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On a Certain Value of the Kauffman Polynomial*

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Abstract. If $F_L(a, x)$ is the Kauffman polynomial of a link L we show that $F_L(1, 2\cos 2\pi/5)$ is determind up to a sign by the rank of the homology of the 2-fold cover of the complement of L. This value corresponds to a certain Wenzl subfactor defined by the Birman-Wenzl algebra, which we describe in simple terms. There also corresponds a "solvable" model in statistical mechanics similar to the 5-state Potts model. It is the 5-state case of a general model of Fateev and Zamolodchikov.

Introduction

This paper is intended to demonstrate the fruitfulness of a correspondence which is now emerging between knot theory, von Neumann algebras, and statistical mechanics. We begin by describing a simpler example of this correspondence which is precisely generalized in this paper. It was already largely present in [J1].

If $V_{\mathbf{L}}(t)$ is the polynomial of [J1] for a tame oriented link $\mathbf{L}$ in $\mathbb{R}^3$ then $V_{\mathbf{L}}$ can be calculated as the (normalized) trace of a braid α whose closure $\hat{\alpha}$ is $\mathbf{L}$, in representations of the braid groups that arose in von Neumann algebras. In order to construct interesting subfactors of II_1 factors the author in [J2] used what was essentially the following device: Find a suitable Hilbert space representation π of the infinite braid group $B_\infty = \langle \sigma_1, \sigma_2, \ldots;\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqq 2\rangle$ such that $\pi(B_\infty)$ generates a type II_1 factor. The subgroup of B_∞ generated by $\sigma_2, \sigma_3, \ldots$ should then generate a subfactor whose index can, under the right circumstances, be calculated. It was some representations that made the subfactor construction work which were used to construct the link invariant $V_{\mathbf{L}}(t)$.

In the case where the subfactor had *integer index* (for the definition of index, see [J2], it was possible to use a braid group representation that had already been (essentially) discovered by Temperley and Lieb in [TL] in their proof of the equivalence of the Potts and ice-type models in 2-dimensional square lattice

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equilibrium statistical mechanics. Of special interest was the value index $=3$, for it had been noticed in [J3] that the images of the braid groups B_3 and B_4 in the corresponding representations were finite. Complete understanding of this phenomenon came in the work of D. Goldschmidt and the author [GJ] where it is shown that all the B_n's have finite images, essentially the finite groups $Sp(2n, 3)$. The clue to this result was the special nature of the braid group representation coming from the three-state Potts model. This model also made it clear that the subfactor of index 3 coming from the braid group construction has the following very special form: choose an outer action of the symmetric group S_3 on hyperfinite type II_1 factor R. Then the fixed point algebras $R^{S_3} \subseteq R^{S_2}$ give a subfactor of index 3 isomorphic to the braid group one. Ocneanu has since shown that this subfactor and another one which also arises naturally in the Potts model context are the only subfactors of index 3 of R.

Thus we have a pretty and useful correspondence between the 3-state Potts model on the one hand and the subfactors of index 3 on the other (for more detailed information on the Potts model, see Sect. 4).

On the knot theory side, the value of t corresponding to index 3 is $e^{i\pi/3}$, and it was noticed by Birman that $V_{\mathbf{L}}(e^{i\pi/3})$ is always $\pm i$ times a power of $\sqrt{3}$. The exponent of $\sqrt{3}$ was understood by Lickorish and Millet in [LM] as the rank of the homology mod 3 of the (most obvious) 2-fold branched cover of S^3 branched over $\mathbf{L}$. Lipson in [Li] interpreted the $\pm i$ factor and in [GJ] a natural formula giving the whole value was found as (normalization) $\sum_{v \in H_1(S, \mathbb{Z}/3)} \omega^{\langle v, v \rangle}$, where S is a Seifert surface for $\mathbf{L}$, ω is a cube root of unity, and $\langle\, ,\, \rangle$ is the symmetrized Seifert pairing.

The knot theory picture presents an abvious generalization: replace "3" by any positive odd integer. In this paper we begin by showing that if this integer is 5, the knot invariant is a specialization of the Kauffman polynomial, namely $F_{\mathbf{L}}(1, 2\cos 2\pi/5)$. In particular, $|F_{\mathbf{L}}(1, 2\cos 2\pi/5)| = (\sqrt{5})^n$, n being the rank of the homology mod 5 of the 2-fold branched cover. Note that this is more surprising than the $V_{\mathbf{L}}(e^{i\pi/3})$ case since there is no a priori reason for the values of $F_{\mathbf{L}}(1, 2\cos 2\pi/5)$ to even be discrete!

