On The Structure of Unitary Conformal Field Theory.
I. Existence of Conformal Blocks

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Abstract. We study the general mathematical structure of unitary rational conformal field theories in two dimensions, starting from the Euclidean Green functions of the scaling fields. We show that, under certain assumptions, the scaling fields of such theories can be written as sums of products of chiral fields. The chiral fields satisfy an algebra whose structure constants are the matrix elements of Yang–Baxter- or braid-matrices whose properties we analyze. The upshot of our analysis is that two-dimensional conformal field theories of the type considered in this paper appear to be constructible from the representation theory of a pair of chiral algebras.

1. Introduction

In this paper we study the general structure of unitary rational conformal field theories in two dimensions. The starting point of our analysis is motivated by concepts of two-dimensional statistical mechanics: The basic properties of a statistical system are coded into its thermodynamic and correlation functions. The correlation functions are expectations of products of local order- and disorder variables in a Gibbs equilibrium state. If the system is at a critical point its correlation functions tend to exhibit asymptotic Euclidean- and scale invariance, as one learns from the study of exactly solved models and the renormalization group. Scaling limits of the correlation functions then exist. They turn out to be the Euclidean Green functions of some Euclidean field theory. If the underlying statistical system has a self-adjoint transfer matrix, the scaling limits of its correlation functions satisfy reflection positivity. A variant of Osterwalder–Schrader reconstruction then permits us to associate with the sequence of scaling limits of correlation functions of such a system a unitary relativistic quantum field theory. At a critical point the scaling limits of correlation functions of scaling operators are Möbius-invariant. This invariance property, combined with reflection

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positivity, permits us to associate a quantum field theory with every parametrized disk on the Riemann sphere. Our construction proves, in particular, that standard Osterwalder–Schrader quantization and radial quantization provide equivalent descriptions of the quantum field theory. See Sect. 2.

Points in the two-dimensional Euclidean domain are conveniently parametrized by complex numbers $z = t + ix$, $z^* = t - ix$, where t is the time- and x the space component of a point $\underline{x} = (t, x) \in \mathbb{E}^2$. The variables z and z^* are the Euclidean versions of the standard light cone variables.

One of the objectives of our paper is to analytically continue the Euclidean Green functions of two-dimensional conformal field theory in the light cone variables $z, \bar{z}$ to a maximal domain of holomorphy. A point $(z, \bar{z})$ belongs to the Euclidean domain if $\bar{z} = z^* = $ complex conjugate of z. This process of analytic continuation of the Green functions is started in Sect. 2.

Let L_0 and $\bar{L}_0$ be the generators of the transformations $(z, \bar{z}) \mapsto (e^\theta z, \bar{z})$, $(z, \bar{z}) \mapsto (z, e^\theta \bar{z})$, respectively. We show that, in a unitary conformal field theory, L_0 and $\bar{L}_0$ are positive operators on the Hilbert space of radial quantization, under natural regularity assumptions on the Euclidean Green functions.

In Sect. 3, we consider unitary conformal field theories with a symmetric, conserved energy-momentum tensor of dimension 2. We recall the Lüscher–Mack theorem which shows that, in such theories, the energy-momentum tensor has only two independent components $T(z)$ (independent of $\bar{z}$) and $\bar{T}(\bar{z})$ (independent of z) which generate two commuting, unitary representations of Virasoro algebras, Vir and $\overline{\text{Vir}}$, on the Hilbert space, $\mathcal{H}$, of radial quantization. We show that these representations are completely reducible into direct sums (or -integrals) of irreducible, unitary highest-weight representations.

We then proceed to study the notion of chiral algebras: Given some unitary conformal field theory, we consider all those scaling fields which are independent of $\bar{z}$ (independent of z). Among these fields are of course $T(z)$ ($\bar{T}(\bar{z})$, respectively). They generate algebras $\mathcal{A}$, ($\bar{\mathcal{A}}$, respectively) which we call chiral algebras. We define the symmetry algebra, $\mathfrak{A}$, of the conformal field theory to consist of all local operators in $\mathcal{A} \otimes \bar{\mathcal{A}}$.

An important aspect of the notion of rational conformal field theory, as used in this paper, is that the Hilbert space, $\mathcal{H}$, of the theory splits into *finitely many* irreducible subspaces for $\mathfrak{A}$. This assumption is made more precise in Sect. 3. There we also formulate the Ward identities which describe how the symmetry algebra $\mathfrak{A}$ acts on the scaling fields, $\phi_\alpha(z, \bar{z})$, of the theory. The main result of Sect. 3 is the existence of chiral intertwiner fields: We show that under natural assumptions on the structure of the symmetry algebra $\mathfrak{A}$ and the algebra of scaling fields $\{\phi_\alpha(z, \bar{z})\}$ of the theory, every field $\phi_\alpha(z, \bar{z})$ can be written as a sum of products of chiral intertwiner fields $\varphi_a(z)$ (independent of $\bar{z}$) and $\varphi_{\bar{a}}(\bar{z})$ (independent of z).

In Sect. 4, we study the vacuum expectation values of products of chiral intertwiner fields $\varphi_a(z)$ (or of products of fields $\varphi_{\bar{a}}(\bar{z})$) which we call conformal blocks. We then determine the envelope of holomorphy of the conformal blocks. For an n-point conformal block this is the domain

$$M_n = \{(z_1, \ldots, z_n): z_i \neq z_j, \text{ for } i \neq j\},$$

whose fundamental group is the pure braid group, P_n. The conformal blocks are multi-valued analytic functions on M_n. The different branches of these functions are connected to one another by matrix representations of the braid group B_n generated by a Yang–Baxter (or braid-) matrix $R = (R_{ab}^{cd})$. This matrix can be viewed as a matrix of structure constants for the algebra generated by the chiral intertwiner fields: A product, $\varphi_a(z)\varphi_b(w)$, of two such fields is well defined, a priori, only if (for example) $\mathrm{Re}\, z < \mathrm{Re}\, w$. But it has a multivalued analytic continuation to the space M_2. If $\varphi_a(w)\varphi_b(z)$ is defined by analytic continuation of $\varphi_a(z')\varphi_b(w')$ in z' from z to w and in w' from w to z along paths shown in the following figure:
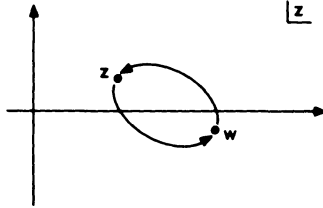
then

$$\varphi_a(w)\varphi_b(z) = R_{ab}^{cd}\varphi_c(z)\varphi_d(w). \qquad (1)$$

This equation captures the basic structure of the algebra of chiral intertwiner fields. In Sect. 4, we specify a class of unitary conformal field theories which we call *rational* theories for which Eq. (1) can be proven. We also derive some of the simplest properties of those R-matrices which can appear as structure constants in Eq. (1). A more systematic study of the properties of R will appear in a separate paper.

   As a consequence of our analysis we are able to determine the envelopes of holomorphy of the Euclidean Green functions of rational, unitary conformal field theories and to calculate their monodromy in terms of the braid-matrices R and $\bar{R}$, where $\bar{R}$ is the matrix of structure constants for the algebra generated by the fields $\varphi_{\bar{a}}(\bar{w})$, i.e.

$$\varphi_{\bar{a}}(\bar{w})\varphi_{\bar{b}}(\bar{z}) = \bar{R}_{\bar{a}\bar{b}}^{\bar{c}\bar{d}}\varphi_{\bar{c}}(\bar{z})\varphi_{\bar{d}}(\bar{w}) \qquad (2)$$

   In the final section (Sect. 5) of this paper, we extract the basic mathematical structure of rational, unitary conformal field theory from the results in Sects. 2, 3 and 4. We show that in a sense to be made more precise in future work on the subject, two-dimensional conformal field theory can be viewed as the representation theory of a pair of abstract chiral algebras $\mathscr{A}, \bar{\mathscr{A}}$. Examples of such algebras are the Virasoro algebra, current algebra, algebras of higher-spin currents, or of parafermions. The chiral intertwiner fields are then viewed as "tensor operators" for a chiral algebra $\mathscr{A}$. Products of such fields are sections of bundles whose base spaces are the spaces M_n and whose fibres consist of tensor operators for $\mathscr{A}$ which intertwine different representations of $\mathscr{A}$. These bundles carry flat connections whose holonomy generates a representation of the braid group B_n. Under suitable hypotheses (which will require further study) these representations are generated

by Yang–Baxter matrices, R, which appear as structure constants in a quadratic relation between chiral intertwiner fields, φ_a, of the form (1). Finally, we show how one can reconstruct local fields $\phi_a(z, \bar{z})$ out of the chiral intertwiner fields $\varphi_a(z)$, $\varphi_{\bar{a}}(\bar{z})$ associated with the algebras $\mathscr{A}$, $\bar{\mathscr{A}}$.

In separate publications the structure described in Sect. 5 will be investigated in more detail and an application to minimal models will be given.

2. Quantum Field Theory on the Riemann Sphere

2.1. In this section we review some fundamental properties of two-dimensional, unitary conformal field theory in a mathematically precise form. In view of the basic significance of conformal field theory for the theory of two-dimensional critical phenomena and string theory, the Euclidean formulation of conformal field theory [1] is an appropriate formalism. It is based on work in [2, 3] which develops the Euclidean description of relativistic quantum field theory. Our analysis will show that, given Euclidean Green functions of a two-dimensional conformal field theory satisfying reflection positivity [2], one can associate with each parametrized disk on the Riemann sphere a conformal quantum theory, or "quantization." Different quantizations are intertwined by isometries which form a representation of the Möbius group, PSL $(2, \mathbb{C})$. Special cases are Osterwalder–Schrader quantization [2, 3] corresponding to the right half plane $\{z = t + ix : t > 0\}$, and radial quantization [1] corresponding to the unit disk $\{z : |z| < 1\}$.

In two dimensions, quantum field theory has peculiar features intimately connected with the fact that the complement of the closure of the light cone is disconnected: The statistics of fields is not limited to Bose- or Fermi statistics—as it was in higher·dimensions. This is related to the property of Euclidean Green functions to be, in general, multi-valued functions on the space

$$M_n = \{\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n) : \underline{x}_i \in \mathbb{E}^2, \ \underline{x}_i \neq \underline{x}_j, \ \text{for } i \neq j\} \tag{2.1}$$

corresponding to single-valued functions on the universal cover, $\tilde{M}_n$, of M_n. [The fundamental group of M_n is the pure braid group on n strings [4].] Different branches of a Green function are connected to each other by a matrix representation of the braid group on n strings, [4]. In statistical mechanics, multi-valued Green functions appear as order-disorder and parafermion correlation functions. These features are discussed in some detailed in [5]. In the following, we shall assume that Euclidean Green functions are single-valued functions on M_n, symmetric under permutations of their arguments. This will merely simplify text and notations. The general case will be discussed elsewhere; see also [5].

It will be convenient to write points, $\underline{x} = (t, x) \in \mathbb{E}^2$, as complex numbers, $z = t + ix$, $z^* = t - ix = $ complex conjugate of z. Here x is the space component of $\underline{x}$ and t its (imaginary-) time component. Both parametrizations will be used.

Next, we describe some basic properties of Euclidean Green functions of unitary conformal field theory. These properties are variants of the Osterwalder–Schrader axioms [2]. In order to describe them, we require some notation and definitions: Let

$$M_n^+ = \{\underline{z} \in M_n : t_i = \operatorname{Re} z_i > 0, \ \text{for } i = 1, \ldots, n\}. \tag{2.2}$$

We set

$$\mathscr{S}_0^+ = \mathbb{C},$$

and

$$\mathscr{S}_n^+ = \{f \in \mathscr{S}(\mathbb{E}^{2n}): \operatorname{supp} f \subset M_n^+\}. \tag{2.3}$$

Here $\mathscr{S}(\mathbb{E}^{2n})$ is the Schwartz test function space over $\mathbb{E}^{2n}$. We also introduce some transformations of $\mathbb{E}^2$:

$$\theta(t, x) = (-t, x), \quad \text{i.e.} \quad \theta z = -z^*, \tag{2.4}$$

$$\text{(time reflection)}$$

$$\pi(t, x) = (t, -x), \quad \text{i.e.} \quad \pi z = z^*. \tag{2.5}$$

$$\text{(space reflection)}$$

Möbius transformations are denoted by

$$w: z \mapsto w(z) = \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}), \tag{2.6}$$

with $z^* \mapsto w(z)^*$. Special cases are

$$t_a: z \mapsto z + a, \quad a \in \mathbb{C}, \tag{2.7}$$

$$\text{(space-time translations)}$$

$$r_\varphi: z \mapsto e^{i\varphi} z, \quad 0 \leq \varphi < 2\pi, \tag{2.8}$$

$$\text{(rotations = Euclidean boosts)}$$

and

$$.d_\tau: z \mapsto e^{-\tau} z. \tag{2.9}$$

$$\text{(dilatations)}$$

We shall study theories given in terms of a sequence, $\{G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n)\}_{n=0}^\infty$, of Euclidean Green functions of scaling fields with the following properties which are motivated by the analysis of models, like the two-dimensional Ising-, Potts- or six-vertex models, at a critical point.

