Nonlinear Poisson Structures and r-Matrices

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Abstract. We introduce quadratic Poisson structures on Lie groups associated with a class of solutions of the modified Yang–Baxter equation and apply them to the Hamiltonian description of Lax systems. The formal analog of these brackets on associative algebras provides second structures for certain integrable equations. In particular, the integrals of the Toda flow on generic orbits are shown to satisfy recursion relations. Finally, we exhibit a third order Poisson bracket for which the r-matrix approach is feasible.

1. Introduction

The classical r-matrices were first introduced by E. Sklyanin in [17] and [18] as limits of their quantum counterparts. Subsequently, this has led V. G. Drinfel'd to introduce a new geometric concept, that of a Poisson Lie group [7]. The relevance of these notions in the study of classical integrable systems was recently explained in two fundamental papers of M. Semenov–Tian–Shansky [15, 16]. By abandoning the classical Yang–Baxter equation in favor of the modified Yang–Baxter equation (mYB), the result is a unification of the generalized Adler–Kostant–Symes procedure and the method of the Riemann problem [15]. Furthermore, in the second half of [15] and in [16], it was revealed that the r-matrix approach is naturally associated with a class of quadratic Poisson structures commonly referred to as the Sklyanin brackets. These quadratic Poisson structures on Lie groups (and modifications thereof [16]) are associated with skew symmetric solutions of (mYB) and give rise to a geometrical theory of Lax systems and dressing transformations. On the other hand, their formal analog on associative algebras provides an abstract version of the "second Hamiltonian structure" for equations of KdV type as conjectured by Adler [2] and proved in [9] by Gelfand and Dikii.

It is the purpose of this paper to extend the theory of Lax systems and the construction of "second Poisson structures" in [15] and [16] to a wider class of r-matrices. This will be carried out in Sects. 3 and 4 below. Instead of assuming the r-matrix $R \in \mathrm{End}\, \mathfrak{g}$ to be skew symmetric, we shall assume that R and $A = \frac{1}{2}(R - R^*)$ are solutions of (mYB). Here, the choice of this particular class of

r-matrices is motivated by applications (see, for example, [12] and Proposition 4.2 below). In Sect. 3, we construct a theory of Lax systems for r-matrices which satisfy our assumption. This is accomplished by means of a twisted Poisson structure $\{\ ,\ \}_\tau$ on the group G, where τ is an orthogonal map on $\mathfrak{g}$ which commutes with R. In particular, for $\tau = 1$, the corresponding Poisson structure is an analog of the Sklyanin bracket. In contrast to the theory in [15], when the r-matrix is not skew-symmetric but satisfies our assumption, the Poisson structure for the associated Lax systems is not a product structure (cf. Theorem 11 of [16] and Theorem 3.6 below). In Sect. 4, we consider the formal analog of the Poisson structure $\{\ ,\ \}_\tau$, $\tau = 1$, on associative algebras. As in [15], the bracket is compatible with the Lie Poisson structure associated with the generalized Adler–Kostant–Symes scheme and gives rise to equations in Lax form. If we take $\mathfrak{g}$ to be $gl(n, \mathbb{R})$, and let R be the r-matrix for the Toda flow [6], then R verifies our assumption and the quadratic bracket provides a second Poisson structure for that system. In [6], the Toda flow was shown to be completely integrable on generic coadjoint orbits with $\frac{1}{2}n(n-1)$ independent integrals. The rest of the section is devoted to proving recursion relations for this collection of integrals. Here, we find n recursion relations, one for each of the n coadjoint orbit invariants, and the result implies the involution of the integrals in both structures. With a little more work, one can in fact establish the integrability on the generic symplectic leaves of the second structure. Finally, in Sect. 5, we make an attempt to construct higher order structures on associative algebras. The main result is a cubic Poisson structure such that the Hamilton's equations associated to ad-invariant functions are in Lax form. Moreover, this cubic structure is compatible with the quadratic bracket of Sect. 4 and the Lie Poisson structure mentioned earlier. In contrast to the quadratic case, the r-matrix here is only assumed to satisfy (mYB). The investigations in Sects. 4 and 5 naturally lead to the following question: Is there a natural hierarchy of Poisson structures in the r-matrix approach and if so, what implications does it have towards the complete integrability of the Lax equations?

Some of the results proved in this paper have been announced in [13].

2. Preliminaries

In this section, we provide the reader with some basic results and constructs which will be used throughout the paper. The material is based mainly on the work of Drinfel'd [7] and Semenov–Tian–Shansky [16]. In the sequel, we consider Lie algebras $\mathfrak{g}$ which are equipped with nondegenerate invariant pairings $(\cdot, \cdot)$. We allow $\dim \mathfrak{g} = \infty$, in which case we assume, wherever necessary, that there exists a corresponding local Lie group. This assumption is valid, for example, if $\mathfrak{g}$ is a Banach Lie algebra [4].

A. The Modified Yang–Baxter Equation and Squares of Baxter Algebras

Definition 1. A linear operator $R \in \mathrm{End}\,\mathfrak{g}$ is called a classical r-matrix if the formula

$$[X, Y]_R = \tfrac{1}{2}([RX, Y] + [X, RY]), \quad X, Y \in \mathfrak{g}$$

defines a Lie bracket. We shall denote by $\mathfrak{g}_R$ the algebra $\mathfrak{g}$ when equipped with the bracket$[\ ,\ ]_R$ and call the pair $(\mathfrak{g}, \mathfrak{g}_R)$ a double Lie algebra.

An important sufficient condition for $[\ ,\ ]_R$ to define a Lie bracket is given by the modified Yang–Baxter equation

(mYB) $\qquad [RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y].$

This choice is motivated by the following equivalent statement:

$$R_\pm = \tfrac{1}{2}(R \pm 1): \mathfrak{g}_R \to \mathfrak{g} \text{ are a Lie algebra homomorphisms,}$$

which allows one to obtain the solution of certain dynamical systems by factorization problems. Note that a large number of Hamiltonian systems which are related to Lie algebra decompositions [3, 8],

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \text{ (vector space direct sum)}, \quad [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}, \quad [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b},$$

are in this category as $R = \Pi_\mathfrak{a} - \Pi_\mathfrak{b}$ (where $\Pi_\mathfrak{a}, \Pi_\mathfrak{b}$ are the projections relative to $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$) is a solution of (mYB).

If G_{loc} and $(G_R)_{\mathrm{loc}}$ denote the germs of local Lie groups corresponding to the Lie algebra $\mathfrak{g}, \mathfrak{g}_R$ respectively, then there exist homomorphisms $\hat{R}_\pm : (G_R)_{\mathrm{loc}} \to G_{\mathrm{loc}}$ such that $T_e \hat{R}_\pm = R_\pm$. For $h \in (G_R)_{\mathrm{loc}}$ $(X \in \mathfrak{g}_R)$, we shall write $h_\pm = \hat{R}_\pm(h)$ $(X_\pm = R_\pm X)$. Now, consider the map $m: (G_R)_{\mathrm{loc}} \to G_{\mathrm{loc}}: g \mapsto g_+ g_-^{-1}$. Since $T_e m = R_+ - R_- = 1$, this allows us to identify $(G_R)_{\mathrm{loc}}$ with G_{loc}. Thus we shall write $g = g_+ g_-^{-1}$ and if $*$ denotes the group operation in $(G_R)_{\mathrm{loc}}$, then $g * h = g_+ h g_-^{-1}$.

Definition 2. A double Lie algebra $(\mathfrak{g}, \mathfrak{g}_R)$ is called a Baxter Lie algebra if R satisfies (mYB) and $R = -R^*$.

The square of a Baxter Lie algebra $(\mathfrak{g}, \mathfrak{g}_R)$ is defined as follows. First, we put $\delta = \mathfrak{g} \oplus \mathfrak{g}$ and equip it with the ad-invariant pairing

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle = (X_1, X_2) - (Y_1, Y_2).$$

Let $^\delta \mathfrak{g} \subset \delta$ be the diagonal subalgebra and embed $\mathfrak{g}_R \hookrightarrow \delta$ via $X \mapsto (X_+, X_-)$. Then $\delta = {}^\delta \mathfrak{g} \oplus \mathfrak{g}_R$ as a linear space therefore, if $P_{\delta_\mathfrak{g}}$ and $P_{\mathfrak{g}_R}$ denote the projections onto $^\delta \mathfrak{g}$ and $\mathfrak{g}_R$ respectively, it follows that

$$R_\delta = P_{\delta_\mathfrak{g}} - P_{\mathfrak{g}_R} \in \mathrm{End}\, \delta$$

is a solution of (mYB) which is skew-symmetric relative to $\langle\ ,\ \rangle$. The Baxter Lie algebra $(\delta, \delta_{R_\delta})$ is called the square of $(\mathfrak{g}, \mathfrak{g}_R)$. We shall let D_{loc} denote the germ of local Lie group corresponding to δ. Note that $(G_R)_{\mathrm{loc}} \hookrightarrow D_{\mathrm{loc}}$ via $g \mapsto (g_+, g_-)$.