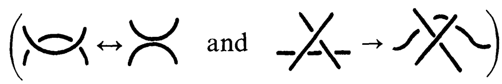
The proof of the knot theory result will use the Birman-Wenzl algebra of [BW]. Wenzl has also used this algebra to construct many interesting subfactors. Using [GJ] we show that the Wenzl subfactor corresponding to $(1, 2\cos 2\pi/5)$ can also be defined as $R^{D_5} \subseteq R^{\mathbb{Z}/2\mathbb{Z}}$ (where D_5 is the dihedral group of order 10 for some (hence any, see [J4]) outer action of D_5 on R.

We also show the existence of a 5-state "solvable" model related to the above knot invariant/subfactor exactly as in the 3-state case. There is also of course a corresponding "integrable" quantum spin chain whose Hamiltonian we give. The model is obtained by "deforming" one of the braid group representations of [GJ]. Here we take advantage of the fact that the braid group representation factors through the Birman-Wenzl algebra and obtain the Boltzmann weights from a "universal" Baxter type formula with nontrivial spectral parameter in the Birman-Wenzl algebra (see [Ba] for the analogous Hecke algebra situation).

1. The Kauffman Polynomial and the Birman-Wenzl Algebra

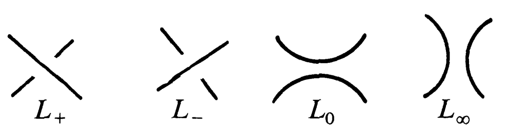
The Kauffman polynomial $F_{\mathbf{L}}(a, x)$ of a tame oriented link $\mathbf{L}$ in $\mathbb{R}^3$ is defined by the following procedure (see [K]). First define an invariant $\mathscr{I}_L$ of regular isotopy of unoriented link diagrams L by the following axioms:

(K0) $\mathscr{I}_L$ does not change under type II or III Reidemeister moves



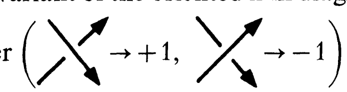
(K1) $\mathscr{I}_{\curvearrowright} = a\mathscr{I}_{\frown}$, $\mathscr{I}_{\curvearrowleft} = a^{-1}\mathscr{I}_{\frown}$.

(K2) If L_+, L_-, L_0, L_∞ are as below



then $\mathscr{I}_{L_+} + \mathscr{I}_{L_-} = x(\mathscr{I}_{L_0} + \mathscr{I}_{L_\infty})$.

(K3) $\mathscr{I}_0 \equiv 1$.

In these axioms we have used the convention that a partial like diagram such as is imagined to be connected to another partial link diagram which remains unchanged when the change or changes are performed on the partial diagrams indicated. Then $F_{\mathbf{L}}(a, x) = a^{+w(\mathbf{L})}\mathscr{I}_L(a, x)$ is an invariant of the oriented link diagram $\mathbf{L}$, where $w(\mathbf{L})$ is the algebraic crossing number  of $\mathbf{L}$.

The Birman-Wenzl algebra $C_n(x, a)$ is a complex algebra with identity $\mathbb{1}$ depending (algebraically) on two complex parameters a and x. It was designed partly to help understand the Kauffman polynomial and has the following (redundant) presentation:

generators $G_1, G_2 \ldots G_{n-1}$, and their inverses, and

$E_1, \ldots E_{n-1}$

relations $G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1}$, $G_i G_j = G_j G_i$ if $|i-j| \geqq 2$,

$G_i + G_i^{-1} = x(\mathbb{1} + E_i)$,

$E_i G_i = G_i E_i = aE_i$,

$E_i^2 = (a + a^{-1} - x)x^{-1} E_i$,

$E_i G_{i\pm 1}^{\pm 1} E_i = a^{\mp 1} E_i$,

$E_i G_{i\pm 1} G_i = E_i E_{i\pm 1}$.