(P1) $G(\varnothing) = 1$; $G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n)$ is a well defined, continuous, polynomially bounded function on M_n, for arbitrary $\alpha_1, \ldots, \alpha_n$ and all $n = 1, 2, 3, \ldots$. [The subscripts $\alpha_1, \ldots, \alpha_n$ label different scaling fields and range over a finite or countably infinite index set A_0.] It is also assumed that

$$G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n) = G_{\alpha_{\pi(1)} \cdots \alpha_{\pi(n)}}(\underline{x}_{\pi(1)}, \ldots, \underline{x}_{\pi(n)}), \tag{2.10}$$

for arbitrary permutations, π, of n elements.

In statistical mechanics, (P1) expresses the property that the scaling limits of order- or disorder correlation functions of a statistical system at a critical point exist and are well defined, symmetric functions on M_n. [Mixed order-disorder correlation functions are discussed in [5].]

(P2) There are real numbers $h(\alpha)$ and $\bar{h}(\alpha), \alpha \in A_0$, called conformal weights, such

that, under a Möbius transformation w, see (2.6),

$$G_{\alpha_1 \cdots \alpha_n}(z_1, z_1^*, \ldots, z_n, z_n^*) = \prod_i \left(\frac{dw}{dz}\right)^{h_i}(z_i) \left(\frac{dw}{dz}\right)^{\bar{h}_i}(z_i)^* G_{\alpha_1 \cdots \alpha_n}(w_1, w_1^*, \ldots, w_n, w_n^*),$$

(2.11)

where $w_i = w(z_i)$, $h_i = h(\alpha_i)$ and $\bar{h}_i = \bar{h}(\alpha_i)$.

Assumption (P2) expresses the property of scaling limits or correlation functions of critical statistical systems to be Möbius-invariant. Actually, full Möbius invariance pre-supposes that one works with order parameters which, in the scaling limit, transform tensorially under Möbius transformations. This is more than what is needed in many parts of our analysis. Often it would be enough to assume

(P2^a) Euclidean Green functions are translation-invariant

(P2^b) Rotation invariance:

$$G_{\alpha_1 \cdots \alpha_n}(e^{i\varphi}z_1, e^{-i\varphi}z_1^*, \ldots, e^{i\varphi}z_n, e^{-i\varphi}z_n^*) = e^{-i\varphi(\sum s_i)} G_{\alpha_1 \cdots \alpha_n}(z_1, z_1^*, \ldots, z_n, z_n^*),$$ (2.12)

where $s_i \equiv s(\alpha_i) = h(\alpha_i) - \bar{h}(\alpha_i)$, $i = 1, \ldots, n$; $s(\alpha)$ is called "spin."

(P2^c) Dilatation invariance:

$$G_{\alpha_1 \cdots \alpha_n}(e^{-\tau}\underline{x}_1, \ldots, e^{-\tau}\underline{x}_n) = e^{\tau(\sum d_i)} G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n),$$ (2.13)

where $d_i \equiv d(\alpha_i) = h(\alpha_i) + \bar{h}(\alpha_i)$ is the so-called "scaling dimension," $i = 1, \ldots, n$.

Next, we formulate a property, *reflection positivity* [2,3], which is somewhat unnatural from the point of view of statistical mechanics, but plays an important role in our analysis of conformal field theory; see also [1]. By $\mathscr{S}^+$ we denote the space of finite sequences of test functions,

$$\{f_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n) \in \mathscr{S}_n^+, \alpha_i \in A_0, i = 1, \ldots, n\}_{n=0,1,2,\ldots}.$$ (2.14)

["Finite" means that $f_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n) = 0$, except for finitely many choices of $(\alpha_1, \ldots, \alpha_n)$ and finitely many n.]

(P3) We assume that there is an involution, $*: A_0 \mapsto A_0, \alpha \mapsto \alpha^*$, such that

$$G_{\alpha_n^* \cdots \alpha_1^*}(\theta \underline{x}_n, \ldots, \theta \underline{x}_1) = G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n)^*,$$ (2.15)

and, for arbitrary sequences $\underline{f} \in \mathscr{S}^+$,

$$\sum_{n,m} \sum_{\alpha,\beta} \int G_{\alpha_n^* \cdots \alpha_1^* \beta_1 \cdots \beta_m}(\theta \underline{x}_n, \ldots, \theta \underline{x}_1, \underline{y}_1, \ldots, \underline{y}_m)$$
$$\cdot f_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n)^* f_{\beta_1 \cdots \beta_m}(\underline{y}_1, \ldots, \underline{y}_m) d^{2n}x \, d^{2m}y \geqq 0.$$ (2.16)

Reflection positivity (P3) can be derived from the selfadjointness of the transfer matrix of an underlying statistical system. This is a frequent, but not a fundamental property of lattice systems. [It fails e.g. in the theory of selfavoiding walks.] But without assumption (P3), it is more difficult to undertake a general analysis of conformal field theory; but see [6].

2.2. Next, we review some important consequences of assumptions (P1)–(P3), (i.e. we sketch Osterwalder–Schrader reconstruction, [2,3]).

Assumption (P3) permits us to define an inner product, $\langle \cdot, \cdot \rangle_{\gamma_+}$, on $\mathscr{S}^+$: For

$\underline{f}$ and $\underline{g}$ in $\mathscr{S}^+$, we define

$$\langle \underline{f},\underline{g} \rangle_{\gamma_+} \equiv \sum_{n,m} \sum_{\underline{\alpha},\underline{\beta}} \int G_{\alpha_n^* \cdots \alpha_1^* \beta_1 \cdots \beta_m}(\theta \underline{x}_n, \ldots, \theta \underline{x}_1, \underline{y}_1, \ldots, \underline{y}_m)$$

$$\cdot f_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n)^* g_{\beta_1 \cdots \beta_m}(\underline{y}_1, \ldots, \underline{y}_m) d^{2n}x \, d^{2m}y. \qquad (2.17)$$

Here γ_+ denotes the right half-plane $\{z : \mathrm{Re}\, z > 0\}$. Let $\mathscr{N}^+$ be the kernel of $\langle \cdot, \cdot \rangle_{\gamma_+}$ in $\mathscr{S}^+$. An equivalence class of a sequence $\underline{f} \in \mathscr{S}^+$, mod, $\mathscr{N}^+$, is denoted by $i(\underline{f}) \equiv i_{\gamma_+}(\underline{f})$. Then

$$\mathscr{H}_{\gamma_+} = \{i_{\gamma_+}(\underline{f}) : \underline{f} \in \mathscr{S}^+\}^-, \qquad (2.18)$$

where the closure is taken in the norm induced by $\langle \cdot, \cdot \rangle_{\gamma_+}$, is a separable Hilbert space. We let $\Omega \equiv \Omega_{\gamma_+}$ denote the image under i_{γ_+} of the sequence $\underline{f}$ with $f(\varnothing) = 1$ and $f_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n) = 0$, for all $n \geq 1$; Ω is called *vacuum*.

Assuming (P1), (P2^a) and (P3), $\mathscr{H}_{\gamma_+}$ can be shown to carry a representation of space-time translations, constructed as follows: Given $\underline{f} \in \mathscr{S}^+$, let $^a\underline{f}$ be given by the sequence

$$\{f_{\alpha_1 \cdots \alpha_n}(\underline{x}_1 - \underline{a}, \ldots, \underline{x}_n - \underline{a}) : \underline{a} = (a^0, a) \in \mathbb{E}^2\}_{n=0,1,2,\ldots}. \qquad (2.19)$$

If $a^0 \geq 0$ then $^a\underline{f} \in \mathscr{S}^+$, for $\underline{f} \in \mathscr{S}^+$, and we define

$$e^{-a^0 H - i a P} i(\underline{f}) = i(^a\underline{f}), \quad a^0 \geq 0. \qquad (2.20)$$

A standard result of Osterwalder–Schrader reconstruction says that (2.20) defines a semigroup on $\mathscr{H}_{\gamma_+}$ generated by selfadjoint operators H and P, and

$$H \geq 0; \qquad (2.21)$$

see [2]. If, in addition, (P2^b) holds we may define an operator M by setting

$$e^{\varphi M} i(\underline{f}) = i(^\varphi\underline{f}), \qquad (2.22)$$

where $^\varphi\underline{f}$ is given by the sequence

$$\{e^{i\varphi(\sum s_i)} f_{\alpha_1 \cdots \alpha_n}(e^{-i\varphi} z_1, e^{i\varphi} z_1^*, \ldots, e^{-i\varphi} z_n, e^{i\varphi} z_n^*)\}_{n=0,1,2,\ldots}.$$

Clearly, for $\underline{f} \in \mathscr{S}^+$, $^\varphi\underline{f}$ is contained in $\mathscr{S}^+$, provided $|\varphi|$ is small enough. A theorem in [3] then says that (2.22) defines a selfadjoint operator M. It is the generator of boosts. It is easy to conclude now that

$$H \pm P \geq 0, \qquad (2.23)$$

which is the *relativistic spectrum condition*, [2, 3]. Assuming also property (P2^c), the equation

$$e^{i\tau D} i(\underline{f}) = i(^\tau\underline{f}), \qquad (2.24)$$

where $\underline{f} \mapsto {}^\tau\underline{f}$ represents dilatations on $\mathscr{S}^+$, defines a selfadjoint operator D generating a unitary representation of dilatations on $\mathscr{H}_{\gamma_+}$.

By construction, Ω is invariant under the operators $e^{-a^0 H - i a P}, e^{\varphi M}$ and $e^{i\tau D}$.[1]

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[1] If (P2^c) holds Ω is the *unique* invariant state in $\mathscr{H}_{\gamma_+}$

Since $\{e^{-a^0H-iaP}:a^0\geqq 0\}$ is a contraction semigroup on $\mathscr{H}_{\gamma+}$, and since $i(\underline{\mathscr{S}}^+)$ is dense in $\mathscr{H}_{\gamma+}$, by construction, the subspace $i(^a\underline{\mathscr{S}}^+)$ is *dense* in $\mathscr{H}_{\gamma+}$, for all $a^0\geqq 0$. Here

$$^a\underline{\mathscr{S}}^+ = \{\underline{f}\in\underline{\mathscr{S}}^+ : {}^{-a}\underline{f}\in\underline{\mathscr{S}}^+\},$$

and, by (2.20),

$$i(^a\underline{\mathscr{S}}^+) = e^{-a^0H-iaP}i(\underline{\mathscr{S}}^+). \tag{2.25}$$

Let f be a test function with

$$\text{supp}\, f \subset \{\underline{x}:0 < x^0 < a^0\},$$

for some $a^0 < \infty$. Let $\underline{f}\in {}^a\underline{\mathscr{S}}^+$, and define $f_\alpha \times \underline{f}\in\underline{\mathscr{S}}^+$ to be the sequence

$$\{f(\underline{x}_1)\delta_{\alpha_1\alpha}f_{\alpha_2\cdots\alpha_{n+1}}(\underline{x}_2,\ldots,\underline{x}_{n+1})\}_{n=0,1,2,\ldots}. \tag{2.26}$$

We define an operator $\phi_\alpha(f)$ by the equation

$$\phi_\alpha(f)i(\underline{f}) = i(f_\alpha \times \underline{f}), \quad \underline{f}\in {}^a\underline{\mathscr{S}}^+. \tag{2.27}$$

Since $i(^a\underline{\mathscr{S}}^+)$ is dense in $\mathscr{H}_{\gamma+}$, $\phi_\alpha(f)$ is densely defined. Since the Green functions $G_{\alpha_1\cdots\alpha_n}(\underline{x}_1,\ldots,\underline{x}_n)$ are continuous on M_n, we may let f approach a δ-function at some point $\underline{x}=(x^0,x)$, $x^0>0$, and obtain a densely defined operator

$$\phi_\alpha(x) = \phi_\alpha(f=\delta_x). \tag{2.28}$$

This is the *Euclidean field operator*.