B. Poisson Lie Groups and Poisson Reduction

Definition 3. A Lie group H equipped with a Poisson structure is called a Poisson Lie group if group multiplication is a Poisson map from $H \times H$ (equipped with the product structure) into H.

Let $\mathfrak{h} = T_e H$. For $\varphi \in C^\infty(H)$, we define the left and right gradients $D\varphi, D'\varphi \in \mathfrak{h}$ by

$$(D\varphi(g), X) = \frac{d}{dt}\bigg|_{t=0} \varphi(e^{tX} g), \quad (D'\varphi(g), X) = \frac{d}{dt}\bigg|_{t=0} \varphi(g e^{tX}).$$

Writing the Poisson bracket on H in the right invariant frame:

$$\{\varphi, \psi\}_H(g) = (\eta(g)D\varphi(g), D\psi(g)),$$

we have $(H, \{\ ,\ \}_H)$ is a Poisson Lie group if and only if the Hamiltonian operator $\eta: H \to \text{End}\,\mathfrak{h}$ is a 1 cocycle of H for the Ad-action, i.e.

$$\eta(gh) = \text{Ad}_g\eta(h)\text{Ad}_{g-1} + \eta(g).$$

In particular, the condition $\eta(e) = 0$ allows one to define the tangent Lie algebra structure $[\ ,\ ]_*$ on $\mathfrak{h}$ (irrespective of whether H is a Poisson Lie group or not):

$$([X, Y]_*, Z) = d\{\varphi, \psi\}_H(e)\cdot Z,$$

where

$$(X, Z) = d\varphi(e)\cdot Z \quad \text{and} \quad (Y, Z) = d\psi(e)\cdot Z, \quad Z \in \mathfrak{h}.$$

Let $R, R' \in \text{End}\,\mathfrak{h}$ be skew-symmetric solutions of (mYB), we have

Theorem 4. $\{\varphi, \psi\}_{(R,R')} = \frac{1}{2}(R(D\varphi), D\psi) + \frac{1}{2}(R'(D'\varphi), D'\psi)$ *defines a Poisson structure on* H.

If we let $H_{(R,R')} = (H, \{\ ,\ \}_{(R,R')})$, then in particular, $H_{(R,-R)}$ is a Poisson Lie group whose tangent Lie algebra is $\mathfrak{h}_R$. The Poisson bracket $\{\ ,\ \}_{(R,-R)}$ is known as the Sklyanin bracket.

Definition 5. Let G be a Poisson Lie group, M a Poisson manifold. A Lie group action $\Phi: G \times M \to M$ is called a Poisson group action if it is a Poisson map from $G \times M$ (equipped with the product structure) into M.

Remark. The notions of Poisson group actions and reduction are different from the more familiar ones considered, for example, in [1] and [11].

A useful proposition in Poisson group reduction is

Proposition 6 [16]. *Let* $\Phi: G \times M \to M$ *be a left (right) Poisson group action. Let* $H \subset G$ *be a connected Lie subgroup. If* $[\mathfrak{h}^\perp, \mathfrak{h}^\perp]_* \subset \mathfrak{h}^\perp$, *then the algebra* C^H *of* H-*invariant functions is a Lie subalgebra of* $C^\infty(M)$, *i.e.* $\Phi|H \times M$ *is admissible. In this case, there exists a unique Poisson structure on the quotient* $H\backslash M$ (M/H) *such that the projection is a Poisson map.*

The Sklyanin bracket admits such a reduction theory [16]. Indeed, the description of its symplectic leaves is based on the construction of a dual pair which involves the symplectic manifold $(D_{\text{loc}})_{(R_\delta,R_\delta)}$. We now give the definition of a dual pair due to Weinstein [19, 20].

Definition 7. A pair of constant rank Poisson maps $P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2$ from the symplectic manifold S to the Poisson manifolds P_1 and P_2 is called a dual pair if either of the following equivalent conditions is satisfied:

(i) $\pi_1^* C^\infty (P_1)$ and $\pi_2^* C^\infty (P_2)$ are mutual centralizers in $C^\infty(S)$,
(ii) at each $x \in S$, $\ker T_x\pi_1 = (\ker T_x\pi_2)^\perp$.

The dual pair is said to be full if π_1, π_2 are submersions onto P_1 and P_2.

3. Twisted Poisson Structures on G and Application to Lax Systems

Let G be a Lie group whose Lie algebra $\mathfrak{g}$ is equipped with a nondegenerate invariant pairing $(\cdot, \cdot)$. In this section, we give a new class of Poisson structures on G associated with r-matrices $R \in \text{End}\,\mathfrak{g}$ which satisfy the basic assumption:

(H) R and $A = \frac{1}{2}(R - R^*)$ are solutions of (mYB).

We then apply these structures to the Hamiltonian description of the associated Lax systems. Clearly, the skew-symmetric solutions of (mYB) are among those which satisfy hypothesis (H). Therefore, our results are extensions of the work of V. G. Drinfel'd [7] and M. Semenov–Tian–Shansky [15, 16].

    Before describing an important consequence of the basic assumption (H), we shall provide the reader with some concrete examples of r-matrices which verify the hypothesis.

Examples Associated with $\mathfrak{g} = gl(n, \mathbb{R})$. In the first two examples, equip $\mathfrak{g}$ with the ad-invariant pairing $(X, Y) = \text{tr}\,XY$, and let

$\mathfrak{l} = $ the algebra of real, lower triangular matrices,
$\mathfrak{u} = $ the algebra of real strictly upper triangular matrices,
$\mathfrak{k} = $ the Lie algebra of real, $n \times n$ skew-symmetric matrices.

$1°$ We have $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$. If $\Pi_\mathfrak{l}, \Pi_\mathfrak{u}$ denote the associated projection operators, then $R = \Pi_\mathfrak{l} - \Pi_\mathfrak{u}$ satisfies the assumption and indeed, $A(X) = X_+ - X_-$, where $X_\pm$ is the strict upper/lower triangular part of X. Thus A is a solution of (mYB).
$2°$ For the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, the corresponding r-matrix is $R = \Pi_\mathfrak{k} - \Pi_\mathfrak{l}$. As in the first example, $A(X) = \frac{1}{2}(R - R^*)(X) = X_+ - X_-$.
$3°$ Let $\tilde{\mathfrak{g}} = \mathfrak{g}_\mathbb{C} \otimes \mathbb{C}[h, h^{-1}]$ equipped with the nondegenerate invariant pairing

$$(X, Y) = \frac{1}{2\pi\sqrt{-1}} \int_{|h|=1} \text{tr}\,(X(h)Y(h)) \frac{dh}{h}, \quad X, Y \in \tilde{\mathfrak{g}}.$$

For any r-matrix $R \in \text{End}\,\mathfrak{g}_\mathbb{C}$ which satisfies the basic assumption (H), we associate the operator $\tilde{R} \in \text{End}\,\tilde{\mathfrak{g}}$:

$$\tilde{R}X(h) = -\sum_{i<0} X_i h^i + RX_0 + \sum_{i>0} X_i h^i, \quad X(h) = \sum_i X_i h^i.$$

Then clearly, $\tilde{R}$ is a solution of (mYB). Since $\tilde{R}^* \in \text{End}\,\tilde{\mathfrak{g}}$ is given by

$$\tilde{R}^*X(h) = \sum_{i<0} X_i h^i + R^*X_0 - \sum_{i>0} X_i h^i, \quad X(h) = \sum_i X_i h^i,$$

we obtain

$$\tilde{A}X(h) = \frac{1}{2}(\tilde{R} - \tilde{R}^*)X(h) = -\sum_{i<0} X_i h^i + AX_0 + \sum_{i>0} X_i h^i.$$

Therefore, $\tilde{A}$ is also a solution of (mYB). This shows $\tilde{R}$ is an r-matrix which verifies hypothesis (H).

