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Functional Determinants on Mandelstam Diagrams*

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Abstract. We investigate the special properties of Mandelstam metrics in regard to changing weights in path integrals and relations between determinants of different spins. Regularizations of determinants are discussed along the lines of Sonoda. Weyl anomalies developing at zeroes of metrics in reparametrization invariant regularizations are evaluated in terms of Arakelov metrics. Holomorphic forms are constructed, and determinant identities for Arakelov and Mandelstam metrics rigorously established for any weight and generic even and odd spin structures.

1. Introduction

Determinants of Laplacians are playing an increasingly important role in diverse areas of physics and geometry. The foundations of their theory have however been developed mostly for compact manifolds and regular metrics. In this paper we wish to examine properties of determinants on certain surfaces with degenerate metrics and punctures which arise as Feynman diagrams in string theory.

It is a fundamental principle of string theory that scattering amplitudes depend only on the conformal class of the Feynman diagrams and not on particular choices of metrics within the class. Nevertheless it is sometimes useful to select a privileged representative to carry out explicit computations. For surfaces, two metrics arise which in some sense are opposites of one another: metrics with constant curvature, and metrics with all curvature concentrated at isolated points. The former are familiar from hyperbolic geometry, and their determinants now well understood in the compact case, as special values of the Selberg zeta function [1]. Examples of the latter are $|\nu_+|^4$ or $|\omega_z|^2$, where ν_+ and ω_z are respectively a meromorphic spinor [2] and a meromorphic form [3, 4]. Their poles can be viewed as punctures on the surface indicating external string states, and their zeroes as interaction

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points in a Feynman diagram representation. Such surfaces are known as Mandelstam diagrams, and we will refer to their metrics as Mandelstam metrics. For Mandelstam diagrams with at least two punctures, physical parameters given by interaction times, twist angles, internal and external momenta in their natural ranges provide a single copy of the corresponding moduli space. This is basic to a light-cone gauge formulation of string perturbation theory [3, 5].

Relations between determinants of different spins for Mandelstam diagrams are central to the issue of unitary of the Polyakov string. At a formal level, a relation between scalar and spin 2 ghost determinants, as well as the equivalence between light-cone gauge and Polyakov approaches for the bosonic string was derived in [6]. A careful treatment of this case together with modifications dictated by regularization procedures was provided later by Sonoda [7]. Formal relations between Dirac and spin 3/2 superghost determinants have been obtained in [8].

In this paper we investigate this type of relations for any spin. We provide a heuristic argument for them in the spirit of Witten's arguments for multiplicative Ward identities [9]. In the present context however, the conformal anomaly is crucial since we deal with path integrals over fields of different spins. Thus the identities hold only for Mandelstam metrics, as it should be expected. At the rigorous level, we follow Sonoda in defining the determinant ratio

$$\frac{\det' \Delta_n^-}{\det \langle \phi_a^n | \phi_b^n \rangle \det \langle \phi_a^{1-n} | \phi_b^{1-n} \rangle}(|\omega_z|^2) \tag{1.1}$$

for Mandelstam metrics by scalings to a regular metric. We show that the reparametrization invariant regularization of the Liouville action develops a Weyl anomaly at each zero of the Mandelstam metric. Elimination of the Weyl anomaly leads to equivalence with Sonoda's coordinate-dependent regularization, and the proper definition for (1.1) is thus completely unambiguous. Bosonization [10, 11, 12] and Sonoda's methods can then be applied to establish the conjectured identities, for any spin and both odd and even spin structures. For surfaces with punctures, coordinate-dependent regularization is the only reasonable procedure, and the same analysis can be applied to produce similar identities.

Finally we note that for spin 1/2 the determinants of zero modes in (1.1) is finite, so we get a genuine notion of determinants for Laplacians on spinors with respect to Mandelstam metrics. It would be valuable to know whether this notion admits spectral theoretic interpretations or other characterizations besides the identities discussed here. For hyperbolic surfaces with cusps, an interesting candidate for a determinant based on scattering data has been suggested in [13].

2. Formal Considerations

We shall be mainly concerned with determinants of Laplacians acting on fields b, c of ranks n and $1-n$ on a Riemann surface M. More precisely the Cauchy–Riemann operator $\partial_{\bar z}$ sends $b(dz)^n$ and $c(dz)^{1-n}$ to $\partial_{\bar z} b d\bar z dz^n$ and $\partial_{\bar z} c d\bar z dz^{1-n}$. If we choose a metric $ds^2 = 2g_{z\bar z} d\bar z dz$ to represent the complex structure of M, then the covariant derivatives ∇_z^n and ∇_n^z can be written as

$$\nabla_n^z(b(dz)^n) = g^{z\bar{z}}\partial_z b dz^{n-1}, \quad \nabla_z^n(b(dz)^n) = g_{z\bar{z}}^n \partial_z((g^{z\bar{z}})^n b) dz^{n+1}. \tag{2.1}$$

With respect to the pairing

$$\| b \|^2 = \int d^2z (g^{z\bar{z}})^{n-1} \bar{b} b, \quad \| c \|^2 = \int d^2z (g^{z\bar{z}})^{-n} \bar{c} c, \tag{2.2}$$

it is readily seen that $(\nabla_n^z)^\dagger = -\nabla_z^{n-1}$, $(\nabla_z^n)^\dagger = -\nabla_{n+1}^z$, and we can form the Laplacians

$$\Delta_n^+ = -2\nabla_{n+1}^z \nabla_z^n, \quad \Delta_n^- = -2\nabla_z^{n-1}\nabla_n^z. \tag{2.3}$$

It is evident that the spectra of Δ_n^-, Δ_{-n}^+, and Δ_{1-n}^- are identical, so henceforth we shall restrict to determinants of Δ_n^- for $1/2 \leqq n$. For n half-integer, proper definition of rank n tensors requires selection of a spin structure on M, and we always assume this has been done. The genus of M will be denoted by h.

