bar{u}}{4!}L^{\bar{d}}z^4\right] d\mu$$
$$\times \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\,s)^4+(\mathrm{Re}\,s)^2)\right]. \tag{2.57}$$

For $|z|\geqq\frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,s|$, one observes that for exactly one of the signs $+$ or $-$, $|\mathrm{Re}\,S_\pm| > \frac{3}{2}L^{-\alpha/2}\sqrt{1-L^{-\alpha}}|\log\varepsilon|$, and thus (2.56) holds. For the other sign use that $|g|\leqq 2$. Then,

$$\left|\int_{|z|\geqq\frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,s|} g_+^{L^{\bar{d}}/2} g_-^{L^{\bar{d}}/2} d\mu\right|$$
$$\leqq 2^{L^{\bar{d}}/2} \exp\left[-\frac{\bar{u}}{96}(L^{\bar{d}}/2)((\tfrac{3}{4})^4 L^{2\alpha}(\mathrm{Re}\,s)^4+(\tfrac{3}{2})^2 L^{\alpha}(\mathrm{Re}\,s)^2)\right]\cdot \int_{|z|\geqq\frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,s|} d\mu(z)$$
$$\leqq 2^{L^{\bar{d}}/2} \exp\left[-\frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\,s)^2\right] \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\,s)^4+(\mathrm{Re}\,s)^2)\right]\cdot O(e^{-C|\log\varepsilon|^2})$$
$$= \exp\left[-\frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\,s)^2\right] \int \exp\left[-\frac{\bar{u}}{4!}L^{\bar{d}}z^4\right] d\mu$$
$$\times [1+O(e^{-C|\log\varepsilon|^2})] \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\,s)^4+(\mathrm{Re}\,s)^2)\right]. \tag{2.58}$$

Combining the contributions in (2.57) and (2.58) gives

$$\left|\int g_+^{L^d/2} g_-^{L^d/2} d\mu\right| \leqq \exp\left[-\frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\,s)^2\right] \int \exp\left[-\frac{\bar{u}}{4!}L^{\bar{d}}z^4\right] d\mu$$
$$\times [1+O(e^{-C|\log\varepsilon|^2})] \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\,s)^4+(\mathrm{Re}\,s)^2)\right]. \tag{2.59}$$

On the other hand,

$$\int g(z)^{L^{\bar{d}}} \mu(z) \geqq \exp[O(\varepsilon^2|\log\varepsilon|^6 \log^2 L)] \int \exp\left[-\frac{\bar{u}}{4!}L^{\bar{d}}z^4\right] d\mu \tag{2.60}$$

and

$$|g'(s)| \leqq \exp\left[O(\varepsilon^2|\log\varepsilon|^6 \log^2 L) - \frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\, s)^2\right]$$
$$\times \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\, s)^4 + (\mathrm{Re}\, s)^2)\right]$$
$$\leqq \exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\, s)^4 + (\mathrm{Re}\, s)^2)\right] \quad \text{for} \quad \varepsilon < \varepsilon_0, \tag{2.61}$$

which is (c′).

To complete the proof of the theorem statement, we must only demonstrate mathematically that r may be fine-tuned. This is established also by arguing along rather standard lines [7–8]. We note first the iteration equation for $\tilde{r}$ which follows from (2.3), (2.8):

$$\tilde{r}' = L^2[\tilde{r} + \alpha_3 u^3 + O(\varepsilon^4|\log\varepsilon|^{24} \log^4 L)] \tag{2.62}$$

with $\alpha_3 = O(L^{2\bar{d}})$. Let $\mathscr{B}_0$ be the closed disk in $\mathbb{C}$ about O with radius $B\varepsilon^3 L^{-\bar{d}} \log^3 L$ and assume chosen closed subsets $\mathscr{B}_k, \ldots, \mathscr{B}_0$ such that $\mathscr{B}_l \subseteqq \mathscr{B}_{l+1}$ and $\mathscr{R}_l[\mathscr{B}_l] = \mathscr{B}_0$, $k \leqq l \leqq 0$, where $\tilde{r}_k = \mathscr{R}_k[\tilde{r}]$. Now if the constant B is taken sufficiently large, say, $B > 4L^{\bar{d}}\alpha_3/\beta_2^3$, then for L sufficiently large, and small ε, it follows from (5.78) that $\mathscr{R}_{k-1}[\mathscr{B}_k] \supset \mathscr{B}_0$, and one may chose inductively the closed set $\mathscr{B}_{k-1} \equiv \mathscr{R}_{k-1}^{-1}[\mathscr{B}_0] \cap \mathscr{B}_k$. Of course, the set $\mathscr{B} \equiv \bigcap_{k \leqq 0} \mathscr{B}_k$ is nonempty and for $\tilde{r}^c \in \mathscr{B}$, the bound $|\tilde{r}_k| \leqq B\varepsilon^3 L^{-\bar{d}} \log^3 L$ is obeyed for all $k \leqq 0$.

This completes the proof. We only remark that, following the same arguments as were used above, one can show that $g(s) = e^{-w(s)}$ with $w(s)$ given by (2.12) obeys (c). Obviously it obeys (a), (b) and this shows that the domain $\mathscr{D}$ is nonempty. Similarly, one can show that (2.13) obeys (c).