We define

$$K = \tfrac{1}{2}(H+P), \quad \bar{K} = \tfrac{1}{2}(H-P). \tag{2.29}$$

By (2.19) and (2.20), $e^{-\zeta K}$, $\text{Re}\,\zeta\geqq 0$, represents the transformation $z\mapsto z+\zeta, z^*\mapsto z^*$ on $\mathscr{H}_{\gamma+}$, and by (2.23) the operator norm of $e^{-\zeta K}$ is bounded by 1, for $\text{Re}\,\zeta\geqq 0$. Similarly, $e^{-\bar{\zeta}\bar{K}}$ represents the transformation $z\mapsto z$, $z^*\mapsto z^*+\bar{\zeta}$ on $\mathscr{H}_{\gamma+}$ and is bounded in norm by 1, for $\text{Re}\,\bar{\zeta}\geqq 0$.

We define some subspaces of $\mathbb{C}^{2n}$:

$$\mathscr{E}_n = \{\underline{z},\bar{\underline{z}}:\bar{z}_i = z_i^*, i=1,\ldots,n\}, \tag{2.30}$$

$$M_n^> = \{\underline{z}:\text{Re}\,z_n > \cdots > \text{Re}\,z_1\},$$

$$M_n^{>,\pi} = \{\underline{z}:\text{Re}\,z_{\pi^{-1}(n)} > \cdots > \text{Re}\,z_{\pi^{-1}(1)}\}, \tag{2.31}$$

where π is a permutation of $\{1,\ldots,n\}$,

$$M_n^{\pi,\pi} = M_n^{>,\pi} \times M_n^{>,\pi}, \tag{2.32}$$

and

$$M_n^{\pi,\pi}(w) = M_n^{>,\pi}(w) \times M_n^{>,\pi}(w^*), \tag{2.33}$$

where

$$M_n^{>,\pi}(w) = \{\underline{z}:(w(z_1),\ldots,w(z_n))\in M_n^{>,\pi}\}, \tag{2.34}$$

with $w:z\mapsto w(z)$ a Möbius transformation.

Proposition 2.1. *Let* $\{G_{\alpha_1\cdots\alpha_n}(\underline{x}_1,\ldots,\underline{x}_n)\}_{n=0}^{\infty}$ *be a sequence of Euclidean Green functions satisfying properties* (P1), (P2^a), (P2^b) *and* (P3). *Then* $G_{\alpha_1\cdots\alpha_n}(z_1,z_1^*,\ldots,z_n,z_n^*)$ *is the restriction of a function*

$$H_{\alpha_1\cdots\alpha_n}(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n)$$

holomorphic in $(z_1,\ldots,z_n)$ *and* $(\bar{z}_1,\ldots,\bar{z}_n)$ *on the domain*

$$\bigcup_{\pi\in S_n}\ \bigcup_{\substack{w:\text{Euclidean}\\ \text{motion}}}M_n^{\pi,\pi}(w) \tag{2.35}$$

to the Euclidean domain $\mathscr{E}_n$. *If the Green functions are Möbius invariant, see* (2.11), *then* w *in* (2.35) *can be an arbitrary Möbius transformation.*

Sketch of Proof. A complete proof of Proposition 2.1 can easily be inferred from [2]. Here we sketch the heuristic ideas on which the proof is based. From (2.27), (2.28) and translation invariance, (P2^a), we conclude that if $\text{Re}\,z_n > \cdots > \text{Re}\,z_1$,

$$G_{\alpha_1\cdots\alpha_n}(z_1,z_1^*,\ldots,z_n,z_n^*) = \langle\Omega, \phi_{\alpha_1}(z_1,z_1^*)\cdots\phi_{\alpha_n}(z_n,z_n^*)\Omega\rangle. \tag{2.36}$$

From (2.19), (2.20), (2.27), (2.28) and (2.29), i.e. $K \geqq 0$, $\bar{K} \geqq 0$, it follows that

$$\phi_\alpha(z+\varepsilon,\bar{z}+\varepsilon) = e^{-zK-\bar{z}\bar{K}}\phi_\alpha(\varepsilon)e^{zK+\bar{z}\bar{K}}; \tag{2.37}$$

for $\text{Re}\,z \geqq 0$, $\text{Re}\,\bar{z} \geqq 0$, $\varepsilon > 0$, as an operator equation on the dense domain $i(^a\mathscr{S}^+)$, where $a^0 > \max(\text{Re}\,z, \text{Re}\,\bar{z}) + \varepsilon$. We have set $\phi_\alpha(\varepsilon,\varepsilon) \equiv \phi_\alpha(\varepsilon)$. Hence, using (P2^a),

$$G_{\alpha_1\cdots\alpha_n}(z_1,z_1^*,\ldots,z_n,z_n^*) = \langle\Omega, \phi_{\alpha_1}(\varepsilon)e^{(z_1-z_2)K+(z_1^*-z_2^*)\bar{K}}\phi_{\alpha_2}(\varepsilon)$$
$$\cdot e^{(z_2-z_3)K+(z_2^*-z_3^*)\bar{K}}\cdots\phi_{\alpha_n}(\varepsilon)\Omega\rangle \tag{2.38}$$

if $\text{Re}\,z_n > \cdots > \text{Re}\,z_1$. By (2.23) the norm of $e^{-zK-\bar{z}\bar{K}}$ is bounded by 1, for $\text{Re}\,z \geqq 0$, $\text{Re}\,\bar{z} \geqq 0$. Hence, formally, the right-hand side of (2.38) extends to a function

$$H_{\alpha_1\cdots\alpha_n}(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n) = \langle\Omega, \phi_{\alpha_1}(\varepsilon)e^{(z_1-z_2)K+(\bar{z}_1-\bar{z}_2)\bar{K}}\phi_{\alpha_2}(\varepsilon)$$
$$\cdot e^{-(z_2-z_3)K-(\bar{z}_2-\bar{z}_3)\bar{K}}\cdots\phi_{\alpha_n}(\varepsilon)\Omega\rangle \tag{2.39}$$

holomorphic in $(\underline{z},\underline{\bar{z}})$ on $M_n^> \times M_n^>$. Due to difficulties with domains of definition of the unbounded operators $\phi_{\alpha_i}(\varepsilon)$, the formal arguments leading to (2.39) are untenable. But the considerations in [2] show that (2.39) is correct anyway. [Our formal arguments would be correct if the operators $\phi_\alpha(\varepsilon)e^{-\varepsilon'H}$, $\varepsilon > 0$, $\varepsilon' > 0$, were bounded operators. This would follow from a sharper version of property (P1) sketched in [7].]

In order to complete the proof of Proposition 2.1, we note that, by (P2^a) and (P2^b), the domain of definition of $H_\alpha(\underline{z},\underline{\bar{z}})$ extends to

$$\bigcup_w M_n^{1,1}(w), (M_n^{1,1}(w) = M_n^>(w)\times M_n^>(w^*)),$$

where w is an arbitrary Euclidean motion. But $\left(\bigcup_w M_n^{1,1}(w)\right)\cap\left(\bigcup_{\pi\neq 1}\bigcup_{w'}M_n^{\pi,\pi}(w')\right)$ is non-empty. Since $G_{\alpha_1\cdots\alpha_n}(z_1,z_1^*,\ldots,z_n,z_n^*)$ is symmetric under arbitrary permutations of $\{1,\ldots,n\}$, $H_{\alpha_1\cdots\alpha_n}(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n)$ is symmetric on $\left(\bigcup_w M_n^{1,1}(w)\right)\cap$

$\left(\bigcup_{\pi \neq 1} \bigcup_{w'} M_n^{\pi, \pi}(w') \right)$. By a little geometrical argument (see also Sect. 4) it then follows

that $H_\alpha(\underline{z}, \bar{\underline{z}})$ extends to a function that is holomorphic in $(\underline{z}, \bar{\underline{z}}) \in \bigcup_w \bigcup_\pi M_n^{\pi, \pi}(w)$, where

the w's are Euclidean motions.   ∎

Our goal will be to extend $H_\alpha(\underline{z}, \bar{\underline{z}})$ to a multi-valued holomorphic function on the domain $M_n \times M_n$, corresponding to a single-valued holomorphic function on $\tilde{M}_n \times \tilde{M}_n$. [Recall that $M_n = \{\underline{z} : z_i \neq z_j, \text{ for } i \neq j\}$, and $\tilde{M}_n$ is the universal cover of M_n.] This will require further assumptions on the Green functions $G_\alpha(\underline{z}, \underline{z}^*)$ and a considerable amount of additional work.

Let $\overset{\circ}{M}_n$ be a non-empty open subset of $M_n^+ = \{\underline{z} \in M_n : \text{Re } z_i > 0\}$, $n = 1, 2, 3, \ldots$, and let $\overset{\circ}{\underline{M}} = \{\overset{\circ}{M}_n\}_{n=1}^\infty$ be a sequence of such subsets. We define a subspace $\mathcal{S}^+(\overset{\circ}{\underline{M}})$ of $\mathcal{S}^+$ to consist of all sequences $\underline{f} \in \mathcal{S}^+$ with the property that $\text{supp} f_{\alpha_1 \cdots \alpha_n} \subset \overset{\circ}{M}_n$, for all $n \geq 1$. It is an elementary consequence of Proposition 2.1 (see e.g. [18]) that

$$i\mathcal{S}^+(\overset{\circ}{\underline{M}})) \text{ is dense in } \mathcal{H}_{\gamma_+}, \tag{2.40}$$

for arbitrary sequences $\overset{\circ}{\underline{M}}$ with the properties specified above.

We may now exploit consequences of full Möbius invariance, Eq. (2.11), assumption (P2). [This was irrelevant for Proposition 2.1 which required only translation- and rotation invariance.] Because of (2.11), it is natural to view test functions, $f_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n)$, as the components of a tensor with conformal weights $(1 - h_i, 1 - \bar{h}_i)$, $h_i = h(\alpha_i)$, $\bar{h}_i = \bar{h}(\alpha_i)$. For a Möbius transformation w, we define

$$^w f_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n) = \prod_{i=1}^n \left(\frac{dw}{dz}\right)^{1-h_i}(z_i)\left(\frac{dw}{dz}\right)^{1-\bar{h}_i}(z_i)^* f_{\alpha_1 \cdots \alpha_n}(w(z_1), \ldots, w(z_n)).$$
$$\tag{2.41}$$

Given a Möbius transformation w close to the identity, let $\overset{\circ}{\underline{M}}$ be a sequence of non-empty subsets, $\overset{\circ}{M}_n$, of M_n^+ with the property that

$$\overset{\circ}{M}_n(w) = \{\underline{z} : (w(z_1), \ldots, w(z_n)) \in \overset{\circ}{M}_n\}$$

is contained in M_n^+, for all $n = 1, 2, 3, \ldots$.

For $\underline{f} \in \mathcal{S}^+(\overset{\circ}{\underline{M}})$, we define

$$U(w)i(\underline{f}) = i(^{w^{-1}}\underline{f}). \tag{2.42}$$

Thanks to property (2.40), this determines a densely defined operator $U(w)$ on $\mathcal{H}_{\gamma_+}$.