The relations between determinants of different weights we are looking for are suggested by a change of variables in the chirally symmetric path integrals over anti-commuting fields b and c

$$Z_n(z_1, \dots, w_M) = \int D(b\bar{b}c\bar{c}) \exp(-I_n(b,c)) \left| \prod_{i=1}^{M+\Upsilon_n} b(z_i) \prod_{j=1}^{M} c(w_j) \right|^2. \tag{2.4}$$

Here the action is given by

$$I_n(b,c) = \frac{1}{2\pi} \int d^2z (b\partial_{\bar{z}} c + \text{c.c}). \tag{2.5}$$

and Υ_n is the index of the Laplacian Δ_n^- which gives the violation of fermion number. Recall that the Riemann–Roch theorem asserts that

$$\Upsilon_n = (\#b\,\text{zero modes}) - (\#c\,\text{zero modes}) = -\tfrac{1}{2}(2n-1)\chi(M), \tag{2.6}$$

where the dimensions are taken over complex numbers. To pass from a rank n tensor b to a rank $n+1$ tensor b' we need a holomorphic one-form ω_z so as to set

$$b' = \omega_z b, \quad c' = \omega_z^{-1} c. \tag{2.7}$$

For this transformation to be an isometry with respect to the pairing (2.2), the metric $ds^2 = 2g_{z\bar{z}} dz d\bar{z}$ cannot be any metric but must be chosen to be the singular metric

$$2|\omega_z|^2 |dz|^2. \tag{2.8}$$

We shall refer to metrics of the form (2.8) as Mandelstam metrics. A holomorphic form will have $2h-2$ zeroes, and we shall assume for simplicity that the zeroes of ω_z are simple, and denote them by R_a. To preserve locality of the path integrals, we must require c' to be continuous, b' to vanish at R_a. Vanishing of the fermionic field b' at R_a can be insured by an insertion of $b'(R_a)$, and we are thus led to

$$\int D(b'\bar{b}'c'\bar{c}') \exp(-I_{n+1}(b',c')) \left| \prod_1^{M+\Upsilon_n} b'(z_i) \prod_1^{M} c'(\omega_j) \prod_1^{2h-2} b'(R_a) \right|^2 \left| \frac{\prod_1^{M} \omega_{w_j}}{\prod_1^{M+\Upsilon_n} \omega_{z_i}} \right|^2. \tag{2.9}$$

Note that the fermion number violation is now $\Upsilon_n + (2h-2) = \Upsilon_{n+1}$, as it should be. However the expression (2.9) cannot agree with the original expression (2.4) since their tensor types at R_a are different. To remedy the situation, we introduce powers of the non-vanishing rank two tensor at R_a,

$$\partial\omega(R_a). \tag{2.10}$$

The naive powers to go with (2.9) would seem to be $|\partial\omega(R_a)|^{-(n+1)}$, and they would be correct if the determinants of Δ_n^- were scalars as in the case of regular metrics. Here the requirement that (2.7) be an isometry has forced us to take singular Mandelstam metrics (2.8). We shall see in Sect. 4 that any regularization consistent with Weyl scalings will force the determinant ratio

$$W_n(|\omega_z|^2) = \frac{\det' \Delta_n^-}{\det\langle\phi_a^n|\phi_b^n\rangle \det\langle\phi_a^{1-n}|\phi_b^{1-n}\rangle} \tag{2.11}$$

to transform as a tensor of rank $(c_n/12, c_n/12)$ at each zero R_a of the metric. Here

$$c_n = 6n^2 - 6n + 1$$

is the familiar coefficient of the conformal anomaly, and ϕ_a^n are a fixed basis of zero modes for ∇_n^z. It follows that the proper correction factor is $|\partial\omega(R_a)|^{-(n+1)-(c_{n+1}-c_n)/12} = |\partial\omega(R_a)|^{-2n-1}$. Thus the change of weight formula in path integrals should be

$$\begin{aligned}\int D(b\bar{b}c\bar{c})\exp(-I_n)\left|\prod_1^{M+\Upsilon_n} b(z_i)\prod_1^M c(w_j)\right|^2 &= \int D(b\bar{b}c\bar{c})\exp(-I_{n+1})\\ &\cdot\left|\prod_1^{M+\Upsilon_n} b(z_i)\prod_1^{2h-2} b(R_a)\prod_1^M c(w_j)\right|^2\\ &\cdot\left|\frac{\prod_1^M \omega(w_j)}{\prod_1^{M+\Upsilon_n}\omega(z_i)}\right|^2 \left|\prod_1^{2h-2}\partial\omega(R_a)\right|^{-2n-1}.\end{aligned} \tag{2.12}$$

To derive relations between determinants we consider (2.12) with the minimal number of insertions to absorb all zero modes. For simplicity we discuss now only the case of weight $n>1$, so that there are no c-zero mode, the number of b-zero modes is Υ_n, and we may set $M=0$. The result is

$$\begin{aligned}\frac{\det'\Delta_n^-}{\det\langle\phi_a^n|\phi_b^n\rangle}|\det\phi_a^n(z_b)|^2 &= \frac{\det'\Delta_{n+1}^-}{\det\langle\phi_a^{n+1}|\phi_b^{n+1}\rangle}|\det\phi_a^{n+1}(z_b,R_a)|^2\\ &\cdot\prod_1^{2h-2}|\partial\omega(R_a)|^{-2n-1}\prod_1^{\Upsilon_n}|\omega(z_i)|^{-2}.\end{aligned} \tag{2.13}$$

It is worth observing that with the regularization we shall adopt for the determinant ratio $W_n(|\omega_z)|^2)$ will scale as

$$W_n(e^{2\mu}|\omega_z|^2) = W_n(|\omega_z|^2)\exp\left(\tfrac{2}{3}(h-1)c_n\mu\right) \tag{2.14}$$

for constant μ and fixed choice of zero modes, so that (2.13) is indeed independent of any normalization for the holomorphic form ω_z. It would otherwise carry little information.

The relation (2.13) can be simplified considerably if we choose the bases ϕ_a^n and ϕ_a^{n+1} appropriately. First we need a propagator, i.e., a rank n form meromorphic form in z with a simple pole at w. Evidently two such propagators will differ by a holomorphic n form, and thus such propagators are parametrized by Υ_n parameters. If we choose these parameters to be Υ_n generic points on the Riemann surface M, we can construct the corresponding propagator explicitly as

$$S^n(z, w; z_1, \ldots, z_{\Upsilon_n}) = \frac{\int D(bc) \exp(-I_n(b,c)) b(z) c(w) \prod_1^{\Upsilon_n} b(z_i)}{\int D(bc) \exp(-I_n(b,c)) \prod_1^{\Upsilon_n} b(z_i)}$$
$$= \frac{\left\langle b(z) c(w) \prod_1^{\Upsilon_n} b(z_i) \right\rangle}{\left\langle \prod_1^{\Upsilon_n} b(z_i) \right\rangle}. \tag{2.15}$$

Then $S^n(z, \omega; z_1, \ldots, z_{\Upsilon_n})$ is a rank n tensor in z with a unique pole at w and zeroes at z_i. Given a basis ϕ_a^n of holomorphic rank n tensors, we can construct a basis for holomorphic rank $n+1$ tensors by letting

$$\phi_a^{n+1}(z) = \omega_z \phi_a^n(z), \quad a = 1, \ldots, \Upsilon_n,$$
$$\phi_{\Upsilon_n+b}^{n+1}(z) = \omega_z S^n(z, R_b; z_1, \ldots, z_{\Upsilon_n}) (\partial\omega(R_b))^{(n-1)/2}. \tag{2.16}$$