3. Existence of Critical Trajectories and a Non-Gaussian Fixed-Point Theory

We now go on to establish the existence of critical trajectories and a non-Gaussian fixed point for the hierarchical recursion. This is an old result [1–3, 5] and we shall follow closely the proof of Proposition 2 in the paper [5] of Gawedzki and Kupiainen. The results are improved to third-order and the technique is refined, using methods of [6]. Most of the modifications can be inferred from those made to their Proposition 1 in our Theorem 1. Therefore, we shall content ourselves with a statement of the results and a sketch of the proof. Then, we point out the implications of the result for the existence of a scale-invariant, non-Gaussian field-theory.

Theorem 2. *For any weight function g in the domain* $\mathscr{D}$ *obeying conditions* (a), (b), (c) *of Theorem 1, it is possible to choose the parameter* $\tilde{r} = \tilde{r}^c$ *with all other parameters fixed, so that for all* $k \leqq 0$, g_k^c *exists and belongs to* $\mathscr{D}$, *and further, the limit of* g_k^c *as* $k \to -\infty$ *exists absolutely uniformly for s in the strip* $|\mathrm{Im}\, s| < L^{-\alpha/2}|\log\varepsilon|$. *The limit*

$$g^*(s) = \lim_{k \to -\infty} g_k^c(s) \tag{3.1}$$

belongs to $\mathscr{K}$ and is a non-Gaussian RG *fixed-point with*

$$u^* = \hat{u} + O(\varepsilon^3 |\log\varepsilon|^{24} \log^3 L), \tag{3.2}$$

where $\hat{u}$ is the approximate fixed point coupling of 3^{rd} order perturbation theory.

More specifically, with quantities $\delta f_k = f_{k-1} - f_k$, we establish for all $k \leqq 0$:

(A_k) For l in the range $k \leqq l \leqq 0$, one can choose compacts $\mathscr{B}'_{l-1}$ such that $\mathscr{B}'_{l-1} \subset \mathscr{B}'_l$ and such that $\delta\tilde{r}_l$ ranges over the closed ball $\delta\mathscr{B}'_l = \{z : |z| < \frac{1}{4} B\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}+l\varepsilon} \log^4 L\}$ as $\tilde{r} = \tilde{r}_0$ ranges over $\mathscr{B}'_{l-1}$. The constant B is the same as in Theorem 1, and $\mathscr{B}'_l \subseteq \mathscr{B}_l$.

(B_k) For $\tilde{r} \in \mathscr{B}_k$, one has also for some constants A, C, D, E:

$$|\delta u_k| \leqq 2A\varepsilon^2 L^{-\bar{d}+k\varepsilon} \log^2 L, \tag{3.3}$$

$$|\delta\tilde{t}_k| \leqq 2C\varepsilon^3 L^{-\bar{d}+k\varepsilon} \log^3 L, \tag{3.4}$$

$$|\delta h_k| \leqq 2D\varepsilon^3 L^{-\bar{d}+k\varepsilon} \log^3 L, \tag{3.5}$$

$$\sup_{|s| < |\log\varepsilon|} |\delta\tilde{w}_k(s)| \leqq 2E\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}+k\varepsilon} \log^4 L. \tag{3.6}$$

(C_k) For $\tilde{r} \in \mathscr{B}_k$, iff $|s| \geqq |\log\varepsilon|$, $|\mathrm{Im}\, s| < L^{-\alpha/2} |\log\varepsilon|$, then

$$|\delta g_k(s)| \leqq L^{k\varepsilon} (\mathrm{Re}\, s)^6 \exp\left[-\frac{\bar{u}}{96} ((\mathrm{Re}\, s)^4 + (\mathrm{Re}\, s)^2) \right]. \tag{3.7}$$

Then, for $\tilde{r}^c \in \mathscr{B}' \equiv \bigcap_{k \leqq 0} \mathscr{B}'_k$, the existence and convergence claims made above are true.

Proof. The proof of assertions (A_k), (B_k), (C_k) for $k \leqq 0$ is by induction on k. There is no difficulty in verifying (B_0), (C_0). For (A_0), one defines

$$\delta\mathscr{B}'_0 \equiv \{z : |z| < \tfrac{1}{4} B\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}} \log^3 L\} \tag{3.8}$$

uses

$$\delta\mathscr{R}_0[\tilde{r}_0] \equiv \delta\tilde{r}_0 = \tilde{r}_{-1} - \tilde{r}_0 = (L^2 - 1)\tilde{r}_0 + O(\varepsilon^3 L^{-\bar{d}} \log^3 L) \tag{3.9}$$

and defines

$$\mathscr{B}'_{-1} \equiv \delta\mathscr{R}_0^{-1}[\delta\mathscr{B}'_0] \cap \mathscr{B}'_0. \tag{3.10}$$