As shown in [BW], $C_n(x,a)$ embeds in $C_{n+1}(x,a)$ in the obvious way and there is a trace tr: $\bigcup_n C_n(x,a)\to\mathbb{C}$ uniquely defined by the following four axioms:

(Tr 1) $\operatorname{tr}(1)=1$,

(Tr 2) $\operatorname{tr}(AB)=\operatorname{tr}(BA)$,

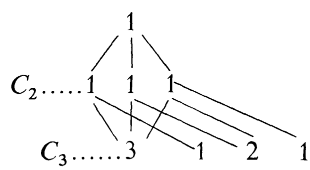
(Tr 3) $\operatorname{tr}(AG_n)=\dfrac{a^{-1}}{\delta}\operatorname{tr}(A)$ for $a\in C_n(x,a)$,

(Tr 4) $\operatorname{tr}(AE_n)=\dfrac{1}{\delta}\operatorname{tr}(A)$ for $A\in C_n(x,a)$

$$(\delta=(a+a^{-1}-x)/x) .$$

It is then true that if α is a braid in B_n with closure $\hat{\alpha}$ and $\pi(\alpha)$ is the image of α in $C_n(x,a)$ under the obvious representation $\sigma_i\mapsto G_i$ (see [BW]) then $F_{\hat{\alpha}}(x,a)=\delta^{n-1}a^{w(\hat{\alpha})}\operatorname{tr}(\pi(\alpha))$.

The structure of the Birman-Wenzl algebra for generic parameter values is determined in [BW]. The Bratteli diagram for the first three algebras is given below



The "1, 2, 1" part corresponds to the Hecke algebra quotient obtained by putting $E_i=0$. See [BW] for details and an elegant formula for the 3-dimensional irreducible representation of C_3.

2. The Metaplectic Representation

We consider a special case of the situation of [GJ]. For each odd integer $p\geqq 3$ we let ω be a primitive p^{th} root of unity and $ES(\omega,n)$ be the algebra over $\mathbb{C}$ with presentation $\langle u_1,u_2,\dots,u_n : u_j^P=\mathbb{1},\ u_ju_{j+1}=\omega^2u_{j+1}u_j,\ u_iu_j=u_ju_i \text{ if } |i-j|\geqq 2\rangle$. Obviously $\dim_{\mathbb{C}}(GS(\omega,n))=p^n$ and $ES(\omega,n)$ embeds in $ES(\omega,n+1)$, so let $ES(\omega,\infty)$ be the inductive limit. The trace tr on $ES(\omega,\infty)$ defined by $\operatorname{Tr}(1)=1$, $\operatorname{Tr}(w)=0$ if w is a monomial in the u_i's not proportional to $\mathbb{1}$, has the property $\operatorname{Tr}(xy)=\operatorname{Tr}(x)\operatorname{Tr}(y)$ if $x\in ES(\omega,n)$, $y=\sum_{j=0}^{p-1}c_ju_{n+1}^j$.

There are two reasonably obvious braid group representations inside $ES(\omega,\infty)$. The first is the "Potts" representation defined by sending σ_i to $(t+1)e_i-\mathbb{1}$, where $e_i=\frac{1}{p}\sum_{j=0}^{p-1}u_i^j$ and $2+t+t^{-1}=p$. The second is the "Gaussian" defined by sending σ_i to $\sum_{j=0}^{p-1}\omega^{j^2}u_i^j$. The (normalized) trace Tr of a braid in the Potts representation gives

$V_{\mathbf{L}}(t)$, $\mathbf{L}$ being the closure of the braid. It is shown in [GJ] that the trace Tr of a braid in the Gaussian representation gives (constant) $\sum_{v \in H_1(S, \mathbb{Z}/p\mathbb{Z})} \omega^{\langle v,v \rangle}$, where S is any Seifert surface for the closed braid $\mathbf{L}$ and $\langle \, , \rangle$ is the Seifert form.

The group theorist will have no difficulty recognizing the algebras $ES(\omega, n)$ and the Gaussian braid group representation as an algebraic version of the extra-special p-group and the action of the symplectic group on it. Taking some irreducible representation of $ES(\omega, 2n)$ we obtain the metaplectic representation of $Sp(2n, p)$, as always, up to normalization. See [GJ] for details.