In (2.16) the factor $(\partial\omega(R_b))^{(n-1)/2}$ is necessary to insure that $\phi_{\Upsilon_n+b}^{n+1}$ be a rank $n+1$ tensor. The reason is that $S^n(z, w; z_1, \ldots, z_{\Upsilon_n})$ is a rank $1-n$ tensor in w, so that $\lim_{z \to R_b} \omega_z S^n(z, R_b; z_1, \ldots, z_{\Upsilon_n})$ is a rank 2 tensor at R_b for any weight n. It is now immediately seen that the matrix $\phi_c^{n+1}(z_i, R_b)$ is block diagonal, and

$$\frac{\det \phi_c^{n+1}(z_i, R_b)}{\det \phi_a^n(z_i)} = \prod_1^{2h-2} \partial\omega(R_b)^{(n+1)/2} \prod_1^{\Upsilon_n} \omega(z_i). \tag{2.17}$$

Thus we obtain the remarkable relation

$$\frac{\det' \Delta_n^-}{\det \langle \phi_a^n | \phi_b^n \rangle} = \frac{\det' \Delta_{n+1}^-}{\det \langle \phi_a^{n+1} | \phi_b^{n+1} \rangle} \left| \prod_1^{2h-2} \partial\omega(R_a) \right|^{-n}. \tag{2.18}$$

With the regularizations described in Sect. 4, we shall establish later by explicit computations that (2.18) does hold for Mandelstam metrics $ds^2 = 2|\omega_z|^2 |dz|^2$, together with its analogues for weights $n=1$ and $n=1/2$, and for both types of spin structures. The case $n=1$ is the one treated in Sonoda [7]. The presence of the factor involving $\partial\omega(R_a)$ in (2.18) is required by the fact that $W_n(|\omega_z|^2)$ is a tensor of rank $(c_n/12, c_n/12)$ at each R_a. Perhaps more important, the above discussion shows that this factor also leads to invariance of (2.18) under constant scalings of ω_z. For rank $n=1$ the relation (2.16) between bases of holomorphic differentials and quadratic differentials simplifies, and this key scale invariance in ω_z of (2.18) can be checked directly, assuming the reasonable behavior of $W_n(|\omega_z|^2)$ given by (2.14).

3. Arakelov Green's Functions and Arakelov Metrics

To regularize determinants with respect to the metric $|\omega_z|^2$, we shall scale $|\omega_z|^2$ to a regular metric. Conformal factors between metrics of given curvatures and their Liouville actions are most conveniently expressed in terms of Arakelov Green's functions, so we begin with a brief study of their basic properties and scaling behavior.

Let $ds^2 = 2g_{z\bar z}dzd\bar z$ be a fixed metric on the surface M. The usual scalar Green's function $G(z,w)$ is symmetric and characterized by

$$-\partial_z\partial_{\bar z}G(z,w) = 2\pi\delta(z-w) - \frac{2\pi g_{z\bar z}}{\int d^2z\sqrt{g}}, \tag{3.1}$$

$$\int d^2z g_{z\bar z}G(z,w) = 0. \tag{3.2}$$

The Arakelov Green's function $G^A(z,w)$ on the other hand will be defined by

$$-\partial_z\partial_{\bar z}G^A(z,w) = 2\pi\delta(z-w) + \frac{g_{z\bar z}R}{2(h-1)}, \tag{3.3}$$

$$\int d^2z g_{z\bar z}RG^A(z,w) = 0, \tag{3.4}$$

where $R = -g^{z\bar z}\partial_z\partial_{\bar z}\ln g_{z\bar z}$ is the curvature scalar. Equation (3.4) is just a normalization condition. We note that $G^A(z,w)$ is invariant if we scale $g_{z\bar z}$ by a constant. Solving for $G^A(z,w)$ in terms of the usual Green's function $G(z,w)$ gives

$$\begin{aligned} G^A(z,w) = G(z,w) &+ \frac{1}{4\pi(h-1)}\int d^2vG(z,v)g_{v\bar v}R + \frac{1}{4\pi(h-1)}\int d^2vG(v,w)g_{v\bar v}R \\ &+ \frac{1}{16\pi^2(h-1)^2}\int d^2vd^2ug_{v\bar v}RG(v,u)g_{u\bar u}R \end{aligned} \tag{3.5}$$

which shows that the Arakelov Green's function is symmetric,

$$G^A(z,w) = G^A(w,z). \tag{3.6}$$

Let now $d\hat s^2 = 2\hat g_{z\bar z}dzd\bar z$ be another metric conformally equivalent to ds^2, with

$$g_{z\bar z} = \exp(2\lambda)\hat g_{z\bar z}.$$

If $G^A(z,w)$ denotes the corresponding Arakelov Green's function, it is readily seen that the same equation (3.5) will hold, with $G(z,w)$ replaced by $\hat G^A(z,w)$ on the right-hand side. Since the curvatures of the two metrics are related by

$$g_{z\bar z}R = \hat g_{z\bar z}\hat R - 2\partial_z\partial_{\bar z}\lambda. \tag{3.7}$$

we can evaluate the right-hand side explicitly and obtain

$$G^A(z,w) = \hat G^A(z,w) + \frac{1}{(h-1)}(\lambda(z)+\lambda(w)) + \frac{6}{(h-1)^2}S(\hat g,\lambda). \tag{3.8}$$

Here $S(\hat{g}, \lambda)$ is the Liouville action

$$S(\hat{g}, \lambda) = \frac{1}{12\pi} \int d^2z (\lambda \partial_z \partial_{\bar{z}} \lambda + \hat{g}_{z\bar{z}} \hat{R} \lambda). \tag{3.9}$$

At coincident points, Green's functions will diverge as $-\ln d^2(z,w)$, where $d(z,w)$ is the reparametrization invariant distance with respect to the metric ds^2. Thus we may regularize the Green's function $G^A(z,w)$ by

$$G^A(z,z) = \lim_{w \to z} (G^A(z,w) + \ln d^2(z,w)). \tag{3.10}$$

Note that $G^A(z,z)$ is a reparametrization invariant scalar. For coincident points the scaling law (3.8) becomes

$$G^A(z,z) = \hat{G}^A(z,z) + \frac{2h}{(h-1)} \lambda(z) + \frac{6}{(h-1)^2} S(\hat{g}, \lambda), \tag{3.11}$$

since $d^2(z,w) = \hat{d}^2(z,w) e^{2\lambda(z)}$ up to higher order terms for z near w.