(B_{k-1}) The essential tools to establish (3.3–6) for $k-1$ are a set of iteration equations (with $\delta f_{k-1} \equiv \delta f'$, $\delta f_k \equiv \delta f$)

$$\delta\tilde{r}' = L^2[\delta\tilde{r} + O(\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}+k\varepsilon} \log^4 L)], \tag{3.11}$$

$$\delta\tilde{u}' = L^{-\varepsilon}\delta\tilde{u} + O(\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}+k\varepsilon} \log^4 L), \tag{3.12}$$

$$\delta\tilde{t}' = L^{-2(1-\varepsilon)}[\delta\tilde{t} + O(\varepsilon^4 |\log\varepsilon|^{24} L^{-\bar{d}+k\varepsilon} \log^4 L)], \tag{3.13}$$

$$\delta h' = L^{-4+3\varepsilon}[\delta h + O(\varepsilon |\log\varepsilon|^{24} L^{-\bar{d}+k\varepsilon} \log^4 L)], \tag{3.14}$$

from which the stated bounds iterate rather directly. These can be derived by writing

$$\delta w'(s) = \delta y'(s) - \delta y'(0), \tag{3.15}$$

where

$$\delta y'(s) = y''(s) - y'(s) \tag{3.16}$$

and

$$y'(s) = -\log \int d\mu e^{-L^{\bar{d}} q}, \qquad y''(s) = -\log \int d\mu e^{-L^{\bar{d}} q'}, \tag{3.17}$$

so that

$$\delta y'(s) = -\log[\int d\mu e^{-L^{\bar{d}} q'} / \int d\mu e^{-L^{\bar{d}} q}]. \tag{3.18}$$

One can then decompose $\delta y'(s)$ as

$$\delta y'(s) = \delta y'_\chi(s) + \delta y'_\perp(s) \tag{3.19}$$

with

$$\delta y'_\chi(s) = -\log[\int e^{-L^{\bar{d}} \delta q(s,z)} d\nu(s,z)], \tag{3.20}$$

where

$$d\nu(s,z) = e^{-L^{\bar{d}} q(s,z)} \chi(z) d\mu(z) / \int e^{-L^{\bar{d}} q(s,z)} \chi(z) d\mu(z), \tag{3.21}$$

and with

$$\begin{aligned} \delta y'_\perp(s) = & -\log[1 + (\int e^{-L^{\bar{d}} q'} \chi^\perp d\mu / \int e^{-L^{\bar{d}} q'} \chi d\mu)] \\ & + \log[1 + (\int e^{-L^{\bar{d}} q} \chi^\perp d\mu / \int e^{-L^{\bar{d}} q} \chi d\mu)]. \end{aligned} \tag{3.22}$$

The same arguments applied to (2.27) to derive (2.43–47), apply also to (3.20) to derive (3.11–14). The derivation of (B_k) is completed by showing that

$$\sup_{|s| < |\log \varepsilon|} |\delta y'_\perp(s)| = O(e^{-C|\log \varepsilon|^2} L^{k\varepsilon}). \tag{3.23}$$

First, as a consequence of (B_k) and (C_k) it is seen that

$$|\delta g| \leqq 2C(L) L^{k\varepsilon} / \varepsilon^{3/2} \tag{3.24}$$

with

$$C(L) = O(L^{3\bar{d}/2}). \tag{3.25}$$

Then, it is not difficult to show also that

$$\delta y'_\perp = O(I' \delta J) + O(\delta I) \tag{3.26}$$

with

$$I = \int e^{-L^{\bar{d}} q} \chi^\perp d\mu / \int e^{-L^{\bar{d}} q} \chi d\mu, \tag{3.27}$$

$$\delta J = 1 - (\int e^{-L^{\bar{d}} q'} \chi d\mu / \int e^{-L^{\bar{d}}} \chi d\mu) = O(\varepsilon^2 |\log \varepsilon|^4 L^{k\varepsilon} \log^2 L) \tag{3.28}$$

and

$$I' = \int e^{-L^{\bar{d}} q'} \chi^\perp d\mu / \int e^{-L^{\bar{d}} q} \chi d\mu = O(e^{-C|\log \varepsilon|^2}). \tag{3.29}$$

One then verifies $\delta I = I' - I = O(e^{-C|\log \varepsilon|^2} L^{k\varepsilon})$ and (3.23) follows.

(A_{k-1}) One uses (3.11) to show that

$$\delta \mathcal{R}_{k-1}[\mathcal{B}'_{k-1}] \supseteqq \delta \mathcal{B}'_{k-1} \tag{3.30}$$

and defines

$$\mathcal{B}'_{k-2} \equiv \delta \mathcal{R}'_{k-1}[\delta \mathcal{B}'_{k-1}] \cap \mathcal{B}'_{k-1}. \tag{3.31}$$

(C_{k-1}) We write

$$\begin{aligned} \delta g' = g'' - g' &= (\int f'(s,z) d\mu(z) / \int f'(0,z) d\mu(z)) - (\int f(s,z) d\mu(z) / \int f(0,z) d\mu(z)) \\ &= (\int \delta f(s,z) d\mu(z) / \int f'(0,z) d\mu(z)) - g'(s) (\int \delta f(0,z) d\mu(z) / \int f'(0,z) d\mu(z)), \end{aligned} \tag{3.32}$$

where

$$f(s,z)\equiv g(L^{-\alpha/2}s+z)^{L^{\bar{d}}/2}g(L^{-\alpha/2}s-z)^{L^{\bar{d}}/2}=e^{-L^{\bar{d}}q(s,z)}. \tag{3.33}$$

