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# Hidden $SL(n)$ Symmetry in Conformal Field Theories

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*This paper is dedicated to the memory of Vadik G. Knizhnik*

**Abstract.** We prove that an irreducible representation of the Virasoro algebra can be extracted from an irreducible representation space of the $SL(2,\mathscr{R})$ current algebra by putting a constraint on the latter using the Becchi–Rouet–Stora–Tyutin formalism. Thus there is a $SL(2,\mathscr{R})$ symmetry in the Virasoro algebra, but it is gauged and hidden. This construction of the Virasoro algebra is the quantum analogue of the Hamiltonian reduction. We then are naturally lead to consider a constrained $SL(2,\mathscr{R})$ Wess–Zumino–Witten model. This system is also related to quantum field theory of coadjoint orbit of the Virasoro group. Based on this result, we present a canonical derivation of the $SL(2,\mathscr{R})$ current algebra in Polyakov's theory of two-dimensional gravity; it is a manifestation of the $SL(2,\mathscr{R})$ symmetry in conformal field theory hidden by the quantum Hamiltonian reduction. We also discuss the quantum Hamiltonian reduction of the $SL(n,\mathscr{R})$ current algebra and its relation to the $W_n$-algebra of Zamolodchikov. This makes it possible to define a natural generalization of the geometric action for the $W_n$-algebra despite its non-Lie-algebraic nature.

## 1. Introduction

Among various favourable properties of string theory as a candidate for the unified theory of everything, the uniqueness of target spacetime dimensions is one of the most appealing. It is therefore crucial to know whether string theory is possible off the critical dimensions. This question is also relevant in understanding the large-$N_c$ limit of QCD in four dimensions, and many attempts have been made to solve string theory below criticality. Kazakov and Migdal [1] have studied various statistical models on triangulated random surfaces, and computed scaling dimensions. Last year Polyakov examined the two-dimensional gravity induced

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by conformal field theory, and found that there is an $SL(2,\mathscr{R})$ current algebra [2]. This result opened the way to solve the off-critical string theory. Later Knizhnik, Polyakov and Zamolodchikov exploited this observation to evaluate scaling dimensions of planar random surfaces [3]. Their result shows complete agreement with previous computations by Kazakov and Migdal. This seems to suggest the validity of Polyakov's observation. Still, the way the $SL(2,\mathscr{R})$ current algebra emerges is like a bolt out of the blue. He computed correlation functions of metrics using the Ward identity of the energy-momentum tensor, and showed that they contain the $SL(2,\mathscr{R})$ current algebra. One of the motivations of this paper is to obtain a canonical derivation of this current algebra and to understand the structure of the off-critical string theory.

It has been suspected by several people that there should be hidden relations between the Virasoro algebra and the $SL(2,\mathscr{R})$ current algebra, and in general, between the $W_n$-algebra and the $SL(n,\mathscr{R})$ current algebra. The $W_n$-algebra is an extension of the Virasoro algebra with additional chiral operators of spin-$n$ [4]. For example, Fateev and Lykyanov [5] computed highest weights of completely degenerate representations of the $W_n$-algebra, and found that they can be expressed in terms of highest weights of the $SL(n,\mathscr{R})$ algebra. There is also an intriguing connection with the classical version of these algebras observed in the context of the Korteweg-de Vries type Eqs. [6]. Consider a dual space of the $SL(n,\mathscr{R})$ loop algebra. This space is endowed with a natural Poisson bracket, and we may regard it as a classical phase space. This phase space has a certain symmetry, and one may consider the reduced phase space with respect to this symmetry. The Poisson bracket for the reduced phase space turns out to be the classical version of the $W_n$-algebra. This procedure is called the Hamiltonian reduction.

In this paper, we develop the quantum analogue of the Hamiltonian reduction. We replace the Poisson bracket by a commutation relation of operators, and the classical phase space by an irreducible representation of the algebra. An attempt in this direction was initiated by Belavin [7]. The irreducible representation spaces of the Virasoro algebra are extracted from those of the $SL(2,\mathscr{R})$ current algebra by imposing a certain constraint on the latter. Consider an irreducible representation space of the current algebra. In classical mechanics, we put a constraint $J^-(z)=1$ to reduce the phase space of the loop algebra. Quantum mechanically, we introduce a set of ghosts and define the Becchi–Rouet–Stora–Tyutin (BRST) operator associated with this constraint. It is then proved that a quotient $\mathrm{Ker}(Q_{\mathrm{BRST}})/\mathrm{Im}(Q_{\mathrm{BRST}})$ is isomorphic to an irreducible representation space of the Virasoro algebra. The idea of our proof is the following. Both the Virasoro algebra [8] and the $SL(2,\mathscr{R})$ current algebra [10] have realizations in terms of free bosons. Although such realizations are highly reducible, there are BRST-like operators whose cohomologies are isomorphic to irreducible representations of these algebras [12, 13]. The point is that the BRST-like operators for these algebras are equivalent modulo trivial operators with respect to the BRST operator $Q_{\mathrm{BRST}}$ for the constraint $J^-(z)=1$.

We are then naturally lead to consider the $SL(2)$ Wess–Zumino–Witten (WZW) model with one of its right-moving currents, $J^-(z)$, being gauged as

$$\int \frac{[g^{-1}dg, d\bar{A}]}{(\text{gauge volume})} \exp\left( ikS(g) + i\int \frac{d^2z}{8\pi} \bar{A}(J^- - 1) \right).$$

The physical Hilbert space of its right-moving sector gives irreducible representations of the Virasoro algebra. Thus there is a $SL(2)$ symmetry in the Virasoro representations, but it is gauged and is not observable. This system is also equivalent to quantum field theory of the coadjoint orbit of the Virasoro group. The geometric quantization of the Virasoro group was previously discussed by Witten [14]. Based on this result, we present a canonical derivation of the $SL(2)$ current algebra in the induced gravity; it is a manifestation of the hidden $SL(2)$ symmetry in conformal field theory.

In understanding various aspects of conformal field theories, it has proved fruitful to explore the interplay between the Virasoro algebra and the complex geometry of Riemann surfaces. Is there also some geometrical structure behind the $W_n$-algebra? To answer this question, we must understand what kind of symmetry the $W_n$-algebra implies. The Virasoro algebra is the consequence of reparametrization and Weyl scaling invariance of a field theory, and the structure of these symmetries is encoded into the geometric action of the Virasoro algebra. Thus the first step to appreciate the geometric aspect of the $W_n$-algebra would be to construct a geometric action for this algebra. The $W_n$-algebra is not a Lie algebra, but an algebra with quadratic relations. Usually a geometric action is defined for a Lie group, and one might suspect that there should be no such action for the $W_n$-algebra. Still the quantum Hamiltonian reduction makes it possible to define a natural generalization of the geometric action for the $W_n$-algebra.

The paper is organized as follows. In Sect. 2, we study an effective theory of gauge fields coupled to the WZW model. This gives a prototype of our construction of induced gravity in later sections. Section 3 is devoted to proving the quantum Hamiltonian reduction. In this section, we first recapitulate the classical Hamiltonian reduction following the result of Drinfeld and Sokolov [6]. A reader may wish to skip this part in the first reading. We then prove the quantum Hamiltonian reduction in the case of $SL(2,\mathscr{R})$ exploiting the free boson realizations of the Virasoro and the current algebras. We also discuss the quantum Hamiltonian reduction of the $SL(n,\mathscr{R})$ current algebra and its relation to the $W_n$-algebra. In Sect. 4, we consider the constrained WZW model. Due to the quantum Hamiltonian reduction, this system gives irreducible representations of the Virasoro algebra. At the classical level, the constrained WZW model is equivalent to the field theory of the coadjoint orbits of the Virasoro group. We then discuss the quantization of the Virasoro group. These results are applied to the induced gravity in Sect. 5. It turns out that the quantum gravity is equivalent to the quantum field theory of the coadjoint orbits of the Virasoro group, which, in turn, is related to the constrained $SL(2,\mathscr{R})$ WZW model. In the last section, we consider a generalized geometric action for the $W_n$-algebra, and discuss its symmetries.

*Notation and Conventions.* In this paper, we employ the Lorentzian signature metric in two-dimensions. To avoid complication in notation, we denote light-cone coordinates by $z = t + x$ and $\bar{z} = t - x$, where $x$ and $t$ are space and time coordinates respectively. The $z$-dependent sector is often called the right-mover and the $\bar{z}$-dependent sector is the left-mover. Throughout this paper, $SL(n)$ is meant to be $SL(n, \mathbb{R})$.

## 2. Gauge Field Coupled to Wess–Zumino–Witten Model

In this section, we describe an effective theory of gauge fields coupled to the WZW model following Polyakov [15]. The cocycle condition of WZW action plays a central role in understanding the dynamics of the effective theory. This is a prototype of our construction of induced gravity in later sections.

The effective action $\Gamma(\bar{A})$ for the gauge field $\bar{A}$ coupled to the WZW model is given by

$$\begin{aligned} \exp(i\Gamma(\bar{A})) &= \left\langle \exp\left( + i\int \frac{d^2z}{8\pi} \bar{A}^a J^a \right) \right\rangle_{\text{WZW model}} \\ &= \int [g^{-1}dg] \exp\left( ikS_{\text{wzw}}(g) + i\int \frac{d^2z}{8\pi} \bar{A}^a J^a \right) \qquad (1) \\ J^a &= -\frac{k}{2} \operatorname{tr}(t^a \partial g g^{-1}), \end{aligned}$$

where $t^a$ is a generator of the gauge group. Since $\Gamma(\bar{A})$ is also a generating functional for correlation functions of the currents $J^a$, the operator product expansion

$$J^a(z)J^b(w) \sim \frac{f^{abc}}{z-w} J^c(w) + \frac{k/2}{(z-w)^2}\delta^{ab},$$

when applied to Eq. (1), implies the following functional differential equation for $\Gamma(\bar{A})$:

$$(\delta^{ac}\bar{\partial} + f^{abc}\bar{A}^b(z,\bar{z}))\frac{\delta\Gamma(\bar{A})}{\delta\bar{A}^c(z,\bar{z})} = -\frac{k/2}{8\pi}\partial\bar{A}^a(z,\bar{z}). \qquad (2)$$

Let us now quantize the gauge field $\bar{A}$ with this action $\Gamma(\bar{A})$. The correlation function is given by

$$\langle \bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_n}(\bar{z}_n)\rangle = \int [d\bar{A}]\bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_n}(\bar{z}_n)e^{i\Gamma(\bar{A})}. \qquad (3)$$

From the functional differential Eq. (2) for $\Gamma(\bar{A})$, one can derive the following identity,

$$\begin{aligned} \left\langle \bar{A}^a(\bar{z}) \prod_j \bar{A}^{b_j}(\bar{w}_j) \right\rangle &= \sum_i \frac{f^{ab_ic}}{\bar{z}-\bar{w}_i} \left\langle \bar{A}^c(\bar{w}_i) \prod_{j\neq i} \bar{A}^{b_j}(w_j) \right\rangle \\ &\quad - \frac{k+2c_V}{2} \sum_i \frac{\delta^{ab_i}}{(\bar{z}-\bar{w}_i)^2} \left\langle \prod_{j\neq i} \bar{A}^{b_j}(\bar{w}_j) \right\rangle, \qquad (4) \end{aligned}$$

after changing normalization of the gauge field, $\bar{A} \to (k + 2c_V)\bar{A}$. Here $c_V$ is the dual

Coxeter number of the gauge group. Thus the correlation function of gauge fields make the current algebra of level $\tilde{k} = -(k + 2c_V)$. Deriving Eq. (4), we performed integration by parts in the functional integral (3). There we used the anomalous transformation property of the functional integral measure

$$(\delta^{ac}\bar{\partial} + f^{abc}\bar{A}^b)\frac{\delta}{\delta\bar{A}^c}[d\bar{A}] = -\frac{c_V}{8\pi}\partial\bar{A}^a[d\bar{A}]. \tag{5}$$

This is where the shift $-k \to -(k + 2c_V)$ of the level comes from.

We can also compute the effective action $\Gamma(\bar{A})$ of the gauge fields directly. Using the cocycle condition of the WZW action [16, 17],

$$S_{\mathrm{wzw}}(Ug) = S_{\mathrm{wzw}}(g) + S_{\mathrm{wzw}}(U) - \int \frac{d^2z}{16\pi}\mathrm{tr}(U^{-1}\bar{\partial}U)(\partial g g^{-1}), \tag{6}$$

the right-hand side of Eq. (1) can be rewritten as

$$\exp(i\Gamma(\bar{A})) = \exp(-ikS_{\mathrm{wzw}}(U))\int [g^{-1}dg]\exp(ikS_{\mathrm{wzw}}(Ug)), \tag{7}$$

where $U$ is related to $\bar{A}$ as $\bar{A} = U^{-1}\bar{\partial}U$. Since the measure $[g^{-1}dg]$ is invariant under the left action of the gauge group $g \to U^{-1}g$, the result of the $g$-integration is independent of $U$. Thus we obtain

$$\Gamma(\bar{A}) = -kS_{\mathrm{wzw}}(U), \quad \bar{A} = U^{-1}\bar{\partial}U. \tag{8}$$

It is easy to check that $\Gamma(\bar{A})$ in the above solves the functional differential Eq. (2). In fact Eq. (2) is the infinitesimal version of the cocycle condition of the WZW action. Now we can compute the correlation function (3) of the gauge fields as

$$\begin{aligned}\langle \bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_n}(\bar{z}_n)\rangle &= \int [dA]\bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_n}(\bar{z}_n)\exp(-ikS_{\mathrm{wzw}}(U))\\ &= \int [U^{-1}dU]\bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_n}(\bar{z}_n)\exp(-i(k+2c_V)S_{\mathrm{wzw}}(U)).\end{aligned} \tag{9}$$

The shift of the factor $-k \to -(k + 2c_V)$ in front of the WZW action is due to the Jacobian under the change of variable $\bar{A} \to U$.

$$[d\bar{A}] = \det(\delta^{ac}\bar{\partial} + f^{abc}\bar{A}^b)[U^{-1}dU] = \exp(-i2c_V S_{\mathrm{wzw}}(U))[U^{-1}dU]. \tag{10}$$

From the above Eq. (9), it is clear that the correlation functions of the gauge fields make the current algebra of level $\tilde{k} = -(k + 2c_V)$.