Remark. The first two examples above are clearly related the the root space

decomposition and the Iwasawa decomposition, and therefore the results can be adapted to other classical reductive Lie algebras as well. For further examples, we refer the reader to the Appendix.

Lemma 1. *Let* $S = \frac{1}{2}(R + R^*)$, *then under the basic assumption* (H), $\frac{1}{2}S:\mathfrak{g}_A \to \mathfrak{g}$ *is a Lie algebra homomorphism.*

Proof. We must show $[\frac{1}{2}SX, \frac{1}{2}SY] = \frac{1}{2}S[X, Y]_A$, i.e., $[(R + R^*)X, (R + R^*)Y] = (R + R^*)([X, (R - R^*)Y] + [(R - R^*)X, Y])$. From the identities

(a) $(R + R^*)([X, (R - R^*)Y] + [(R - R^*)X, Y]) = (R - R^*)([X, (R - R^*)Y]$

$\qquad + [(R - R^*)X, Y]) + 2R^*([X, RY] - [R^*X, Y]) + R^*([RX, Y] - [X, R^*Y])$

(b) $[(R + R^*)X, (R + R^*)Y] = [(R - R^*)X, (R - R^*)Y] + 2[R^*X, RY]$

$\qquad + 2[RX, R^*Y],$

and the assumption that A is a solution of (mYB), we have

$$[(R + R^*)X, (R + R^*)Y] - (R + R^*)([X, (R - R^*)Y] + [(R - R^*)X, Y])$$
$$= -4[X, Y] + 2([R^*X, RY] - R^*[X, RY] - R^*[R^*X, Y])$$
$$+ 2([RX, R^*Y] - R^*[RX, Y] + R^*[X, R^*Y]).$$

To complete the proof, it suffices to show

$$[R^*X, RY] - R^*[X, RY] + R^*[R^*X, Y] = [X, Y].$$

But this is just an equivalent way of asserting that R is a solution of (mYB), as can be easily verified. $\quad\square$

Remark. Conversely, if $J \in \mathrm{End}\,\mathfrak{g}$ is a skew-symmetric solution of (mYB) and $\frac{1}{2}\tilde{S}:\mathfrak{g}_J \to \mathfrak{g}$ is a Lie algebra homomorphism which is symmetric, then $J + \tilde{S}$ is a solution of (mYB).

Theorem 2. *Let* $\tau \in \mathrm{Aut}(G)$ *whose induced map on* $\mathfrak{g}$ *(denoted by the same letter) is orthogonal and commutes with* R.

(a) *The formula*

$$\{\varphi, \psi\}_\tau = \frac{1}{2}(A(D'\varphi), D'\psi) - \frac{1}{2}(A(D\varphi), D\psi) + \frac{1}{2}(\tau \circ S(D\varphi), D'\psi)$$
$$- \frac{1}{2}(\tau^{-1} \circ S(D'\varphi), D\psi), \quad \varphi, \psi \in C^\infty(G)$$

defines a Poisson bracket on G. *If* $\tau^2 = 1$, *its tangent Lie algebra is* $\mathfrak{g}_{A + \tau \circ S}$.
(b) *If* φ *is invariant under twisted conjugation* $g \to hg(\tau(h))^{-1}, g, h \in G$, *the equation of motion defined by the Hamiltonian* φ *in the structure* $\{\ ,\ \}_\tau$ *is given by*

$$\dot{g} = \frac{1}{2}T_e R_g(R(D\varphi(g))) - \frac{1}{2}T_e L_g(\tau \circ R(D\varphi(g))).$$

(c) *Let* $h_\pm(t)$ *be the solution of the factorization problem*

$$\exp(-tD\varphi(g_0)) = h_+(t)^{-1}h_-(t), \quad (h_+(t), h_-(t)) \in G_R,$$

for those values of t for which the left-hand side is in the image of the map m in Sect. 2A. Then the solution of the initial value problem associated with the equation in

(b) *is given by*

$$g(t) = h_+(t)g_0\tau(h_+(t)^{-1}) = h_-(t)g_0\tau(h_-(t)^{-1}).$$

(d) *Functions which are invariant under twisted conjugation commute in* $\{\ ,\ \}_\tau$.

Proof.

(a) The formula clearly defines a skew-symmetric bilinear form on $C^\infty(G)$ which is a derivation in each argument. To complete the proof, it remains to verify the Jacobi identity. This makes use of the basic assumption together with Lemma 1, but we will omit the details here as we are going to describe the reduction theory in the next theorem. Writing the bracket in the right invariant frame, we find that the Hamiltonian operator is given by $\eta(g) = \frac{1}{2}\mathrm{Ad}_g \circ A \circ \mathrm{Ad}_{g^{-1}} - \frac{1}{2}A + \frac{1}{2}\mathrm{Ad}_g \circ \tau \circ S - \frac{1}{2}\tau^{-1} \circ S \circ \mathrm{Ad}_{g^{-1}}$. If $\tau^2 = 1$, then $\eta(e) = 0$. From Sect. 2B, it follows that the tangent Lie algebra structure can be defined on $\mathfrak{g}$ and a direct computation shows that it is given by $[\ ,\ ]_{A + \tau \circ S}$.

(b) If φ is invariant under twisted conjugation, we have $D'\varphi = \tau \circ D\varphi$. Therefore,

$$-(D\psi(g), T_g R_{g^{-1}} X_\varphi(g)) = \{\varphi, \psi\}_\tau(g) = \frac{1}{2}(D\psi(g), \mathrm{Ad}_g \circ A \circ \tau(D\varphi(g)) - A(D\varphi(g))$$
$$+ \mathrm{Ad}_g \circ \tau \circ S(D\varphi(g)) - S(D\varphi(g)))$$
$$= \frac{1}{2}(D\psi(g), \mathrm{Ad}_g \circ \tau \circ R(D\varphi(g)) - R(D\varphi(g))),$$

from which it follows that $X_\varphi(g) = \frac{1}{2}T_e R_g(R(D\varphi(g))) - \frac{1}{2}T_e L_g(\tau \circ R(D\varphi(g)))$.

(c) This can be checked by direct differentiation. See, however, the remark after Corollary 5.

(d) This follows easily from the assumptions on τ and the invariance properties. $\square$

Remarks.

(a) When $\tau = 1$, the Poisson bracket $\{\ \}_\tau$ is the analog of the Sklyanin bracket for the class of r-matrices which satisfy hypothesis (H). It provides first of all a Hamiltonian description of the Lax equation $\dot{g} = \frac{1}{2}T_e R_g(R(D\varphi(g))) - \frac{1}{2}T_e L_g(R(D\varphi(g)))$ corresponding to a central function φ. Furthermore, when $R = -R^*$, it reduces to the Sklyanin bracket.

(b) In general, the group G equipped with the twisted Poisson structure $\{\ ,\ \}_\tau$ is not a Poisson Lie group. This fact can be obtained from the explicit form of the Hamiltonian operator η in the proof of (a).

(c) The inversion map $\iota: g \mapsto g^{-1}$ satisfies $\{\varphi \circ \iota, \psi \circ \iota\}_\tau = -\{\varphi, \psi\}_{\tau^{-1}} \circ \iota, \varphi, \psi \in C^\infty(G)$. In particular, when $\tau^2 = 1, \iota$ is an anti-Poisson map.

We now describe the reduction theory of the twisted Poisson structure. In the next lemma and theorem, we shall deal exclusively with germs of local Lie group. To simplify notations, the image of a Lie group under the localization functor will be denoted by the same symbol. Thus, from Lemma 1, the Lie algebra homomorphism $\frac{1}{2}S:\mathfrak{g}_A \to \mathfrak{g}$ can be lifted up to a group homomorphism $\sigma: G_A \to G$. Modifying the action of the Poisson Lie group G_A on $D_{(A_\delta, A_\delta)}$ ($D = G \times G$ and A_δ is defined as in Sect. 2A) by left and right translations, we consider the maps

$p, q : G_A \times D \to D$ defined by

$$p(g, (x, y)) = (\tau^{-1}(\sigma(g)^{-1}) x g_+, \tau^{-1}(\sigma(g)^{-1}) y g_-),$$
$$q(g, (x, y)) = (g_+ x \tau(\sigma(g)^{-1}), g_- y \tau(\sigma(g)^{-1})).$$

Then p is a right action and q is a left action.