3. $F_{\mathbf{L}}(1, 2\cos 2\pi/5)$

We consider the construction of Sect. 2 with $p=5$. Let $\delta = (2-\omega-\omega^{-1})/(\omega+\omega^{-1})$ $(=\pm\sqrt{5})$ and define $G_i = \frac{1}{\delta} \sum_{j=0}^{4} \omega^{j^2} u_i^j$, $E_i = \frac{1}{\delta} \sum_{j=0}^{4} u_i^j$. Then G_j, G_i satisfy the Birman-Wenzl relations for $a=1$, $x=\omega+\omega^{-1}$. Moreover, the trace on $ES(\omega, \infty)$ defined in Sect. 2 satisfies Tr 1→Tr 4 of Sect. 2 when restricted to the algebra generated by the G_i's. We conclude from [GJ] that for a knot K,

$$F_K(1, 2\cos 2\pi/5) = \left(\tfrac{1}{5}\right)^{\mathrm{genus}(S)} \sum_{v \in H_1(S; \mathbb{Z}/5\mathbb{Z})} e^{2\pi i/5 \langle v,v \rangle},$$

where S is any Seifert surface for K, and

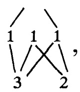
$$F_K(1, 2\cos 4\pi/5) = \left(-\tfrac{1}{5}\right)^{\mathrm{genus}(S)} \sum_{v \in H_1(S; \mathbb{Z}/2\mathbb{Z})} e^{4\pi i/5 \langle v,v \rangle}.$$

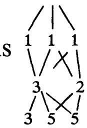
In particular $|F_K(1, 2\cos 2\pi/5)| = |F_K(1, 2\cos 4\pi/5)| = (\sqrt{5})^r$, where r is the rank of the first homology mod 5 of the 2-fold branched cover of S^3, branched over K. Similar formulae hold for links.

4. Description of a Wenzl Subfactor

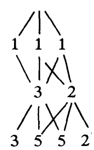
We begin by determining the Bratteli diagram for the algebra generated by the G_i's we defined in Sect. 3, inside $ES(\omega, \infty)$. Note that if we define $u_i^* = u_i^{-1}$ there is a unique C^*-norm on $ES(\omega, \infty)$, so it can be completed to a C^*-algebra. In fact an iterated crossed product argument or direct computation show that $ES(\omega, 2n)$ is a $5^n \times 5^n$ matrix algebra so $ES(\omega, \infty)$ is a 5^∞ UHF algebra in standard operator algebra terminology. Since the G_i's are unitary, the quotient of the Birman-Wenzl algebra at $x=2\cos 2\pi/5$, $a=1$ is thus a C^*-algebra, the trace being positive definite since it is on $ES(\omega, 2n)$.

The Bratteli diagram for $\mathbb{C} \subseteq \mathrm{alg}\{G_1\}$ is obviously $\begin{smallmatrix} & 1 & \\ / & | & \backslash \\ 1 & 1 & 1 \end{smallmatrix}$, the traces of the corresponding minimal projections being 1/5, 2/5 and 2/5. Then by the fact that $E_2 \, \mathrm{alg}\{G_1\} E_2 \subseteq \mathbb{C}E_2$ and the analysis of [J2] or [Wl] (see also [GHJ]) we know that the Bratteli diagram for $\mathbb{C} \subseteq \mathrm{alg}(G_1) \subseteq \mathrm{alg}(G_1, G_2)$ contains $\begin{smallmatrix} & 1 & \\ / & | & \backslash \\ 1 & 1 & 1 \\ \backslash & | & / \\ & 3 & \end{smallmatrix}$ as a subdiagram.

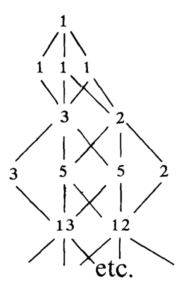
It is easy to exhibit a 2×2 matrix quotient of $\mathrm{alg}(G_1, G_2)$, so since it is contained in the 5×5 matrices we see that the full Bratteli diagram for $\mathbb{C} \subseteq \mathrm{alg}(G_1) \subseteq \mathrm{alg}(G_2, G_1)$ is 

, where the trace vectors are (1/5, 1/5, 1/5) and (1/5, 1/5) respectively. As before the diagram for $\mathrm{alg}(G_1, G_2, G_3)$ contains 

as a subdiagram, and the 2-dimensional representation of $\mathrm{alg}(G_1, G_2)$ extends to $\mathrm{alg}(G_1, G_2, G_3)$ so that the full Bratteli diagram up to G_1, G_2, G_3 is



The last trace vector is $\frac{1}{25}(1, 2, 2, 1)$. The whole Bratteli diagram is now determined by iterating the basic construction of [J2] since the vector (1/5, 1/5) is an eigenvector of the matrix $\Lambda\Lambda^t$, Λ being $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Thus we obtain the diagram



There are many ways to determine the above diagram. The finite group theorist could obtain it by analyzing the metaplectic representation. Note how we see on the diagram the splitting of the metaplectic representation of $Sp(2n, 5)$ into two irreducible direct summands of dimensions $\dfrac{5^n \pm 1}{2}$.