The same methods applied in (c′) of Theorem 1 to $g'(s)$ apply here to give

$$|\int \delta f d\mu / \int df_0' d\mu| \leqq L^{\bar{d}-3\alpha}(\tfrac{3}{2})^6 \exp\left[O(\varepsilon^2|\log\varepsilon|^6\log^2 L)-\frac{\bar{u}}{96}(L^2-1)(\mathrm{Re}\,s)^2\right]$$
$$\times L^{k\varepsilon}(\mathrm{Re}\,s)^6\exp\left[-\frac{\bar{u}}{96}((\mathrm{Re}\,s)^4+(\mathrm{Re}\,s)^2)\right] \tag{3.34}$$

and

$$|\int \delta f_0 d\mu / \int f_0' d\mu| \leqq L^{\bar{d}-3\alpha}\exp[O(\varepsilon^2|\log\varepsilon|^6\log^2 L)]L^{k\varepsilon} \tag{3.35}$$

which, with (2.61), give (3.7). Hence, (A_k), (B_k), (C_k) are inductively established. The convergence claims follow rather directly from the established bounds. That the limit function is a fixed point can be seen by taking the limit as $k\to+\infty$ of both sides of the recursion formula (1.1). Finally, (3.2) is a consequence of (2.9). □

We now observe the following

Corollary 1. *For either of the following choices of sequences (renormalization schemes):*

(i) $$g^{(M)}(s)\equiv g^c_{-M}(s),\quad M\geqq 0, \tag{3.36}$$

(ii) $$g^{(M)}(s)\equiv g^*(s),\quad M\geqq 0, \tag{3.37}$$

the continuum limit or equivalent scaling limit

$$G^*(x_1,\dots,x_p;N)=\lim_{M\to\infty}G^{(M)}(x_1,\dots,x_p;N)$$
$$=\lim_{M\to\infty}L^{M\alpha p/2}C([\![L^Mx_1]\!],\dots,[\![L^Mx_p]\!];g^{(M)},N+M) \tag{3.38}$$

exists pointwise for non-coincident points $(x_1,\dots,x_p)\in({}^*\mathbb{F}^N)^p$ *and is the same limit for* (i) *or* (ii). *Furthermore, the Green's functions* G^* *obey the scaling law*

$$G^*(Lx_1,\dots,Lx_p;N)=L^{-\alpha p/2}G^*(x_1,\dots,x_p;N-1). \tag{3.39}$$

This corollary follows essentially directly from Theorem 2 and our earlier remarks in Sect. 4. Equation (3.39) is easily derived from the explicit expression

$$G^*(x_1,\dots,x_p;N)=L^{k\alpha p/2}C([\![L^kx_1]\!],\dots,[\![L^kx_p]\!];g^*,N+k). \tag{3.40}$$

We also note that the continuum limits of k-scale Green's functions

$$G_k(n_1,\dots,n_p;N)=\lim_{M\to\infty}G_k^{(M)}(n_1,\dots,n_p;N)$$
$$=\lim_{M\to\infty}L^{M\alpha p/2}C([\![L^Mn_1]\!],\dots,[\![L^Mn_p]\!];g_k^{(M)},N+M) \tag{3.41}$$

exist for non-coincident points $(n_1,\dots,n_p)\in({}^*\mathbb{F}_k^N)^p$ and obey a difference RG equation:

$$\delta G_k^*(n_1,\dots,n_p;N)=0. \tag{3.42}$$

4. Massive Theory with Non-Gaussian UV Fixed Point

We finally present the construction of the massive theory with the Wilson-Fisher non-Gaussian fixed-point function as its UV attractor. This result is new, although the proofs are not technically much different from the preceding. Again, we state the results, sketch the proofs and then draw the conclusions for the quantum field-theory models.

For each $k \geqq 0$, define a domain $\mathscr{D}^{(k)}$ of weight functions g by the following conditions $(\mathrm{a^k})$, $(\mathrm{b^k})$, $(\mathrm{c^k})$:

$(\mathrm{a^k})$ For $|\mathrm{Im}\, s| < |\log \varepsilon|$, $g(s)$ is an even analytic function, positive for real s and $g(0)=1$.

$(\mathrm{b^k})$ For $|s| < |\log \varepsilon|$, $g(s) = e^{-w(s)}$ with $w(s)$ analytic and

$$w(s) = c + \frac{1}{2} r : s^2 : + \frac{1}{4!} u : s^4 : + \frac{1}{6!} t : s^6 : + \tilde{w}(s) \tag{4.1}$$

with $d^i \tilde{w}(0)/ds^i = 0$, $i = 0, 2, 4, 6$ and c fixed by $w(0) = 0$. Furthermore, for some arbitrary but fixed choice of θ, $0 < \theta < 1$:

$$u = u^* + \Delta u, \qquad |\Delta u| \leqq A \varepsilon L^{-\bar{a} - 2k\varepsilon} \log L, \tag{4.2}$$

$$r = r^* + 2 r_{2c} u^* \Delta u + r_{2c} (\Delta u)^2 + \Delta r, \qquad |\Delta r| \leqq B \varepsilon^2 L^{-\bar{a} - 2k\theta} \log^2 L, \tag{4.3}$$

$$t = t^* + 2 t_{2a} u^* \Delta u + t_{2a} (\Delta u)^2 + \Delta t, \qquad |\Delta t| \leqq C \varepsilon^2 L^{-\bar{a} - 2k\theta} \log^2 L, \tag{4.4}$$

$$\tilde{w} = \tilde{w}^* + \Delta \tilde{w}, \qquad \sup_{|s| < |\log \varepsilon|} |\Delta \tilde{w}(s)| \leqq D \varepsilon^3 |\log \varepsilon|^{18} L^{-\bar{a} - 2k\theta} \log^3 L. \tag{4.5}$$