Lemma 3. *The actions p and q are admissible, i.e. the invariant functions on $D_{(A_\delta, A_\delta)}$ for these actions form subalgebras.*

Proof. We shall prove the assertion for p. First of all, by combining left and right translations, we get a Poisson group action:

$$D_{(A_\delta, A_\delta)} \times (D_{(-A_\delta, A_\delta)} \times D_{(-A_\delta, A_\delta)}) \to D_{(A_\delta, A_\delta)}$$
$$(g_1, g_2) : x \mapsto g_1^{-1} x g_2, \quad (g_1, g_2) \in D \times D.$$

Now, embed G_A into $D_{(-A_\delta, A_\delta)} \times D_{(-A_\delta, A_\delta)}$ via $g \mapsto ((\tau^{-1}(\sigma(g)), \tau^{-1}(\sigma(g))), (g_+, g_-))$ and embed $\mathfrak{g}_A$ into $\delta_{A_\delta} \oplus \delta_{A_\delta}$ via the differential of this map. By Proposition 2.6, to show p is admissible, it is enough to show $\mathfrak{g}_A^\perp \subset \delta_{A_\delta} \oplus \delta_{A_\delta}$ is a Lie subalgebra. Now,

$$\mathfrak{g}_A^\perp = \{((\xi, \xi) + (\eta_+, \eta_-), (\xi', \xi') + (\eta'_+, \eta'_-)) | \xi, \xi' \in \mathfrak{g}, \eta, \eta' \in \mathfrak{g}_A, \tfrac{1}{2} S \circ \tau(\eta) = -\xi'\}.$$

Since $\delta_{A_\delta} = {}^\delta\mathfrak{g} \ominus \mathfrak{g}_A$ (Lie algebra antidirect sum), it suffices to show that

$$\tfrac{1}{2} S \circ \tau(\eta_1) = -\xi'_1 \quad \text{and} \quad \tfrac{1}{2} S \circ \tau(\eta_2) = -\xi'_2$$

implies $\tfrac{1}{2} S \circ \tau(-[\eta_1, \eta_2]_A) = -[\xi'_1, \xi'_2]$. But this follows from Lemma 1 and the assumptions on τ. □

From Lemma 3, there exist unique Poisson structures on D/G_A and $G_A \backslash D$ such that the projections $\rho_r : D \to D/G_A$, $\rho_l : D \to G_A \backslash D$ are Poisson maps. If we now identify the quotients with G, then

$$\rho_r(x, y) = \tau^{-1}(\sigma(x^{-1} y)^{-1}) x (x^{-1} y)_+ \quad \text{and} \quad \rho_l(x, y) = (x y^{-1})_+^{-1} x \tau(\sigma(x y^{-1})).$$

Theorem 4. *Under the assumptions of Theorem 2,*

(a) *The reduced Poisson structure on $D/G_A \simeq G$ is the twisted Poisson structure $\{\ ,\ \}_\tau$.*
(b) *Left G_A-invariant functions and right G_A-invariant functions commute in $D_{(A_\delta, A_\delta)}$.*

Proof.

(a) Let $\varphi, \psi \in C^\infty(G)$ and set $\hat{\varphi} = \varphi \circ \rho_r$, $\hat{\psi} = \psi \circ \rho_r$. Then $\hat{\varphi}, \hat{\psi}$ are right G_A-invariant functions on D and the reduced bracket on $D/G_A \simeq G$ is given by

$$\{\varphi, \psi\}_{\text{red}}(x) = \langle A_\delta(D\hat{\varphi}(x, x)), D\hat{\psi}(x, x) \rangle + \langle A_\delta(D'\hat{\varphi}(x, x)), D'\hat{\psi}(x, x) \rangle.$$

To simplify notations, let $X = D\varphi(x)$, $X' = D'\varphi(x)$, $Y = D\psi(x)$, $Y' = D'\psi(x)$. Since the subspaces ${}^\delta\mathfrak{g}$ and $\mathfrak{g}_A$ of δ are isotropic with respect to $\langle\ ,\ \rangle$, we have

$$\langle A_\delta(D'\hat{\varphi}(x, x)), D'\hat{\psi}(x, x) \rangle = \tfrac{1}{2}(\tau \circ S(X), Y') - \tfrac{1}{2}(X', \tau \circ S(Y)).$$

On the other hand, using the properties of A, S and τ, we find

$$\langle A_\delta(D\hat{\varphi}(x, x)), D\hat{\psi}(x, x) \rangle = (A(X'), Y') - (A(X), Y) + \tfrac{1}{2}(\tau \circ S(X), Y')$$
$$- \tfrac{1}{2}(\tau \circ S(Y), X').$$

Putting the calculations together yields $\{\varphi, \psi\}_{\text{red}} = \{\varphi, \psi\}_\tau$, as asserted.

(b) Suppose $\varphi \in C^\infty(D)$ is right G_A-invariant and $\psi \in C^\infty(D)$ is left G_A-invariant, then

$(*)$ $\frac{1}{2}\tau^{-1} \circ S(Y_1 - X_1) - X'_{1-} + Y'_{1+} = 0$, $\quad \frac{1}{2}\tau \circ S(Y'_2 - X'_2) - X_{2-} + Y_{2+} = 0$,

where $D\varphi(x, y) = (X_1, Y_1)$, $D'\varphi(X, Y) = (X'_1, Y'_1)$, $D\psi(x, y) = (X_2, Y_2)$ and $D'\psi(x, y) = (X'_2, Y'_2)$. Now,

$$2\{\varphi, \psi\}_{(A_\delta, A_\delta)} = \langle P_{\delta_g}(D\varphi), D\psi \rangle - \langle P_{g_A}(D\varphi), D\psi \rangle + \langle P_{\delta_g}(D'\varphi), D'\psi \rangle$$
$$- \langle P_{g_A}(D'\varphi), D'\psi \rangle,$$

and on using $(*)$, we find

$$- \langle P_{g_A}(D\varphi), D\psi \rangle + \langle P_{\delta_g}(D'\varphi), D'\psi \rangle = 0.$$

Similarly,

$$\langle P_{\delta_g}(D\varphi), D\psi \rangle - \langle P_{g_A}(D'\varphi), D'\psi \rangle = 0.$$

This completes the proof. □

Corollary 5. *If G is finite dimensional, then*

$$G_A \backslash D \xleftarrow{\rho_l} D_{(A_\delta, A_\delta)} \xrightarrow{\rho_r} D/G_A \text{ is a full dual pair,}$$

and the symplectic leaf of $(G, \{ \ , \ \}_\tau)$ passing through $x \in G$ is given by $\{\tau^{-1}\sigma((\tau \circ \sigma(h))^{-1}x^{-1}hx\tau \circ \sigma(h))^{-1}h_+^{-1}x\tau \circ \sigma(h)((\tau \circ \sigma(h))^{-1}x^{-1}hx\tau \circ \sigma(h))_+ \mid h \in G_A\}$.

Proof. The maps ρ_l and ρ_r are submersions as $T_{(x,x)}\rho_l(Z, Z) = T_{(x,x)}\rho_r(Z, Z) = Z$, $Z \in T_x G$ and the rank is constant on orbits. Since $\dim D = \dim G_A \backslash D + \dim D/G_A$, the first assertion is a consequence of Theorem 4(b). To prove the second assertion, we apply a general result in [19], according to which symplectic leaves are obtained by blowing up points in the double fibering $G_A \backslash D \leftarrow D \rightarrow D/G_A$. From the explicit expression for ρ_l, we obtain

$$\rho_l^{-1}(x) = \{(h_+ x(\tau \circ \sigma(h))^{-1}, h_- x(\tau \circ \sigma(h))^{-1}) \mid h \in G_A\},$$

from which the result follows. □

Remarks.

(a) When $R = -R^*$, we have $\sigma(G) = \{e\}$. In this case, the actions p and q reduce to left and right G_R-translations considered by Semenov–Tian–Shansky and the symplectic leaf passing through x is the orbit of an action, known as the dressing transformation [16].