We can now eliminate the explicit dependence on the genus by forming combinations of Arakelov Green's functions. The simplest such expression is

$$(4 g_{z\bar{z}} g_{w\bar{w}})^{-1/2} \exp(-G^A(z,w) + \tfrac{1}{2} G^A(z,z) + \tfrac{1}{2} G^A(w,w)),$$

which is immediately seen to agree with the similar expression in terms of the Green's function $G(z,w)$ and its reparametrization invariant regularization $G(z,z) = \lim_{w \to z} (G(z,w) + \ln d^2(z,w))$. This is known to be the same as the $(-1/2, -1/2) \times (-1/2, -1/2)$ form on $M \times M$

$$F(z,w) = \exp\left(-2\pi \sum_{IJ} \operatorname{Im} \int_w^z \omega_I (\operatorname{Im} \Omega)^{-1}_{IJ} \operatorname{Im} \int_w^z \omega_J \right) |E(z,w)|^2, \tag{3.12}$$

where $E(z,w)$ is the prime form, ω_I a basis of abelian differentials, and Ω_{IJ} the corresponding period matrix. A more unusual combination is obtained by selecting any $(2h-2)$ points R_a on the surface M, and considering

$$\sum_{a,b=1}^{2h-2} G^A(R_a, R_b). \tag{3.13}$$

Unless stated explicitly otherwise, coincident points are included at which the Arakelov Green's functions are taken to be their regularized values. The expression (3.13) now transforms as

$$\sum_{a,b=1}^{2h-2} G^A(R_a, R_b) = \sum_{a,b=1}^{2h-2} \hat{G}^A(R_a, R_b) + 6 \sum_{a,b=1}^{2h-2} \lambda(R_a) + 24 S(\hat{g}, \lambda). \tag{3.14}$$

This equation will be needed when evaluating the Weyl anomaly developing at each zero R_a of a Mandelstam metric $|\omega_z|^2 \sim |\partial\omega(R_a)|^2 |z - R_a|^2$.

So far we have discussed Arakelov Green's functions with respect to any metric $ds^2 = 2 g_{z\bar{z}} dz d\bar{z}$. Some calculations simplify if we make use of some particular metric on the surface M. Recall that M can be imbedded in its Jacobian $C^h/(Z^h + \Omega Z^h)$ by the Abel map

$$z \to \int_P^z \omega_I,$$

where P is a fixed base point. The induced metric on M from the flat metric on the Jacobian is

$$j_{z\bar{z}} = \sum_{IJ} \omega_I(z)(\mathrm{Im}\,\Omega)^{-1}_{IJ}\,\bar{\omega}_J(z). \tag{3.15}$$

The Arakelov metric on M is a metric $g^A_{z\bar{z}}$ whose curvature form $g^A_{z\bar{z}}R^A$ is equal to $2\pi(1-h)\,j_{z\bar{z}}/h$. This condition determines $g^A_{z\bar{z}}$ only up to a multiplicative constant which we shall choose later. It is then evident that the Arakelov Green's function $G^A(z,w)$ with respect to the Arakelov metric is the same as the usual Green's function with respect to the metric $j_{z\bar{z}}$ induced from the Jacobian.

This process of passing from a metric to another metric whose curvature is proportional to the volume form of the former can in principle be repeated indefinitely. With suitable normalizations of the volume, it will converge to constant curvature metrics.

Returning to the Arakelov metric proper, we note that its Arakelov Green's function is actually rather simple. Since the form $F(z,w)$ of (3.12) satisfies the equation

$$\partial_z\partial_{\bar{z}} \ln F(z,w) = 2\pi\delta(z,w) - \pi j_{z\bar{z}} \tag{3.16}$$

the Arakelov Green's function for the Arakelov metric must be equal to

$$G^A(z,w) = -\ln F(z,w) - \tfrac{1}{2}\ln g^A_{z\bar{z}} - \tfrac{1}{2}\ln g^A_{w\bar{w}} \tag{3.17}$$

up to an additive constant independent of both z and w. This implies that up to an additive constant we have

$$-\ln g^A_{z\bar{z}} = \lim_{w\to z}\,(G^A(z,w) + \ln|z-w|^2). \tag{3.18}$$

The initial constant ambiguity in the definition of the Arakelov metric can now be eliminated by requiring that (3.18), and hence (3.17), hold exactly as they are written there. The relations (3.17), (3.18) are sometimes referred to as residue formulas. The usefulness of Arakelov metrics and their Arakelov Green's functions in bosonization can now be traced to the fact that $F(z,w)$ is essentially the correlation function of two normal-ordered vertex operators

$$(2g_{z\bar{z}})^{k^2/2}(2g_{w\bar{w}})^{(k')^2/2}\langle :e^{ikx(z)}::e^{ik'x(w)}:\rangle = \delta(k+k')F(z,w)^{-k^2} \tag{3.19}$$

in a bose theory of scalar fields x with lagrangian $\partial_z x\partial_{\bar{z}} x/4\pi$. In Arakelov's original work [14] norms on line bundles are defined by setting $\|1_w\| = \exp(-G^A(z,w))$ for the canonical section 1_w of the point bundle $O(w)$. The above relations then express the fact that for the canonical bundle the new metric defined this way will give back the Arakelov metric.

The right-hand side of (3.18) is a regularization procedure for $G^A(z,w)$ at coincident points which is coordinate dependent, but does not require a choice of metric within the conformal class. We shall sometimes denote it by $:G^A(z,z):$ to distinguish it from the $G^A(z,z)$ of (3.10). Note that $\exp :G^A(z,z):$ is a $(-1,-1)$

tensor. For the Arakelov metric, the reparametrization invariant regularization $G^A(z,z)$ vanishes

$$G^A(z,z) = 0. \tag{3.20}$$

This follows from (3.18) and the fact that the distance $d^A(z,w)$ equals $|z-w|(2g^A_{z\bar{z}})^{1/2}$ up to higher order terms.

This completes our survey of Arakelov Green's functions.