If w_1 and w_2 are two Möbius transformations close to the identity, then it follows directly from (2.42) and (2.41) that

$$U(w_1)U(w_2) = U(w_1 \circ w_2), \tag{2.43}$$

as an equation between densely defined operators on $\mathcal{H}_{\gamma_+}$. Thus U defines what in [3] is called a *virtual representation* of the universal cover of the Möbius group, $SL(2, \mathbb{C})$. The generators of infinitesimal Möbius transformations are denoted by $L_{-1}, L_0, L_1, \bar{L}_{-1}, \bar{L}_0, \bar{L}_1$. We define

$$w_\theta(z) = \theta w^{-1}(\theta z) = -w^{-1}(-z^*)^*.$$

Returning to the definition (2.17) of the scalar product $\langle \cdot, \cdot \rangle_{\gamma_+}$, changing variables

and using (2.11) we easily verify that

$$\langle \underline{f}, {}^{w}\underline{g} \rangle_{\gamma_+} = \langle {}^{w_\theta}\underline{f}, \underline{g} \rangle_{\gamma_+}. \tag{2.44}$$

This identity and (2.42) show that

$$U(w)^* = U(w_\theta) \tag{2.45}$$

on some domain dense in $\mathscr{H}_{\gamma_+}$. One can choose the generators $L_{-1}, L_0, L_1, \bar{L}_{-1}, \bar{L}_0, \bar{L}_1$ in such a way that (2.45) and Proposition 2.1 imply that

$$\overset{(-)}{L_0^*} = \overset{(-)}{L_0}, \quad \overset{(-)}{L_1^*} = \overset{(-)}{L_{-1}}, \quad \text{and} \quad \overset{(-)}{L_{-1}^*} = \overset{(-)}{L_1}.$$

This will be discussed in more detail below.

Equations (2.20), (2.22) and (2.24) are special cases of (2.42), (2.43). Obviously, the vacuum Ω is invariant under U. It follows from results in [3] and [9] that U can be analytically continued to a unitary representation of the group of pseudo-Möbius transformations on $\mathscr{H}_{\gamma_+}$.

2.3. We now show how to associate a quantization consisting of a Hilbert space $\mathscr{H}_\gamma$, a vacuum $\Omega_\gamma \in \mathscr{H}_\gamma$, a virtual representation U_γ of $SL(2, \mathbb{C})$ on $\mathscr{H}_\gamma$ leaving Ω_γ invariant, and Euclidean field operators $\phi_\alpha^\gamma, \alpha \in A_0$, with every parametrized disk, γ, on the Riemann sphere. Let γ be the image of $\gamma_+ = \{z : \text{Re}\, z > 0\}$ under a Möbius transformation $w^{-1} \equiv w_\gamma^{-1}$. Let $f \mapsto {}^{w}f$ be given by (2.41). We define a reflection θ_γ at the boundary, $\partial\gamma$, of γ by setting

$$\theta w(z) = w(\theta_\gamma z), \quad \text{i.e.}$$
$$\theta_\gamma z = w^{-1}(\theta w(z)) = w^{-1}(-w(z)^*); \tag{2.46}$$

see (2.4) for the definition of $\theta \equiv \theta_{\gamma_+}$.

We define

$$\mathscr{L}_\gamma^+ = \{ \underline{f} : {}^{w}\underline{f} \in \mathscr{L}^+ \}. \tag{2.47}$$

The space $\mathscr{L}_\gamma^+$ carries an inner product defined by

$$\langle \underline{f}, \underline{g} \rangle_\gamma = \sum_{n,m} \sum_{\alpha,\beta} \int G_{\alpha_n^* \cdots \alpha_1^* \beta_1 \cdots \beta_m}(\theta_\gamma z_n, (\theta_\gamma z_n)^*, \ldots, z_m', z_m'^*)$$

$$\cdot \prod_{i=1}^{n} \left(-\frac{\partial}{\partial z^*} \theta_\gamma z \right)^{h_i} \left[\left(-\frac{\partial}{z^*} \theta_\gamma z \right)^{\bar{h}_i} \right]^* f_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n)^*$$

$$\cdot g_{\beta_1 \cdots \beta_m}(z_1', \ldots, z_m') \prod_{i=1}^{n} d^2 z_i \prod_{j=1}^{m} d^2 z_j'. \tag{2.48}$$

Here $d^2 z \equiv d^2 x = dt dx$. By a careful change of variables, $w = w(z)$, and by using that w is a Möbius transformation, one finds that

$$\langle \underline{f}, \underline{g} \rangle_\gamma = \langle {}^{w}\underline{f}, {}^{w}\underline{g} \rangle_{\gamma_+}. \tag{2.49}$$

Since the image of $\mathscr{L}_\gamma^+$ under w is $\mathscr{L}^+$, see (2.47), $\langle {}^{w}\underline{f}, {}^{w}\underline{f} \rangle_{\gamma_+} \geqq 0$, by (2.17) and (2.16). Hence $\langle \cdot, \cdot \rangle_\gamma$ is positive semi-definite on $\mathscr{L}_\gamma^+$. A Hilbert space $\mathscr{H}_\gamma$ can now be constructed by the same reasoning that gave $\mathscr{H}_{\gamma_+}$. The injection of $\mathscr{L}_\gamma^+$

into $\mathcal{H}_\gamma$ is denoted by i_γ. The map $I_{\gamma\gamma_+}$ defined by

$$I_{\gamma\gamma_+} i(\underline{f}) = i_\gamma(^{w^{-1}}\underline{f}), \qquad (2.50)$$

where w is the Möbius transformation mapping γ to γ_+, defines an isomorphism from $\mathcal{H}_{\gamma_+}$ onto $\mathcal{H}_\gamma$ which by (2.49) preserves scalar products. We set

$$\Omega_\gamma = I_{\gamma\gamma_+} \Omega_{\gamma_+}.$$

Then Ω_γ is the image under i_γ of the sequence $\underline{f} \in \mathscr{L}_\gamma^+$ given by $f(\varnothing) = 1$, $f_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n) = 0$, for all $n \geq 1$. We also define

$$\phi_\alpha^\gamma(z, z^*) I_{\gamma\gamma_+} = I_{\gamma\gamma_+} \left(\frac{dw}{dz} \right)^{h(\alpha)}(z) \left(\frac{dw}{dz} \right)^{\bar{h}(\alpha)}(z)^* \phi_\alpha(w, w^*). \qquad (2.51)$$

This definition is consistent with Eqs. (2.27), (definition of ϕ_α on $\mathcal{H}_{\gamma_+}$), (2.41), (definition of $f \mapsto {}^w f$), and (2.50), (definition of $I_{\gamma\gamma_+}$).

Every Hilbert space $\mathcal{H}_\gamma$ carries a virtual representation, U_γ, of the universal cover of the Möbius group, given by

$$U_\gamma(w) i_\gamma(\underline{f}) = i_\gamma(^{w^{-1}}\underline{f}), \qquad (2.52)$$

for $\underline{f}$ in a subspace of $\mathscr{L}_\gamma^+$ whose image under w is still contained in $\mathscr{L}_\gamma^+$. Let w_γ be the Möbius transformation taking γ to γ_+. Combining (2.50) and (2.52), we find

$$U_\gamma(w) I_{\gamma\gamma_+} i(\underline{f}) = U_\gamma(w) i_\gamma(^{w_\gamma^{-1}}\underline{f}) = i_\gamma(^{(w\circ w_\gamma)^{-1}}\underline{f}) = i_\gamma(^{w_\gamma^{-1}\circ w\circ w_\gamma)^{-1}\circ w_\gamma^{-1}}\underline{f})$$

$$= I_{\gamma\gamma_+} i(^{(w_\gamma^{-1}\circ w\circ w_\gamma)^{-1}}\underline{f}) = I_{\gamma\gamma_+} U_{\gamma_+}(w_\gamma^{-1}\circ w\circ w_\gamma) i(\underline{f}).$$

Hence

$$U_\gamma(w) I_{\gamma\gamma_+} = I_{\gamma\gamma_+} U_{\gamma_+}(w_\gamma^{-1}\circ w\circ w_\gamma) \qquad (2.53)$$

as an operator equation on a dense domain in $\mathcal{H}_{\gamma_+}$.

We conclude this section by discussing a special example: $\gamma = \gamma_0 \equiv \{z:|z| < 1\}$. In this example, which corresponds to *radial quantization* [1], the transformation $w_{\gamma_0} = w_0$ is given by

$$w_0(z) = -\frac{z-1}{z+1}, \qquad (2.54)$$

mapping the unit disk to the half plane $\{\operatorname{Re} w_0 > 0\}$ with $z = 1 \mapsto w_0 = 0$, $z = -1 \mapsto \hat{w} = i\infty$. The space $\mathscr{L}_{\gamma_0}^+$ is given by sequences of test functions, $f_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n)$, with support in $M_n \cap \{z:|z_i| < 1, i = 1, \ldots, n\}$, $n = 0, 1, 2, \ldots$, and the scalar product $\langle \cdot, \cdot \rangle_{\gamma_0}$ is

$$\langle \underline{f}, \underline{g} \rangle_{\gamma_0} = \sum_{n,m} \sum_{\underline{\alpha},\underline{\beta}} \int G_{\alpha_n^* \cdots \alpha_1^* \beta_1 \cdots \beta_m}(z_n^{*-1}, z_n^{-1}, \ldots, z_m', z_m'^*)$$

$$\cdot \left(\prod_{i=1}^n (z_i^*)^{-2h_i} z_i^{-2\bar{h}_i} \right) f_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n)^*$$

$$\cdot g_{\beta_1 \cdots \beta_m}(z_1', \ldots, z_m') \prod_{i=1}^n d^2 z_i \prod_{j=1}^m d^2 z_j'. \qquad (2.55)$$

The Hilbert space $\mathcal{H}_{\gamma_0}$ carries a virtual representation, U_{γ_0} of $SL(2, \mathbb{C})$. The

generators $L^{\gamma_0}_{-1}, L^{\gamma_0}_0, L^{\gamma_0}_1, \bar{L}^{\gamma_0}_{-1}, \bar{L}^{\gamma_0}_0$ and $\bar{L}^{\gamma_0}_1$ of U_{γ_0} are chosen as follows; (we drop the superscript γ_0):

$$e^{\tau L_0} \text{ represents } z \mapsto e^{\tau}z, \quad z^* \mapsto z^*, \tag{2.56}$$

$$e^{\tau L_{-1}} \text{ represents } z \mapsto z + \tau, \quad z^* \mapsto z^*, \tag{2.57}$$

$$e^{\tau L_1} \text{ represents } z \mapsto \frac{z}{1 - \tau z}, \quad z^* \mapsto z^*. \tag{2.58}$$

The action of the generators $\bar{L}_{-1}, \bar{L}_0$ and $\bar{L}_1$ is obtained by exchanging z and z^* in the above formulas. Using the analyticity properties of the Green functions established in Proposition 2.1, one verifies that the operators $e^{\tau L_0}$, $e^{\tau L_{-1}}$ and $e^{\tau L_1}$ are densely defined operators on $\mathscr{H}_{\gamma_0}$, for $|\tau|$ small enough. Using (2.55), one shows that

$$L^*_0 = L_0, \quad L^*_1 = L_{-1}, \quad L^*_{-1} = L_1, \tag{2.59}$$

and similarly for $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$. From the definition of the generators and (2.59) one may conclude that

$$[L_n, \bar{L}_m] = 0, \quad n, m = -1, 0, 1 \tag{2.60}$$

in the sense that the spectral projections of the selfadjoint generators $L_0, L_1 + L_{-1}$, $i(L_1 - L_{-1})$, commute with those of $\bar{L}_0, \bar{L}_1 + \bar{L}_{-1}, i(\bar{L}_1 - \bar{L}_{-1})$. It also follows easily from (2.56)–(2.58) and Proposition 2.1 that on some natural domain $\mathscr{D}$ dense in $\mathscr{H}_{\gamma_0}$,

$$L_n e^{i\omega L_0} = e^{i\omega(L_0 + n)} L_n, \quad \text{Im } \omega \geq 0. \tag{2.61}$$

See [3] for techniques useful to prove these claims.

Note that, by (2.56), $e^{-\tau(L_0 + \bar{L}_0)}$ represents the dilatation $z \mapsto e^{-\tau}z, z^* \mapsto e^{-\tau}z^*$ on $\mathscr{H}_{\gamma_0}$. It is shown in [10] that

$$L_0 + \bar{L}_0 \geq 0. \tag{2.62}$$

This can also be proven by using arguments of [2]. First one notices that the scaling dimensions, $d(\alpha), \alpha \in A_0$, are all positive. This follows from the fact that, by assumptions (P2) and (P3),

$$G_{\alpha^*\alpha}(z, z^*, 0, 0) \sim |z|^{-2d(\alpha)} e^{-2i \arg z \, s(\alpha)}$$

must tend to 0, as $|z| \to \infty$. It is then consistent to sharpen assumption (P1) by requiring the bounds

$$|G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_j, e^{-\tau}\underline{x}_{j+1}, \ldots, e^{-\tau}\underline{x}_n)| e^{-\tau(\sum^n_{i=j+1} d_i)} \leq C(\underline{\alpha}) (\min_{i \neq j} |\underline{x}_i - \underline{x}_j|)^{-N(\underline{\alpha})}, \tag{2.63}$$

uniformly in $\tau \geq 0$, provided $|\underline{x}_1| > |\underline{x}_2| > \cdots > |\underline{x}_n|$, $n = 1, 2, 3, \ldots$; (here $C(\underline{\alpha})$ and $N(\underline{\alpha})$ are some finite constants). Using (2.52) and the Schwartz inequality with respect to $\langle \cdot, \cdot \rangle_{\gamma_0}$ repeatedly, as in [2], one shows that (2.63) implies (2.62).