(b) Under additional assumptions on the r-matrix, we can obtain Theorem 2(c) via reduction technique. Indeed, by modifying the argument in [16], we can consider the action

$$G_R \times D_{(^\tau A_\delta, A_\delta)} \rightarrow D_{(^\tau A_\delta, A_\delta)}$$
$$g : (x, y) \mapsto (g_+ x g_-^{-1}, \tau(g_+)yg_-^{-1}),$$

where $\hat{\tau}(X, Y) = (X, \tau(Y))$, $\ ^\tau A_\delta = \hat{\tau} \circ A_\delta \circ \hat{\tau}^{-1}$, $\ g_\pm = \hat{R}_\pm(g)$. If we assume $R_+^*[R_-^* X, R_-^* Y]_A = R_-^*[R_+^* X, R_+^* Y]_A$ (this is satisfied by Examples $1°$ and $2°$

above), the action is admissible and the unique Poisson structure on $G_R \backslash D \simeq G$ coincides with $\{\ ,\ \}_\tau$ when $R^2 = 1$. In this case, the integral curves of the generalized Lax equation in Theorem 2(b) are images under the reduction map $\rho : D \to G_R \backslash D \simeq G$, $(x, y) \mapsto \tau^{-1}(y_+^{-1}) x y_-$ of an associated Hamiltonian system on D. Such a description, however, is not available to us without the extra assumption. So the general case remains open.

For r-matrices which satisfy the assumption (H), we now describe the Hamiltonian character of the associated Lax systems. For this purpose, let $G_N = G \times \cdots \times G$ (N copies) and $\mathfrak{G}_N = \bigoplus_1^N \mathfrak{g}$. Equip $\mathfrak{G}_N$ with the ad-invariant pairing

$$(X, Y) = \sum_{i=1}^N (X_i, Y_i), \quad X = (X_1, \ldots, X_N), \quad Y = (Y_1, \ldots, Y_N) \in \mathfrak{G}_N,$$

and let $\tau : G_N \ni (g_1, \ldots, g_N) \mapsto (g_2, \ldots, g_N, g_1)$, as in [16]. Now, extend the operators R, R^*, etc. componentwise to $\mathfrak{G}_N$ and denote them by the same symbols. Then obviously, both $R \in \operatorname{End} \mathfrak{G}_N$ and $\tau \in \operatorname{Aut}(G_N)$ satisfy the assumptions of Theorem 2, so that we can equip G_N with the twisted Poisson structure. Thus we obtain

Theorem 6. *If $R \in \operatorname{End} \mathfrak{g}$ satisfies (H) and τ is the map defined above, then*

(a) *The twisted Poisson structure on G_N takes the form*

$$\{\varphi, \psi\}_\tau(g_1, \ldots, g_N) = \tfrac{1}{2} \sum_{j=1}^N ((A(D_j'\varphi), D_j'\psi) - (A(D_j\varphi), D_j\psi) + (S(D_j\varphi), D_{j-1}'\psi)$$

$$- (S(D_{j-1}'\varphi), D_j\psi)),$$

where $D_j\varphi$ ($D_j'\varphi$) denotes the jth component of $D\varphi$ ($D'\varphi$)
(b) *Let φ be a central function on G, ψ_m, $T : G_N \to G$ be maps defined by $\psi_m(g) = g_1 \cdots g_{m-1}$, $T(g) = \psi_{N+1}(g)$, $g = (g_1, \ldots, g_N)$. Then the Lax system*

$$\dot{g}_j = \tfrac{1}{2} T_e R_{g_j} R(\operatorname{Ad}_{\psi_j(g)^{-1}} D\varphi(T(g))) - \tfrac{1}{2} T_e L_{g_j} R(\operatorname{Ad}_{\psi_{j+1}(g)^{-1}} D\varphi(T(g))), \quad j = 1, \ldots, N$$

is the Hamilton's equation defined by $h_\varphi(g) = \varphi(T(g))$ in the Poisson structure $\{\ ,\ \}_\tau$ on G_N.
(c) *Let $(h_j)_\pm(t)$ be the solution of the factorization problem*

$$\exp(-t\operatorname{Ad}_{\psi_j(g_0)^{-1}} D\varphi(T(g_0))) = (h_j)_+(t)^{-1}(h_j)_-(t), \quad j = 1, \ldots, N, \ ((h_j)_+(t),$$

$(h_j)_-(t)) \in G_R$ *for those values of t for which the left-hand side lies in the image of the map m in Sect. 2A. Then the solution of the initial value problem associated with the Lax system in (b) is given by*

$$g_j(t) = (h_j)_\pm(t)(g_0)_j(h_{j+1})_\pm(t)^{-1}, \quad j = 1, \ldots, N.$$

In (a), (b) *and* (c), *the subscripts j are taken* $\operatorname{mod} N$. □.

Remarks. (a) In contrast to the theory in [16], when the r-matrix is not skew-symmetric but satisfies our assumption, the Poisson structure (in Theorem 6(a)) for the associated Lax systems is not a product structure. (b) For an application of the above theorem, the reader is referred to [12], where the equations in [5]

are reformulated as a Lax system. Moreover, the complete integrability of the equations on generic symplectic leaves is established.

4. Quadratic Poisson Structures on Associative Algebras

We now consider the formal analog of the bracket $\{\ ,\ \}_\tau$ with $\tau = 1$ on associative algebras.

Theorem 1. *Let $\mathfrak{g}$ be the Lie algebra of an associative algebra for which multiplication is symmetric with respect to some fixed nondegenerate pairing $(\cdot,\cdot)$, i.e. $(XY,Z) = (Y,ZX)$, $X, Y, Z \in \mathfrak{g}$. If $R \in \mathrm{End}\,\mathfrak{g}$ satisfies (H), we have*

(a) *The formula*

$$\{F,H\}(X) = \tfrac{1}{2}(A(\mathrm{grad}\,F(X)X), \mathrm{grad}\,H(X)X) - \tfrac{1}{2}(A(X\,\mathrm{grad}\,F(X)), X\,\mathrm{grad}\,H(X))$$
$$+ \tfrac{1}{2}(S(X\,\mathrm{grad}\,F(X)), \mathrm{grad}\,H(X)X)$$
$$- \tfrac{1}{2}(S(\mathrm{grad}\,F(X)X), X\,\mathrm{grad}\,H(X)),$$

$F, H \in C^\infty(\mathfrak{g})$, defines a Poisson structure on $\mathfrak{g}$.
(b) *The Poisson structure in* (a) *is compatible with the Lie Poisson structure $\{F,H\}_R(X) = (X, [\mathrm{grad}\,F(X), \mathrm{grad}\,H(X)]_R)$.*
(c) *ad-invariant functions commute in $\{\ ,\ \}$ and the Hamilton's equation generated by an ad-invariant function H is in Lax form*

$$\dot{X} = \tfrac{1}{2}[R(X\,\mathrm{grad}\,H(X)), X] = [R_\pm(X\,\mathrm{grad}\,H(x)), X].$$

(d) *If* $\quad [X, \mathrm{grad}\,F(X)] \in (\mathrm{Im}(R+1))^\perp, \quad [X, \mathrm{grad}\,H(X)] \in (\mathrm{Im}\,(R-1))^\perp, \quad$ *then* $\{F,H\}(X) = 0$.

Remarks.

(a) If there exist local Lie groups G, G_R corresponding to the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_R$, then the hypothesis in part (d) are just the infinitesimal versions of the invariance properties

$$F(\mathrm{Ad}_{g_+}X) = F(X), \quad H(\mathrm{Ad}_{g_-}X) = H(X), \quad g \in G_R, \quad X \in \mathfrak{g}.$$

(b) From the symmetry of the multiplication, it is clear that $(\cdot,\cdot)$ is ad-invariant.
(c) Under the assumptions of part (d), we also have $\{F,H\}_R(X) = 0$. An application of this fact can be found in [6].

Proof of Theorem 1.

(a) Clearly, $\{\ ,\ \}$ defines a skew-symmetric bilinear form on $C^\infty(\mathfrak{g})$ which is a derivation in each argument. To verify the Jacobi identity, write $\{\ ,\ \} = \{\ ,\ \}_A + \{\ ,\ \}_S$ with the obvious meaning and let $\mathrm{grad}\,F_i(x) = L_i$, $F_i \in C^\infty(\mathfrak{g})$, $i = 1,2,3$. Then

$$4\{F_1, \{F_2, F_3\}_A\}_A(X) + \text{c.p.}$$
$$= (L_1 X, [A(L_2 X), A(L_3 X)]) + (X L_1, [A(X L_3), A(X L_2)]) + \text{c.p.}$$
$$= 0 \quad (\text{by mYB}).$$

In a similar way,

$$4\{F_1, \{F_2, F_3\}_S\}_S(X) + \text{c.p.}$$
$$= (XL_1, [S(L_3X), S(L_2X)]) + (L_1X, [S(XL_2), S(XL_3)]) + \text{c.p.}$$

Finally, by using Lemma 3.1,

$$4\{F_1, \{F_2, F_3\}_A\}_S(X) + 4\{F_1, \{F_2, F_3\}_S\}_A(X) + \text{c.p.}$$
$$= -(XL_1, [S(L_3X), S(L_2X)]) - (L_1X, [S(XL_2), S(XL_3)]) + \text{c.p.}$$

This completes the verification.