It is also possible to consider the gauge field in the presence of several primary fields $\Phi(z)$,

$$\mathscr{Z}(\bar{A}; w_1\cdots w_n) = \int [g^{-1}dg]\Phi_1(w_1)\cdots\Phi_n(w_n)\exp\left(ikS_{\mathrm{wzw}}(g) + i\int\frac{d^2z}{8\pi}\bar{A}^aJ^a\right), \tag{11}$$

where $\Phi_i(w_i)$ has the operator product expansion with the current as

$$J^a(z)\Phi_i(w_i) \sim \frac{t_i^a}{z - w_i}\Phi_i(w_i, \bar{w}_i). \tag{12}$$

As in the case of the effective action, we can derive the functional differential equation for $\mathscr{L}$ as

$$(\delta^{ac}\bar{\partial} + f^{abc}\bar{A}^b(z,\bar{z}))\frac{\delta\mathscr{L}}{\delta\bar{A}^c(z,\bar{z})} = \frac{i}{8\pi}\left(-\frac{k}{2}\partial\bar{A}^a(z,\bar{z}) + \sum_{i=1}^{n} t_i^a\delta(z-w_i)\right)\mathscr{L}. \tag{13}$$

From this equation, it is easy to show that the correlation function of the gauge fields

$$\langle \bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_m}(\bar{z}_m)\rangle_{w_1\cdots w_n} = \int [d\bar{A}]\bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_m}(\bar{z}_m)\mathscr{L}(\bar{A};w_1\cdots w_n) \tag{14}$$

behaves as

$$\langle \bar{A}^{a_1}(\bar{z}_1)\cdots\bar{A}^{a_m}(\bar{z}_m)\rangle_{w_1\cdots w_n} \sim \frac{-t_j^{a_1}}{\bar{z}_1-\bar{w}_j}\langle \bar{A}^{a_2}(\bar{z}_2)\cdots\bar{A}^{a_m}(\bar{z}_m)\rangle_{w_1\cdots w_n}, \quad \bar{z}_1 \to \bar{w}_j. \tag{15}$$

With slight abuse of notation, one may express the above as

$$\bar{A}^a(\bar{z})\Phi(w,\bar{w}) \sim \frac{-t^a}{\bar{z}-\bar{w}}\Phi(w,\bar{w}). \tag{16}$$

It is clear that the highest weight state of the *right-moving* current algebra of level $k$ corresponds to the lowest weight state[1] of the induced *left-moving* current algebra of level $-(k+2c_V)$.

## 3. $SL(n)$ Symmetry Hidden in $W_n$-Algebra

In this and the next section, we develop tools to study an effective theory of gravity coupled to conformal field theory. In this section, we point out that there is an $SL(n)$ symmetry hidden in the $W_n$-algebra. The relation between classical versions of the $W_n$ and the $SL(n)$ current algebras has been known in the context of the Korteweg-de Vries type equations. Consider a dual space of the $SL(n)$ loop algebra. This space has a natural Poisson bracket, and can be regarded as a classical phase space. This phase space has a certain symmetry, and one may reduce it with respect to this symmetry. The Poisson bracket of the reduced phase space turns out to be the classical version of the $W_n$-algebra. We review this classical Hamiltonian reduction briefly following the paper by Drinfeld and Sokolov [6]. The rest of this paper is understandable without knowing the classical Hamiltonian reduction, and the reader may wish to skip this part for the first reading. We then develop the quantum analogue of this Hamiltonian reduction.

*3.1 Classical Hamiltonian Reduction.* For the purpose of illustration, let us start with the finite dimensional situation. Let $M$ be a finite dimensional phase space. This means that $M$ is implemented with a non-degenerate symplectic form $\omega$, which defines a Poisson bracket $\{,\}_{\mathrm{PB}}$. Suppose that a group $G$ acts on $M$ while preserving the symplectic form $\omega$. The group $G$ is then the symmetry of the phase space $M$. Now we are going to define the reduced phase space with respect to this symmetry.

[1] By the lowest weight state, we mean the one with respect to the zero mode of the current algebra

Each element $e$ of $\mathscr{G}$, the Lie algebra of the symmetry group $G$, defines a vector field $v_e$ on the phase space. Assume that for each $e \in \mathscr{G}$ there is a function $H_e(x)$ on $M$ such that

$$v_e F(x) = \{H_e(x), F(x)\}_{\mathrm{PB}} \quad \text{for any function } F(x) \text{ on } M. \tag{17}$$

In this case the symmetry is called Hamiltonian. Associated with this Hamiltonian structure of the symmetry $G$, there is a canonical momentum mapping $P$ from the phase space $M$ into the dual of the Lie algebra $\mathscr{G}^*$ defined as

$$x \in M \to P(x) \in \mathscr{G}^* : \langle P(x), e \rangle = H_e(x). \tag{18}$$

For some $e^* \in \mathscr{G}^*$ consider a level set of the momentum $P^{-1}(e^*) = M_{e*} \subset M$, and let $G_{e*}$ be a stationary subgroup of $G$ mapping $M_{e*}$ into itself. One may then consider the quotient space $M_{e*}/G_{e*}$, which has a natural symplectic structure (for a proof see for example Appendix 5 of a book by Arnold [18]). This defines the reduced phase space.

Now we are going to apply this construction to the case when $M$ is a dual space of $SL(n)_k$, the level-$k$ central extension of $SL(n)$-loop algebra. In this case, the symmetry algebra $\mathscr{G}$ will be a subalgebra of $SL(n)_k$ given below. To be more explicit, an element of $SL(n)_k$ is a pair $(A(z), a_0)$ where $A(z)$ is the mapping from circle into $SL(n)$ and $a_0$ is a constant number. The commutator of this algebra is given by

$$[(A(z), a_0), (B(z), b_0)] = ([A(z), B(z)], \quad k \oint \mathrm{tr}\,(A(z)\partial_z B(z))dz). \tag{19}$$

The dual space of $SL(n)_k$ is defined with respect to the following pairing:

$$\begin{aligned} &\langle x, a \rangle = \oint \mathrm{tr}\,(J(z)A(z))dz + x_0 a_0 \\ &\qquad x = (J(z), x_0) \in (SL(n)_k)^*, \quad a = (A(z), a_0) \in SL(n)_k. \end{aligned} \tag{20}$$

The Poisson bracket of the phase space $M = (SL(n)_k)^*$ is defined as

$$\begin{aligned} &\{J^a(z), J^b(w)\}_{\mathrm{PB}} = f^{abc} J^c(w)\delta(z-w) + \frac{k}{2}\delta^{ab}\delta'(z-w), \\ &\{x_0, J^a(z)\}_{\mathrm{PB}} = 0, \quad \{x_0, y_0\}_{\mathrm{PB}} = 0, \end{aligned} \tag{21}$$

where

$$J^a(z) = \mathrm{tr}\,(t^a J(z)).$$

We consider a subalgebra $\mathscr{G}$ of $SL(n)_k$ consisting of pairs $e = (E(z), e_0)$ with $E(x)$ in the Borel subalgebra of $SL(n)$ (subalgebra generated by strictly upper triangular matrices). The action of $\mathscr{G}$ on $M = (SL(n)_k)^*$ is coadjointly defined as

$$x \in M \to \mathrm{Ad}_e^*(x) \in M \quad (e \in \mathscr{G}), \tag{22}$$

where

$$\langle \mathrm{Ad}_e^*(x), a \rangle = -\langle x, \mathrm{Ad}_e(a) \rangle, \quad \forall a \in SL(n)_k, \tag{23}$$

and $\mathrm{Ad}_e$ is the adjoint action of $e \in \mathscr{G}$ on $SL(n)_k$.

Now that we have the phase space $M = (SL(n)_k)^*$ with the symmetry $\mathscr{G} =$

(subalgebra of $SL(n)_k$), we can define a reduced phase space with respect to this symmetry. We are going to show that the Poisson bracket in the reduced phase space gives the classical $W_n$-algebra. The dual space $\mathscr{G}^*$ is isomorphic to the quotient $(SL(n)_k)^*/(U(x), 0)$ where $U(x)$ belongs to the Borel subgroup of $SL(n)$. As a representative of a point in $\mathscr{G}^*$, one may take $e^* = (E^*(z), 1) \in (SL(n)_k)^*$, where $E^*(z)$ is in the form

$$\begin{pmatrix} * & 1 & 0 & 0 & \cdots & 0 \\ * & * & 1 & 0 & \cdots & 0 \\ * & * & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ * & * & & & * & 1 \\ * & * & \cdots & \cdots & * & * \end{pmatrix} . \tag{24}$$

For such choice of $e^*$, the level set $M_{e^*}$ of the momentum mapping $P$ consists of $(J(z), 1)$ where $J(z)$ is also of the form (24). This condition is expressed in terms of the currents $J^{-a}(z)$ as

$$J^{-a} = \begin{cases} 1, & \text{if } a \text{ is simple root,} \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

for any positive root $a$. The stationary subgroup $G_{e^*}$ of $M_{e^*}$ consists of pairs $(U(z), 1)$, where $U(z)$ belongs to the Borel subgroup of $SL(n)$. It acts on $M_{e^*}$ by a coadjoint action. To be explicit,

$$\mathrm{Ad}_U^*(J(z)) = UJ(z)U^{-1} - \frac{k}{2}\partial_z U U^{-1}. \tag{26}$$

In order to describe the Poisson bracket on reduced phase space let us consider some specific coordinate system on $M_{e^*}/G_{e^*}$. Exploiting the gauge symmetry (26) one can always put $J(z)$ into the form

$$J(z) = \begin{pmatrix} \phi_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \phi_2 & 1 & 0 & & 0 \\ 0 & 0 & \phi_3 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & & & 0 & \phi_{n-1} & 1 \\ 0 & 0 & \cdots & 0 & 0 & \phi_n \end{pmatrix} \tag{27}$$

where $\sum \phi_j = 0$. It is convenient to introduce the fields $\varphi_j$ defined as

$$\begin{aligned} &\phi_i = \varphi_i - \varphi_{i-1} \quad (i = 2, \ldots, n-1), \\ &\phi_1 = \varphi_1, \quad \phi_n = -\varphi_{n-1}, \end{aligned} \tag{28}$$

and to use them as coordinates over the reduced phase space $M_{e^*}/G_{e^*}$. In terms of fields $\varphi_j$, the Poisson brackets acquire the following simple form:

$$\{\varphi_i(x)\varphi_j(y)\}_{\mathrm{PB}} = \frac{k}{2}\delta'(x-y)K_{ij}, \tag{29}$$

where $K_{ij}$ is Cartan matrix of $SL(n)$. They are the Poisson brackets of the free scalar fields.

What happens if one chooses another choice of gauge slice? For example, instead of Eq. (24), one may also reduce $J(z)$ to the following form:

$$J(z)=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ u_n & u_{n-1} & u_{n-2} & \cdots & u_2 & 0 \end{pmatrix}. \tag{30}$$

In order to clarify the relation between these two sets of coordinates, let us consider the differential equation $((k/2)\partial_z - J(z))\vec{v}(z)=0$, where $\vec{v}(z)$ is a $n$-dimensional vector. Due to the following form of this equation

$$\left(\frac{k}{2}\partial_z - \begin{pmatrix} * & 1 & 0 & 0 & \cdots & 0 \\ * & * & 1 & 0 & \cdots & 0 \\ * & * & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ * & * & & & * & 1 \\ * & * & \cdots & \cdots & * & * \end{pmatrix}\right)\begin{pmatrix} v_1(z) \\ v_2(z) \\ \vdots \\ \vdots \\ v_{n-1}(z) \\ v_n(z) \end{pmatrix}=0, \tag{31}$$

one can eliminate all the components of the vector $\vec{v}(z)$ but $v_1(z)$ and obtain an $n^{\text{th}}$ order differential equation $\mathscr{L}[J]v_1(z)=0$ for $v_1(z)$. It is easy to show that the differential operator $\mathscr{L}[J]$ is invariant under the gauge transformation (26). Thus by computing $\mathscr{L}[J]$ in two cases, (27) and (30), and identifying them, we get the relation in a compact form as

$$\prod\left(\frac{k}{2}\partial_z - \phi_j(z)\right) = -\sum u_{n-j}(z)\left(\frac{k}{2}\partial_z\right)^j, \quad u_0=-1, \quad u_1=0. \tag{32}$$

This relation is known as the Miura transformation in the theory of the Korteweg-de Vries type equations. Using this relation we can rewrite the Poisson bracket (29) in terms of $u_j$. The fields $u_j$ make an associative algebra with quadratic relations known as the Gelfand–Dickey algebra,

$$\{u_i(z)u_j(w)\}_{\text{PB}} = C_{ij}(z-w) + F_{ij}^k(z-w)u_k + D_{ij}^{kl}(z-w)u_k u_l. \tag{33}$$

For the case of $SL(2)$ the algebra (33) reduces to the standard Lie algebra

$$\{u(z)u(w)\}_{\text{PB}} = (\partial u(w) + 2u(w)\partial + k^2\partial^3)\delta(z-w), \tag{34}$$

and this is the classical version of the Virasoro algebra. The case of $SL(3)$ gives us the first non-trivial example of an algebra with a quadratic relation which is the classical version of $W_3$-algebra discovered by Zamolodchikov [4]. In general algebras (33) correspond to $W_n$-algebras of spin $n$. For more details we refer the reader to the original papers [6, 7, 19].