4. Liouville Actions and Regularization of Determinants

We turn now to the problem of defining a determinant for the surface M with a Mandelstam metric $|\omega_z|^2$, where $\omega_z dz$ is a holomorphic abelian differential. Following Sonoda [7] we introduce a regular (i.e. with neither zeros nor poles) metric $g_{z\bar{z}}$ conformally equivalent to $|\omega_z|^2$,

$$|\omega_z|^2 = \exp(2\sigma(z))g_{z\bar{z}}. \tag{4.1}$$

Taking the curvatures for both sides gives an equation for $\sigma(z)$, which can be solved using the Arakelov Green's function $G^A(z,w)$ for the metric $g_{z\bar{z}}$

$$\sigma(z) = -\tfrac{1}{2}\sum_{a=1}^{2h-2} G^A(z,R_a) - \tfrac{1}{2}c. \tag{4.2}$$

Here R_a, $a = 1,\ldots,2h-2$ are the zeroes of ω_z which are assumed to be simple, and c is a constant measuring the relative normalizations of $|\omega_z|^2$ and $g_{z\bar{z}}$:

$$c = \frac{1}{2(h-1)}\left(-\sum_{a,b=1}^{2h-2} G^A(R_a,R_b) + \ln\left|\prod_{a=1}^{2h-2}\sqrt{2g_{R_a\bar{R}_a}}/\partial\omega(R_a)\right|^2\right). \tag{4.3}$$

Formally Weyl scalings of determinants (more precisely of ratios $W_n(g)$ of (2.11)) would suggest the following definition for $W_n(|\omega_z|^2)$:

$$W_n(|\omega_z|^2) = W_n(g)\exp(-2c_n S(g,\sigma)) \tag{4.4}$$

for fixed zero modes, $S(g,\sigma)$ the Liouville action of (3.9), and

$$c_n = 6n^2 - 6n + 1$$

the coefficient of the conformal anomaly for tensors of rank n. However the Liouville action diverges logarithmically for scaling factors $\sigma(z)$ of the type (4.2) which have logarithmic singularities at the points R_a. There are then two ways of regularizing the Liouville action which we shall examine in turn:

- Cutting out a reparametrization invariant, but metric dependent disk $D_g(R_a) = \{d(z,R_a) < \varepsilon\}$ around each R_a,

$$S_{\mathrm{cov}}(g,\sigma) = \lim_{\varepsilon\to 0}\int_{M(\varepsilon,g)} d^2z(\partial_z\sigma\partial_{\bar{z}}\sigma - (\partial_z\partial_{\bar{z}}\ln g_{z\bar{z}})\sigma) - \frac{1}{24}(2h-2)\ln\varepsilon^2. \tag{4.5}$$

The surface $M(\varepsilon,g)$ is the surface M with the disks $D_g(R_a)$ removed.

The regularized Liouville action (4.5) can actually be computed explicitly in terms of Arakelov Green's functions using the expression (4.2) for the conformal factor $\sigma(z)$:

$$S_{\text{cov}}(g,\sigma)=\lim_{\varepsilon\to 0}\frac{1}{12\pi}\int_{M(\varepsilon,g)} d^2z\sum_{a,b=1}^{2h-2}\partial_z G^A(z,R_a)\partial_{\bar z}G^A(z,R_b)$$
$$-\frac{1}{24}(2h-2)\ln\varepsilon^2+\frac{1}{6}(h-1)c.$$

An integration by parts produces

$$S_{\text{cov}}(g,\sigma)=\lim_{\varepsilon\to 0}\frac{1}{12\pi}\int_{M(\varepsilon,g)} d^2z\frac{1}{4}\sum_{a,b=1}^{2h-2}(-\partial_{\bar z}\partial_z G^A(z,R_a))G^A(z,R_b)$$
$$+\frac{i}{48\pi}\sum_{a=1}^{2h-2}\int_{\partial D_g(R_a)}\partial_z G^A(z,R_a)\sum_{b=1}^{2h-2}G^A(z,R_b)$$
$$-\tfrac{1}{24}(2h-2)\ln\varepsilon^2+\tfrac{1}{6}c(h-1). \tag{4.6}$$

The first term on the right-hand side vanishes due to the normalization condition (3.4) and the fact that $M(\varepsilon,g)$ does not contain any of the sources R_a. As for the second we may replace $G^A(z,R_a)$ effectively by its asymptotic $-\ln|z-R_a|^2$. For $b\neq a$ $G^A(z,R_b)$ can be replaced by $G^A(R_a,R_b)$, while for $b=a$ it should be replaced by $-\ln\varepsilon^2+G^A(R_a,R_a)$, with $G^A(R_a,R_a)$ the reparametrization invariant regularized Arakelov Green's function at coincident points. The result is

$$S_{\text{cov}}(g,\sigma)=\frac{1}{24}\sum_{a,b=1}^{2h-2}G^A(R_a,R_b)+\tfrac{1}{6}(h-1)c \tag{4.7}$$

with c given by (4.3). In particular we can read off the scaling behavior of the Liouville action, under a change from g to $\hat g=\exp(-2\lambda)g$ with λ a regular function. Since the normalization constants will scale as

$$\tfrac{1}{6}\hat c(h-1)-2S(\hat g,\lambda)-\frac{1}{6}\sum_{a=1}^{2h-2}\lambda(R_a)=\tfrac{1}{6}c(h-1) \tag{4.8}$$

and Arakelov Green's functions scale as in (3.14), we see that the usual additive rule for Liouville actions in the case of regular metrices $g,\hat g$ and regular scalings $\lambda,\hat\sigma,\sigma=\hat\sigma-\lambda$,

$$S(\hat g,\hat\sigma)=S(\hat g,\lambda)+S(g,\sigma) \tag{4.9}$$

gets modified to

$$S_{\text{cov}}(\hat g,\hat\sigma)=S(\hat g,\lambda)+S_{\text{cov}}(g,\sigma)-\frac{1}{12}\sum_{a=1}^{2h-2}\lambda(R_a) \tag{4.10}$$

for scalings σ of the form (4.2). This particular relation can also be derived directly from (4.5). It can be interpreted as a Weyl anomaly developing at each singular point R_a in a metric.

The net outcome is that the naive definition (4.4), (4.5) for the determinant ratio $W_n(|\omega_z|^2)$ leads to a scalar which depends on the choice of the regularizing metric $g_{z\bar z}$. We can restore independence from $g_{z\bar z}$ by defining instead

$$W_n(|\omega_z|^2) = W_n(g_{z\bar{z}}) \exp(-2c_n S(g,\sigma)) \left(\prod_{a=1}^{2h-2} g_{R_a \bar{R}_a} \right)^{c_n/12}$$

$$= W_n(g_{z\bar{z}}) \exp\left(\frac{c_n}{12} \sum_{a,b=1}^{2h-2} G^A(R_a, R_b) \right)$$

$$\cdot \left(\prod_{a=1}^{2h-2} (\sqrt{2} g_{R_a \bar{R}_a})^{-1/4} \prod_{a=1}^{2h-2} |\partial\omega(R_a)|^{1/3} \right)^{c_n}, \tag{4.11}$$

which is now a tensor of rank $(c_n/12, c_n/12)$ at each $R_a, a = 1, \ldots, 2h-2$.