$(\mathrm{c^k})$ For $|s| \geqq |\log \varepsilon|$, $|\mathrm{Im}\, s| < L^{-\alpha/2} |\log \varepsilon|$,

$$|g(s)| \leqq \exp\left[-\frac{u}{96} ((\mathrm{Re}\, s)^4 + (\mathrm{Re}\, s)^2) \right] \tag{4.6}$$

and

$$|\Delta g(s) \equiv |g(s) - g^*(s)| \leqq L^{-2k\theta} (\mathrm{Re}\, s)^6 \exp\left[-\frac{u}{96} ((\mathrm{Re}\, s)^4 + (\mathrm{Re}\, s)^2) \right]. \tag{4.7}$$

Then, the following holds:

Theorem 3. *For each* $k \geqq 0$ *if* $g \in \mathscr{D}^{(k)}$, *then the* RG *transform* g' *exists and also obeys* $(\mathrm{a^k})$, $(\mathrm{b^k})$, $(\mathrm{c^k})$ *except possibly condition* (4.3) *of* $(\mathrm{b^k})$, *and a fortiori all of* $(\mathrm{a^{k-1}})$, $(\mathrm{b^{k-1}})$, $(\mathrm{c^{k-1}})$ *except possibly* (4.3) *of* $(\mathrm{b^{k-1}})$.

Furthermore, for some constants A', B', C', independent of k,

$$|\Delta u' - (1 - \beta_2 u) \Delta u| \leqq A' \varepsilon^3 |\log \varepsilon|^{18} L^{-2k\theta} \log^3 L, \tag{4.8}$$

$$|\Delta r' - L^2 \Delta r| \leqq B' \varepsilon^3 |\log \varepsilon|^{18} L^{-2k\theta} \log^3 L, \tag{4.9}$$

$$|\Delta t' - L^{-2} \Delta t| \leqq C' \varepsilon^3 |\log \varepsilon|^{18} L^{-2k\theta} \log^3 L. \tag{4.10}$$

Proof. The proof is essentially the same as that of Theorem 1. Note the major changes in the statements: rather than the bound (2.6) on $\tilde{w}$ one requires now a bound on $\Delta \tilde{w}$, in the large-field bounds (4.6), (4.7) the running coupling appears rather than the fixed point coupling; and, there is a second large-field bound, on

Δg, as well as on g. To derive the bound on $\Delta w'$, one employs the expression

$$y'(s) = L^{\bar{d}}\langle q_0\rangle - \frac{1}{2!}L^{2\bar{d}}\langle q_0^2\rangle^T + \frac{1}{2!}L^{3\bar{d}}\int_0^1 dt(1-t)^2\langle q_0^3\rangle_t^T$$
$$+ L^{\bar{d}}\int_0^1 dt\langle\tilde{q}\rangle_{v_0,t} + O(e^{-C|\log\varepsilon|^2}) \tag{4.11}$$

giving the perturbative contributions to second order. Writing

$$y' = y^* + \Delta y \tag{4.12}$$

and so forth, one obtains at once

$$\Delta y'(s) = L^{\bar{d}}\langle\Delta q_0\rangle - \frac{1}{2!}L^{2\bar{d}}(2\langle q_0^*;\Delta q_0\rangle^T + \langle(\Delta q_0)^2\rangle^T)$$
$$+ \frac{1}{2!}L^{3\bar{d}}\int_0^1 dt(1-t)^2(3\langle q_0^*;q_0^*;q_0\Delta\rangle_t^T + \langle(\Delta q_0)^3\rangle_t^T)$$
$$+ L^{\bar{d}}\int_0^1 dt\langle\Delta\tilde{q}\rangle_{v_0,t} + O(e^{-C|\log\varepsilon|^2}) \tag{4.13}$$

which, analyzed along the lines of Theorem 1, provides the desired bound. The proofs of the iterability of (4.6), (4.7) proceed in tandem. For (4.6) one argues as in Theorem 1, but uses

$$y' = y^* + \Delta y \tag{4.14}$$

to replace u in the analogue of (2.61) (with $\bar{u}\to u$) by u' and so proceeds to derive the bound

$$|g'(s)| \leqq \exp\left[-\frac{u'}{96}((\mathrm{Re}\,s)^4 + (\mathrm{Re}\,s)^2)\right]. \tag{4.15}$$

For (4.7), one argues essentially the same as for (3.7) of Theorem 2, noting that

$$\Delta y'(s) = L^{\bar{d}}\langle\Delta q_0\rangle - \frac{1}{2!}L^{2\bar{d}}(2\langle q_0^*;\Delta q_0\rangle^T + \langle(\Delta q_0)^2\rangle^T)$$
$$+ \frac{1}{2!}L^{3\bar{d}}\int_0^1 dt(1-t)^2(3\langle q_0^*;q_0^*;\Delta q_0\rangle_t^T + 3\langle q_0^*;\Delta q_0;\Delta q_0\rangle_t^T + \langle(\Delta q_0)^3\rangle_t^T)$$
$$+ L^{\bar{d}}\int_0^1 dt\langle\Delta\tilde{q}\rangle_{v_0,t} + 0(e^{-C|\log\varepsilon|^2}). \tag{4.16}$$