*3.2 Quantum Hamiltonian Reduction; Case of SL(2).* Now we are going to develop the quantum analogue of the Hamiltonian reduction. To motivate our construction, let us start with some numerology in the case of $SL(2)$ current algebra. For the level-$k$ $SL(2)$ current algebra,

$$J^+(z)J^-(w) \sim \frac{2}{z-w}J^3(w) + \frac{k}{(z-w)^2},$$

$$J^3(z)J^\pm(w) \sim \frac{\pm 1}{z-w}J^\pm(w), \quad J^3(z)J^3(w) \sim \frac{k/2}{(z-w)^2}, \tag{35}$$

there is a canonical definition of an energy-momentum tensor (the Sugawara construction):

$$T_{SL(2)}(z) = \frac{1}{k+2}\sum :J^a(z)J^a(z):. \tag{36}$$

With respect to this energy-momentum tensor, the currents $J^\pm, J^3$ behave as conformal fields of weight 1. In order to put the constraint $J^-(z) = 1$ consistently with the conformal invariance, this property of $T_{SL(2)}$ is not convenient, and we wish to deform the energy-momentum tensor so that the conformal weight of $J^-$ vanishes [7, 20],

$$T_{\text{improved}}(z) = T_{SL(2)}(z) - \partial J^3(z). \tag{37}$$

The central charge for this improved energy-momentum tensor is

$$c^k_{SL(2)} = \frac{3k}{k+2} - 6k = 15 - \frac{6}{k+2} - 6(k+2). \tag{38}$$

On the other hand, a conformal anomaly for a degenerate representation of the Virasoro algebra is given by the formula

$$c_{\text{vir}} = 1 - 6\frac{(p-q)^2}{pq} = 13 - \frac{6}{p/q} - 6p/q. \tag{39}$$

Substituting $p/q = k+2$ in Eq. (39), we obtain the relation

$$c^{(k)}_{\text{vir}} = 13 - \frac{6}{k+2} - 6(k+2) = c^{(k)}_{SL(2)} - 2. \tag{40}$$

We note that the difference of $c^{(k)}_{SL(2)}$ and $c^{(k)}_{\text{vir}}$ is independent of $k$. In fact $-2$ is equal to the conformal anomaly of a ghost system $(b(z), c(z))$ of weights $(0, 1)$. Ghosts of such weights naturally emerge if we put the constraint $J^-(z) = 1$ using the Becchi–Rouet–Stora–Tyutin (BRST) formalism.

This observation leads us to the following conjecture. Consider an irreducible representation space of the level-$k$ $SL(2)$ current algebra $\mathscr{H}^{(k)}_{SL(2)}$ and the Fock space of the ghost system $\mathscr{H}_{b,c}$. The BRST operator defined by

$$Q_{\text{BRST}} = \oint \frac{dz}{2\pi i}(J^-(z) - 1)c(z) \tag{41}$$

is nilpotent $Q^2_{\text{BRST}} = 0$, and one may consider the cohomology $H_{Q_{\text{BRST}}}$ with respect to the BRST charge,

$$H_{Q_{\mathrm{BRST}}}(\mathscr{H}_{SL(2)} \otimes \mathscr{H}_{b,c}) = \mathrm{Ker}\,(Q_{\mathrm{BRST}})/\mathrm{Im}\,(Q_{\mathrm{BRST}}). \tag{42}$$

The claim is that this cohomology is isomorphic to an irreducible representation space $\mathscr{H}^{(k)}_{\mathrm{vir}}$ of the Virasoro algebra with $c = c^{(k)}_{SL(2)} - 2$,[2]

$$H_{Q_{\mathrm{BRST}}}(\mathscr{H}_{SL(2)} \otimes \mathscr{H}_{b,c}) \simeq \mathscr{H}^{(k)}_{\mathrm{vir}}. \tag{43}$$

The rest of this subsection is devoted to proving this theorem.

In the classical Hamiltonian reduction discussed in the previous section, we considered a subspace of the total phase space restricted by $J^-(z) = 1$. The *reduced phase space* was then defined as a space of orbits in this subspace generated by $J^-(z)$ through Poisson bracket. Here quantum mechanically, the physical subspace is defined by the constraint $Q_{\mathrm{BRST}}|\Psi\rangle = 0$ in the total Hilbert space, and the *reduced Hilbert space* is the space of orbits of the BRST charge.

As a matter of fact, the total energy-momentum tensor

$$T^{\mathrm{total}}(z) = T_{\mathrm{improved}}(z) + \partial b(z)c(z) \tag{44}$$

acting on $\mathscr{H}^{(k)}_{SL(2)} \otimes \mathscr{H}_{b,c}$ commutes with the BRST charge since the BRST current $(J^-(z) - 1)c(z)$ is a field of weight 1 with respect to $T^{\mathrm{total}}(z)$. It is also easy to convince oneself that this total energy-momentum tensor is not a BRST exact operator, $T^{\mathrm{total}}(z) \neq \{Q_{\mathrm{BRST}}, *\}$. One can, for example, examine the grade-2 physical subspace in the descendants of the vacuum state $|0\rangle_{SL(2)} \otimes |0\rangle_{b,c}$. The BRST cohomology of this subspace is one-dimensional and generated by $L^{\mathrm{total}}_{-2}$ acting on the vacuum. Therefore it is clear that the Virasoro algebra with $c = c^{(k)}_{SL(2)} - 2$ acts on the reduced Hilbert space $H_{Q_{\mathrm{BRST}}}(\mathscr{H}_{SL(2)} \otimes \mathscr{H}_{b,c})$. The issue is whether the representation is irreducible.

To prove the irreducibility of the reduced Hilbert space, it is useful to employ the realization of the $SL(2)$ current algebra in terms of a scalar field $\varphi(z)$ and a set of bosonic ghosts $(\beta, \gamma)$ with weights $(0, 1)$

$$\varphi(z)\varphi(w) \sim \log\left(\frac{1}{z-w}\right), \quad \beta(z)\gamma(w) \sim \frac{1}{z-w} \tag{45}$$

as

$$\begin{aligned} J^+(z) &= -\beta(z)(\gamma(z))^2 + i\alpha_+\gamma(z)\partial\varphi(z) + k\partial\gamma(z), \\ J^3(z) &= \beta(z)\gamma(z) - \frac{i}{2}\alpha_+\partial\varphi(z), \\ J^-(z) &= \beta(z) \\ \alpha_+ &= \sqrt{2k+4}. \end{aligned} \tag{46}$$

This free boson realization was introduced by Wakimoto [9] (in the case of $k = 1$) and by Zamolodchikov (for general $k$).

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[2] We are informed by A. A. Beilinson and T. Eguchi that a similar construction was suggested by B. L. Feigin

In this realization, the ghosts $(b,c)$ and $(\beta,\gamma)$ make a pair of BRST doublets, which is called the Kugo–Ojima quartet [11]. We are going to prove the following lemma.

**Lemma (Quartet Confinement).**

$$H_{Q_{\mathrm{BRST}}}(\mathscr{H}_{\varphi}\otimes\mathscr{H}_{\beta,\gamma}\otimes\mathscr{H}_{b,c})\simeq\mathscr{H}_{\varphi}. \tag{47}$$

This lemma means that the ghosts simply decouple from the physical subspace. The proof of this lemma goes as follows. Consider the following projection operator $\mathscr{P}$,

$$\mathscr{P}:\mathscr{H}_{\varphi}\otimes\mathscr{H}_{\beta,\gamma}\otimes\mathscr{H}_{b,c}\to\mathscr{H}_{\varphi}\otimes|0\rangle_{\beta,\gamma}\otimes|0\rangle_{b,c}. \tag{48}$$

Here the vacua $|0\rangle_{\beta,\gamma}$ and $|0\rangle_{b,c}$ are defined by

$$\begin{aligned}\beta_n|0\rangle_{\beta,\gamma}=0,\quad b_n|0\rangle_{b,c}=0\quad(n\geqq 0)\\ \gamma_m|0\rangle_{\beta,\gamma}=0,\quad c_m|0\rangle_{b,c}=0\quad(m\geqq 1).\end{aligned} \tag{49}$$

The zero modes of the bosonic ghosts, $\beta_0$ and $\gamma_0$ obey the commutation relation

$$[\beta_0,\gamma_0]=1. \tag{50}$$

If $\gamma_0$ is diagonalized, $\beta_0$ can be regarded as a differential operator,

$$\beta_0=\frac{\partial}{\partial\gamma_0}. \tag{51}$$

Though the projection operator $\mathscr{P}$ does not commute with $Q_{\mathrm{BRST}}$, we can modify it as

$$P^{(0)}=e^{\gamma_0}\mathscr{P}e^{-\gamma_0}=e^{\gamma_0}\mathscr{P} \tag{52}$$

so that $P^{(0)}$ commute with $Q_{\mathrm{BRST}}$.

Now we show that $1-P^{(0)}$ is BRST exact. Following the paper by Kugo and Ojima [11], we introduce a set of operators $P^{(N)}$ $(N=1,2,3,\dots)$ defined inductively as

$$\begin{aligned}P^{(N)}=&\frac{1}{N}\sum_{n\geqq 1}(b_{-n}P^{(N-1)}c_n-\beta_{-n}P^{(N-1)}\gamma_n)\\ &+\frac{1}{N}\sum_{n\geqq 0}(c_{-n}P^{(N-1)}b_n+\gamma_{-n}P^{(N-1)}(\beta_n-\delta_{n0})).\end{aligned} \tag{53}$$

These operators $P^{(N)}$ commute with $Q_{\mathrm{BRST}}$. In fact, for $N\geqq 1$, $P^{(N)}$ is a BRST exact operator,

$$\begin{aligned}P^{(N)}&=\{Q_{\mathrm{BRST}},R^{(N)}\}\\ R^{(N)}&=-\frac{1}{N}\sum_{n\geqq 1}(b_{-n}P^{(N-1)}\gamma_n+c_{-n}P^{(N-1)}\gamma_n)\\ &\quad+\frac{1}{N}\sum_{n\geqq 0}(\gamma_{-n}P^{(N-1)}b_n+\gamma_{-n}P^{(N-1)}b_n).\end{aligned} \tag{54}$$

It is easy to see that they are complete,

$$\sum_{N \geqq 0} P^{(N)} = 1. \tag{55}$$

Therefore any physical state $|\Psi\rangle$ annihilated by $Q_{\mathrm{BRST}}$ is written as

$$|\Psi\rangle = \sum_N P^{(N)}|\Psi\rangle = P^{(0)}|\Psi\rangle + Q_{\mathrm{BRST}}\left(\sum_{N \geqq 1} R^{(N)}|\Psi\rangle\right). \tag{56}$$

This means that the physical state $|\Psi\rangle$ is equivalent to its projection onto $\mathscr{H}_\varphi \otimes e^{\gamma_0}|0\rangle_{\beta,\gamma} \otimes |0\rangle_{b,c}$ modulo the BRST operator. This completes the proof of the lemma.

On the other hand, the total energy-momentum tensor is expressed as

$$\begin{aligned} T^{\mathrm{total}}(z) &= T_{\mathrm{FF}}(z) - \{Q_{\mathrm{BRST}}, \gamma(z)\partial b(z)\} \\ T_{\mathrm{FF}}(z) &= -\tfrac{1}{2}(\partial\varphi(z))^2 + i\alpha_0\partial^2\varphi(z), \quad \alpha_0 = \frac{k+1}{\sqrt{2k+4}}, \end{aligned} \tag{57}$$

i.e. upto a BRST exact operator, $T^{\mathrm{total}}$ is equal to $T_{\mathrm{FF}}(z)$, which is in the same form as that in the free boson realization of the Virasoro algebra developed by Dotsenko and Fateev [8] (also called the Feigin-Fuchs realization). Thus one may suspect that there is a close connection between these free boson realizations of the Virasoro and the $SL(2)$ current algebras.[3]

Let us recall the free boson realization of the Virasoro algebra by Dotsenko and Fateev. They assume that the scalar field $\varphi(z)$ winds around a torus of a radius $\alpha_+ = \sqrt{2k+4}$. Then the Fock space $\mathscr{H}_\varphi$ is decomposed into subspaces with definite $U(1)$ charges with respect to $-i\partial\varphi$,

$$\begin{gathered} \mathscr{H}_\varphi = \bigotimes_{r,s} \mathscr{H}_\varphi^{r,s}, \\ \mathscr{H}_\varphi^{r,s}\text{:subspace with charge } (1-r)\alpha_+ + (1-s)\alpha_-\left(\alpha_- = -\frac{2}{\alpha_+}\right). \end{gathered} \tag{58}$$

In this realization, primary fields interwine subspaces of different $U(1)$ charges, and their correlation functions vanish in general due to the conservation of the $U(1)$ charge. To get meaningful results, we must introduce charge screening operators $\psi_{\mathrm{vir}}^{\pm}(z)$,

$$\psi_{\mathrm{vir}}^{\pm}(z) = e^{i\alpha_\pm\varphi(z)}. \tag{59}$$

Since their operator product expansion with $T_{\mathrm{FF}}(z)$ is total derivative,

$$T_{\mathrm{FF}}(z)\psi_{\mathrm{vir}}^{\pm}(w) \sim \partial_w\left(\frac{1}{z-w}\mathcal{O}^{\pm}(w)\right), \tag{60}$$

[3] D. Bernard and G. Felder have examined degenerate representations of the $SL(2)$ current algebra using the free boson realization (46). They also found the intriguing relation between the representations of the Virasoro and the $SL(2)$ current algebra, which is potentially related to our observations here. We thank them for informing as of their result before publication

we can insert their contour integrals in correlation functions to satisfy the total charge conservation without spoiling the Ward identities.