Since L_0 and $\bar{L}_0$ commute, the joint spectrum of $(L_0, \bar{L}_0)$ is a subset of $\mathbb{R}^2$, and by (2.62)

$$\text{spec}(L_0, \bar{L}_0) \subseteq \{(h, \bar{h}) : h + \bar{h} \geq 0\}.$$

Let $\{E_0(\Delta)\}$ denote the joint spectral projections of $(L_0, \bar{L}_0)$. It follows from (2.61) by Fourier transformation that, on the dense domain $\mathscr{D}$,

$$L_n E_0(\Delta) = E_0(\Delta(n,0))L_n, \quad \bar{L}_n E_0(\Delta) = E_0(\Delta(0,n))\bar{L}_n, \tag{2.64}$$

for $n = -1, 0, 1$, where

$$\Delta(n, m) = \{(h, \bar{h}) : (h + n, \bar{h} + m) \in \Delta\}. \tag{2.65}$$

We now prove the following general result.

Proposition 2.2. *Let $\mathscr{H}$ be a separable Hilbert space carrying two commuting representations $\{L_n\}_{n=-1}^1$, $\{\bar{L}_n\}_{n=-1}^1$ of the Möbius algebra $sl(2, \mathbb{C})$ with the following properties*:

(a) *$L_0^* = L_0$, $L_1^* = L_{-1}$, $\bar{L}_0^* = \bar{L}_0$, $\bar{L}_1^* = \bar{L}_{-1}$,*
(b) *Eq. (2.64) holds on some domain $\mathscr{D}$ dense in $\mathscr{H}$,*
(c) *$L_0 + \bar{L}_0 \geqq 0$.*

Then $L_0 \geqq 0$ and $\bar{L}_0 \geqq 0$.

Proof. Let Λ be the joint spectrum of $(L_0, \bar{L}_0)$. By hypothesis (c), Λ is contained in the set $\{(h, \bar{h}) : h + \bar{h} \geqq 0\}$. Hence we may find a non-empty subset Δ of Λ such that

$$\Delta(1, 0) = \{(h, \bar{h}) : (h + 1, \bar{h}) \in \Delta\} \cap \Lambda = \varnothing. \tag{2.66}$$

Let ψ be an arbitrary vector in $\mathscr{D}$ with $E_0(\Delta)\psi = \psi$. We claim that

$$L_1 \psi = 0. \tag{2.67}$$

To prove (2.67), we note that

$$L_1 \psi = L_1 E_0(\Delta)\psi = E_0(\Delta(1,0))L_1\psi,$$

by hypothesis (b). But by (2.66), $\Delta(1,0) \cap \Lambda = \varnothing$, so $E_0(\Delta(1,0))L_1\psi = 0$.

Next, let $h_{\max} = \max\{h : (h, \bar{h}) \subset \Delta\}$. Then

$$0 \leqq \langle L_{-1}\psi, L_{-1}\psi \rangle = \langle \psi, L_1 L_{-1}\psi \rangle, \quad \text{by (a)}$$

$$= \langle \psi, [L_1, L_{-1}]\psi \rangle, \quad \text{by (2.67)}$$

$$= 2\langle \psi, L_0\psi \rangle \leqq 2h_{\max}\langle \psi, \psi \rangle.$$

Hence $h_{\max} \geqq 0$. Clearly, given any $\varepsilon > 0$, we can find a set Δ with all the properties stated above such that $\min\{h : (h, \bar{h}) \in \Delta\} \geqq h_{\max} - \varepsilon \geqq -\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, it follows that

$$\min\{h : (h, \bar{h}) \in \Lambda\} \geqq 0.$$

The same arguments apply to $\bar{L}_0, \bar{L}_1$. Thus

$$\Lambda \subseteq \{(h, \bar{h}) : h \geqq 0, \bar{h} \geqq 0\}. \quad \blacksquare$$

Remarks. 1. The properties of the representation U_{γ_0} of $SL(2, \mathbb{C})$ established above can be transferred to quantizations associated with arbitrary parametrized disks, γ, on the Riemann sphere. Let $w_{\gamma\gamma_0}$ be a Möbius transformation mapping γ to γ_0,

and let $I_{\gamma\gamma_0}$ be the corresponding isomorphism from $\mathscr{H}_{\gamma_0}$ to $\mathscr{H}_\gamma$. As in (2.53) one finds

$$U_\gamma(w)I_{\gamma\gamma_0} = I_{\gamma\gamma_0} U_{\gamma_0}(w_{\gamma\gamma_0}^{-1} \, owow_{\gamma\gamma_0}).$$

It is compatible with this equation to define the generators L_n^γ, $\bar{L}_n^\gamma$, $n = -1, 0, 1$, by setting

$$L_n^\gamma I_{\gamma\gamma_0} = I_{\gamma\gamma_0} L_n^{\gamma_0}, \quad \bar{L}_n^\gamma I_{\gamma\gamma_0} = I_{\gamma\gamma_0} \bar{L}_n^{\gamma_0}, \tag{2.68}$$

for $n = -1, 0, 1$.

2. Using Proposition 2.2, one can prove an analogue of Proposition 2.1 in radial quantization by working with the semigroups z^{L_0} and $\bar{z}^{\bar{L}_0}$ which are contractions for $|z| \leq 1$ and $|\bar{z}| \leq 1$. One notices that by (2.56) and property (P2), Eq. (2.11),

$$\phi_\alpha(z, \bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0} \phi_\alpha(1, 1) z^{-L_0 - h(\alpha)} \bar{z}^{\bar{L}_0 - \bar{h}(\alpha)}, \quad \alpha \in A_0, \tag{2.69}$$

provided $-\pi < \arg z < \pi$, $-\pi < \arg \bar{z} < \pi$. As in (2.39), one then finds that $G_\alpha(z, z^*)$ is the restriction of a function $H_\alpha(z, \bar{z})$ holomorphic and single-valued in z and $\bar{z}$ on $K_n^> \times K_n^>$, where

$$K_n^> = \{\underline{z} : |z_1| > \cdots > |z_n|, \ -\pi < \arg z_i < \pi, \ i = 1, \ldots, n\}. \tag{2.70}$$

The function $H_\alpha(z, \bar{z})$ can obviously be extended to the domain $\bar{K}_n^> \times \bar{K}_n^>$, where

$$\bar{K}_n^> = \{\underline{z} : |z_1| > \cdots > |z_n|\}.$$

But since $\bar{K}_n^>$ is not contractible, $H_\alpha(z, \bar{z})$ may and does have *non-trivial monodromy* on $\bar{K}_n^> \times \bar{K}_n^>$; see Sect. 4. Its monodromy can be removed by passing to the covering space $\tilde{K}_n^> \times \tilde{K}_n^>$, where

$$\tilde{K}_n^> = \{\underline{z} : |z_1| > \cdots > |z_n|, \ -\infty < \arg z_i < \infty, \ i = 1, \ldots, n\},$$

and extending the definition of $\phi_\alpha(z, \bar{z})$ to the domain $\{z, \bar{z} : -\infty < \arg z, \arg \bar{z} < \infty\}$. These features were a source of confusion in the early days of conformal field theory which was resolved e.g. in [9].

3. The Chiral Structure of Conformal Field Theory

The goal of this section is to show that, under certain additional assumptions concerning the existence of a conserved energy-momentum tensor and possibly further conserved "currents," every field $\phi_\alpha(z, \bar{z})$ has a holomorphic factorization

$$\phi_\alpha(z, \bar{z}) = \sum f_{\alpha\mu\bar{\nu}} \varphi_\mu(z) \otimes \varphi_{\bar{\nu}}(\bar{z}) \tag{3.1}$$

for some complex coefficients $f_{\alpha\mu\bar{\nu}}$ and chiral fields $\varphi_\mu(z)$, $\varphi_{\bar{\nu}}(\bar{z})$. The sum in (3.1) extends over multi-indices μ and $\bar{\nu}$. If the theory is a so-called *rational* conformal field theory that sum is *finite*, for all $\alpha \in A_0$, and Eq. (3.1) becomes an extremely powerful tool in the study of conformal field theory.

   The derivation of (3.1) rests on first finding all chiral fields already contained in the operator algebra generated by $\{\phi_\alpha(z, \bar{z})\}_{\alpha \in A_0}$. The most prominent example of such a field is the *energy–momentum tensor* which we now study. Its existence is guaranteed by the following *additional assumption* typically made in conformal field theory [10]:

(P4) In the operator algebra generated by the fields $\{\phi_\alpha(\underline{x})\}_{\alpha \in A_0}$ there are *local* fields $T_{\mu\nu}(\underline{x})$, $\mu, \nu = 0, 1$, with the following properties:

$$T_{\mu\nu}(\underline{x}) = T_{\nu\mu}(\underline{x}), \quad T_{\mu\nu}^*(t, \underline{x}) = T_{\mu\nu}(-t, x), \tag{3.2}$$

$$\partial_0 T_{\mu 0} - \partial_1 T_{\mu 1} = 0, \quad \text{(i.e. } T_{\mu\nu} \text{ is conserved),} \tag{3.3}$$

$$d(T_{\mu\nu}) = 2, \quad s(T_{00} - T_{11} \pm 2iT_{01}) = \pm 2, \tag{3.4}$$

where d is the scaling dimension and s the spin.

It is assumed, moreover, that the generators $L_n, \bar{L}_n$, $n = -1, 0, 1$, can be expressed in terms of $T_{\mu\nu}$, in particular, in Osterwalder–Schrader quantization, $(\gamma = \gamma_+ = \{z : \operatorname{Re} z > 0\})$,

$$H = \int dx T_{00}(0, x), \quad \text{and}$$

$$P = \int dx T_{01}(0, x). \tag{3.5}$$

Finally, it is assumed (temporarily) that the Green functions (vacuum expectation values) of $T_{\mu\nu}$ are parity-invariant.

Remarks. 1. Eqations (3.2), (3.3) and (3.5) hold in the sense of densely defined sesqui-linear forms on $\mathscr{H} \times \mathscr{H}$.

2. Assumption (P4) can be rewritten as an assumption on Green functions $G_{\alpha_1 \cdots \alpha_n}(\underline{x}_1, \ldots, \underline{x}_n)$, $n = 0, 1, 2, \ldots$, in the Euclidean domain. But such a formulation is more cumbersome; (see also [1]).

It follows from assumptions (P1)–(P3), Sect. 2, and (P4) that $T_{\mu\nu}$ is *traceless*, i.e.

$$T_\mu^\mu = 0, \tag{3.6}$$

and hence, using in addition (3.3), that $T_{\mu\nu}$ has only two independent components,

$$T \equiv T_{00} + iT_{01}, \quad \text{only depending on} \quad z = t + ix,$$

$$\bar{T} \equiv T_{00} - iT_{01}, \quad \text{only depending on} \quad z^* = t - ix. \tag{3.7}$$

In Osterwalder–Schrader quantization $(\gamma = \gamma_+)$, we define operators

$$L_n = \frac{(-1)^n}{4\pi} \int dx (x - i)^{1-n}(x + i)^{1+n} T(0, x),$$

$$\bar{L}_n = \frac{(-1)^n}{4\pi} \int dx (x - i)^{1-n}(x + i)^{1+n} \bar{T}(0, x). \tag{3.8}$$

Then $\{L_n\}_{n \in \mathbb{Z}}$ and $\{\bar{L}_n\}_{n \in \mathbb{Z}}$ satisfy two commuting Virasoro algebras

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

$$[\bar{L}_n, \bar{L}_m] = (n - m)\bar{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{3.9}$$

for some central charge $c \geq \frac{1}{2}$.