• Cutting out Weyl invariant, but coordinate dependent disks $D(R_a) = \{|z - R_a| < \varepsilon\}$. This is the choice of Sonoda, and letting $M(\varepsilon) = M - (D(R_a))$, the Liouville action is regularized by

$$S(g,\sigma) = \lim_{\varepsilon \to 0} \frac{1}{12\pi} \int_{M(\varepsilon)} d^2 z (\partial_z \partial_{\bar{z}} \sigma - (\partial_z \partial_{\bar{z}} \ln g_{z\bar{z}}) \sigma) - \tfrac{1}{24}(2h-2) \ln \varepsilon^2. \tag{4.12}$$

It is easy to see that the additive rule (4.9) is now unchanged, and $\exp(-2c_n S(g,\sigma))$ is a rank $(c_n/12, c_n/12)$ tensor at each R_a. This means that $W_n(|\omega_z|^2)$ defined by (4.4) and (4.12) is again a rank $(c_n/12, c_n/12)$ tensor which is independent of the choice of the regular metric $g_{z\bar{z}}$. The same calculation leading to (4.7) now gives

$$S(g,\sigma) = \frac{1}{24} \sum_{a \neq b} G^A(R_a, R_b) + \frac{1}{24} \sum_{a=1}^{2h-2} :G^A(R_a, R_a): + \tfrac{1}{6}(h-1)c \tag{4.13}$$

from which the additive rule can also be verified using (3.8). We can then show that the two regularizations (4.11) and (4.4), (4.12) lead to identical notions. Since both are independent of $g_{z\bar{z}}$, we may choose $g_{z\bar{z}}$ to be the Arakelov metric $g^A_{z\bar{z}}$, upon which $G^A(R_a, R_a) = 0$ and $:G^A(R_a, R_a): = -\ln g^A_{z\bar{z}}$. We then obtain for both regularizations the same answer

$$W_n(|\omega_z|^2) = W_n(g^A_{z\bar{z}}) \exp\left(-\frac{c_n}{12} \sum_{a,b=1}^{2h-2} G^A(R_a, R_b) + \frac{c_n}{3}(1-h)c \right) \left(\prod_{a=1}^{2h-2} g^A_{R_a \bar{R}_a} \right)^{c_n/12}. \tag{4.14}$$

Thus Weyl scaling rules lead to an unambiguous notion of determinants for Mandelstam metrics.

5. Evaluations in Terms of Theta Functions

The goal of this section is to justify the formal rules of Sect. 2 by evaluating explicitly the regularized determinants of Sect. 4, using bosonization formulas.

Bosonization expresses correlation functions of the form (2.4) in terms of the function $F(z,w)$ of (3.12) and (3.19), or equivalently in view of (3.17), (3.18),

$$(g^A_{z_i \bar{z}_i})^{-n} (g^A_{w_j \bar{w}_j})^{-1+n} Z_n(z_1, \ldots, w_M)$$

in terms of Arakelov Green's functions for the Arakelov metric [10–12],

$$\prod (g^A_{z_i\bar z_i})^{-n}(g^A_{w_j\bar w_j})^{-1+n} Z_n(z_1,\dots,w_M)$$
$$=\left[\frac{\det' \Delta_0^-}{\int d^2z\sqrt{g^A_{z\bar z}}\det \operatorname{Im}\Omega}\right]^{-1/2} |\theta[\delta](0,\Omega)|^2$$
$$\cdot\exp\left(-\frac{1}{2}\sum_{i,j} G^A(z_i,z_j)-\frac{1}{2}\sum_{i,j} G^A(w_i,w_j)+\sum_{i,j} G^A(z_i,w_j)\right). \tag{5.1}$$

The theta characteristics $\delta=(\delta',\delta'')$ are given by

$$\Omega\delta'+\delta''=\sum_i \int_P^{z_i}\omega_I-\sum_j\int_P^{w_j}\omega_I-(2n-1)\Delta+\Omega\alpha'+\alpha'', \tag{5.2}$$

where (α',α'') define the spin structure of the $b(dz)^n, c(dz)^{1-n}$, and Δ is the Riemann vector of constants. A more precise chiral version of (5.1) is also in [12], which is of special interest since it shows that the meromorphic propagators $S^n(z,w;z_1,\dots,z_{Y_n})$ of (2.15) can all be written in terms of theta functions:

$$\left\langle\prod_{i=1}^{M+Y_n} b(z_i)\prod_{j=1}^{M} c(w_j)\right\rangle = Z_\Delta^{-1}\,\theta[\alpha]\left(\sum_i\int_P^{z_i}\omega_I-\sum_j\int_P^{w_j}\omega_I-(2n-1)\Delta\right)$$
$$\cdot\frac{\prod_{i<j}E(z_i,z_j)\prod_{i<j}E(w_i,w_j)}{\prod_{i,j}E(z_i,w_j)}\left(\frac{\prod\sigma(z_i)}{\prod\sigma(w_j)}\right)^{2n-1}. \tag{5.3}$$

Here $\sigma(z)$ is a multi-valued nowhere vanishing holomorphic form of rank $h/2$. The ratio $\sigma(z)/\sigma(w)$ can be expressed as

$$\frac{\sigma(z)}{\sigma(w)}=\frac{\theta\left(z-\sum_1^h\int_P^{p_i}\omega_I-\Delta\right)}{\theta\left(w-\sum_1^h\int_P^{p_i}\omega_I-\Delta\right)}\prod_i\frac{E(w,p_i)}{E(z,p_i)} \tag{5.4}$$

for any choice of h points p_i on M. The factor Z_Δ^{-1} is the partition function of a chiral scalar. The expression (5.3) is anomalous, but the anomalies will cancel in all relevant formulas. The passage from (5.1) to (5.3) requires extraction of the correct conformal and gravitational anomalies from the theta factors and Arakelov Green's functions to produce the holomorphic forms $\sigma(z_i)$ and $\sigma(w_j)$.

As a consequence the propagator of (2.15) is given by (say $n>1$)

$$S^n(z,w;z_1,\dots,z_{Y_n})=\frac{1}{E(z,w)}\frac{\theta[\alpha]\left(z-w+\sum_i z_i-(2n-1)\Delta\right)}{\theta[\alpha]\left(\sum_i z_i-(2n-1)\Delta\right)}$$
$$\cdot\frac{\prod_j E(z,z_j)}{\prod_j E(w,z_j)}\left(\frac{\sigma(z)}{\sigma(w)}\right)^{2n-1}. \tag{5.5}$$

Explicit expressions for holomorphic n-forms can be obtained as well. In fact the operator product expansions of the b, c system imply that

$$\partial_{\bar{w}} S^n(z, w; z_1, \ldots, z_{\Upsilon_n}) = -2\pi\delta(z - w) + 2\pi \sum_1^{\Upsilon_n} \delta(z_k - w)\phi_k^n(z) \tag{5.6}$$

with $\phi_k^n(z)$ a basis of holomorphic n-forms. Applying (5.5) gives then

$$\phi_j^n(z) = \frac{\theta[\alpha]\left(z - z_j + \sum_k z_k - (2n-1)\Delta\right)}{\theta[\alpha]\left(\sum_k z_k - (2n-1)\Delta\right)} \frac{\prod_{k \neq j} E(z, z_k)}{\prod_{k \neq j} E(z_j, z_k)} \left(\frac{\sigma(z)}{\sigma(z_j)}\right)^{2n-1}. \tag{5.7}$$