The realization of the Virasoro algebra in $\mathscr{H}_\varphi^{(r,s)}$ is highly reducible. It was pointed out by Felder [12] that one can extract an irreducible representation out of $\mathscr{H}_\varphi^{r,s}$ by using the charge screening operators. Following Thorn [21], he introduced an operator $Q_{\text{vir}}$ defined by contour integrals of the screening operators. It is nilpotent $Q_{\text{vir}}^2 = 0$, and generates the following spectral sequence:

$$\xrightarrow{Q_{\text{vir}}} \mathscr{H}_\varphi^{(r,2q-s)} \xrightarrow{Q_{\text{vir}}} \mathscr{H}_\varphi^{(r,s)} \xrightarrow{Q_{\text{vir}}} \mathscr{H}_\varphi^{(r,-s)} \xrightarrow{Q_{\text{vir}}}. \tag{61}$$

He then proved that this spectral sequence is exact, $\text{Ker}(Q_{\text{vir}}) = \text{Im}(Q_{\text{vir}})$, except at the middle Fock space $\mathscr{H}_\varphi^{(r,s)}$ with

$$1 \leqq r \leqq p-1, \quad 1 \leqq s \leqq q-1, \quad qs < pr. \tag{62}$$

The cohomology at the middle Fock space is isomorphic to an irreducible representation space of the Virasoro algebra $\mathscr{H}_{\text{vir}}^{(r,s)}$ with a highest weight $\Delta_{r,s} = \frac{1}{2}((\alpha_+ r + \alpha_- s)^2 - (\alpha_+ + \alpha_-)^2)$,

$$\mathscr{H}_{\text{vir}}^{(r,s)} \simeq H_{Q_{\text{vir}}}(\mathscr{H}_\varphi^{(r,s)}). \tag{63}$$

For the $SL(2)$ current algebra, there are also two screening operators[4]

$$\begin{aligned} \psi_{SL(2)}^-(z) &= \beta(z) e^{i\alpha_- \varphi(z)}, \\ \psi_{SL(2)}^+(z) &= (\beta(z))^{-(k+2)} e^{i\alpha_\pm \varphi(z)}, \end{aligned} \tag{64}$$

which satisfy

$$J^a(z)\psi_{SL(2)}^\pm(w) \simeq \partial_w\left(\frac{1}{z-w}\mathcal{O}^\pm(w)\right). \tag{65}$$

Since all the singular terms in the operator product expansions are total derivatives, we can insert contour integrals of these screening operators into correlation functions without spoiling their Ward identities. As in the case of the Virasoro algebra, one can construct a nilpotent operator $Q_{SL(2)}$ from the screening operators. In the case of the Virasoro algebra, the spectral sequence (61) is exact except at one point $\mathscr{H}^{(r,s)}$, where the cohomology $H_{Q_{\text{vir}}}$ is isomorphic to an irreducible representation of the Virasoro algebra. The corresponding statement in the case of the $SL(2)$ current algebra, i.e. exactness of the spectral sequence of $Q_{SL(2)}$ except at $\mathscr{H}_\varphi^{(r,s)} \otimes \mathscr{H}_{\beta,\gamma}$ and

$$\begin{aligned} \mathscr{H}_{SL(2)}^{(r,s)} &= (\text{irreducible representation of the current algebra}) \\ &\simeq H_{Q_{SL(2)}}(\mathscr{H}_\varphi^{(r,s)} \otimes \mathscr{H}_{\beta,\gamma}), \end{aligned} \tag{66}$$

is now being worked out by Bernard and Felder [13].

[4] The definition $\psi_{SL(2)}^+$ may be subtle, for it involves a negative power of $\beta(z)$ when $k$ is greater than $-2$. In extracting an irreducible representation from $\mathscr{H}_\varphi^{(n,m)}$ as described below, however, one needs to use only one of these screening operators, say $\psi_{SL(2)}^-$ which is well-defined [13]. (This is also the case for the Virasoro algebra [12].) Thus this subtlety is not relevant to our construction here

The screening operators for the Virasoro and the $SL(2)$ current algebra are equivalent to each other modulo BRST exact operators,

$$\psi^-_{SL(2)}(z) = \psi^-_{\text{vir}}(z) + \{Q_{\text{BRST}}, b(z)e^{i\alpha_-\varphi(z)}\}$$
$$\psi^+_{SL(2)}(z) = \psi^+_{\text{vir}}(z) + \left\{Q_{\text{BRST}}, \frac{1-\beta(z)^{-(k+2)}}{1-\beta(z)} b(z)e^{i\alpha_+\varphi(z)}\right\}. \tag{67}$$

This implies that the BRST-like operators $Q_{\text{vir}}$ and $Q_{SL(2)}$ are also related as

$$Q_{SL(2)} = Q_{\text{vir}} + \{Q_{\text{BRST}}, *\}. \tag{68}$$

In deriving these relations, it is crucial that $J^-(z)$ is constrained to be a non-vanishing constant (in our convention $J^-(z)=1$). If we had chosen $Q_{\text{BRST}} = \oint(dz/2\pi i)J^-(z)c(z)$, $Q_{SL(2)}$ would have been a BRST exact operator.

Now we are ready to prove the main theorem of this subsection (43). The quartet confinement (47) and the result by Felder (63) implies

$$\mathscr{H}^{(r,s)}_{\text{vir}} \simeq H_{Q_{\text{vir}}}(\mathscr{H}^{(r,s)}_{\varphi}) \simeq H_{Q_{\text{vir}}} \circ H_{Q_{\text{BRST}}}(\mathscr{H}^{(r,s)}_{\varphi} \otimes \mathscr{H}_{\beta,\gamma} \otimes \mathscr{H}_{b,c}). \tag{69}$$

Since $Q_{SL(2)}$ commutes with $Q_{\text{BRST}}$, $H_{Q_{SL(2)}} \circ H_{Q_{\text{BRST}}}$ is also well-defined. We now show that it is equivalent to the right-hand side in the above. Consider kernels of $Q_{\text{vir}}$ and $Q_{SL(2)}$ on $\text{Ker}(Q_{\text{BRST}})$. Thanks to Eq. (68),

$$Q_{SL(2)}|\Psi\rangle \in \text{Im}(Q_{\text{BRST}}) \leftrightarrow Q_{\text{vir}}|\Psi\rangle \in \text{Im}(Q_{\text{BRST}}). \tag{70}$$

On the other hand, due to the quartet confinement of ghosts, images of $Q_{\text{vir}}$ and $Q_{SL(2)}$ are equivalent modulo $Q_{\text{BRST}}$ when considered on $\text{Ker}(Q_{\text{BRST}})$. Hence we obtain

$$\mathscr{H}_{\text{vir}} \simeq H_{Q_{\text{vir}}} \circ H_{Q_{\text{BRST}}}(\mathscr{H}^{(r,s)}_{\varphi} \otimes \mathscr{H}_{\beta,\gamma} \otimes \mathscr{H}_{b,c})$$
$$\simeq H_{Q_{SL(2)}} \circ H_{Q_{\text{BRST}}}(\mathscr{H}^{(r,s)}_{\varphi} \otimes \mathscr{H}_{\beta,\gamma} \otimes \mathscr{H}_{b,c}). \tag{71}$$

The next step is to relate $H_{SL(2)} \circ H_{Q_{\text{BRST}}}$ in the above to $H_{Q_{\text{BRST}}}(\mathscr{H}_{SL(2)})$. Following the result of Bernard and Felder (66), we may identify $\mathscr{H}^{(r,s)}_{SL(2)}$ with $H_{Q_{SL(2)}}$. Thus what we need to prove is

$$H_{Q_{SL(2)}} \circ H_{Q_{\text{BRST}}} \simeq H_{Q_{\text{BRST}}} \circ H_{Q_{SL(2)}}. \tag{72}$$

For general nilpotent operators $Q_1$ and $Q_2$ which commute with each other, $H_{Q_1} \circ H_{Q_2} \simeq H_{Q_2} \circ H_{Q_1}$ is not neccessarily valid. To find a sufficient condition for this to be valid, let us consider the following spectral sequence of the double complex generated by $Q_1$ and $Q_2$,

$$\begin{array}{ccccccc}
 & \downarrow Q_2 & & \downarrow Q_2 & & \downarrow Q_2 & \\
\xrightarrow{Q_1} & V_{n-1,m-1} & \xrightarrow{Q_1} & V_{n,m-1} & \xrightarrow{Q_1} & V_{n+1,m-1} & \xrightarrow{Q_1} \\
 & \downarrow Q_2 & & \downarrow Q_2 & & \downarrow Q_2 & \\
\xrightarrow{Q_1} & V_{n-1,m} & \xrightarrow{Q_1} & V_{n,m} & \xrightarrow{Q_1} & V_{n+1,m} & \xrightarrow{Q_1} .
\end{array} \tag{73}$$

$$
\begin{array}{ccccccc}
 & \downarrow Q_2 & & \downarrow Q_2 & & \downarrow Q_2 & \\
\xrightarrow{Q_1} & V_{n-1,m+1} & \xrightarrow{Q_1} & V_{n,m+1} & \xrightarrow{Q_1} & V_{n+1,m+1} & \xrightarrow{Q_1} . \\
 & \downarrow Q_2 & & \downarrow Q_2 & & \downarrow Q_2 &
\end{array}
$$

Here a pair of indices $(n,m)$ of $V_{n,m}$ denotes a double-grading with respect to $Q_1$ and $Q_2$; where $Q_1 = Q_{\mathrm{BRST}}$ and $Q_2 = Q_{SL(2)}$, $n$ is the ghost number and $m$ is related to the $U(1)$ charge. Now we prove the following lemma.

**Lemma**

(I) *Assume* (1) *the horizontal sequence is exact except at* $V_{n=0,m}$ *and the vertical sequence is exact except at* $V_{n,m=0}$, *and* (2) *the horizontal sequence is finite*, i.e. *for a sufficiently large* $N$, $V_{N+1,m} = \{0\}$ *and* $V_{-N-1,m} = \{0\}$ *for any* $m$. *Then*

$$\mathrm{Ker}\,(Q_1 Q_2) = \mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2) \tag{74}$$

*holds on* $V_{n,-m}$ *and* $V_{-m,n}$ *with* $n \geqq 0$, $m > 0$.
(II) *Under these assumptions*, $H_{Q_1} \circ H_{Q_2}(V_{0,0}) = H_{Q_2} \circ H_{Q_1}(V_{0,0})$.

When $Q_1 = Q_{\mathrm{BRST}}$ and $Q_2 = Q_{SL(2)}$, the assumptions of this lemma are satisfied. The assumption (1) is the consequence of the quartet confinement of the ghosts and the result of Bernard and Felder [13]. Concerning the assumption (2), we note that both $Q_{\mathrm{BRST}}$ and $Q_{SL(2)}$ commute with the total energy-momentum tensor $T^{\mathrm{total}}(z)$ given by Eq. (44). Thus we may restrict $V_{n,m}$'s to be in the same eigenspace with respect to $L_0^{\mathrm{total}}$. In this case, the ghost numbers of states are bounded below and above, and the horizontal sequence terminates beyond these bounds.

The first part of the lemma is proved by mathematical induction. We first show that, if the lemma holds at $V_{n+1,-(m+1)}(V_{-(n+1),m+1})$, so does $V_{n,-m}(V_{-n,m})$. By repeating this procedure finite times, we arrive at an edge of the horizontal sequence, where the lemma can be easily checked explicitly.

Since $\mathrm{Ker}\,(Q_1 Q_2) \supset \mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2)$ is obvious, we just need to show $\mathrm{Ker}\,(Q_1 Q_2) \subset \mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2)$.

$$
\begin{array}{ccc}
 & \longrightarrow V_{n+1,-m-1} \longrightarrow & \\
 & \downarrow & \\
V_{n,-m} \longrightarrow & V_{n+1,-m} & \longrightarrow . \\
 & \downarrow & \\
 & V_{n+1,-m+1} &
\end{array}
\tag{75}
$$

Take an arbitrary element $v$ of $\mathrm{Ker}\,(Q_1 Q_2)|_{V_{n,-m}}$. Since $Q_2 Q_1 v = 0$, $Q_1 v$ belongs to $\mathrm{Ker}\,(Q_2)|_{V_{n+1,-m}}$. By the assumption (1), there is some element $\lambda$ of $V_{n+1,-(m+1)}$ such that $Q_1 v = Q_2 \lambda$. Such $\lambda$ should satisfy $Q_1 Q_2 \lambda = 0$. According to the assumption of the induction, $\lambda$ belongs to $\mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2)$, and it can be written as $\lambda = Q_1 v_1 + Q_2 v_2$ by the assumption (2). Substituting this into $Q_1 v = Q_2 \lambda$, we obtain $Q_1(v - Q_2 v_1) = 0$, i.e. $v - Q_2 v_1 \in \mathrm{Ker}\,(Q_1)$ and $v \in \mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2)$.Thus we proved $\mathrm{Ker}\,(Q_1 Q_2) \subset \mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2)$. This is what we wanted to show. By interchanging $Q_1$ and $Q_2$, we can also prove $\mathrm{Ker}\,(Q_1 Q_2) = \mathrm{Ker}\,(Q_1) \oplus \mathrm{Ker}\,(Q_2)$ on $V_{-n,m}$.

Let us have a look at the right edge of the horizontal sequence,

$$\to V_{N,-m} \to 0. \tag{76}$$

It is obvious that $V_{N,-m} = \mathrm{Ker}(Q_1)|_{V_{N,-m}}$. Thus we obtain

$$\begin{aligned}\mathrm{Ker}(Q_1 Q_2)|_{N,-m} \subset V_{N,-m} &= \mathrm{Ker}(Q_1)|_{V_{N,-m}} \\ &\subset \mathrm{Ker}(Q_1) \oplus \mathrm{Ker}(Q_2)|_{V_{N,-m}}.\end{aligned} \tag{77}$$

Thus $\mathrm{Ker}(Q_1 Q_2) = \mathrm{Ker}(Q_1) \oplus \mathrm{Ker}(Q_2)$ holds at the right edge. At the left edge

$$\begin{array}{ccc} 0 \longrightarrow & V_{-N,m} & \longrightarrow \\ & \downarrow & \\ 0 \longrightarrow & V_{-N,m+1} & \longrightarrow \\ & \downarrow & \end{array} \tag{78}$$

the assumption of the induction is trivially satisfied for $V_{-N-1,m+1} = 0$. This completes the proof of the first part of the lemma.