(b) We use a device in [15]. Assume first that $\mathfrak{g}$ is an algebra with identity. Pick $\lambda \in \mathbb{C}$ and associate with $F \in C^\infty(\mathfrak{g})$ the function F^λ defined by $F^\lambda(X + \lambda I) = F(X)$. Then $\operatorname{grad} F^\lambda(X + \lambda I) = \operatorname{grad} F(X)$. A straightforward calculation shows that we have the relation $\{F^\lambda, H^\lambda\}(X + \lambda I) = \{F, H\}(X) + \lambda\{F, H\}_R(X)$. This shows the right-hand side is a Poisson bracket. If $\mathfrak{g}$ does not have an identity element, then we adjoin an identity I to it and note that the formula in (a) defines a Poisson bracket on the extended algebra $\mathfrak{g} + \mathbb{C}I$, which is uniquely determined by its restriction to a hyperplane $\lambda = \text{const}$.

(c) The first part of the assertion is a special case of (d) which will be proved below. For the other part, note that ad-invariance implies $X \operatorname{grad} H(X) = \operatorname{grad} H(X)X$. Therefore,

$$\{F, H\}(X) = \tfrac{1}{2}(A[\operatorname{grad} F(X), X], X \operatorname{grad} H(X))$$
$$+ \tfrac{1}{2}(S[X, \operatorname{grad} F(X)], X \operatorname{grad} H(X))$$
$$= \tfrac{1}{2}((A - S)[\operatorname{grad} F(X), X], X \operatorname{grad} H(X))$$
$$= \tfrac{1}{2}(\operatorname{grad} F(X), [R(X \operatorname{grad} H(X)), X]).$$

(d) The assertion follows by noting that the bracket in (a) can be rewritten in the form

$$\{F, H\}(X) = \tfrac{1}{2}(R_-(\operatorname{grad} F(X)X), [\operatorname{grad} H(X), X])$$
$$- \tfrac{1}{2}([\operatorname{grad} F(X), X], R_+(\operatorname{grad} H(X)X))$$
$$+ \tfrac{1}{2}(R_-(X \operatorname{grad} F(X)), [\operatorname{grad} H(X), X])$$
$$- \tfrac{1}{2}([\operatorname{grad} F(X), X], R_+(X \operatorname{grad} H(X))). \quad \square$$

As an application of Theorem 1, we take $\mathfrak{g} = gl(n, \mathbb{R})$ with the pairing $(X, Y) = \operatorname{tr} XY$ and let R be the r-matrix in Example 2°, Sect. 3. Then Theorem 1 is valid. In particular, if we take the Hamiltonian to be $H_1(M) = \operatorname{tr} M$, then the equation of motion it generates is given by

$$\dot{M} = \tfrac{1}{2}[(\Pi_{\mathfrak{t}} - \Pi_{\mathfrak{l}})(M), M] = [\Pi_{\mathfrak{t}}(M), M].$$

In other words, we have proved

Proposition 2. *The Toda flow* $\dot{M} = [\Pi_{\mathfrak{t}}(M), M]$ *is Hamiltonian relative to the Poisson structures* $\{\ ,\ \}_R$ *and* $\{\ ,\ \}$ *with Hamiltonians given by* $H_2(M) = \tfrac{1}{2}\operatorname{tr} M^2$ *and* $H_1(M) = \operatorname{tr} M$ *respectively.* $\quad \square$

Caveat. Due to a different choice of pairing, what we call the Toda flow here is different from the one in [6], which is given by $\dot{M} = [\Pi_f(M^T), M]$. The necessary changes one has to make in the construction of integrals is straightforward, as we shall describe below.

Now, if a dynamical system is Hamiltonian relative to a pair of compatible symplectic structures, then it was proved in [10] and [14] that commuting integrals can be generated sequentially by means of a recursion operator and as such are connected by a recursion relation. Although the result makes no claim about the integrability of the system, nevertheless, it does give a simple and practical means to obtain integrals. Unfortunately, with a pair of compatible (degenerate) Poisson structures, such a simple procedure does not seem to be available. In the rest of the section, we shall demonstrate, however, that in the case of the Toda flow, the integrals in [6] (which were used to establish its integrability on generic coadjoint orbits) are indeed connected by n recursion relations. Aside from proving the involution of the integrals in both structures, this information also provides us with a better geometric picture of the situation.

In the following, we introduce (with modifications) the variables that were constructed in [6]. For an $n \times n$ matrix M, let $(M)_k$ be the $(n-k) \times (n-k)$ matrix obtained by deleting the last k rows and the first k columns of M and define

$$P_k(M, \lambda) = \det (M - \lambda)_k = \sum_{r=0}^{n-2k} E_{rk}(M)\lambda^{n-2k-r}, \quad 0 \leq k \leq \left[\frac{n}{2}\right].$$

Also, set

$$J(M, h, z) = \det (M_h - z) = \sum_{r=0}^{n} E_r(M_h)z^{n-r} = \sum_{r=0}^{n} \sum_{k=0}^{[r/2]} J_{rk}(M)h^k(1-h)^k z^{n-r},$$

where

$$M_h = M + h(M^T - M).$$

Lemma 3. *The sign of E_{0k}, $0 \leq k \leq [n/2]$ is constant on the connected components of the symplectic leaves of $\{\ ,\ \}_R$ and $\{\ ,\ \}$.*

Proof. The symplectic leaves of $\{\ ,\ \}_R$ are the coadjoint orbits of the group G_R, where $G = GL(n, \mathbb{R})$ and the proof of the assertion is in [6]. For the other part, note that

$$2\{E_{0k}, H\}(M) = (\Pi_1(\nabla^T H(M)M + M\nabla^T H(M)), [\nabla^T E_{0k}(M), M])$$
$$+ ([\nabla^T H(M), M], \Pi_f(\nabla^T E_{0k}(M)M + M\nabla^T E_{0k}(M)))^1.$$

Now, it is easy to see from the definition of E_{0k} that

$$M\nabla^T E_{0k}(M) = \begin{pmatrix} E_{0k}(M)I_k & 0 \\ * & 0 \end{pmatrix} \quad \text{and} \quad \nabla^T E_{0k}(M)M = \begin{pmatrix} 0 & 0 \\ * & E_{0k}(M)I_k \end{pmatrix}.$$

---

[1] If $F : gl(n, \mathbb{R}) \to \mathbb{R}$, $\nabla F(M) \equiv (\partial F/\partial m_{ij})$, and so grad $F(M) = \nabla^T F(M)$

This immediately implies the vanishing of the second term and leads to

$$2\{E_{0k}, H\}(M) = E_{0k}(M)\left(\sum_{i=n-k+1}^{n} (\nabla^T H(M)M + M\nabla^T H(M))_{ii} \right.$$
$$\left. - \sum_{i=1}^{k} (\nabla^T H(M)M + M\nabla^T H(M))_{ii} \right).$$

Thus the sign of E_{0k} is constant along the trajectories of Hamiltonian vectorfields. Since any two points on a connected component of a symplectic leaf can be joined by a piecewise smooth curve consisting of segments of trajectories of Hamiltonian vectorfields, this completes the proof. $\square$

From the above lemma, the set $W = \{M \in gl(n, \mathbb{R}) \mid E_{0k}(M) \neq 0, k = 1, \ldots, [n/2]\}$ is foliated by the symplectic leaves of the two compatible Poisson structures. For $M \in W$, we define $I_{rk}(M) = E_{rk}(M)/E_{0k}(M)$, $0 \leq k \leq [\frac{1}{2}(n-1)]$, $1 \leq r \leq n - 2k$.

Remark. From [6], the I_{rk}'s and the J_{rk}'s have the invariance properties

$$I_{rk}(lMl^{-1}) = I_{rk}(M), \quad l \in L(n, \mathbb{R}), \quad J_{r'k'}(OMO^T) = J_{r'k'}(M), \quad O \in O(n, \mathbb{R}),$$

where $L(n, \mathbb{R})$ is the lower triangular group and $O(n, \mathbb{R})$ is the orthogonal group.