If one now takes $\delta\to\Delta$, $g'\to g$, $g\to g^*$, $L^{k\varepsilon}\to L^{-2k\theta}$, $\bar{u}\to u$, the same proof as of (3.7) carries through except for a few minor modifications. □

One next defines, for each $M \geqq 0$, certain subdomains $\mathscr{D}_m^{(M)}$, $\tilde{\mathscr{D}}_m^{(M)}$ of $\mathscr{D}^{(M)}$ by adopting, for $\mathscr{D}_m^{(M)}$, the conditions

$$\Delta r^{(M)} = L^{-2M}m^2 + \Delta\tilde{r}^{(M)}, \tag{4.17}$$
$$|\Delta\tilde{r}^{(M)}| \leqq E\varepsilon^3|\log\varepsilon|^{18}L^{-\bar{d}-2M\theta}\log^3 L, \tag{4.18}$$
$$|m^2| \leqq \tfrac{1}{2}B\varepsilon^2 L^{-\bar{d}-2M\theta}\log^2 L \tag{4.19}$$

in place of the bound in (4.3). The definition of $\tilde{\mathscr{D}}_m^{(M)}$ os identical, but with (4.18) replaced by

$$|\Delta\tilde{r}^{(M)}| \leqq \tfrac{1}{3}E\varepsilon^3|\log\varepsilon|^{18}L^{-\bar{d}-2M\theta}\log^3 L, \tag{4.20}$$

The results of Theorem 3 carry over with ease to the subdomain $\mathscr{D}_m^{(M)}$. If one defines $\Delta\tilde{r}_k^{(M)}$ in general by the equation

$$\Delta r_k^{(M)} = L^{-2k}m^2 + \Delta\tilde{r}_k^{(M)}, \quad k \leqq M, \tag{4.21}$$

then one derives

$$|\Delta\tilde{r}_{M-1}^{(M)} - L^2 \cdot \Delta\tilde{r}^{(M)}| \leqq B\varepsilon^3|\log\varepsilon|^{18}L^{-2M\theta}\log^3 L. \tag{4.22}$$

The fundamental result is then:

Theorem 4. *For each sequence* $(g^{(M)} \in \tilde{\mathscr{D}}_m^{(M)} : M \geqq 0)$ *there exists a choice of sequence* $(\Delta\tilde{r}^{(M)} : M \geqq 0)$ *such that for all* $M \geqq 0$ *and all* k, $0 \leqq k \leqq M$, $g_k^{(M)}$ *exists and belongs to* $\mathscr{D}_m^{(k)}$. *Furthermore, for all* $k \geqq 0$, *there exists a* $g_k \in \mathscr{D}_m^{(k)}$ *such that as* $M \to +\infty$,

$$g_k^{(M)}(s) \to g_k(s), \tag{4.23}$$

absolutely uniformly on the strip $|\mathrm{Im}\, s| < L^{-\alpha/2}|\log\varepsilon|$. *The sequence* $(g_k : k \geqq 0)$ *is an* RG *trajectory:*

$$g_k' = g_{k-1} \tag{4.24}$$

which has effective mass m^2 *at unit length scale, or, more generally,*

$$\Delta r_k = L^{-2k}m^2 + O(\varepsilon^3|\log\varepsilon|^{18}L^{-2k\theta}\log^3 L). \tag{4.25}$$

Finally, g_k *has* g^* *as* UV *fixed point: as* $k \to +\infty$,

$$g_k(s) \to g^*(s), \tag{4.26}$$

absolutely uniformly on the strip $|\mathrm{Im}\, s| < L^{-\alpha/2}|\log\varepsilon|$.

More specifically, with $\delta f_k^{(M)} \equiv f_k^{(M)} - f_k^{(M)}$ and for some arbitrary but fixed θ, $0 < \theta < 1$ we prove for each $M \geqq 0$ that $\Delta\tilde{r}^{(0)} \in \mathscr{B}^{(0)}, \ldots, \Delta\tilde{r}^{(M)} \in \mathscr{B}^{(M)}$ may be chosen – where $\mathscr{B}^{(J)}$ is the disk in $\mathbb{C}$ centered at 0 with radius $\frac{1}{3}E\varepsilon^3|\log\varepsilon|^{18}L^{-2J\theta}\log^3 L$ – such that for each $0 \leqq k \leqq M$:

(A_k^M) For all J, $0 \leqq J < M$ and all l, $0 \leqq l \leqq J$,

$$|\delta\Delta\tilde{r}_l^{(J)}| \leqq \tfrac{1}{3}E\varepsilon^3|\log\varepsilon|^{18}L^{-\bar{d}-2J\theta}\log^3 L. \tag{4.27}$$

Furthermore, there exists a descending chain of compacts, $\mathscr{B}_l^{(M+1)} \subseteq \mathscr{B}_{l+1}^{(M+1)}$, $k \leqq l \leqq M$, $\mathscr{B}_{(M+1)}^{(M+1)} \equiv \mathscr{B}^{(M+1)}$ such that as $\Delta\tilde{r}^{(M+1)}$ ranges over $\mathscr{B}_l^{(M+1)}$, $\delta\Delta\tilde{r}_l^{(M)}$ ranges over $\mathscr{B}^{(M)}$.