Next we prove the following equality at $V_{0,0}$:

$$H_{Q_1} \circ H_{Q_2}(V_{0,0}) = \frac{\mathrm{Ker}(Q_1) \cap \mathrm{Ker}(Q_2)}{(\mathrm{Im}(Q_1) \oplus \mathrm{Im}(Q_2)) \cap (\mathrm{Ker}(Q_1) \cap \mathrm{Ker}(Q_2))}. \tag{79}$$

Since the right-hand side is symmetric with respect to $Q_1$ and $Q_2$, the second part of the lemma follows from this equality. A class $[v]$ in $H_{Q_1} \circ H_{Q_2}$ is given by $v \in V_{0,0}$ satisfying

$$Q_1 v \in Q_2(V_{1,-1}), \quad Q_2 v = 0, \tag{80}$$

and taken modulo $Q_1(\mathrm{Ker}(Q_2)|_{V_{-1,0}}) \oplus Q_2(V_{0,-1})$.

$$\begin{array}{ccccc} & & V_{0,-1} & \longrightarrow & V_{1,-1} \\ \downarrow & & \downarrow & & \downarrow \\ V_{-1,0} & \longrightarrow & V_{0,0} & \longrightarrow & V_{1,0} \\ \downarrow & & \downarrow & & \\ & & V_{0,1} & & \end{array} . \tag{81}$$

Let us rewrite the first condition on $v$; $Q_1 v = Q_2 \lambda$ for some $\lambda$ in $V_{1,-1}$. Since $\lambda$ belongs to $\mathrm{Ker}(Q_1 Q_2)|_{V_{1,-1}}$, it can be written as $\lambda = Q_1 \varepsilon_1 + Q_2 \varepsilon_2$ thanks to the first part of the lemma. Thus the first condition becomes

$$Q_1(v - Q_2 \varepsilon_1) = 0 \quad \text{for some} \quad \varepsilon_1 \in V_{0,-1}. \tag{82}$$

Since the representative $v$ of the class $[v]$ is chosen modulo $Q_2(V_{0,-1})$, we can exploit this freedom to set

$$Q_1 v = 0. \tag{83}$$

We have shown the following equality

$$H_{Q_1} \circ H_{Q_2}(V_{0,0}) = \frac{\mathrm{Ker}(Q_1) \cap \mathrm{Ker}(Q_2)}{Q_1(\mathrm{Ker}(Q_2)|_{V_{-1,0}}) \oplus (Q_2(V_{0,-1}) \cap \mathrm{Ker}(Q_1)|_{V_{0,0}})}. \tag{84}$$

Finally we prove

$$Q_1\,\mathrm{Ker}(Q_2)|_{V_{-1,0}} = Q_1(V_{-1,0}) \cap \mathrm{Ker}(Q_2)|_{V_{0,0}}. \tag{85}$$

It is clear that

$$Q_1(\mathrm{Ker}(Q_2)|_{V_{-1,0}}) \subset Q_1(V_{-1,0}) \cap \mathrm{Ker}(Q_2)|_{V_{0,0}}. \tag{86}$$

On the other hand, any element of $Q_1(V_{-1,0})\cap\mathrm{Ker}(Q_2)|_{V_{0,0}}$ is written as $Q_1\varepsilon$ with $\varepsilon\in\mathrm{Ker}(Q_1Q_2)|_{V_{-1,0}}$. Thanks to the first part of the lemma, $\varepsilon$ belongs to $\mathrm{Ker}(Q_1)\oplus\mathrm{Ker}(Q_2)$, and $Q_1\varepsilon$ is in $Q_1$ $(\mathrm{Ker}(Q_2)|_{V_{-1,0}})$. Thus Eq. (85) holds. Combining this with Eq. (84) we obtain Eq. (79). This completes the proof of the lemma.

This completes the proof of the quantum Hamiltonian reduction.

**Theorem (Quantum Hamiltonian Reduction).**

$$V_{\mathrm{vir}} \simeq H_{Q_{\mathrm{BRST}}}(V_{SL(2)} \otimes V_{b,c}). \tag{87}$$

We would like to make two comments. The completely degenerate representations of the Virasoro algebra are parametrized by a set of two integers $(r,s)$ with $1 \leqq r \leqq p-1$, $1 \leqq s \leqq q-1$, and there are corresponding representations of the $SL(2)$ current algebra. According to Bernard and Felder, the $SL(2)$ current algebra has another class of representations, which corresponds to the case of $r=0$. In this case, it is not yet checked whether the assumptions of our theorem (87) hold. Most probably, the cohomology of the nilpotent operator $Q_{\mathrm{vir}}$ may not give irreducible representations of the Virasoro algebra. This existence of such representations for the current algebra does not contradict our proof. We also remark that this construction may not work if one replaces $SL(2,\mathscr{R})$ by $SU(2)$. First of all, it is not clear whether the constraint $J^- = 1$ makes sense in $SU(2)$, for $J^+ = (J^-)^*$. Another important point is that $SU(2)$ is simply connected while $SL(2,\mathscr{R})$ is not. The level $k$ of the current algebra related to a completely degenerate representation of the Virasoro algebra is in general fractional ($k+2=p/q$), and so is its highest weight. This is possible only for $SL(2)$,

*3.3 Quantum Hamiltonian Reduction; Case of SL(n).* The above result can be extended to the case of $SL(3)$ current algebra though it requires more elaborate computations. The $SL(3)$ algebra is generated by six charged currents, $J_1^\pm, J_2^\pm, J_3^\pm$, and two neutral currents, $H_1, H_2$. The basis is chosen in such a way that these currents correspond to $SL(3)$ generators as

$$J_1^+ \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad J_2^+ \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad J_3^+ \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H_1 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{88}$$

In order to put the constraints

$$J_1^-(z) = J_2^-(z) = 1, \quad J_3^-(z) = 0 \tag{89}$$

consistently with the conformal invariance, we deform the Sugawara energy-momentum tensor $T_{SL(3)}$ for the $SL(3)$ current algebra as,

$$T_{\text{improved}}(z) = T_{SL(3)} - \partial(H_1(z) + H_2(z)). \tag{90}$$

With respect to this improved energy-momentum tensor, $J_3^-$ has weight $-1$ while $J_1^-$ and $J_2^-$ have weight 0. To put the constraints (89) in the BRST formalism, we must introduce three sets of ghosts, $(b_1, c_1)$, $(b_2, c_2)$ and $(b_3, c_3)$, with weights $(0,1), (0,1)$ and $(-1,2)$ respectively. The BRST charge defined by

$$Q_{\text{BRST}} = \oint \frac{dz}{2\pi} [(J_1^-(z) - 1)c_1(z) + (J_2^-(z) - 1)c_2(z) + J_3^-(z)c_3(z) + c_1(z)c_2(z)b_3(z)] \tag{91}$$

acts nilpotently on $\mathscr{H}^{(k)}_{SL(3)} \otimes \mathscr{H}_{b_{1,2,3},c_{1,2,3}}$. The total energy-momentum commuting with $Q_{\text{BRST}}$ is

$$T^{\text{total}}(z) = T_{\text{improved}} + \partial b_1(z)c_1(z) + \partial b_2(z)c_2(z) + 2\partial b_3(z)c_3(z) + b_3(z)\partial c_3(z). \tag{92}$$

The free boson realization (46) of Wakimoto and Zamolodchikov can be extended to the case of the $SL(3)$ current algebra as follows. Let us employ two scalar fields $\varphi_1$ and $\varphi_2$ and three sets of bosonic ghosts, $(\beta_1, \gamma_1), (\beta_2, \gamma_2)$ and $(\beta_3, \gamma_3)$, with weights $(0,1)$, $(0,1)$ and $(-1,2)$ respectively.

$$\varphi_a(z)\varphi_b(w) \sim \delta_{ab} \log\left(\frac{1}{z-w}\right), \quad \beta_i(z)\gamma_j(w) \sim \frac{\delta_{ij}}{z-w}.$$
$$(a,b = 1,2,\ i,j = 1,2,3) \tag{93}$$

It is straightforward to check that the following is a realization of the $SL(3)$ current algebra:

$$\begin{aligned}
J_1^+(z) &= -\beta_1(z)(\gamma_1(z))^2 + \beta_2(z)\gamma_3(z) + (k+1)\partial\gamma_1(z) + i\alpha'_+\gamma_1(z)\vec{e}_1\cdot\partial\vec{\varphi}(z),\\
J_2^+(z) &= \beta_1(z)(\gamma_1(z)\gamma_2(z) - \gamma_3(z)) - \beta_2(z)\gamma_2(z)^2 - \beta_3(z)\gamma_2(z)\gamma_3(z)\\
&\quad + i\alpha'_+\gamma_2(z)\vec{e}_2\cdot\partial\vec{\varphi}(z) + k\partial\gamma_2(z),\\
J_3^+(z) &= \beta_1(z)((\gamma_1(z))^2\gamma_2(z) - \gamma_1(z)\gamma_3(z)) - \beta_2(z)\gamma_2(z)\gamma_3(z)\\
&\quad - \beta_3(z)(\gamma_3(z))^2 + k\partial\gamma_3(z) - (k+1)\partial\gamma_1(z)\gamma_2(z)\\
&\quad + i\alpha'_+\gamma_3(z)\vec{e}_1\cdot\partial\vec{\varphi}(z) + i\alpha'_+(\gamma_3(z) - \gamma_1(z)\gamma_2(z))\vec{e}_2\cdot\partial\vec{\varphi}(z),\\
H_1(z) &= 2\beta_1(z)\gamma_1(z) - \beta_2(z)\gamma_2(z) + \beta_3(z)\gamma_3(z) - i\alpha'_+\vec{e}_1\cdot\partial\vec{\varphi}(z),\\
H_2(z) &= -\beta_1(z)\gamma_1(z) + 2\beta_2(z)\gamma_2(z) + \beta_3(z)\gamma_3(z) - i\alpha'_+\vec{e}_2\cdot\partial\vec{\varphi}(z),\\
J_1^-(z) &= \beta_1(z) + \gamma_2(z)\beta_3(z),\\
J_2^-(z) &= \beta_2(z),\\
J_3^-(z) &= \beta_3(z),\\
&\alpha'_+ = \sqrt{2k+6}, \quad \vec{e}_1 = \tfrac{1}{2}(1,\sqrt{3}), \quad \vec{e}_2 = \tfrac{1}{2}(1,-\sqrt{3}),\\
&\vec{\varphi}(z) = (\varphi_1(z), \varphi_2(z)).
\end{aligned} \tag{94}$$

Substituting the above bosonized expressions of the $SL(3)$ currents into the total energy-momentum tensor (92), we obtain

$$T^{\rm total}(z) = T_{W_3}(z) - \{Q_{\rm BRST}, t(z)\},$$
$$t(z) = \gamma_1(z)\partial b_1(z) + \gamma_2(z)\partial b_2(z) + 2\gamma_3(z)\partial b_3(z) + \partial\gamma_3(z)b_3(z) - \gamma_1\partial(\gamma_2(z)b_3(z)), \quad (95)$$

with

$$T_{W_3}(z) = -\tfrac{1}{2}((\partial\varphi_1(z))^2 + (\partial\varphi_2(z))^2) + i2\alpha'_0\partial^2\varphi_1(z),$$
$$\alpha'_0 = \frac{k+2}{\sqrt{2k+6}}. \quad (96)$$

The energy-momentum tensor $T_{W_3}(z)$ in the above is in the same form as that in the free boson realization of the $W_3$ algebra (chiral algebra generated by the energy-momentum tensor and a spin-3 chiral operator $W_3(z)$) developed by Fateev and Zamolodchikov [22].

The charge screening operators for $W_3$ algebra are given by

$$\psi^{(a,\pm)}_{W_3}(z) = \exp(i\alpha'_\pm \vec{e}_a\cdot\vec{\varphi}(z)) \quad (a = 1, 2), \quad (97)$$

where

$$\alpha'_- = -\frac{4}{\alpha'_+}.$$

The structure of representations of the $W_3$ algebra has been examined by Mizoguchi [23] using these screening operators. For the $SL(3)$ current algebras, there are also four screening operators

$$\psi^{(1,\pm)}_{SL(3)} = (\beta_2(z) + \gamma_1(z)\beta_3(z))^{n'_\pm}\exp(i\alpha_\pm\vec{e}_1\cdot\vec{\varphi}(z)),$$
$$\psi^{(2,\pm)}_{SL(3)} = (\beta_1(z))^{n'_\pm}\exp(i\alpha_\pm\vec{e}_2\cdot\vec{\varphi}(z)), \quad (98)$$

with

$$n'_+ = -(k+3), \quad n_- = 1. \quad (99)$$

It is easy to check that the screening operators of the $W_3$ and the $SL(3)$ current algebras coincide modulo BRST exact operators. Although the representation theory of $SL(2)$ current algebra has not yet been worked out, it is plausible that the construction of irreducible representations using the screening operators (98) along the line of refs. (12, 13) extends to the case of $SL(3)$. If this is the case, one can employ the argument used for the relation between the Virasoro and the $SL(2)$ current algebra to show that the BRST cohomology in the physical subspace of $\mathscr{H}^{(k)}_{SL(3)}\otimes\mathscr{H}_{b_{1,2,3},c_{1,2,3}}$ is isomorphic to an irreducible representation space of the $W_3$ algebra.

We have explored the relation between the Virasoro and the $SL(2)$ algebras and the $W_3$ and the $SL(3)$ current algebras. It is then natural to expect that such relation between the $W_n$ and the $SL(n)$ current algebras persists for an arbitrary value of $n$. There are some suggestive feature to support this expectation. The central charge for the Sugawara energy-momentum tensor $T_{SL(n)}$ for the level-$k$

$SL(n)$ current algebra is $(n^2-1)k/(k+n)$. Let us deform the energy-momentum tensor as

$$T_{\text{improved}}(z) = T_{SL(n)}(z) - \vec{\delta}\cdot\partial\vec{H}(z) \tag{100}$$

($\vec{H}(z)$ is an $(n-1)$-dimensional vector of Cartan generators and $\vec{\delta}$ is a sum of positive roots) so that we can put constraints on the Borel subalgebra of the $SL(n)$ consistently with the conformal invariance. The central charge of this improved energy-momentum tensor is

$$c^{(k)}_{SL(n)} = n^4 - 1 - n(n^2-1)\left(\frac{1}{k+n} + k + n\right). \tag{101}$$

On ther other hand, the conformal anomaly for a completely degenerate representation of the $W_n$ algebra discussed by Fateev and Lykyanov is given by

$$c_{W_n} = (n-1)\left(1 - n(n+1)\frac{(p-q)^2}{pq}\right). \tag{102}$$

Substituting $p/q = k+n$, we obtain

$$\begin{aligned} c^{(k)}_{W_n} &= 2n^3 - n - 1 - n(n^2-1)\left(\frac{1}{k+n} + k + n\right) \\ &= -(n^4 - 2n^3 + n) + c^{(k)}_{SL(n)}. \end{aligned} \tag{103}$$

In order to put constraints on the Borel subalgebra using the BRST formalism, we need $j$-sets of ghost systems with weights $(-n+j+1, n-j)$ for $j = 1, 2, \ldots, n-1$. The sum of conformal anomalies for these ghosts is $-(n^4 - 2n^3 + n)$ and coincides with the difference between $c^{(k)}_{W_n}$ and $c^{(k)}_{SL(n)}$.