Results (3.6)–(3.9) form the contents of a general theorem due to Lüscher and

Mack [11]; see also [12] for some earlier partial result. It is easy to show, using (3.4) and (3.7), that

$$L_n \Omega = 0, \quad \text{for} \quad n \geq -1. \tag{3.10}$$

The vacuum expectation values $\langle \Omega, L_{n_1} \cdots L_{n_k} \Omega \rangle$ can be computed recursively from (3.9) and (3.10); see [11, 1]. We now extend the intertwining relations (2.68) from $n = -1, 0, 1$ to all $n \in \mathbb{Z}$. For example,

$$L_n^{\gamma+} I_{\gamma + \gamma 0} = I_{\gamma + \gamma 0} L_n^{\gamma 0}, \quad n \in \mathbb{Z}, \tag{3.11}$$

where $\{L_n^{\gamma+}\}_{n \in \mathbb{Z}}$ are the generators introduced in (3.8); (similar relations are required for $\bar{L}_n^{\gamma+}, \bar{L}_n^{\gamma 0}, n \in \mathbb{Z}$). Using that, by (3.4), (3.7) and (3.9), $T(z)$ transforms tensorially under Möbius transformations with conformal weights $h(T) = 2$, $\bar{h}(T) = 0$, we conclude from (3.11) that

$$L_n^{\gamma 0} = \oint_{|z|=1} z^{n+1} T(z) dz, \quad n \in \mathbb{Z}, \tag{3.12}$$

in radial quantization, and similarly for $\bar{L}_n^{\gamma 0}$.

More generally, if

$$\varepsilon(w) = \sum_{n=-\infty}^{\infty} \varepsilon_n w^{n+1}$$

is an infinitesimal conformal transformation and γ is a disk on the Riemann sphere, we set

$$T^\gamma(\varepsilon) = \oint_{\partial \gamma} \varepsilon(w) T(w) dw.$$

If $w_{\gamma'\gamma}$ is a Möbius transformation mapping a disk γ' to γ then

$$T^\gamma(\varepsilon) I_{\gamma\gamma'} = I_{\gamma\gamma'} T^{\gamma'}((\varepsilon o w_{\gamma'\gamma}) \cdot (w'_{\gamma'\gamma})^{-1}). \tag{3.13}$$

This can be derived from (3.11) or, more simply, from the transformation law of T,

$$T(w) = \left(\frac{dw}{dz}\right)^{-2} (z) U(w) T(z) U(w)^{-1} \tag{3.14}$$

under Möbius transformations. [If ε is an infinitesimal Möbius transformation then (3.13) is consistent with (2.53) (with γ_+ replaced by γ') and with (2.68)!].

From now on we shall drop the superscripts, γ, from $L_n^\gamma, \bar{L}_n^\gamma, T^\gamma, \mathscr{H}^\gamma, \ldots$, whenever they are clear from the context.

We now study the representation theory of the Lie algebra $\text{Vir} \oplus \overline{\text{Vir}}$ with generators $\{L_n, \bar{L}_n\}_{n \in \mathbb{Z}}$ on the Hilbert space $\mathscr{H}$ of some conformal field theory satisfying assumptions (P1)–(P4). In the following discussion it may be convenient to think of radial quantization ($\gamma = \gamma_0 = \{z : |z| < 1\}$). The generators $L_n, \bar{L}_n$ are then given by Eq. (3.12). Using the facts that $T(z)$ and $\bar{T}(\bar{z})$ are local fields and using the analyticity properties of the Green functions $H_{\alpha_1 \cdots \alpha_n}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ (transferred to radial quantization; with $\alpha = 1$ being the index for T and $\alpha = \bar{1}$ being the index for $\bar{T}$), one can easily prove that Vir and $\overline{\text{Vir}}$ have a common *invariant* domain, $\mathscr{D}$, dense in $\mathscr{H}$ on which all the generators, $L_n, \bar{L}_n$, are defined, satisfy the

unitarity condition

$$L_n^* = L_{-n}, \quad \bar{L}_n^* = \bar{L}_{-n}, \quad n \in \mathbb{Z}, \tag{3.15}$$

and the Virasoro algebra (3.9).

By (2.56) and assumption (P4), $e^{\tau L_0}$ represents the transformation $z \to e^\tau z, \bar{z} \to \bar{z}$, and $e^{\tau \bar{L}_0}$ represents $z \to z$, $\bar{z} \to e^\tau \bar{z}$. From this and Proposition 2.2 one derives that the domain $\mathscr{D}$ can be chosen to be invariant under $e^{i\omega L_0}$ and, using the Virasoro algebra (3.9), that

$$L_n e^{i\omega L_0} = e^{i\omega(L_0 + n)} L_n, \tag{3.16}$$

for Im $\omega \geq 0$, as an operator equation on $\mathscr{D}$. Similar observations hold for the generators $\bar{L}_n$ of $\overline{\mathrm{Vir}}$. [Detailed proofs of these claims are somewhat lengthy, but follow from Proposition 2.1 and the contour integral techniques in [1].]

Proposition 3.1. *Let $\mathscr{H}$ be a separable Hilbert space, and let* Vir *and* $\overline{\mathrm{Vir}}$ *be two commuting Virasoro algebras with central charge c, defined on a common invariant domain, $\mathscr{D}$, dense in $\mathscr{H}$. Suppose that L_0 and $\bar{L}_0$ are positive operators and that (3.15) and (3.16) hold on $\mathscr{D}$.*

Then the representation of Vir $\oplus \overline{\mathrm{Vir}}$ *on $\mathscr{H}$ is completely reducible, i.e. $\mathscr{H}$ is a direct sum or integral of spaces $\mathscr{H}_h \otimes \mathscr{H}_{\bar{h}}$, where $\mathscr{H}_h$ is the completion of an irreducible, unitary highest-weight module.*

Proof. Let Δ_1 be a non-empty open subset of $\Lambda \equiv \mathrm{spec}(L_0, \bar{L}_0)$ such that

$$\Delta_1(n,m) \equiv \{(h,\bar{h}) : (h+n, \bar{h}+m) \in \Delta_1\} \cap \Lambda = \varnothing, \tag{3.17}$$

for all $n \geq 1$ or $m \geq 1$. Since L_0 and $\bar{L}_0$ are positive operators, by Proposition 2.2, such a set Δ_1 exists. Let $\mathscr{D}_1 = E_0(\Delta_1)\mathscr{D} \subset \mathscr{D}$, where $E_0(\cdot)$ are the joint spectral projections of $(L_0, \bar{L}_0)$. Let $\psi \in \mathscr{D}_1$ and $n > 0$. Then, by (3.16) and (3.17),

$$L_n \psi = L_n E_0(\Delta_1)\psi = E_0(\Delta_1(n,0))L_n \psi = 0, \tag{3.18}$$

for all $n \geq 1$. Similarly, $\bar{L}_n \psi = 0$, for all $n \geq 1$. Hence ψ is a direct sum or integral of highest-weight vectors for Vir $\oplus \overline{\mathrm{Vir}}$, (labelled by points $(h, \bar{h})$ in the support of the measure $d\langle \psi, E_0(h,\bar{h})\psi \rangle$). Let $\mathscr{H}_1$ be the closure of the linear span of

$$\{L_{-n_1} \cdots L_{-n_k} \bar{L}_{-m_1} \cdots \bar{L}_{-m_l} \psi\},$$

for arbitrary positive $n_1, \ldots, n_k, m_1, \ldots, m_l$, and arbitrary $\psi \in \mathscr{D}_1$. Clearly, $\mathscr{H}_1$ is invariant under Vir $\oplus \overline{\mathrm{Vir}}$. We note that since the scalar product on $\mathscr{H}$ is positive definite and by (3.15), a singular vector contained in $\mathscr{H}_1$ is necessarily the zero vector. Thus $\mathscr{H}_1$ is a direct sum or integral of (completions of) irreducible, unitary highest weight modules.

Now, consider the orthogonal complement, $\mathscr{H} \ominus \mathscr{H}_1$, of $\mathscr{H}_1$. Let $\Lambda_2 \subseteq \Lambda \backslash \Delta_1$ be the joint spectrum of $(L_0, \bar{L}_0)$ on $\mathscr{H} \ominus \mathscr{H}_1$. Let Δ_2 be a non-empty open subset of Λ_2 such that

$$\Delta_2(n,m) \cap \Lambda_2 = \varnothing, \quad \text{for all} \quad n \geq 1 \quad \text{or} \quad m \geq 1.$$

Again, such a set Δ_2 exists, unless $\mathscr{H}_1 = \mathscr{H}$. We define $\mathscr{D}_2 = E_0(\Delta_2)(\mathscr{H} \ominus \mathscr{H}_1) \subset \mathscr{D}$. The construction described above can now be repeated, and we obtain a closed subspace, $\mathscr{H}_2$, of $\mathscr{H} \ominus \mathscr{H}_1$, invariant under Vir $\oplus \overline{\mathrm{Vir}}, \ldots$. This process can be

continued inductively at most countably many times. This completes the proof of the proposition. ∎

Remarks. 1. If the spectrum of $(L_0, \bar{L}_0)$ is discrete, then the arguments used in the proof of Proposition 3.1 yield a direct sum decomposition

$$\mathcal{H} = \bigoplus_{(i,\bar{\imath})} \mathcal{H}_i \otimes \mathcal{H}_{\bar{\imath}}, \tag{3.19}$$

where $\mathcal{H}_i$ is isomorphic to the completion of an irreducible, unitary highest-weight module for Vir.

2. Unitary, irreducible highest-weight Virasoro modules, $M_{h,c}$, have been classified in [13, 14, 15]. They are uniquely specified, up to isomorphism, by the central charge, c, and the highest weight, h, (the smallest eigenvalue of L_0 on $M_{h,c}$). If $c > 1$ $M_{h,c}$ is isomorphic to the Verma module with highest weight h and central charge c. If $c < 1$ only a discrete series of values of c,

$$c = 1 - \frac{6}{p(p+1)}, \quad p = 3, 4, \ldots,$$

and of values of h, depending on c, is possible [14], and $M_{h,c}$ is a quotient of the corresponding Verma module by a maximal submodule generated by two singular vectors.

3. The unitarity assumption (3.15) is essential for complete reducibility. Non-unitary Fock space representations which are not completely reducible are known [13].

4. It is natural to ask whether the representation of $\text{Vir} \oplus \overline{\text{Vir}}$ on $\mathcal{H}$ can be integrated to a virtual representation of $\Gamma \otimes \bar{\Gamma}$, where Γ is a central extension of the group of conformal transformations, $z \to w(z)$, which can be continued to a projective, unitary representation of $\text{Diff } S^1 \otimes \text{Diff } S^1$. The answer is affirmative, see [16]. By Proposition 3.1, it is enough to study the integrability problem on irreducible subspaces for $\text{Vir} \oplus \overline{\text{Vir}}$.