Lemma 4. *The following provide Casimir function of the quadratic Poisson structure on* W:

(a) $I_{n-2k,k}$, $1 \leq k \leq \left[\dfrac{n-1}{2}\right]$,

(b) $J_{n,k}$, $0 \leq k \leq \left[\dfrac{n}{2}\right]$.

Proof.

(a) $2\{I_{n-2k,k}, H\}(M) = (\Pi_1(\nabla^T H(M)M + M\nabla^T H(M)), [\nabla^T I_{n-2k,k}(M), M])$
$$+ ([\nabla^T H(M), M], \Pi_1(\nabla^T I_{n-2k,k}(M)M + M\nabla^T I_{n-2k,k}(M))).$$

From the invariance property $I_{n-2k,k}(lMl^{-1}) = I_{n-2k,k}(M)$, $l \in L(n, \mathbb{R})$, the first term is equal to zero. On the other hand, it follows from the definition of $I_{n-2k,k}$ that both $M\nabla^T I_{n-2k,k}(M)$ and $\nabla^T I_{n-2k,k}(M)M$ are lower triangular matrices. So the second term vanishes as well.

(b) Let $F(M) = E_n(M_h)$, then by the invariance property $F(OMO^T) = F(M)$, $O \in O(n, \mathbb{R})$, the matrix $[\nabla^T F(M), M]$ is symmetric. Consequently, the bracket simplifies to

$$2\{F, H\}(M) = -(\Pi_1(\nabla^T E_n(M_h)M + M\nabla^T E_n(M_h)), [\nabla^T H(M), M]).$$

Now, for $h \neq \frac{1}{2}$, we have $M = (1 - h/1 - 2h)M_h - (h/1 - 2h)M_h^T$, this implies
$$M\nabla^T E_n(M_h) = E_n(M_h)[I + (h(1-h)/1 - 2h)(M_h M_h^{-T} - M_h^T M_h^{-1})].$$

Similarly,

$$\nabla^T E_n(M_h)M = E_n(M_h)[I + (h(1-h)/1 - 2h)(M_h^{-T} M_h - M_h^{-1} M_h^T)].$$

Adding the expressions and applying $\Pi_\mathfrak{l}$ to both sides, we find $\Pi_\mathfrak{l}(M\nabla^T E_n(M_h) + \nabla^T E_n(M_h)M) = 2E_n(M_h)I$. As a result,

$$2\{F, H\}(M) = -2E_n(M_h)\,\mathrm{tr}\,([\nabla^T H(M), M]) = 0. \quad \square$$

Remark. From [6], the Casimir functions of the Lie Poisson structure $\{\ ,\ \}_R$ on W are given by I_{1k}, $1 \leq k \leq [n-1/2]$, J_{10} and $J_{2k,k}$, $0 \leq k \leq [n/2]$.

For a function H on $\mathfrak{g}$, let $X_H^{(1)}$ and $X_H^{(2)}$ denote the Hamiltonian vectorfields generated by H relative to $\{\ ,\ \}_R$ and $\{\ ,\ \}$ respectively. We have the following theorem.

Theorem 5   (*Recursion relations*). *On the set* W,

(a) $X_{I_{rk}}^{(2)} = X_{I_{r+1,k}}^{(1)}$, i.e. $[\Pi_\mathfrak{l}(M\nabla^T I_{rk}(M)), M] = [\Pi_\mathfrak{l}(\nabla^T I_{r+1,k}(M)), M]$.

(b) $X_{J_{rk}}^{(2)} = X_{J_{r+1,k}}^{(1)}$, i.e. $\frac{1}{2}[M, \Pi_\mathfrak{l}(M\nabla^T J_{rk}(M) + \nabla^T J_{rk}(M)M)]$
$\qquad = [M, \Pi_\mathfrak{l}(\nabla^T J_{r+1,k}(M))]$.

Proof.

(a) Applying ∇^T to both sides of the relation

$$\frac{\det(M-\lambda)_k}{E_{0k}(M)} = \sum_{r=0}^{n-2k} I_{rk}(M)\lambda^{n-2k-r},$$

and then multiplying both sides on the left by $(M - \lambda)$, we find

$$(\ast) \qquad \sum_{r=0}^{n-2k} M\nabla^T I_{rk}(M)\lambda^{n-2k-r} - \sum_{r=0}^{n-2k-1} \nabla^T I_{r+1,k}(M)\lambda^{n-2k-r}$$

$$= \frac{\det(M-\lambda)_k}{E_{0k}(M)}\begin{pmatrix} I_{n-k} & 0 \\ \ast & 0 \end{pmatrix} - \frac{\det(M-\lambda)_k}{E_{0k}^2(M)}(M-\lambda)\nabla^T E_{0k}(M).$$

But from the definition of E_{0k}, $(M-\lambda)\nabla^T E_{0k}(M)$ is a lower triangular matrix. Therefore, when we apply $\Pi_\mathfrak{l}$ to both sides of $(\ast)$, the right-hand side vanishes and on comparing coefficients of λ, we obtain

$$\Pi_\mathfrak{l}(M\nabla^T I_{rk}(M)) = \Pi_\mathfrak{l}(\nabla^T I_{r+1,k}(M)).$$

(b) Applying ∇^T to both sides of $E_r(M_h) = \sum_{k=0}^{[r/2]} J_{rk}(M)h^k(1-h)^k$, we find

$$(1-h)(\mathrm{grad}\,E_r)(M_h) + h(\mathrm{grad}\,E_r)^T(M_h) = \sum_{k=0}^{[r/2]} \nabla^T J_{rk}(M)h^k(1-h)^k.$$

In particular, this gives

$$\Pi_\mathfrak{l}(\mathrm{grad}\,E_r)(M_h) = \sum_{k=0}^{[r/2]} \Pi_\mathfrak{l}(\nabla^T J_{rk}(M))h^k(1-h)^k.$$

Also,

$$\sum_{k=0}^{[r/2]} (M\nabla^T J_{rk}(M) + \nabla^T J_{rk}(M)M)h^k(1-h)^k = (1-h)(M(\mathrm{grad}\,E_r)(M_h)$$

$$+ (\mathrm{grad}\,E_r)(M_h)M) + h(M(\mathrm{grad}\,E_r)^T(M_h) + (\mathrm{grad}\,E_r)^T(M_h)M).$$

For $h \neq \frac{1}{2}$, $M = (1 - h/1 - 2h)M_h - (h/1 - 2h)M_h^T$. Substituting into the above expression and then applying Π_1 to both sides, we obtain

$$\sum_{k=0}^{[r/2]} \Pi_1(M\nabla^T J_{rk}(M) + \nabla^T J_{rk}(M)M)h^k(1-h)^k = 2\Pi_1(M_h(\text{grad } E_r)(M_h)),$$

which is valid for all h by continuity. Now, from the definition of E_r, it is easy to derive the recurrence relation $M_h(\text{grad } E_r)(M_h) = (\text{grad } E_{r+1})(M_h) + E_r(M_h)I_n$ from which it follows that

$$\sum_{k=0}^{[r/2]} \Pi_1(M\nabla^T J_{rk}(M) + \nabla^T J_{rk}(M)M)h^k(1-h)^k$$

$$= 2\Pi_1(\text{grad } E_{r+1})(M_h) + E_r(M_h)I_n$$

$$= 2\sum_{k=0}^{[r/2]} \Pi_1(\nabla^T J_{r+1,k}(M)h^k(1-h)^k)$$

$$+ E_r(M_h)I_n.$$

Therefore, $\frac{1}{2}[M, \Pi_1(M\nabla^T J_{rk}(M) + \nabla^T J_{rk}(M)M)] = [M, \Pi_1(\nabla^T J_{r+1,k}(M))]$. $\quad\square$

Corollary 6 (*Involution*). *On the set* W, *the* I_{rk}'s *and the* J_{rk}'s *Poisson commute in both the linear and the quadratic structures.*

Proof. We shall do this for the I_{rk}'s in the quadratic structure, the other cases can be done in a similar way. By Theorem 5, we have

$$\{I_{rk}, I_{r'k'}\}(M) = \{I_{r+1,k}, I_{r'k'}\}_R(M) = \{I_{r+1,k}, I_{r'-1,k'}\}(M).$$

Repeating, we would eventually obtain either $\{I_{n-2k,k}, I_{r+r'-n+2k,k'}\}(M)$ or $\{I_{r+r',k}, I_{1k'}\}_R(M)$. By Lemma 4 and the remark which follows, we obtain zero in either case. $\quad\square$

Remarks.