Another piece of evidence for the relation between the $W_n$ and the $SL(n)$ current algebras comes from their highest weights. The conformal weight in a completely degenerate representation of the $W_n$ algebra is given by

$$\Delta_{r,s} = \frac{1}{2pq}\left(\sum_{i=1}^{n-1}(pr_i - qs_i)\vec{\omega}_i\right)^2 - \frac{n(n^2-1)(p-q)^2}{24pq}, \tag{104}$$

where $\vec{\omega}_i$ $(i = 1, 2, \ldots, n-1)$ are the fundamental weights of $SL(n)$ normalized as

$$\vec{\omega}_i\cdot\vec{\omega}_j = \frac{i(n-j)}{n} \text{(for } i \leqq j). \tag{105}$$

This expression for the highest weight can be rewritten as

$$\begin{aligned} \Delta_{r,s} &= \frac{1}{2(k+n)}\vec{\Lambda}\cdot(\vec{\Lambda} + 2\vec{\delta}) - \vec{\delta}\cdot\vec{\Lambda}, \\ \vec{\Lambda} &= \sum_{i=1}^{n-1}((1-r_i)(k+n) - (1-s_i))\vec{\omega}_i. \end{aligned} \tag{106}$$

This is in the same form as the conformal weight of the spin $-\vec{\Lambda}$ primary field of the $SL(n)$ current algebra with respect to the improved energy-momentum tensor

(100). Thus one may suspect that the following relation holds:

$$T_{SL(n)}(z) - \vec{\delta}\cdot\partial H(z) + \sum_{j=1}^{n-1}\sum_{i=1}^{j}((n-j)\partial b_{ij}(z)c_{ij}(z) + (n-j-1)b_{ij}(z)\partial c_{ij}(z))$$
$$= T_{W_n}(z) + \{Q_{\mathrm{BRST}}, *\}. \tag{107}$$

These observations support the validity of the quantum Hamiltonian reduction of the $SL(n)$ current algebras down to the $W_n$ algebra for general value of $n$.

## 4. Quantization of the Virasoro Group

The analysis of the previous section makes it possible to construct a quantum field theory such that its right-movers give irreducible representation spaces of the $W_n$-algebra. In the case of $n=2$, its classical action turns out to be a geometric action for coadjoint orbit of the Virasoro group. We then discuss quantization of this system.

*4.1 Constrained Wess–Zumino–Witten Model.* We are going to show that the reduced Hilbert space discussed above naturally emerges if we consider the $SL(n)$ Wess–Zumino–Witten (WZW) model and couple the gauge field to the $SL(n)$ current belonging to the Borel subalgebra. For simplicity we discuss the case of $SL(2)$, but extension to the case of $SL(n)$ is straightforward. Let us consider the following system

$$S_{\mathrm{gauged}}(g,\bar{A}^+) = kS_{\mathrm{wzw}}(g) + \int \frac{d^2z}{8\pi}\bar{A}^+(J^- - 1),$$
$$J(z) = -\frac{k}{2}\partial g\cdot g^{-1}, \tag{108}$$

where $S_{\mathrm{wzw}}(g)$ is the action of the WZW model. The WZW action obeys the cocycle condition.

$$S_{\mathrm{wzw}}(Ug) = S_{\mathrm{wzw}}(g) + S_{\mathrm{wzw}}(U) - \int \frac{d^2z}{16\pi}\mathrm{tr}(U^{-1}\bar{\partial}U)(\partial g g^{-1}). \tag{109}$$

If we restrict $U$ to be in the Borel subgroup of $SL(2), S_{\mathrm{wzw}}(U)$ in the above vanishes. Therefore the gauged WZW action $S_{\mathrm{gauged}}(g,\bar{A}^+)$ is invariant under the transformation

$$g\to Ug, \quad \bar{A}^+\to\bar{A}^+ + \mathrm{tr}(U^{-1}\bar{\partial}U\cdot t^+), \tag{110}$$

where $U$ belongs to the Borel subgroup, $U=\exp(\varepsilon t^-)$.

In order to do the functional integration over $g$ and $\bar{A}^+$, we must divide the measure by the volume of this Borel gauge symmetry.

$$\int \frac{[g^{-1}dg, d\bar{A}^+]}{(\text{gauge volume})}\exp(iS_{\mathrm{gauged}}(g,\bar{A}^+)). \tag{111}$$

Let us fix this gauge invariance using the BRST formalism. The BRST transforms of $g$ and $\bar{A}^+$ are defined by replacing the parameter $\varepsilon$ of the infinitesimal gauge transformation with the Faddev–Popov ghost field $c$,

$$\delta_{\mathrm{BRST}}(g) = ct^- g, \quad \delta_{\mathrm{BRST}}(\bar{A}^+) = -\bar{\partial} c. \tag{112}$$

The ghost and anti-ghost transform as

$$\delta_{\mathrm{BRST}}(c) = 0, \quad \delta_{\mathrm{BRST}}(b) = B, \quad \delta_{\mathrm{BRST}}(B) = 0. \tag{113}$$

Here we introduced the Nakanishi–Lautrup auxiliary field $B$ as the BRST transform of $b$, which will serve as the Laugrange multiplier to impose the gauge-fixing condition.

The BRST gauge fixing is done by adding the BRST exact operator to the gauged WZW action $S_{\mathrm{gauged}}$. The original gauged WZW action is clearly invariant under the BRST transformation, for the BRST transformation for $g$ and $\bar{A}^+$ is in the same form as the Borel gauge transformation. On the other hand the nilpotency of the BRST transformation implies that the BRST exact operator itself is also BRST invariant. Choice of the BRST exact operator defines a gauge-fixing condition. Here we choose the following gauge-fixing condition:

$$\begin{aligned} &S_{\mathrm{gauged}}(g, \bar{A}^+) + \delta_{\mathrm{BRST}}\left( \int \frac{d^2 z}{8\pi} \bar{A}^+ b \right) \\ &= kS_{\mathrm{wzw}}(g) + \int \frac{d^2 z}{8\pi} b\bar{\partial} c + \int \frac{d^2 z}{8\pi} \bar{A}^+ (J^- - 1 - B). \end{aligned} \tag{114}$$

Integration over $B$ imposes the gauge-fixing condition $\bar{A}^+ = 0$, while integration over $\bar{A}^+$ puts the constraint $B = J^- - 1$. Thus we obtain the relation

$$\begin{aligned} &\int \frac{[g^{-1} dg, d\bar{A}^+]}{(\text{gauge volume})} \exp(iS_{\mathrm{gauged}}(g, \bar{A}^+)) \\ &= \int [g^{-1} dg, db, dc] \exp\left( ikS_{\mathrm{wzw}}(g) - i\int \frac{d^2 z}{8\pi} b\bar{\partial} c \right), \end{aligned} \tag{115}$$

with the on-shell BRST transformation

$$\begin{aligned} &\delta_{\mathrm{BRST}}(g) = ct^- g, \\ &\delta_{\mathrm{BRST}}(c) = 0, \quad \delta_{\mathrm{BRST}}(b) = J^- - 1. \end{aligned} \tag{116}$$

This is the system we have discussed in Sect. 3 in the Hamiltonian formalism. Thus the maximal chiral algebra in the right-moving sector of the constrained WZW model is reduced to the Virasoro algebra.

In the path integral (111), we can also integrate over $\bar{A}^+$ first. We then obtain

$$\int \frac{[g^{-1} dg]}{(\text{gauge volume})} \delta(J^-(z) - 1) \exp(ikS_{\mathrm{wzw}}(g)). \tag{117}$$

Using the composition law (109), it is easy to show that the above path integral still has the Borel gauge invariance. Since the $SL(2)$ group is three-dimensional, the original WZW model has three degrees of freedom. The constrained WZW model (117) has only one degree of freedom since two are killed by the constraint $J^-(z)=1$ and the Borel gauge invariance.

*4.2 Geometric Action for the Virasoro Group.* We have seen that the physical Hilbert space of the gauged WZW model with the constraint $J^-(z)=1$ contains irreducible representation spaces of the Virasoro algebra. Recently Alekseev and Shatashvili [24] also examined the constrained WZW model from a different point of view. Let us make a digression to convey their idea. They start with solving the constraint $J^-(z)=1$. If we parametrize the element $g$ of the $SL(2)$ by the Gauß product

$$g=\begin{pmatrix}1 & 0\\ \Phi & 1\end{pmatrix}\cdot\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\cdot\begin{pmatrix}1 & F\\ 0 & 1\end{pmatrix}, \tag{118}$$

the Borel gauge invariance $g\to Ug$ allows us to put the gauge-fixing condition

$$\Phi=0.$$

In this guage, the constraint takes the form

$$\frac{k}{2}\lambda^2\partial F+1=0. \tag{119}$$

On the other hand the WZW action in this gauge becomes

$$kS_{\mathrm{wzw}}(g)=\frac{k}{2\pi}\int d^2z(\lambda^{-1}\partial\lambda)(\lambda^{-1}\bar\partial\lambda). \tag{120}$$

Solving the constraint with respect to $\lambda$ and substituting it into the WZW action, we obtain the effective action for $F$ as

$$kS_{\mathrm{vir}}(F)=-\frac{k}{8\pi}\int d^2z\frac{\bar\partial F}{(\partial F)^3}(\partial F\partial^3F-2(\partial^2F)^2). \tag{121}$$

Thus the constraint of the gauged WZW model is solved,

$$\int\frac{[g^{-1}dg]}{(\text{gauge volume})}\delta(J^-(z)-1)\exp(ikS_{\mathrm{wzw}}(g))=\int[dF]\exp(ikS_{\mathrm{vir}}(F)). \tag{122}$$

The measure $[dF]$ is derived from the Haar measure $[g^{-1}dg]$ by reduction of degrees of freedom.

The energy-momentum tensor for the theory $S_{\mathrm{vir}}(F)$ can be derived using the Nöther procedure as

$$T=-\frac{k}{4\pi}\left(\frac{\partial^3F}{\partial F}-\frac{3}{2}\left(\frac{\partial^2F}{\partial F}\right)^2\right)=-\frac{k}{4\pi}\{F,z\}. \tag{123}$$

Remarkably the right-hand side is the Schwarzian derivative of $F$. In the conformal field theory, the Schwarzian derivative usually appears as in inhomogeneous term in a coordinate transformation $z\to w(z)$ of an energy-momentum tensor.

$$T(z)\to T(w(z))(w'(z))^2-\frac{c}{24\pi}\{w(z),z\}. \tag{124}$$

This situation is reminiscent of the following fact about the WZW model. In the WZW model, the current $J$ is given by

$$J = -\frac{k}{2}\partial g g^{-1},$$

and this is the inhomogeneous term in a chiral gauge transformation $g \to Ug$ of the current

$$J(z) \to UJ(z)U^{-1} - \frac{k}{2}\partial U(z)U^{-1}(z). \tag{125}$$

In mathematical terms, Eqs. (124) and (125) give the coadjoint orbits of the Virasoro and the Kac–Moody groups. Wiegmann [26] and independently Alekseev, Faddeev and Shatashvili [25] have developed a procedure to define a geometric action for a quantum field theory of coadjoint orbits, which in the case of the Kac–Moody group reproduces the ordinary WZW action. Alekseev and Shatashvili [24] applied this procedure to the Virasoro group and found that the geometrical action is precisely given by (121), i.e. $S_{\mathrm{vir}}(F)$ is an analogue of the WZW action for the Virasoro group.

Let us have a look at the symmetry of the action $S_{\mathrm{vir}}(F)$. The original WZW model has both left- and right-moving chiral symmetries,

$$S_{\mathrm{wzw}}(U(z)g(z,\bar{z})V(\bar{z})) = S_{\mathrm{wzw}}(g(z,\bar{z})). \tag{126}$$

For the constrained WZW model, the left-moving chiral transformation $g(z,\bar{z}) \to g(z,\bar{z})V(\bar{z})$ is still a symmetry of the system since it keeps the constraint $J_z^- = 1$ invariant. Thus the left-moving sector has the $SL(2)$ current algebra. On the other hand, the right-moving chiral transformation $g(z,\bar{z}) \to U(z)g(z,\bar{z})$ either changes the constraint or is absorbed into the Borel gauge symmetry of the system. Therefore the constrained WZW model has no current algebra in the right-moving sector. In fact according to the analysis in Sects. 2 and 3, the maximum chiral algebra for the right-mover is just the Virasoro algebra. The left-moving chiral symmetry of the constrained WZW model is reflected to the invariance of the action $S_{\mathrm{vir}}(F)$ under the following transformation:

$$F(z,\bar{z}) \to \frac{a(\bar{z})F(z,\bar{z}) + b(\bar{z})}{c(\bar{z})F(z,\bar{z}) + d(\bar{z})}, \quad U(\bar{z}) = \begin{pmatrix} a(\bar{z}) & b(\bar{z}) \\ c(\bar{z}) & d(\bar{z}) \end{pmatrix} \in SL(2). \tag{127}$$

*4.3 Functional Integral over the Virasoro Group.* We have seen that the system defined by the functional integral

$$\int [dF] \exp(ikS_{\mathrm{vir}}(F)) \tag{128}$$

is equivalent to the constrained WZW model. The maximal chiral algebra in the right-moving sector is the Virasoro algebra of $c_k = 13 - 6/(k+2) - 6(k+2)$, while the left-mover has the $SL(2)$ current algebra of level-$k$. In this subsection, we examine the property of this functional integral in more detail. Let us consider a generating functional for correlation functions of energy-momentum tensors defined by

$$\exp(i\Gamma(h)) = \int [dF] \exp\left( ikS_{\rm vir}(F) + i\int d^2z \frac{k}{4\pi} \{F(z,\bar{z}),z\} h(z,\bar{z}) \right). \tag{129}$$

The operator product expansion of the energy-momentum tensors

$$T(z)T(w) \sim \left( \frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) T(w) + \frac{c_k/2}{(z-w)^4} \tag{130}$$

implies the functional differential equation for the generating functional,

$$(\bar{\partial} - h(z,\bar{z})\partial - 2\partial h(z,\bar{z})) \frac{\delta}{\delta h(z,\bar{z})} \Gamma(h) = \frac{c_k}{24\pi} \partial^3 h(z,\bar{z}). \tag{131}$$

As in the case of the WZW model discussed in Sect. 2, the geometric action $S_{\rm vir}(F)$ for the Virasoro group satisfies the cocycle condition under a diffeomorphism,

$$\begin{aligned} S_{\rm vir}(F_1 \circ F_2) &= S_{\rm vir}(F_1) + S_{\rm vir}(F_2) + \int \frac{d^2z}{4\pi} \{F_1 \circ F_2, F_2\} \partial F_2 \bar{\partial} F_2, \\ F_1 \circ F_2(z,\bar{z}) &= F_1(F_2(z,\bar{z}),\bar{z}). \end{aligned} \tag{132}$$

After the change of variable, $z \to w = F_2(z,\bar{z})$, $\bar{z} \to \bar{w} = \bar{z}$, in the integral in the right-hand side, this cocycle condition becomes

$$\begin{aligned} S_{\rm vir}(F_1 \circ F_2) &= S_{\rm vir}(F_1) + S_{\rm vir}(F_2) + \int \frac{d^2w}{4\pi} \{F_1(w,\bar{w}), w\} \frac{\bar{\partial} f_2(w,\bar{w})}{\partial f_2(w,\bar{w})} \\ f_2 &= F_2^{-1}, \quad \text{i.e.} \quad F_2(f_2(z,\bar{z}),\bar{z}) = z. \end{aligned} \tag{133}$$

It is worthwhile to note that the cocycle condition (133) can be derived from the cocycle condition of the WZW model (6). Since the Virasoro algebra comes from the $SL(2)$ current algebra by reduction of degrees of freedom, it is natural that the cocycle conditions for these symmetries are related. We will come back to this point in Sect. 6.