One may now ask whether the fields $\{\phi_\alpha(z, \bar{z})\}_{\alpha \in A_0}$ transform tensorially under $\Gamma \otimes \bar{\Gamma}$, i.e.

$$U(w, \bar{w})\phi_\alpha(z, \bar{z})U(w, \bar{w})^{-1} = \left(\frac{dw}{dz}\right)^{h(\alpha)}(z)\left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}(\alpha)}(\bar{z})\phi_\alpha(w, \bar{w}), \tag{3.20}$$

if w and $\bar{w}$ are conformal transformations close to the identity in a neighborhood of $z, \bar{z}$, respectively. This does not follow automatically from assumptions (P1)–(P4). It is customary to make the additional assumption [1] that $\{\phi_\alpha(z, \bar{z})\}_{\alpha \in A_0}$ contains a subset of so-called *primary* fields $\{\phi_\alpha(z, \bar{z})\}_{\alpha \in A_1}$, $A_1 \subsetneq A_0$ for which (3.20) holds. [Note that T and $\bar{T}$ are *not* primary, since the central charge c is non-zero [1, 14].] The infinitesimal version of (3.20) is [1]

$$[L_n, \phi_\alpha(z, \bar{z})] = z^{n+1}\frac{\partial}{\partial z}\phi_\alpha(z, \bar{z}) + (n+1)z^n h(\alpha)\phi_\alpha(z, \bar{z}),$$

$$[\bar{L}_n, \phi_\alpha(z, \bar{z})] = \bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}\phi_\alpha(z, \bar{z}) + (n+1)\bar{z}^n \bar{h}(\alpha)\phi_\alpha(z, \bar{z}), \tag{3.21}$$

for all $\alpha \in A_1$, $n \in \mathbb{Z}$. Note that if $\phi_\alpha(z, \bar{z})$ is primary the vector

$$\phi_\alpha(0)\Omega = \lim_{z, \bar{z} \to 0} \left(\frac{z}{\lambda}\right)^{L_0 - h(\alpha)} \left(\frac{\bar{z}}{\lambda}\right)^{\bar{L}_0 - \bar{h}(\alpha)} \phi_\alpha(\lambda, \lambda)\Omega, \tag{3.22}$$

$\lambda < 1$, is a highest-weight state for $\text{Vir} \oplus \overline{\text{Vir}}$, with

$$\overset{(-)}{L_0} \phi_\alpha(0)\Omega = \overset{(-)}{h}(\alpha) \phi_\alpha(0)\Omega,$$

as follows from (3.21). For $\alpha \in A_1$, we define $\mathscr{H}_\alpha$ to be the closure of

$$\{L_{-n_1} \cdots L_{-n_k} \bar{L}_{-m_1} \cdots \bar{L}_{-m_l} \phi_\alpha(0)\Omega\},$$

for arbitrary positive integers $n_1, \ldots, m_l$.

One may now ask whether

$$\mathscr{H} = \bigoplus_{\alpha \in A_1} \mathscr{H}_\alpha. \tag{3.23}$$

A theory is called $(\text{Vir} \oplus \overline{\text{Vir}})$—*minimal* if (3.23) holds, with A_1 a *finite* set. The minimal models analyzed in [1] are $(\text{Vir} \oplus \overline{\text{Vir}})$—minimal theories. Unfortunately, most theories are *not* $(\text{Vir} \oplus \overline{\text{Vir}})$—minimal, A_1 will be infinite. For example, the Wess–Zumino–Witten models are not $(\text{Vir} \oplus \overline{\text{Vir}})$—minimal theories [17].

Structure analysis of conformal field theory is simple for $(\text{Vir} \oplus \overline{\text{Vir}})$—minimal theories. However, we do not wish to confine our analysis to this class. Our strategy is to look for a larger "symmetry algebra," $\mathfrak{A}$, containing $\text{Vir} \oplus \overline{\text{Vir}}$, with the property that $\mathscr{H}$ splits into a *finite* direct sum of irreducible subspace for $\mathfrak{A}$. Accordingly, such a theory is called $\mathfrak{A}$-*minimal*. The construction and classification of appropriate symmetry algebras will be the subject of another publication. Here we just describe some basic features of $\mathfrak{A}$ relevant for our purpose.

Let $\mathscr{F}$ be the operator algebra generated by $\{\phi_\alpha(z, \bar{z})\}_{\alpha \in A_0}$. By assumption (P4), $\mathscr{F}$ contains $\text{Vir} \oplus \overline{\text{Vir}}$. Let $\mathscr{A}_0$ be the subalgebra of all fields in $\mathscr{F}$ commuting with $\overline{\text{Vir}}$, and let $\overline{\mathscr{A}}_0$ be the subalgebra of fields commuting with Vir. More precisely, we define $\mathscr{A}_0$ to be that subalgebra of $\mathscr{F}$ generated by fields, $J^i(z)$, which are independent of $\bar{z}$; $\overline{\mathscr{A}}_0$ is defined similarly. Clearly $\mathscr{A}_0$ contains Vir. Depending on the theory we study, $\mathscr{A}_0$ may contain further *currents*, $J^a(z)$, of spin $s = 1$ [17] (current algebra) and/or of higher spin $s = 3, 4, \ldots$ [18]. A simple lemma says that if $\mathscr{A}_0$ contains a primary current, $J(z)$, of spin 2 then the theory is *reducible*, in the sense that the Virasoro generators, L_n, can be decomposed,

$$L_n = L_n^1 + \cdots + L_n^k, \quad n \in \mathbb{Z}, \quad k \geq 2, \tag{3.24}$$

where $\{L_n^i\}_{n \in \mathbb{Z}}$, $i = 1, \ldots, k$, are commuting Virasoro algebras with central charges $c_i, i = 1, \ldots, k$, and the central charge, c, of $\{L_n\}_{n \in \mathbb{Z}}$ is given by

$$c = \sum_{i=1}^{k} c_i. \tag{3.25}$$

We may limit our study to irreducible theories. Then $T(z)$ is the only current of spin 2.

Our first candidate for $\mathfrak{A}$ is $\mathscr{A}_0 \otimes \overline{\mathscr{A}}_0$. However, it may happen that

$\mathscr{A}_0 \otimes \bar{\mathscr{A}}_0$ is too small a symmetry algebra for our purposes. It can be further enlarged by a construction that we briefly sketch here; (details will appear elsewhere): In radial quantization, the algebra $\mathscr{A}_0$ carries a representation, $\{\alpha_\tau\}_{\tau\in\mathbb{R}}$, of rotations which, for $|z| = 1$, reduce to light-cone-translations. They are defined on the generators of $\mathscr{A}_0$ by

$$\mathscr{A}_0 \ni J^i(z) \mapsto \alpha_\tau(J^i(z)) \equiv e^{i\tau h} J^i(e^{i\tau}z), \tag{3.26}$$

where $h = h(J^i(z))$. It is easily seen that $\{\alpha_\tau\}$ is an abelian group of $*$-automorphisms. A representation of $\mathscr{A}_0$ on a Hilbert space $\tilde{\mathscr{H}}$ is called a *positive-energy representation* [19] if α_τ is unitarily implemented on $\tilde{\mathscr{H}}$ by operators $e^{i\tau L_0}$, i.e.

$$\alpha_\tau(A) = e^{i\tau L_0} A e^{-i\tau L_0}, \quad \text{for all} \quad A \in \mathscr{A}_0, \tag{3.27}$$

where L_0 is a *positive* operator on $\tilde{\mathscr{H}}$.

We then study the representation of $\mathscr{A}_0$ on the total Hilbert space

$$\mathscr{H}_L = \bigoplus_{j\in\mathscr{J}_0} \mathscr{H}_j, \tag{3.28}$$

where $\mathscr{J}_0$ is the set of all inequivalent irreducible positive-energy representations of $\mathscr{A}_0$. It is assumed that there is exactly one representation, $\mathscr{H}_1$, which contains the vacuum Ω. Clearly, $\mathscr{H}_1$ has a natural embedding in the physical Hilbert space, $\mathscr{H}$, of the conformal field theory. According to the analysis in [5], one attempts to construct intertwiner fields, $\chi^j(z)$, mapping a dense domain in $\mathscr{H}_1$ to a dense domain in $\mathscr{H}_j$. These fields are, in general, *non-local*, i.e. the conformal weight h_j ($=$ spin s_j) of χ^j need neither be an integer nor a half-integer, [5]. We define a *chiral algebra* $\mathscr{A}$ to be a maximal extension of the algebra $\mathscr{A}_0$ by intertwiner fields $\chi^j(z)$, $j\in\mathscr{J}_0(\mathscr{A}) \subseteq \mathscr{J}_0$, such that the vacuum Ω is a *separating vector* for $\mathscr{A}$, i.e. if A is an arbitrary polynomial in the fields $J^i(z)\in\mathscr{A}_0$ and $\chi^j(z)$, $j\in\mathscr{J}_0(\mathscr{A})$, then

$$A\Omega = 0 \quad \text{implies} \quad A = 0 \quad \text{on} \quad \mathscr{H}_L. \tag{3.29}$$

This turns out to be a very powerful constraint that permits one to essentially classify all possible chiral algebras $\mathscr{A}$. They turn out to be slight generalizations of algebras consisting of fields of integer spin [17, 18], half-integer spin [20] and parafermions [21]. This will be discussed in more detail elsewhere.

We denote the generating fields of $\mathscr{A}$ by $\{\psi_m(z)|m\in I\}$; they may be taken to be $T(z)$, $J^i(z)$, $\chi^j(z)$, $j\in\mathscr{J}_0(\mathscr{A})$. Implicitly it is assumed that they are quasi-primary, which in turn guarantees that the algebraic structure which is developed is Möbius-covariant. Instead of working with the fields $\psi_m(z)$, we may consider their Fourier-Laurent coefficients, $\psi_{m,a}$, $a\in\mathbb{R}$. In radial quantization, $\psi_{m,a}$ is given by

$$\psi_{m,a} = \oint_{|z|=1} z^{a+h_m-1} \psi_m(z)\,dz, \tag{3.30}$$

where h_m equals the spin of $\psi_m(z)$, and the real number a in general depends both on $\psi_m(z)$ and on v, where v is some vector in the domain of definition of $\psi_m(z)$, $|z| = 1$. We assume that there is an involution $*$ on the index set I such that, in radial quantization,

$$(\psi_m(z))^* = \left(\frac{1}{z^*}\right)^{2h_m} \psi_{m^*}\left(\frac{1}{z^*}\right).$$

Consequently one deduces from (3.30) that

$$(\psi_{m,a})^* = \psi_{m^*,-a}. \tag{3.31}$$

Concerning the algebraic relations obeyed by $\mathcal{A}$, we assume that the set of generators of $\mathcal{A}, \{\psi_{m,a}\}$, can be written as the disjoint union of sets $g_>, g_0, g_<$: $g_>$ is the set of those generators which strictly raise the spectral value, h, of L_0, the generators in $g_<$ strictly lower h, and g_0 does not alter the conformal dimension.

With regard to the representation theory, it is supposed that a non-empty set, $\mathcal{J}$, is the index set for all inequivalent, irreducible positive-energy representations of $\mathcal{A}$ on spaces $\mathcal{H}_J, J \in \mathcal{J}$. In addition, if $J \in \mathcal{J}$, then $\mathcal{H}_J$ should contain at most a countably infinite number of $\mathcal{A}$-invariant vectors $w_{J,j}$, i.e. of vectors which obey $g_< w_{J,j} = 0$.

The same analysis can be repeated for $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}$. The *symmetry algebra* $\mathfrak{A}$ is defined by

$$\mathfrak{A} := [\mathcal{A} \otimes \bar{\mathcal{A}}]_{\text{loc}}, \tag{3.32}$$

i.e. $\mathfrak{A}$ is generated by monomials $A \otimes \bar{A}$, $A \in \mathcal{A}$, $\bar{A} \in \bar{\mathcal{A}}$, which must have the following commutation relations with $L_0 - \bar{L}_0$:

$$[L_0 - \bar{L}_0, A \otimes \bar{A}] = n \cdot A \otimes \bar{A}, \quad n \in \mathbb{Z}. \tag{3.33}$$

Therefore, $\mathfrak{A}$ is a subalgebra of $\mathcal{A} \otimes \bar{\mathcal{A}}$ containing $\text{Vir} \oplus \overline{\text{Vir}}$. We should emphasize that the generators of $\mathfrak{A}$, or those of $\mathcal{A}$, do, in general, not form a Lie algebra. The examples discussed in [18, 21] are not Lie algebras.

The Hilbert space, $\mathcal{H}$, of the conformal field theory is supposed to split into a *finite* direct sum

$$\mathcal{H} = \bigoplus_{\sigma \in \Sigma} \mathcal{H}_\sigma \tag{3.34}$$

of subspaces, $\mathcal{H}_\sigma$, each carrying an irreducible positive-energy representation of $\mathfrak{A}$. The conditions imposed on $\mathcal{H}_\sigma$ are:

1. There exists a pair of indices $(J, \bar{J}) \in \mathcal{J} \times \bar{\mathcal{J}}$ such that $\mathcal{H}_\sigma \equiv \mathcal{H}^\sigma_{J\bar{J}} \subset \mathcal{H}_J \otimes \mathcal{H}_{\bar{J}}$.