(a) The involution theorem for the I_{rk}'s and the J_{rk}'s in the Lie Poisson structure $\{\ ,\ \}_R$ was first proved in [6] using invariance properties together with the explicit form of the J_{rk}'s.
(b) The vanishing of the Poisson bracket $\{I_{rk}, J_{r'k'}\}$ also follows from Theorem 1(d).
(c) With more work, one can in fact establish the integrability of the Toda flow on generic symplectic leaves of the quadratic Poisson structure. We leave the details to the interested reader.

5. Cubic Poisson Structures in the r-Matrix Approach

The r-matrix approach to classical "integrable" systems was introduced by M. Semenov–Tian–Shansky to generalize the Adler–Kostant–Symes scheme and to provide a link with the method of the Riemann problem [15]. In [15, 16] and in Sects. 3–4 of the present work, we have seen the role played by linear and quadratic Poisson brackets. The purpose of this last section is to exhibit a third order Poisson structure on associative algebras for which the r-matrix approach is feasible.

Theorem 1. *Let $\mathfrak{g}$ be Lie algebra of an associative algebra for which multiplication is symmetric with respect to some fixed nondegenerate pairing $(\cdot,\cdot)$. If $R\in\mathrm{End}\,\mathfrak{g}$ is a solution of* (mYB), *then*

(a) *The formula*

$$\{F,H\}(X) = \tfrac{1}{2}(X,[\mathrm{grad}\,F(X), R(X\,\mathrm{grad}\,H(X)X)]$$
$$+ [R(X\,\mathrm{grad}\,F(X)X), \mathrm{grad}\,H(X)]),$$

$F,H\in C^{\infty}(\mathfrak{g})$, defines a Poisson structure on $\mathfrak{g}$.
(b) *The Hamilton's equation generated by an ad-invariant function H is in Lax form*

$$\dot{X} = \tfrac{1}{2}[R(X\,\mathrm{grad}\,H(X)X), X)] = [R_{\pm}(X\,\mathrm{grad}\,H(X)X), X].$$

(c) *If $[X, \mathrm{grad}\,F(X)]\in(\mathrm{Im}\,(R+1))^{\perp}$, $[X, \mathrm{grad}\,H(X)]\in(\mathrm{Im}\,(R-1))^{\perp}$, then $\{F,H\}$ $(X) = 0$. In particular, ad-invariant functions commute in $\{\ ,\ \}$.*

Proof. (a) We shall omit the straightforward but lengthy calculations. See, however, Proposition 2(a) below.
(b) Using $X\,\mathrm{grad}\,H(X) = \mathrm{grad}\,H(X)X$,

$$\{F,H\}(X) = \tfrac{1}{2}(X,[\mathrm{grad}\,F(X), R(X\,\mathrm{grad}\,H(X)X)])$$
$$= \tfrac{1}{2}(\mathrm{grad}\,F(X), [R(X\,\mathrm{grad}\,H(X)X), X]).$$

(c) The result follows on noting that the bracket can be rewritten in the form

$$\{F,H\}(X) = \tfrac{1}{2}([X, \mathrm{grad}\,F(X)], (R+1)(X\,\mathrm{grad}\,H(X)X))$$
$$- \tfrac{1}{2}([X, \mathrm{grad}\,H(X)], (R-1)(X\,\mathrm{grad}\,F(X)X)). \qquad \square$$

Denote the linear, quadratic and cubic structures respectively by $\{\ ,\ \}_{(1)}$, $\{\ ,\ \}_{(2)}$ and $\{\ ,\ \}_{(3)}$ and let w_1, w_2, and w_3 be the corresponding Hamiltonian operators. We now give relations between these objects. The proof consists of straightforward verification and will be left to the reader. In what follows, in addition to the hypothesis of Theorem 1, we assume $\mathfrak{g}$ has an identity I.

Proposition 2. *Assume that the group of units $\mathfrak{g}_{\mathrm{inv}}$ is an open subset of $\mathfrak{g}$. Then*

(a) $$\{F\circ\iota, H\circ\iota\}_{(3)}(X) = -\{F,H\}_{(1)}\circ\iota(X),$$

where $X\in\mathfrak{g}_{\mathrm{inv}}$, $F,H\in C^{\infty}(\mathfrak{g})$, and $\iota: X\mapsto X^{-1}$ is the inversion map. If, in addition, we assume R satisfies hypothesis (H), *then also*

(b) $$\{F\circ\iota, H\circ\iota\}_{(2)}(X) = -\{F,H\}_{(2)}\circ\iota(X).$$

Proposition 3. *Let v and 1 be the vectorfields on $\mathfrak{g}$ defined by $v(X) = X^2$, $1(X) = I$ and let R satisfy* (H). *Then*

$$L_v w_1 = -2w_2, \quad L_v w_2 = -w_3, \quad L_v w_3 = 0,$$
$$L_1 w_3 = 2w_2, \quad L_1 w_2 = w_1, \quad L_1 w_1 = 0,$$

where $L_v(L_1)$ denotes the Lie derivative with respect to $v(1)$. In particular, this implies the compatibility of the three structures.

Remark. From many points of view, it is clear that the Lax equations corresponding

to ad-invariant functions can be realized most conveniently as Hamiltonian systems in the linear structure $\{\ ,\ \}$ (e.g. the symplectic leaves are coadjoint orbits). However, the presence of an additional structure (whether isomorphic to $\{\ ,\ \}_R$ or not) with respect to which the equations are Hamiltonian would impose further constraints on the dynamics in most cases. So the following question arises: is there a natural hierarchy of Poisson structures in the r-matrix approach and if so, what implications does it have towards the integrability of the Lax equations in the ground structure $\{\ ,\ \}_R$?

Appendix.

We shall provide the reader with further examples of r-matrices in $\mathfrak{g} = gl(n, \mathbb{R})$ which verify our assumption.

Recall that any r-matrix R which satisfies the basic hypothesis (H) can be written uniquely as $R = A + 2\phi$, where $A = \frac{1}{2}(R - R^*)$ is a solution of (mYB) and $\phi = \frac{1}{4}(R + R^*) : \mathfrak{g}_A \to \mathfrak{g}$ is a Lie algebra homomorphism.

Let us take A to be the operator $X \mapsto X_+ - X_-$, then any Lie algebra homomorphism $\mathfrak{g}_A \to \mathfrak{g}$ which is symmetric will give us an example. For simplicity, let us consider those ϕ's which have the additional properties

$$\phi(\mathfrak{n}_+) \subset \mathfrak{n}_-, \quad \phi(\mathfrak{d}) \subset \mathfrak{d}, \quad \phi(\mathfrak{n}_-) = 0,$$

where

$$\mathfrak{n}_+(\mathfrak{n}_-) = \text{subalgebra of strictly upper (lower) triangular matrices}$$

and

$$\mathfrak{d} = \text{the abelian subalgebra of diagonal matrices.}$$

For this class of homomorphisms, the condition $\phi[X, Y]_A = [\phi(X), \phi(Y)]$ simplifies to

(*) $\qquad \phi_+[X_+, Y_+] = [\phi_+(X_+), \phi_+(Y_+)],$

$\qquad \phi_+[X_+, Y_0] = 2[\phi_+(X_+), \phi_0(Y_0)], \quad \phi_\pm = \phi|\mathfrak{n}_\pm, \quad \phi_0 = \phi|\mathfrak{d}.$

There are many solutions to (*), the one given here is of particular interest.

Example. Let $e_{i,j} \in \mathfrak{g}$ be the matrix having 1 in the (i, j) position and 0 elsewhere. For each $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$, we associate the map $\phi^\lambda : \mathfrak{g}_A \to \mathfrak{g}$ defined by

$$\phi_0^\lambda(e_{ii}) = -\tfrac{1}{2}e_{ii}, \quad \phi_-^\lambda = 0,$$

$$\phi_+^\lambda(e_{i,i+k}) = (-1)^{k-1}\lambda_i \cdots \lambda_{i+k-1}e_{i+k,i}.$$

Then ϕ^λ is symmetric and satisfies (*). Thus we obtain an $n - 1$ parameter family of r-matrices $R^\lambda = J + 2\phi^\lambda$ which verifies (H). This family contains Examples $1°$ and $2°$ of Sect. 4 as special cases, corresponding to $\lambda = 0$ and $\lambda = (-1, \ldots, -1)$ respectively.

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Note added in proof. The authors have recently found an abstract theorem in the context of the theory of Poisson Lie groups which contains Theorem 2(a) of Sect. 3 as a special case. Details of this work will be reported elsewhere.