Let us set $h = \bar{\partial} f_2 / \partial f_2$ in Eq. (129). Then we can exploit the cocycle condition to rewrite the path integral (129) as

$$\begin{aligned} \exp(i\Gamma(h)) &= \exp(-ikS_{\rm vir}(F_2)) \int [dF_1] \exp(ikS_{\rm vir}(F_1 \circ F_2)) \\ &= \exp(-ikS_{\rm vir}(F_2)) \int [dF_1 \circ f_2] \exp(ikS_{\rm vir}(F_1)). \end{aligned} \tag{134}$$

If the measure $[dF_1]$ was invariant under diffeomorphism $F_1 \to F_1 \circ f_2$, we would have gotten $\Gamma(h) = -kS_{\rm vir}(F_2)$. However this is correct only in the classical limit of $k \to \infty$. In fact, using the cocycle condition (133); it is straightforward to check that the solution to the functional differential Eq. (131) is given by $(c_k/6)S_{\rm vir}(F_2)$ (note that $c_k/6 \to -k$ as $k \to \infty$). Thus an additional factor $\exp(i(c_k/6 + k)S_{\rm vir}(F_2))$ should come from the change of the functional integration measure $[dF_1 \circ f_2] \to [dF_1]$.

The measure $[dF]$ has been derived from the Haar measure $[g^{-1}dg]$ of the WZW model by reduction of degrees of freedom. In the next section, we will encounter another type of measure $[[dF]] = [dF \circ f]$ $(F(f(z,\bar{z}),\bar{z}) = z)$. This measure is defined to be diffeomorphism invariant and should be a natural functional integral measure to quantize the Virasoro group. The above observation suggests that these two measures are related as

$$\int [dF] \exp(ikS_{\text{vir}}(F)) = \int [[dF]] \exp\left( -i\frac{c_k}{6} S_{\text{vir}}(F) \right). \tag{135}$$

One can easily check that the right-hand side in the above functional integral correctly reproduces the value of the conformal anomaly $c_k$. We thus claim that the quantum theory of the Virasoro group with the action $-(c_k/6)S_{\text{vir}}(F)$ is equivalent to the constrained $SL(2)$ WZW model of level $k$.

## 5. Two-Dimensional Gravity

*5.1 SL(2) Symmetry in Gravity.* Now we are ready to study the quantum theory of the induced gravity. Consider the gravity coupled to the left-right symmetric conformal field theory. Because of the general covariance, we may choose the light-cone gauge for the metric,

$$d^2s = dz\,d\bar{z} + h(z,\bar{z})\,d\bar{z}\,d\bar{z}. \tag{136}$$

In this gauge, an effective action of the gravity is given as a generating functional for the energy-momentum tensor for the right-movers. Then it should satisfy the functional differential Eq. (131), and the effective action for the light-cone metric is given by the geometric action for the Virasoro group as

$$\Gamma(h) = \frac{c_k}{6} S_{\text{vir}}(F), \quad h(z,\bar{z}) = \frac{\bar{\partial} f(z,\bar{z})}{\partial f(z,\bar{z})}, \quad F(f(z,\bar{z}),\bar{z}) = z. \tag{137}$$

In the light-cone gauge, the Faddeev–Popov determinant does not depend on $h$. The quantum gravity is then defined by the following functional integral [2]:

$$\int [dh] \exp\left( i\frac{c_k}{6} S_{\text{vir}}(F) \right). \tag{138}$$

Let us make a change of variable in the above functional integral. To parametrize an infinitesimal variation of the metric $h$, we may introduce a vector field $\varepsilon(z,\bar{z})$ as

$$h(z,\bar{z}) + \delta h(z,\bar{z}) = \frac{\bar{\partial} f(z+\varepsilon(z,\bar{z}),\bar{z})}{\partial f(z+\varepsilon(z,\bar{z}),\bar{z})}. \tag{139}$$

The Jacobian for the change of measure $[dh] \to [d\varepsilon]$ can be easily computed to be

$$[dh] = \det(\bar{\partial} - h\partial + \partial h)[d\varepsilon] = \exp(-i\tfrac{26}{6} S_{\text{vir}}(F))[d\varepsilon]. \tag{140}$$

It is also straightforward to see

$$[d\varepsilon] = [dF \circ f] = [[dF]], \tag{141}$$

where $[[d\varepsilon]]$ is the diffeomorphism invariant measure for $F$ discussed in the last section. Thus the functional integral (138) is written

$$\int [[dF]] \exp\left( -i\frac{26-c_k}{6} S_{\mathrm{vir}}(F) \right). \tag{142}$$

In Sect. 2, we have shown that the gauge field coupled to the WZW model of level $k$ also makes the WZW model of level $-(k+2c_V)$. We now found that the induced gravity coupled to the conformal field theory with the conformal anomaly $c_k$ gives the quantum field theory of the coadjoint orbit of the Virasoro group with the conformal anomaly $\tilde{c} = 26 - c_k$.

In the last section, we have shown that the constrained $SL(2)$ WZW model with level $k$ is equivalent to the quantum theory of the Virasoro group with the action $-c_k/6 S_{\mathrm{vir}}(F)$. Because of the relation $26 - c_k = c_{-(k+4)}$, the quantum gravity coupled to the conformal field theory with $c_k$ is equivalent to the constrained WZW model of level $\tilde{k} = -(k+4)$,

$$\int [[dF]] \exp\left( -i\frac{26-c_k}{6} S_{\mathrm{vir}}(F) \right) = \int [dF] \exp(-i(k+4) S_{\mathrm{vir}}(F))$$
$$= \int \frac{[g^{-1}dg]}{(\text{gauge volume})} \delta(J^-(z) - 1) \exp(-i(k+4) S_{\mathrm{wzw}}(g)). \tag{143}$$

Thus the *left-moving sector* of the gravity has the $SL(2)$ current algebra of level $\tilde{k}$. This relation between the level $\tilde{k}$ of the current algebra in the induced gravity and the conformal anomaly $c_k$ of the original conformal field theory agrees with the result by Knizhnik, Polyakov and Zamolodchikov [3].

So far we have not looked at constraints implied by the light-cone gauge condition of the metric. Originally Knizhnik, Polyakov and Zamolodchikov derived the relation between $\tilde{k}$ and $c_k$ by requiring such constraints be imposed consistently. Since the notion of constraints in the induced gravity sounds delicate, we would like to explain the situation using the example of the WZW model. Consider the WZW model coupled to a gauge field $A$ and $\bar{A}$ as [16]

$$S(g; A, \bar{A}) = k S_{\mathrm{wzw}}(g) + \int \frac{d^2z}{8\pi} \mathrm{tr}\left( \bar{A}J - A\bar{J} + \frac{k}{2}\bar{A}gAg^{-1} - \frac{k}{2}\bar{A}A \right). \tag{144}$$

This action has the vector gauge invariance

$$\begin{aligned} &g \to ugu^{-1}, \\ &A \to uAu^{-1} + \partial u u^{-1}, \quad \bar{A} \to u\bar{A}u^{-1} + \bar{\partial} u u^{-1}. \end{aligned} \tag{145}$$

By setting the gauge-fixing condition $A = 0$, we obtain the system discussed in Sect. 2. In the BRST formalism, we introduce a set of ghosts $(b, c)$ in the adjoint representation of the gauge group. The total current $\bar{J}_{\mathrm{total}}(\bar{z})$ is given by

$$\bar{J}_{\mathrm{total}} = \bar{J} - \frac{k}{2}(g^{-1}\bar{A}g - \bar{A}) + \bar{J}_{\mathrm{ghost}}. \tag{146}$$

By making a change of variables ($g \to U^{-1}g$ and similar operators on $b$ and $c$, where $\bar{A} = U^{-1}\bar{\partial}U$), an effective action for the gauge field is extracted, as we have seen in Sect. 2. After this $U$-dependent chiral gauge transformation, the total current $\bar{J}_{\text{total}}$ becomes the sum of three currents, $J, ((k+2c_V)/2)U^{-1}\bar{\partial}U$ and $J_{\text{ghost}}$. Since anomalies of these currents are $k$, $-(k+2c_V)$ and $2c_V$ respectively, the total current is anomaly free. The BRST charge for the vector gauge invariance is then nilpotent.

The situation should be the same in the case of the induced gravity, and the total conformal anomaly in the *left-moving* sector should vanish. The original conformal field theory has an anomaly $c_k$, and the ghosts for the light-cone gauge-fixing add $-26-2=-28$. Then the conformal anomaly in the left-moving sector of the gravity must be $28-c_k=3\tilde{k}/\tilde{k}+2-6\tilde{k}$ ($\tilde{k}=-(k+4)$). This seems to indicate that the *left-moving* current algebra of the gravity is also constrained, otherwise the conformal anomaly would be $3\tilde{k}/\tilde{k}+2$. Knizhnik, Polyakov and Zamolodchikov argued that one of the constraints associated with the light-cone gauge fixing is $\bar{J}^-(z)=0$ for the left-moving current algebra of the gravity, and that this constraint shifts the conformal anomaly by $-6\tilde{k}$. We think that this aspect of the theory is not well-understood yet and requires further investigation. It should be emphasized that, in our approach, the relation between $c_k$ and $\tilde{k}$ is derived independently of these considerations on constraints.

We would like to note intriguing numerology concerning the levels of current algebras. In Sect. 2, we have seen that the *right-moving* current algebra of level $k$ induces the *left-moving* current algebra of level $-(k+2c_V)$ ($-(k+4)$ for $SL(2)$). Here we found that the conformal field theory with anomaly $c_k$, whose right-mover has a hidden $SL(2)$ current algebra of level $k$, induces the left-moving current algebra of level $\tilde{k}=-(k+4)$ in the gravity. There should be a way to derive the current algebra in the induced gravity directly from the hidden current algebra in the conformal field theory.

*5.2 Scaling Dimensions of Planar Random Surface.* Let us now discuss how we can compute scaling dimensions of a random surface from the above result. First we would like to remind the reader the definition of scaling dimensions. Consider a partition function for random surfaces with fixed area $\mathscr{A}$,

$$\mathscr{Z}(\mathscr{A})=\sum_{\text{surfaces}} e^{-\Gamma}. \tag{147}$$

Here the weight $\Gamma$ is given by a partition function of some statistical model on the surface. It is expected that the partition function behaves asymptotically as

$$\mathscr{Z}(\mathscr{A})\sim\mathscr{A}^{-3+\gamma}e^{-\kappa\mathscr{A}},\quad \mathscr{A}\to\infty. \tag{148}$$

Although $\kappa$ is cutoff dependent, $\gamma$ depends only on the topology of the surface [27, 28] and it gives the scaling dimensions of the random surface. In the following we restrict ourselves to the case of planar topology. In this case, the scaling dimensions have been computed for various statistical models on triangulated random surfaces by Kazakov and Migdal [1].