These formulas are useful in some multiloop calculations for string scattering. For our purposes, it is better to proceed inductively as in (2.16), and we shall make use only of non-chiral bosonization as stated in (5.1). For $n > 1$ recall that a basis ϕ_a^{n+1} of holomorphic $(n+1)$ forms can be obtained from a basis ϕ_a^n of holomorphic n-forms by

$$\begin{aligned} &\phi_a^{n+1}(z) = \omega_z \phi_a^n(z), \quad a = 1, \ldots, \Upsilon_n, \\ &\phi_{\Upsilon_n + b}^{n+1}(z) = \omega_z S^n(z, R_b; z_1, \ldots, z_{\Upsilon_n})(\partial\omega(R_b))^{(n-1)/2}, \end{aligned} \tag{5.8}$$

where the $(2h-2)$ points R_b are the zeroes of the holomorphic form ω_z. For $n = 1$ the construction (5.8) has to be modified to account for the presence of the c zero mode which is just a constant:

$$\begin{aligned} &\phi_I^{(2)}(z) = \omega_z \omega_I(z), \quad I = 1, \ldots, h, \\ &\phi_{h+a}^{(2)}(z) = \omega_z \partial_z \ln\left(\frac{E(z, R_{2h-2})}{E(z, R_a)}\right), \quad a = 1, \ldots, 2h-3. \end{aligned} \tag{5.9}$$

Here ω_I are a basis of abelian differentials, and $\partial_z (\ln E(z, R_{2h-2})/E(z, R_a))$ is just the abelian differential of the third kind with poles at R_{2h-2} and R_a. Finally for $n = 1/2$ we consider separately the cases of even and odd spin structures. Generically for even spin structures, the b, c fields of rank 1/2 have no zero mode, and the $(2h-2)$ holomorphic 3/2 forms can be constructed as in (5.8),

$$\phi_a^{(3/2)}(z) = \omega_z S^{1/2}(z, R_a)(\partial\omega(R_a))^{-1/4}, \quad a = 1, \ldots, 2h-2 \tag{5.10}$$

with $S^{1/2}(z, w)$ the Szego kernel. On the other hand for generic odd spin structure (α', α'') the b, c fields will have exactly one zero mode h_α given by

$$h_\alpha(z) = \left(\sum_1^h \partial_I \theta[\alpha](0, \Omega)\omega_I(z)\right)^{1/2},$$

and the $(2h-2)$ holomorphic 3/2 forms will be obtained by a construction more akin to (5.9),

$$\begin{aligned} &\phi_a^{3/2}(z) = \omega_z S^{1/2}(z, R_a; R, R_{2h-2})(\partial\omega(R_\alpha))^{-1/4}, \quad a = 1, \ldots, 2h-3, \\ &\phi_{2h-2}^{3/2}(z) = \omega_z h_\alpha(z), \end{aligned} \tag{5.11}$$

and $S^{1/2}(z, w; R, R')$ is the meromorphic 1/2-form in z with poles at w and R', and a zero at R given by

$$S^{1/2}(z,w;R,R') = \frac{\langle b(z)c(w)b(R)c(R')\rangle}{\langle b(R)c(R')\rangle}$$
$$= \frac{1}{E(z,w)}\frac{\theta[\alpha](z+R-w-R')}{\theta[\alpha](R-R')}\frac{E(z,R)E(w,R')}{E(w,R)E(z,R')}. \tag{5.12}$$

Next, bosonization formulas give determinants of Laplacians in terms of arbitrary insertions at a fixed number of points z_i and w_j. As in the case of zero modes, it is useful to choose these points inductively. We shall make the following choices:

tensor rank	*Insertion for n*	*Insertion for n + 1*
$n > 1$	$\prod_1^{r_n} b(z_i)$	$\prod_1^{r_n} b(z_i) \prod_1^{2h-2} b(R_a)$
$n = 1$	$\prod_1^{h} b(z_i)c(w)$	$\prod_1^{h} b(z_i) \prod_1^{2h-3} b(R_a)$
$n = 1/2$ even spin str.	No insertion	$\prod_1^{2h-2} b(R_a)$
$n = 1/2$ odd spin str.	$b(R)c(R_{2h-2})$	$\prod_1^{2h-3} b(R_a)b(R)$ (5.13)

With these choices it is readily seen that

$$\det \phi_a^{n+1}(z_b) = \det \phi_a^n(z_c) \prod_1^{r_n} \omega(z_i) \prod_1^{2h-2} (\partial\omega(R_a))^{(n+1)/2},$$
$$n > 1 \quad or \quad n = 1/2, \text{ even spin structure}$$
$$= \det \phi_a^n(z_i) \prod_1^{h} \omega(z_i) \prod_1^{2h-3} \partial\omega(R_a), \quad n = 1;$$
$$= \omega(R)h_\alpha(R)\left(\prod_1^{2h-3} \partial\omega(R_a)\right)^{3/4}, \quad n = 1/2 \text{ odd spin.} \tag{5.14}$$

Finally we need some basic relations between ω_z, the Arakelov metric, and the Arakelov Green's function which follow readily from the defining equation $g_{z\bar{z}}^A = |\omega_z|^2 \exp(-2\sigma(z))$. At general points z_j we have

$$|\omega(z_j)|^2 = \exp\left(-\prod_1^{2h-2} G^A(z_j,R_a) - c\right)g_{z\bar{z}}^A, \tag{5.15}$$

while at the zeroes R_a of ω_z

$$|\partial\omega(R_a)|^2 = \exp\left(-\sum_{b\neq a} G^A(R_b,R_a) - c\right)(g_{R_a\bar{R}_a}^A)^2. \tag{5.16}$$

Recall that we have chosen the Arakelov metric for convenience, and the constant c as given by (4.3) gives the normalization of ω_z.