(B_k^M) For all J, $0 \leqq J \leqq M$ and all l, $0 \leqq l \leqq J$ if $J < M$ and $k \leqq l \leqq M$ if $J = M$,

$$|\delta\Delta u_l^{(J)}| = |\delta u_l^{(J)}| \leqq 2A\varepsilon L^{-\bar{d}-2J\theta}\log L, \tag{4.28}$$

$$\delta\Delta t_l^{(J)} \leqq 2C\varepsilon^2 L^{-\bar{d}-2J\theta}\log^2 L, \tag{4.29}$$

$$\sup_{|s|<|\log\varepsilon|} |\delta\Delta w_l^{(J)}(s)| = \sup_{|s|<|\log\varepsilon|} |\delta w_l^{(J)}(s)| \leqq 2D\varepsilon^3|\log\varepsilon|^{18}L^{-\bar{d}-2J\theta}\log^3 L. \tag{4.30}$$

(C_k^M) For all J, $0 \leqq J \leqq M$ and all l, $0 \leqq l \leqq J$ if $J < M$ and $k \leqq l \leqq M$ if $J = M$,

$$|g_l^{(J)}(s)| \leqq \exp\left[-\frac{u_l^{(J)}}{96}((\mathrm{Re}\,s)^4 + (\mathrm{Re}\,s)^2)\right] \tag{4.31}$$

and

$$|\delta \Delta g_l^{(J)}(s)| \leqq L^{-2J\theta}(\mathrm{Re}\,s)^6 \exp\left[-\frac{u_l^{(J)}}{96}((\mathrm{Re}\,s)^4 + (\mathrm{Re}\,s)^2)\right]. \tag{4.32}$$

Choosing successively for $M \geqq 0$ some $\Delta\tilde{r}^{(M)} \in \mathscr{B}_0^{(M)}$, the existence and convergence claims made in the theorem statement above are true. Furthermore, for all $M > 0$ and all k, $0 < k \leqq M$, one has for constants A'', B'', C'':

$$|\delta\Delta\tilde{r}_{k-1}^{(M)} - L^2 \cdot \delta\Delta\tilde{r}_k^{(M)}| \leqq B''\varepsilon^3|\log\varepsilon|^{18} L^{-2M\theta}\log^3 L, \tag{4.33}$$

$$\begin{aligned}&|\delta\Delta u_{k-1}^{(M)} - [1 - \beta_2(u^* + \Delta u_k^{(M)} + \Delta u_k^{(M+1)})]\delta\Delta u_k^{(M)}| \\ &\quad \leqq A''\varepsilon^3|\log\varepsilon|^{18} L^{-2M\theta}\log^3 L,\end{aligned} \tag{4.34}$$

$$|\delta\Delta t_{k-1}^{(M)} - L^{-2} \cdot \delta\Delta t_k^{(M)}| \leqq C''\varepsilon^3|\log\varepsilon|^{18} L^{-2M\theta}\log^3 L. \tag{4.35}$$

Proof. The proof is very similar to the proof of Theorem 2. The proof of assertions (A_k^M), (B_k^M), (C_k^M) is by induction on M and, within the induction step, a second induction on k.

The proofs of the induction step assertions (B_0^0), (C_0^0) is direct. We note that the proof of (A_0^0) requires the parameter θ that has been introduced. Indeed, one can easily show that for some arbitrary but fixed $\Delta\tilde{r}^{(0)} \in \mathscr{B}^{(0)}$ as $\Delta\tilde{r}^{(1)}$ ranges over $\mathscr{B}^{(1)}$, $L^2\Delta\tilde{r}^{(1)} - \Delta\tilde{r}^{(0)}$ ranges over the displaced disk $\mathscr{B}^{(1)} - \Delta\tilde{r}^{(0)}$ which contains the disk of radius $(L^{2(1-\theta)} - 1)ER$ centered at the origin. Using (4.22) one can then argue that $\delta\Delta\tilde{r}^{(0)}$ ranges over a set containing $\mathscr{B}^{(0)}$ as $\Delta\tilde{r}^{(1)}$ ranges over $\mathscr{B}^{(1)}$. Defining $\delta\mathscr{R}_k^{(M)} : \mathscr{B}_{(M+1)} \to \mathbb{C}$ for k, $0 \leqq k \leqq M$, by $\delta\mathscr{R}_k^{(M)}[\Delta\tilde{r}^{(M+1)}] \equiv \delta\Delta\tilde{r}_k^{(M)}$, one can then define

$$\mathscr{B}_0^{(1)} \equiv (\delta\mathscr{R}_0^0)^{-1}[\mathscr{B}^{(0)}] \cap \mathscr{B}^{(1)}, \tag{4.36}$$

which gives (A_0^0). Now, assuming for $M-1$ that ($A_k^{(M-1)}$), ($B_k^{(M-1)}$), ($C_k^{(M-1)}$) hold for all k, $0 \leqq k \leqq M-1$, one seeks to prove (A_k^M), (B_k^M), (C_k^M) for all k, $0 \leqq k \leqq M$. The proof is by induction on k.

The proof of the initial step $k = M$ is virtually identical to that for $M = 0$ above.