Let us assume that, in the $\mathscr{A}\to\infty$ limit, the sum over surfaces in Eq. (147) reduces to a functional integral over an intrinsic metric on the surface, and the weight $\Gamma$ is replaced by the effective action $\Gamma(h)$ for the gravity. Then the partition function is given by

$$\mathscr{A}^3 e^{\kappa\mathscr{A}} \mathscr{Z}(\mathscr{A}) \sim \int [dh] \langle \exp(-i\int d^2z h(z,\bar{z}) T(z)) \rangle \quad (\mathscr{A} \to \infty)$$
$$= \int [dF] \exp(-i(k+4) S_{\text{vir}}(F)),$$
$$h = \frac{\bar{\partial} f}{\partial f}, \quad f(F(z,\bar{z}),\bar{z}) = z. \tag{149}$$

In the light-cone gauge, the area $\mathscr{A}$ is given by

$$\mathscr{A} = \int d^2 z, \quad (\sqrt{g} \equiv 1). \tag{150}$$

The value of $\mathscr{A}$ is fixed by the range of coordinates $(z, \bar{z})$ independently of the dynamical variable $h$. Thus we can perform the functional integral (149) without any restriction on $h$. As we noted in Sect. 5, the functional integral (149) has the chiral $SL(2)$ symmetry in the left-moving sector.

$$F(z,\bar{z}) \to \frac{a(\bar{z}) F(z,\bar{z}) + b(\bar{z})}{c(\bar{z}) F(z,\bar{z}) + d(\bar{z})}. \tag{151}$$

Remember that the relation between $F$ and $h$ is non-local as $h = \bar{\partial} f / \partial f$, $f(F(z,\bar{z}),\bar{z}) = z$. With respect to $f(z,\bar{z})$, the above transformation becomes

$$f\left(\frac{a(\bar{z}) z + b(\bar{z})}{c(\bar{z}) z + d(\bar{z})}, \bar{z}\right) \to f(z,\bar{z}). \tag{152}$$

The chiral $SL(2)$ transformation of the constrained WZW model turns into a coordinate transformation on the surface. In fact this is the symmetry of the induced action $\Gamma(h)$ as well as the measure. Among the $SL(2)$ currents $\bar{J}^a(\bar{z})$ in the left-moving sector, $\bar{J}^3(\bar{z})$ generates the scale transformation

$$z \to (1 + \varepsilon(\bar{z})) z, \quad \bar{z} \to \bar{z}. \tag{153}$$

Therefore the scaling dimension $\gamma$ is given by the $SL(2)$ spin of the vacuum state of the induced gravity as

$$\gamma = \lim_{\mathscr{A} \to \infty} \left( \frac{\mathscr{A}}{\mathscr{Z}(\mathscr{A})} \frac{\delta \mathscr{Z}(\mathscr{A})}{\delta \mathscr{A}} + \kappa \mathscr{A} + 3 \right) = \frac{1}{\mathscr{Z}} \int [dF] \bar{Q}^3 \exp(-i(k+4) S_{\text{vir}}(F)),$$
$$\bar{Q}^3 = \oint \bar{J}^3(\bar{z}) d\bar{z}. \tag{154}$$

In order to compute the scaling dimensions, we need a relation between the conformal weight of the right-moving Virasoro algebra in the original conformal field theory and the $SL(2)$ spin of the left-moving current algebra in the induced gravity. Following previous sections, we regard the right-movers as that of the constrained WZW model. The energy-momentum tensor is then

$$T(z) = \frac{1}{k+2} \sum J^a(z) J^a(z) - \partial J^3(z) + \partial b(z) c(z), \tag{155}$$

and the BRST charge

$$Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} c(z) (J^-(z) - 1) \tag{156}$$

is used to define physical states. The physical primary field $\Phi_{-\lambda}(z)$ should be the lowest weight state of the $SL(2)$ algebra

$$J^3(z)\Phi_{-\lambda}(w) \sim \frac{-\lambda}{z-w}\Phi_{-\lambda}(w), \quad J^-(z)\Phi_{-\lambda}(w) \sim \text{regular}. \tag{157}$$

The conformal weight $\Delta$ of $\Phi_{-\lambda}(z)$ is given by

$$\Delta = \frac{\lambda(\lambda-1)}{k+2} + \lambda = \frac{\lambda(\lambda+k+1)}{k+2}. \tag{158}$$

By extending the argument in the last subsection in the presence of primary fields, we can show that the physical field $\Phi_{-\lambda}(z)$ is also the highest weight state of the left-moving current algebra with highest weight $+\lambda$,

$$\bar{J}^3(\bar{z})\Phi_{-\lambda}(w) \sim \frac{+\lambda}{z-w}\Phi_{-\lambda}(w). \tag{159}$$

Thus *right-moving* conformal weight $\Delta$ and the *left-moving* $SL(2)$ spin $\lambda$ are related by Eq. (158). Especially the $SL(2)$ spin of the vacuum state is obtained by putting $\Delta = 0$ in Eq. (158), i.e. either 0 or $-(k+1)$. This gives the scaling dimensions of the surface. Since the level $k$ of the $SL(2)$ current algebra is related to the conformal anomaly of the conformal field theory as

$$c_k = 13 - \frac{6}{k+2} - 6(k+2), \tag{160}$$

we obtain the formula

$$\gamma = \tfrac{1}{12}\left(c_k - 1 \pm \sqrt{(c_k-1)(c_k-25)}\right) \quad \text{or } 0. \tag{161}$$

This formula agrees with that by Knizhnik, Polyakov and Zamolodchikov [3]. If one chooses the branch $(-)$ in the above, it is also consistent with the analytical and numerical computations of triangulated random surface by Kazakov and Migdal [1].

## 6. Geometric Action for $W_n$-Algebra

In this section, we construct a generalization of a geometric action for the $W_n$-algebra. For simplicity, we discuss the case of $n = 3$, but a generalization for $n > 3$ is straightforward. As we have seen in Sects. 3 and 4, irreducible representation spaces of the $W_3$ algebra emerge from the constrained $SL(3)$ WZW model. Following the case of the minimal conformal field theory, we will solve the constraints of the WZW model. The constrained $SL(3)$ WZW model is given by

$$\int \frac{[g^{-1}dg]}{\mathrm{vol}(\mathrm{Borel})}\delta(J_1^-(z)-1)\delta(J_2^-(z)-1)\delta(J_3^-(z))\exp(iS_{\mathrm{wzw}}(g)). \tag{162}$$

Using the Gauß decomposition

$$g = \begin{pmatrix} 1 & 0 & 0 \\ \Phi_1 & 1 & 0 \\ \Phi_3 & \Phi_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda^{-1}\mu^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & F_1 & F_3 \\ 0 & 1 & F_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{163}$$

we may put conditions $\Phi_i = 0$ $(i = 1,2,3)$ to fix the Borel gauge invariance. In this guage, the constraints become

$$\begin{aligned} J_1^- &= \lambda\mu^{-1}\partial F_1 = 1, \quad J_2^- = \lambda\mu^2\partial F_2 = 1 \\ J_3^- &= \lambda^2\mu(\partial F_3 - F_2\partial F_1) = 0. \end{aligned} \tag{164}$$

By solving them with respect to $\lambda, \mu$ and $F_2$ and substituting them into the WZW action, we obtain the effective theory for $F_1$ and $F_3$.

$$\begin{aligned} &\int \frac{[g^{-1}dg]}{(\text{gauge volume})}\delta(J_1^-(z)-1)\delta(J_2^-(z)-1)\delta(J_3^-(z))\exp(iS_{\text{wzw}}(g)) \\ &= \int [dF_1, dF_3]\exp(ikS_{W_3}(F_1,F_3)). \end{aligned} \tag{165}$$

Here the effective action $S_{W_3}$ for $F_1$ and $F_3$ is given by

$$\begin{aligned} S_{W_3}(F_1,F_3) = \frac{1}{2\pi}\int d^2z[&(\lambda^{-1}\partial\lambda)(\lambda^{-1}\bar\partial\lambda) + (\mu^{-1}\partial\mu)(\mu^{-1}\bar\partial\mu) \\ &+ \tfrac{1}{2}(\lambda^{-1}\partial\lambda)(\mu^{-1}\bar\partial\mu) + \tfrac{1}{2}(\lambda^{-1}\bar\partial\lambda)(\mu^{-1}\partial\mu)]. \end{aligned} \tag{166}$$

We have chosen $F_1$ and $F_3$ as dynamical variables of the reduced system, for they do not imply an additional Jacobian factor in the functional integral. We claim that $S_{W_3}$ is a natural generalization of a geometric action for the $W_3$-algebra. Especially when $F_3 \propto (F_1)^2$, this action reduces to the geometric action for the Virasoro group,

$$S_{W_3}(F_1, F_3 \propto (F_1)^2) = 4S_{\text{vir}}(F_1). \tag{167}$$

As in the case of the $SL(2)$ current algebra, the effective action $S_{W_3}(F_1, F_3)$ has the left-moving $SL(3)$ current algebra of level $k$, while the maximal chiral algebra in the right-moving sector is reduced to the $W_3$ algebra with the conformal anomaly

$$c_{W_3} = 50 - \frac{24}{k+3} - 24(k+3). \tag{168}$$

The energy-momentum tensor $T(z)$ and the spin-3 generator $W_3(z)$ in the right-mover are given by substituting Eq. (164) into

$$\begin{aligned} T(z) = -\frac{k}{2\pi}[&(\lambda^{-1}\partial\lambda)^2 + (\mu^{-1}\partial\mu)^2 \\ &+ (\lambda^{-1}\partial\lambda)(\mu^{-1}\partial\mu) + 2\partial(\lambda^{-1}\partial\lambda) + \partial(\mu^{-1}\partial\mu)], \\ W_3(z) \propto [&(\lambda^{-1}\partial\lambda + \mu^{-1}\partial\mu)(\lambda^{-1}\partial\lambda)(\mu^{-1}\partial\mu) \\ &- (\lambda^{-1}\partial\lambda)\partial(\lambda^{-1}\partial\lambda - \mu^{-1}\partial\mu) - \partial^2(\lambda^{-1}\partial\lambda)]. \end{aligned} \tag{169}$$

Let us have a look at the symmetry of the action $S_{W_3}$. After some computations, one can check the following cocycle condition directly.

$$\begin{aligned} &kS_{W_3}(F_1\circ F, F_3\circ F) = kS_{W_3}(F_1,F_3) + 4kS_{\text{vir}}(F) + \int d^2z T(F_1,F_3;z)\frac{\bar\partial f}{\partial f}, \\ &F_i\circ F(z,\bar z) = F_i(F(z,\bar z),\bar z), \quad F(f(z,\bar z),\bar z) = z. \end{aligned} \tag{170}$$

One can derive this cocycle condition from the cocycle condition of the WZW action (6). Suppose we have fields, $U(z,\bar{z}), g(z,\bar{z})$, both satisfying the constraints (164). In general, the product of $U(z,\bar{z})g(z,\bar{z})$ does not satisfy these constraints. However, when $U$ is of the form

$$U = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda^{-1}\mu^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & F & \frac{1}{2}F^2 \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}, \tag{171}$$

we can modify the action of $U$ on $g$ as

$$(U \bullet g)(z,\bar{z}) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda^{-1}\mu^{-1} \end{pmatrix} \cdot g(F(z,\bar{z}),\bar{z}), \tag{172}$$

so that $U \bullet g$ also obeys the constraints. It is easy to see that $S_{\mathrm{wzw}}(U \bullet g) = S_{W_3}(F_1 \circ F, F_3 \circ F)$, and that the cocycle condition for $S_{\mathrm{wzw}}$ implies the cocycle condition for $S_{W_3}$. This relation between the cocycle conditions of $S_{\mathrm{wzw}}$ and $S_{W_3}$ is quite natural, for the Virasoro algebra in the theory with $S_{W_3}$ comes from the $SL(3)$ symmetry in $S_{\mathrm{wzw}}$ by way of the quantum Hamiltonian reduction. Unfortunately, we have not succeeded in defining a modified action of $U$ on $g$, $U \bullet g$, for a general WZW field $U$. Such a multiplication rule, if exists, defines the full $W_3$ symmetry realized in terms of $F_1$ and $F_3$, just as a diffeomorphism is realized by a single function $F$ with a product $\circ$. This will also make it possible to derive a generalized cocycle condition for $S_{W_3}$ corresponding to the full $W_3$-symmetry. Since the $W_3$-algebra is not a Lie algebra, it would not be straightforward to find such a modified action $U \bullet g$. Still, we believe that such a construction is possible. For the $W_3$-symmetry comes from the $SL(3)$ symmetry of $S_{\mathrm{wzw}}$ by the quantum Hamiltonian reduction.

Let us make some more speculations about the theory with $S_{W_3}$. In the functional integral (165), the measure $[dF_1, dF_3]$ is derived from the Haar measure $[g^{-1}dg]$ by reduction of degrees of freedom. We conjecture that, if there is a measure $[[dF_1, dF_3]]$ invariant under the full $W_3$-symmetry, it should be related to the reduced Haar measure as

$$\int [dF_1, dF_3] \exp(ikS_{W_3}(F_1, F_3)) = \int [[dF_1, dF_3]] \exp\left( -i\frac{c_k}{24} S_{W_3}(F_1, F_3) \right). \tag{173}$$

In the case of the $SL(2)$, the geometric action $S_{\mathrm{vir}}(F)$ described the induced gravity, which is gauge-equivalent to the Liouville model. Therefore we suspect that the constrained $SL(3)$ WZW model is related to the field theory of the Toda molecule, a natural generalization of the Liouville model. Previously Bilal and Gervais [29] have also pointed out that the Toda field theory realizes the $W_n$-algebra. It should be interesting to re-examine their result from the point of view of the quantum Hamiltonian reduction.

Proceeding further, we may consider gauge fields coupled to chiral currents in a conformal field theory with the $W_3$-algebra. The induced action for the gauge fields would be $-((100 - c_k)/24) S_{W_3}(F_1, F_3)$. The induced gauge fields will then make the $SL(3)$ current algebra of level $-(k+6)$ in the left-mover. If this is the case, we may compute the scaling dimensions of the surface of the "fluctuating $W_3$-geometry." Conformal weight $\Delta$ and $W_3$-charge $w_0$ of the conformal field theory are given by

$$\Delta = \frac{1}{3(k+3)}(\lambda^2 + \mu^2 + \lambda\mu) + \frac{k+2}{k+3}(\lambda + \mu),$$

$$w_0 \propto \left(\frac{1}{k+3}\right)^{3/2} [9(\lambda+\mu)^2(\lambda-\mu) - (\lambda-\mu)^3 + 72(k+2)^2(2\lambda+\mu)$$
$$+ 9(k+2)(3(\lambda+\mu)^2 + (\lambda-\mu)^2 + 4(\lambda^2-\mu^2))]. \tag{174}$$

Here $\lambda$ and $\mu$ are lowest weights of the hidden $SL(3)$ current algebra. The scaling dimensions $\gamma$ of the surface is given from solutions of $\Delta(\lambda,\mu)=0$, $w_0(\lambda,\mu)=0$ as

$$\begin{aligned} \gamma &= \lambda + \mu \\ &= 0, -(k+2), -3(k+2), -4(k+2), \end{aligned} \tag{175}$$

where

$$-4(k+2) = \tfrac{1}{12}(c - 2 \pm \sqrt{(c-2)(c-98)}). \tag{176}$$

Unfortunately we do not know the classical limit of scaling dimension for "fluctuating $W_3$ geometry" which allows one to choose one of the four solutions.

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