We can now apply the bosonization formulas (5.1) with the above choices of zero modes and insertion points. Since the R_a's are the zeros of an abelian differential, they satisfy $I(R_1+\cdots+R_{2h-2})=2\Delta$, and all theta factors will cancel in ratios W_{n+1}/W_n. Using (5.15) and (5.16) we can also eliminate explicitly all dependence on the insertion points z_j, w, R, and just leave dependence on insertion points $R_a, a=1,\dots,2h-2$. The net results are

$$\frac{W_{n+1}}{W_n}(g^A_{z\bar z})=\exp\left(\frac{n}{2}\sum_{a,b=1}^{2h-2}G^A(R_a,R_b)\right)\exp(3n(h-1)c),\quad n\geqq 1,\quad n=1/2,\text{ even spin}$$
$$=\exp\left(\frac{1}{4}\sum_{a,b=1}^{2h-2}G^A(R_a,R_b)\right)\frac{|h_\alpha(R_{2h-2})|^2}{\langle h_\alpha|h_\alpha\rangle}\exp(3(h-1)c/2),$$
$$n=1/2,\text{ odd spin.}\tag{5.17}$$

These are relations between determinants of different weights for Arakelov metrics. They can be rewritten more explicitly as

$$\frac{\det'\Delta^-_{n+1}}{\det'\Delta^-_n}(g^A_{z\bar z})=\exp\left(-n\sum_{a,b=1}^{2h-2}G^A(R_a,R_b)\right)\prod_1^{2h-2}|\sqrt{2}g^A_{R_a\bar R_a}/\partial\omega(R_a)|^{3n}$$
$$\cdot\frac{\det\langle\phi^{n+1}_a|\phi^{n+1}_b\rangle_{\text{Arakelov}}}{\det\langle\phi^n_a|\phi^n_b\rangle_{\text{Arakelov}}},\quad n\geqq 1\quad\text{or}\quad n=1/2,\text{ even spin.}$$

For $n=1/2$, odd spin, the relation becomes

$$\frac{\det'\Delta^-_{3/2}}{\det'\Delta^-_{1/2}}(g^A_{z\bar z})=\exp\left(-\frac{1}{2}\sum_{a,b=1}^{2h-2}G^A(R_a,R_b)\right)\prod_1^{2h-2}\left|\frac{\sqrt{2}g^A_{R_a\bar R_a}}{\partial\omega(R_a)}\right|^{3/2}$$
$$\cdot\frac{|h_\alpha(R_{2h-2})|^2}{\det\langle\phi^{3/2}_a|\phi^{3/2}_b\rangle_{\text{Arakelov}}},\quad n=1/2.\tag{5.18}$$

To pass to Mandelstam metrics we note that the defining relation (4.14) and Eqs. (5.15), (5.16) imply

$$\frac{W_{n+1}}{W_n}(|\omega_z|^2)=\frac{W_{n+1}}{W_n}(g^A_{z\bar z})\exp\left(-n\sum_{a,b=1}^{2h-2}G^A(R_a,R_b)\right)\left(\prod_1^{2h-2}g^A_{R_a\bar R_a}\right)^n\exp(-4n(h-1)c).\tag{5.19}$$

Combining (5.17), (5.19) gives the desired result,

$$\frac{W_{n+1}}{W_n}(|\omega_z|^2)=\left|\prod_{a=1}^{2h-2}\partial\omega(R_a)\right|^n,\quad n\geqq 1\quad\text{or}\quad n=1/2,\text{ even spin structure,}$$
$$\frac{W_{3/2}}{W_{1/2}}(|\omega_z|^2)=\left|\prod_{a=1}^{2h-2}\partial\omega(R_a)\right|^{1/2}\frac{|h_\alpha(R_{2h-2})|^2}{\langle h_\alpha|h_\alpha\rangle_{\text{Arakelov}}},\qquad\text{odd spin structure.}\tag{5.20}$$

6. Surfaces with Punctures

The above analysis can be extended to the case of surfaces with punctures $M^*=M-\{z_A\}$, i.e., metrics $|\omega_z|^2$, where ω_z has poles at points $z_A, A=1,\dots,N$. As shown by Sonoda, one can still define the ratio $W_n(|\omega_z|^2)$ by regularizing the

Liouville action, the only difference between zeros and poles of ω_z reflecting itself in the relative signs of the Dirac point masses in the curvature. The new subtlety lies rather in the fact that for surfaces with punctures z_A the zero modes should also include meromorphic forms with simple poles at z_A. This difficulty is taken care of by noting that given a coordinate system around each z_A, a basis of meromorphic forms m_A can be singled out by requiring that m_A be orthogonal to holomorphic forms and have singularity $\delta_{AB}/z - z_B$ near z_B. The regularized $W_n(|\omega_z|^2)$ of (2.11) for a choice of holomorphic zero modes ϕ_a^n can then be viewed as the regularized $W_n^*(|\omega_z|^2)$ for the basis (ϕ_a^n, m_A) of holomorphic n-forms on M^*. Altogether $W_n^*(|\omega_z|^2)$ is then a tensor of type $(c_n/12 - n + 1, c_n/12 - n + 1)$ at each z_A. In particular the ratio $W_{n+1}^*/W_n^*(|\omega_z|^2)$ is a scalar in z_A. Its dependence on the scale of $|\omega_z|^2$ dictate that it be proportional to $\left(\prod_A \alpha_A\right)^{-2n}$, where α_A is the residue of ω_z at z_A, which is a scalar. This means that the change of weight formula for Mandelstam metrics with both zeroes and poles should read

$$\int D(b\bar{b}c\bar{c}) \exp(-I_{n+1}) \left| \prod_1^{M+Y_n} b(z_i) \prod_1^M c(w_j) \right|^2$$
$$= \int D(b\bar{b}c\bar{c}) \exp(-I_{n+1}) \left| \prod_1^{M+Y_n} b(z_i) \prod_1^{2h-2+N} b(R_a) \prod_1^M c(w_j) \sum_1^N c(z_A) \right|^2$$
$$\cdot \left| \frac{\prod_1^M \omega(w_j)}{\prod_1^{M+Y_n} \omega(z_i)} \right|^2 \prod_1^{2h-2+N} |\partial\omega(R_a)|^{-2n-1} \left(\prod_1^N \alpha_N \right)^{2n}. \tag{6.1}$$

Careful duplication of the arguments for zeroes of ω_z shows that indeed (6.1) holds with precisely this additional factor on the right-hand side. Thus

$$\frac{W_{n+1}}{W_n}(|\omega_z|^2) = \left| \prod_{a=1}^{2h-2+N} \partial\omega(R_a) \right|^n \left| \prod_{A=1}^N \alpha_A \right|^{-2n}, \quad n \geqq 1\, n = 1/2, \text{ even spin structure},$$
$$\frac{W_{3/2}}{W_{1/2}}(|\omega_z|^2) = \left| \prod_{a=1}^{2h-2+N} \partial\omega(R_a) \right|^{1/2} \frac{|h_\alpha(R_{2h-2})|^2}{\langle h_\alpha | h_\alpha \rangle_{\text{Arakelov}}} \prod_{A=1}^N |\alpha_A|^{-1}, \text{ odd spin}. \tag{6.2}$$

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