If we complete the C^*-algebra $ES(\omega, \infty)$ to obtain the hyperfinite II_1 factor, it is clear from the above diagram that $\mathrm{alg}(G_1, G_2, \ldots)$ has unique trace and so generates a subfactor $(= \overline{\mathrm{alg}(G_2, G_3, \ldots)})$. Wenzl has identified it as being one of a series of subfactors corresponding to the quantized group of $Sp(4, \mathbb{R})$, defined in [W2].

Finally we offer an alternative description of the subfactor. We may define a period two automorphism of $ES(\omega, \infty)$ by $u_i \mapsto u_i^{-1}$ (corresponding to $-id$ in the symplectic mod 5 picture) and it is clear that all the G_i's are in its fixed point algebra. There is no shortage of ways to prove that the fixed point algebra of $ES(\omega, n)$ for this involution is precisely $\mathrm{alg}(G_i, \ldots, G_n)$. Moreover the "dual action" defined by $u_1 \mapsto \omega u_1$, $u_j \to u_j$ for $j \geqq 1$ defines an outer action of $\mathbb{Z}/5\mathbb{Z}$ on $ES(\omega, \infty)$ whose fixed point algebra is generated by $\{u_2, u_3, \ldots\}$. Thus the II_1 factor generated by $G_2, G_3, \ldots$

is the fixed point algebra for the dihedral group D_5 of order 10 acting on the II_1 factor defined by $ES(\omega, \infty)$ so we finally get the model $R^{D_5} \subseteq R^{\mathbb{Z}/2\mathbb{Z}}$ for the Wenzl subfactor coming from the quotient of the Birman-Wenzl algebra at $a=1$, $m=2\cos 2\pi/5$, where the outer action of D_5 is arbitrary by [J4].

5. A "Solvable" Chiral Potts Model and Quantum Spin Chain

According to [AY+], a chiral Potts model is a statistical mechanical model defined by atoms on a 2-dimensional square lattice with q spin states, $\sigma=0,1,\ldots,q-1$ per atom. The system is then specified by Boltzmann weights $w_h(\sigma,\sigma')$ and $w_v(\sigma,\sigma')$ for the horizontal and vertical directions. The property that makes it a "chiral Potts model" is that $w_h(\sigma,\sigma')$ and $w_v(\sigma,\sigma')$ only depend on $(\sigma-\sigma') \bmod q$, written $w_h(\sigma-\sigma')$, $w_v(\sigma-\sigma')$. This suggests a Fourier analysis of the Boltzmann weights and this is the connection with the $ES(\omega,n)$ algebras. Of course the Boltzmann weights must be positive.

The partition function for the model, with periodic boundary conditions, can be calculated as the trace of a power of the diagonal transfer matrix. We now discuss this matrix, for a lattice of width N, in accordance with [AY+, J5]. Define matrices X_j and Z_j on $V\otimes V\otimes V$ by "X" and "Z" acting in the j^{th} tensor component where V is a vector space of dimension q, ω is a primitive q^{th} root of unity, and $X_{a,b}=\delta_{a,b+1}$ $(a,b,c\in\mathbb{Z}/q\mathbb{Z})$, $Z_{a,b}=\delta_{a,b}\omega^{2a}$. Now set $u_{2i-1}=X_i$ and $u_{2i}=Z_iZ_{i+1}^{-1}$. It is easy to check that the u_i's define a faithful representation of $ES(\omega,2N)$. Let us suppose w_h and w_v further depend on a parameter λ and define the elements $R_i(\lambda)$ of $ES(\omega,2)$ *by*

$$R_{2i-1}(\lambda)=\sum_{\sigma\in\mathbb{Z}/q\mathbb{Z}} w_h(\sigma)u_{2i-1}^{\sigma}\ ,$$