For the induction step it suffices to establish (A_k^M), (B_k^M), (C_k^M) only for the case $J = M$. Then, in turn one checks:

(B_{k-1}^M) The proof is very similar to the proof of (B_{k-1}) in Theorem 2. One observes that

$$\begin{aligned}\Delta y_{k-1} &= y_{k-1} - y^* = -\log\int d\mu e^{-L^{\bar{d}}q_k} + \log\int d\mu e^{-L^{\bar{d}}q^*} \\ &= -\log\left[\int d\mu e^{-L^{\bar{d}}q_k} / \int d\mu e^{-L^{\bar{d}}q^*}\right]\end{aligned} \tag{4.37}$$

and analyzes this expression as (3.18) there. In particular, this analysis yields the iteration equations (4.33–35), which allow one to establish the bounds (4.28–29). To iterate the bound (4.30) for $\delta\Delta w$ one uses the irrelevancy factor $L^{-4+3\varepsilon}$ of that variable.

(A^M_{k-1}) The proof is essentially the same as above and again requires the use of the arbitrary parameter θ, $0<\theta<1$. One defines

$$\mathscr{B}^{(M+1)}_{k-1} \equiv (\delta \mathscr{R}^{(M)}_{k-1})^{-1}[\mathscr{B}^{(M)}] \cap \mathscr{B}^{(M+1)}_{k}, \tag{4.38}$$

which is seen to have the correct properties.

(C^M_{k-1}) One notes that

$$\begin{aligned}\delta \Delta g^{(M)}_{k-1}(s) &= \delta g^{(M)}_{k-1}(s) = g^{(M+1)}_{k-1}(s) - g^{(M)}_{k-1}(s)\\ &= (\int f^{(M+1)}_k(s,z)d\mu(z)/\int f^{(M+1)}_k(0,z)\\ &\quad -(\int f^{(M)}_k(s,z)d\mu(z)/\int f^{(M)}_k(0,z)d\mu(z))\\ &= (\int \delta f^{(M)}_k(s,z)d\mu(z)/\int f^{(M+1)}_k(0,z))\\ &\quad - g^{(M)}_{k-1}(s)\,(\int \delta f^{(M)}_k(0,z)d\mu(z)/\int f^{(M+1)}_k(0,z)d\mu(z))\end{aligned} \tag{4.39}$$

which corresponds exactly to (3.32) of Theorem 2 and is employed in a similar fashion as there to iterate the bounds (4.31), (4.32) for $J=M$, $l=k-1$, establishing (C^M_{k-1}).

Therefore, (A^M_k), (B^M_k), (C^M_k) are inductively established and the existence and convergence claims at the beginning of Theorem 4 follow rather directly.

To establish (4.24) one takes the limit as $M\to+\infty$ of both sides of the recursion formula,

$$g^{(M)}_{k-1}(s) = \int g^{(M)}_k(L^{-\alpha/2}s+z)^{L^{\bar{d}}/2} g^{(M)}_k(L^{-\alpha/2}s-z)^{L^{\bar{d}}/2}d\mu(z)/\int g^{(M)}_k(z)^{L^{\bar{d}}}d\mu(z), \tag{4.40}$$

using uniformity of the limit $g^{(M)}_k \to g_k$ on the strip $|\mathrm{Im}\, s| < L^{-\alpha/2}|\log\varepsilon|$, whereas (4.26) follows from the fact that $g_k \in \mathscr{D}^{(k)}_m$ for each $k\geqq 0$ and employing the bounds on $\Delta g = g_k - g^*$ which appear in the definition of $\mathscr{D}^{(k)}_m$. □

We conclude with

Corollary 3. *For any choices of sequences (renormalization schemes)*

$$(g^{(M)} \in \tilde{\mathscr{D}}^M_m \mid M>0) \tag{4.41}$$

with $\Delta\tilde{r}^{(M)}$ adjusted as in Theorem 4, the continuum (scaling) limit

$$\begin{aligned}G(x_1,\ldots,x_p;N,m) &= \lim_{M\to\infty} G^{(M)}(x_1,\ldots,x_p;N,m)\\ &= \lim_{M\to\infty} L^{M\alpha p/2}C([\![L^M x_1]\!],\ldots,[\![L^M x_p]\!];g^{(M)},N+M)\end{aligned} \tag{4.42}$$

exists pointwise for non-coincident points $(x_1,\ldots,x_p)\in({}^*\mathbb{F}^N)^p$. The continuum Green's function obeys the asymptotic scaling relation

$$G(L^{-M}x_1,\ldots,L^{-M}x_p;N-M,m) \sim L^{M\alpha p/2}G^*(x_1,\ldots,x_p;N) \tag{4.43}$$

as $M\to+\infty$.

Note that (4.43) follows from the explicit expression

$$G(x_1,\ldots,x_p;N,m) = L^{k\alpha p/2}C([\![L^k x_1]\!],\ldots,[\![L^k x_p]\!];g_k,N+k) \tag{4.44}$$

for $k \geqq Q(x_1, \ldots, x_p)$. Also, the continuum limits $G_k(n_1, \ldots, n_p; N, m)$ of k-scale blockfield Green's functions exist, obey the RG equation (3.42), and also obey

$$G_k(n_1, \ldots, n_p; N, m) \sim G^*(n_1, \ldots, n_p; N) \tag{4.45}$$

as $k \to +\infty$.

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