$$R_{2i}(\lambda)=\sum_{\sigma\in\mathbb{Z}/q\mathbb{Z}} \hat{w}_v(\sigma)u_{2i}^{\sigma}\ ,$$

where $\hat{w}_v$ is the finite Fourier transform of w_v. Then a sufficient condition for "solvability" or commuting of the diagonal-to-diagonal transfer matrices is:

$$R_1(\lambda)R_2(\lambda+\mu)R_1(\mu)=R_2(\mu)R_1(\lambda+\mu)R_2(\mu)\ .$$

If there is a function $f(\lambda)$ such that $f(\lambda)R(\lambda)$ has a finite invertible limit as $\lambda\to\infty$ we see that this limit will give a braid group representation. In the case of the Potts model, the limit exists and the braid group representation calculates $V_L(t)$, $2+t+t^{-1}=q$. One might try to invert the process and, given a braid group representation, "Baxterize" it by finding R_i's whose limit is the original representation. Our next result shows that this is possible whenever the braid group representation factors through the Birman-Wenzl algebra.

Theorem. *Let G_1, G_2 be the generators of the algebra $C_3(x,a)$ of Sect. 1. Suppose $x=k^2a+k^{-2}a^{-2}$ and define $R_i(\lambda)=(e^{\lambda}-1)kG_i+x(k+k^{-1})\mathbb{1}+(e^{-\lambda}-1)k^{-1}G_i^{-1}$ $(i=1,2)$. Then*

(i) $R_i(0)=x(k+k^{-1})\mathbb{1}$.
(ii) $R_i(\lambda)R_i(-\lambda)=f(\lambda)\mathbb{1}$ *for some* $f(\lambda)$.
(iii) $R_1(\lambda)R_2(\lambda+\mu)R_1(\mu)=R_2(\mu)R_1(\lambda+\mu)R_2(\lambda)$.

This result is simply a computation, albeit a rather long one. It could also be proved by invoking the results of Jimbo [Ji] and the faithfulness of certain Birman-Wenzl representations arising in quantum-group theory.

An immediate consequence for us is that one may obtain Boltzmann weights for a 5-state chiral Potts model satisfying the "solvability" equations simply by using the representation of the Birman-Wenzl algebra inside $ES(\omega,,\infty)$ $(\omega^5=1)$ defined in Sect. 3. But of course as we noted before the Boltzmann weights must be positive for the model to have any statistical mechanical meaning. For this we are forced to choose $\omega=e^{\pm 4\pi i/5}$ and we obtain the following Boltzmann weights (here $k=e^{2\pi i/5}$):

$$w_v(a)=2\left\{\cos\left(\lambda+\frac{2\pi}{5}(1+2a^2)\right)-\cos\left(\frac{2\pi}{5}(1+2a^2)\right)\right\}+\sqrt{5}\,\delta_{a,0}$$

and

$$w_h(a)=-2\sqrt{5}\left\{\cos\left(\lambda+\frac{2\pi}{5}\left(1-\frac{a^2}{2}\right)\right)-\cos\left(\frac{2\pi}{5}\left(1-\frac{a^2}{2}\right)\right)\right\}+\sqrt{5}\,.$$

It is clear that all these Boltzmann weights are positive if λ is small and positive.

Any "solvable" model as above with properties (i), (ii), and (iii) will define an "integrable" quantum spin chain with a local Hamiltonian. As in [AY+] it is given up to constants by $\sum_{j=1}^{N}\left.\frac{d}{d\lambda}\right|_{\lambda=0}(R_i(\lambda))$ which in this case gives $\sum_{j=1}^{N}\left(\sum_{a=0}^{4}\sin\left(\frac{2\pi}{5}(1+2a^2)\right)u_j^a\right)$.

In fact, J. Perk has pointed out to us that our Boltzmann weights are precisely the 5-state case of the model of Fateev and Zamolodchikov in [FZ], which is defined for all integers N. It is easy to show that the appropriate limit of this model gives the Gaussian invariant of [GJ]. Thus the model of Fateev and Zamolodchikov actually Baxterizes the Gaussian. It sems to be an interesting problem to try to Baxterize other knot invariants that can be expressed as the trace of a braid group representation. That the F-Z model Baxterizes the Gaussian was also noticed independently in [KMM].

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