2. In the finite-dimensional linear space of $\mathfrak{A}$-invariant vectors in $\mathcal{H}^\sigma_{J\bar{J}}$ we can choose as a basis a set of factorized vectors $(v^\sigma_{J,j} \otimes \bar{v}^\sigma_{\bar{J},j}) \equiv v^\sigma_{j\bar{j}}, v^\sigma_{J,j} \in \mathcal{H}_J$, $\bar{v}^\sigma_{\bar{J},\bar{j}} \in \mathcal{H}_{\bar{J}}$. Furthermore, $v^\sigma_{J,j}(\bar{v}^\sigma_{\bar{J},j})$ is an eigenvector of $L_0(\bar{L}_0)$ such that $v^\sigma_{j\bar{j}}$ has integer spin; $\Delta^\sigma_{J\bar{J}}$ is the corresponding index set $\{(j, \bar{j})\}$.

The above hypotheses on the structure of the Hilbert space are supplemented by the completeness assumption concerning the field algebra $\mathcal{F}$. We require that for each $v^\sigma_{j\bar{j}} \in \mathcal{H}_\sigma$ there exists an $\mathfrak{A}$-invariant (scaling) *field* $\phi^\sigma_{j\bar{j}}(z, \bar{z})$ defined on some domain dense in $\mathcal{H}$ obeying $\phi^\sigma_{j\bar{j}}(0, 0)\Omega = v^\sigma_{j\bar{j}}$. It follows that $\phi^\sigma_{j\bar{j}}$ is a primary field having conformal dimensions h^σ_j and $\bar{h}^\sigma_j$. And finally, $\{\phi^\sigma_{j\bar{j}} | \sigma \in \Sigma,$ $(j, \bar{j}) \in \Delta^\sigma_{J\bar{J}}\}$, and their $\mathfrak{A}$-descendants should form a set of mutually local fields.

Next, we must specify what we mean by saying that $\mathfrak{A}$ is a symmetry algebra of the theory. This is conveniently expressed in terms of *Ward identities*. Let A and B be polynomials in the generators $\{\psi_{m,a}\}$ of $\mathcal{A}$ and $\bar{A}$ and $\bar{B}$ be polynomials in the generators $\{\bar{\psi}_{\bar{m},\bar{a}}\}$ of $\bar{\mathcal{A}}$ such that $A \otimes \bar{A} \in \mathfrak{A}$, $B \otimes \bar{B} \in \mathfrak{A}$. Consider the

amplitude

$$\left\langle A \otimes \bar{A} v_{i\bar{i}}^{\sigma'}, \prod_{r=1}^{N} \phi_{j_r \bar{j}_r}^{\sigma_r}(z_r, \bar{z}_r) B \otimes \bar{B} v_{k\bar{k}}^{\sigma''} \right\rangle. \tag{3.35}$$

We say that $\mathfrak{A}$ is a *symmetry algebra* for the field algebra $\mathscr{F}$ if (3.35) satisfies the Ward indentity

$$\left\langle A \otimes \bar{A} v_{i\bar{i}}^{\sigma'}, \prod_{r=1}^{N} \phi_{j_r \bar{j}_r}^{\sigma_r}(z_r, \bar{z}_r) B \otimes \bar{B} v_{k\bar{k}}^{\sigma''} \right\rangle = \sum_{\substack{l,\bar{l},n,\bar{n} \\ m,\bar{m}}} P_{AB}\left(\sigma', \underline{\sigma}, \sigma'', \underline{z}, \frac{\partial}{\partial \underline{z}}\right)_{ijk}^{lmn}$$

$$\cdot \bar{P}_{\bar{A}\bar{B}}\left(\sigma', \underline{\sigma}, \sigma'', \bar{\underline{z}}, \frac{\partial}{\partial \bar{\underline{z}}}\right)_{\bar{i}\bar{j}\bar{k}}^{\bar{l}\bar{m}\bar{n}} \cdot \left\langle v_{l\bar{l}}^{\sigma'}, \prod_{r=1}^{N} \phi_{m_r \bar{m}_r}^{\sigma_r}(z_r, \bar{z}_r) v_{n\bar{n}}^{\sigma''} \right\rangle. \tag{3.36}$$

Here $P_{AB}(\sigma', \underline{\sigma}, \sigma'', \underline{z}, (\partial/\partial \underline{z}))_{ijk}^{lmn}$ is a polynomial in $z_1, \ldots, z_N$, $(\partial/\partial z_1), \ldots, (\partial/\partial z_N)$ with coefficients depending on the choices of $\sigma', \underline{\sigma} = (\sigma_1, \ldots, \sigma_N)$, $\sigma'', l, \underline{m} = (m_1, \ldots, m_N)$, $n, i, j = (j_1, \ldots, j_N), k, A, B$, but independent of $\bar{l}, \bar{\underline{m}}, \bar{n}, \bar{i}, \bar{j}, \bar{k}, \bar{A}, \bar{B}$, and similarly for $\bar{P}_{\bar{A}\bar{B}}$. We also assume that

$$\left. \begin{array}{l} P_{AB} \text{ and } \bar{P}_{\bar{A}\bar{B}} \text{ are symmetric under} \\ \text{arbitrary permutations of } \{1, \ldots, N\}, \end{array} \right\} \tag{3.37}$$

for arbitrary choices of $A, B, \bar{A}, \bar{B}, \underline{\sigma}$ and for all $\sigma', i, \bar{i}$ and $\sigma'', k, \bar{k}$. Using representation (3.30), it is straightforward to verify (3.36) and (3.37) for theories where $\mathscr{A}$ is the Virasoro or a spin-1 current algebra. [For higher spin currents and parafermions, the proofs of (3.36) and (3.37) are slightly more involved.]

The Ward identity (3.36), (3.37) has important consequences. By the conformal algebra (3.21)

$$\phi_{j\bar{j}}^{\sigma}(z, \bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0} \phi_{j\bar{j}}^{\sigma}(1, 1) z^{-L_0 - h_j^{\sigma}} \bar{z}^{-\bar{L}_0 - \bar{h}_j^{\sigma}}, \tag{3.38}$$

and since $v_{i\bar{i}}^{\sigma'}$ and $v_{k\bar{k}}^{\sigma''}$ are eigenvectors of L_0 and $\bar{L}_0$ (3.38) yields

$$\langle v_{i\bar{i}}^{\sigma'}, \phi_{j\bar{j}}^{\sigma}(z, \bar{z}) v_{k\bar{k}}^{\sigma''} \rangle = \tilde{D}_{ijk}^{\bar{i}\bar{j}\bar{k}}(\sigma', \sigma, \sigma'') z^{h_i^{\sigma'} - h_j^{\sigma} - h_k^{\sigma''}} \bar{z}^{\bar{h}_i^{\sigma'} - \bar{h}_j^{\sigma} - \bar{h}_k^{\sigma''}}, \tag{3.39}$$

for some structure constants $\tilde{D}_{ijk}^{\bar{i}\bar{j}\bar{k}}(\sigma', \sigma, \sigma'')$. We *assume* that these structure constants have the factorized form

$$\tilde{D}_{ijk}^{\bar{i}\bar{j}\bar{k}}(\sigma', \sigma, \sigma'') = D(\sigma', \sigma, \sigma'') \cdot C_{ijk}(\sigma', \sigma, \sigma'') \cdot \bar{C}_{\bar{i}\bar{j}\bar{k}}(\sigma', \sigma, \sigma''), \tag{3.40}$$

where C and $\bar{C}$ are generalizations of Clebsch–Gordan coefficients. [Equation (3.40) is easily established for the examples of the Virasoro- and current algebras, and, in the latter case, C and $\bar{C}$ *are* Clebsch–Gordan coefficients.] A general analysis of symmetry algebras and Eqs. (3.36), (3.37) and (3.40) must be deferred to another publication.

Let P_σ denote the orthogonal projection onto $\mathscr{H}_\sigma$, $\sigma \in \Sigma$. The main result of Sect. 3 is the following lemma.

Lemma 3.2. *Under the hypotheses on $\mathfrak{A}$ and $\mathscr{H}$ specified above, the operators $P_{\sigma'} \phi_{j\bar{j}}^{\sigma}(z, \bar{z}) P_{\sigma''}$ factorize as follows:*

$$P_{\sigma'} \phi_{j\bar{j}}^{\sigma}(z, \bar{z}) P_{\sigma''} = C_{a\bar{a}} \varphi_a(z) \otimes \varphi_{\bar{a}}(\bar{z}), \tag{3.41}$$

where $a \equiv (\sigma', j, \sigma, \sigma'')$, $\bar{a} \equiv (\sigma', \bar{j}, \sigma, \sigma'')$, $C_{a\bar{a}} = C_{a\bar{a}}(\sigma)$ *is some complex number, and* $\varphi_a(z)$, $\varphi_{\bar{a}}(\bar{z})$ *are operators on the enlarged space* $\tilde{\mathscr{H}}$.

Proof. For purposes of an unambiguous interpretation of (3.41), it is convenient to assume that the operators

$$\phi_{j\bar{j}}^{\sigma}(z, \bar{z}) \lambda^{L_0 + \bar{L}_0}$$

are *bounded* operators, for $\lambda < \min(|z|, |\bar{z}|) \leqq \max(|z|, |\bar{z}|) < 1$.[2] Since $P_{\sigma''}$ commutes with $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$, the operator

$$P_{\sigma'} \phi_{j\bar{j}}^{\sigma}(z, \bar{z}) P_{\sigma''} \lambda^{L_0 + \bar{L}_0} \tag{3.42}$$

is bounded, too, under the same assumptions on λ, z and $\bar{z}$. Hence it is uniquely determined by its matrix elements in some bases of $\mathscr{H}_{\sigma'}$ and $\mathscr{H}_{\sigma''}$. It is therefore enough to calculate the matrix elements of (3.42) between states of the form $A \otimes \bar{A} v_{i\bar{i}}^{\sigma'}$ and $B \otimes \bar{B} v_{k\bar{k}}^{\sigma''}$, with $A \otimes \bar{A}$ and $B \otimes \bar{B}$ in $\mathfrak{A}$. By (3.36), (3.39) and (3.40), these matrix elements are given by

$$D(\sigma', \sigma, \sigma'') \left[\sum_{l,m,n} P_{A, B_\lambda} \left(\sigma', \sigma, \sigma'', z, \frac{\partial}{\partial z} \right)_{ijk}^{lmn} z^{h_i^{\sigma'} - h_j^{\sigma} - h_k^{\sigma''}} C_{lmn}(\sigma', \sigma, \sigma'') \right] \tag{3.43}$$

$$\left[\sum_{\bar{l},\bar{m},\bar{n}} P_{\bar{A}, \bar{B}_\lambda} \left(\sigma', \sigma, \sigma'', \bar{z}, \frac{\partial}{\partial \bar{z}} \right)_{\overline{ijk}}^{\overline{lmn}} \bar{z}^{\bar{h}_i^{\sigma'} - \bar{h}_j^{\sigma} - \bar{h}_k^{\sigma''}} C_{\overline{lmn}}(\sigma', \sigma, \sigma'') \right],$$

where $B_\lambda = \lambda^{L_0} B$, $\bar{B}_\lambda = \lambda^{\bar{L}_0} \bar{B}$. Hence these matrix elements factorize into a function only depending on z and a function only depending on $\bar{z}$.

For $\alpha \in \Sigma$, let $\mathscr{H}_{\alpha, L}$ be the Hilbert space which is obtained by taking the closure of span $\{ A v_{J,j}^\alpha | \forall A \in \mathscr{A}$ such that $\exists \bar{A}(A) \in \bar{\mathscr{A}}$ obeying $A \otimes \bar{A} \in \mathfrak{A}; \forall j$ such that $\exists \bar{j}$ with $(j, \bar{j}) \in \Delta_{J\bar{J}}^\alpha \}$, the closure being taken in the norm of $\mathscr{H}_J$. The Hilbert space $\mathscr{H}_{\alpha, R}$ is defined similarly. We get the inclusions $\mathscr{H}_{J\bar{J}}^\alpha \subset \mathscr{H}_{\alpha, L} \otimes \mathscr{H}_{\alpha, R} \subset \mathscr{H}_J \otimes \mathscr{H}_{\bar{J}}$. Define now

$$\tilde{\mathscr{H}} = \left(\bigoplus_{\alpha \in \Sigma} \mathscr{H}_{\alpha, L} \right) \otimes \left(\bigoplus_{\alpha \in \Sigma} \mathscr{H}_{\alpha, R} \right). \tag{3.44}$$

The two factors in parentheses in (3.43) can now consistently be interpreted as the matrix elements,

$$\langle A v_{I,i}^{\sigma'}, \varphi_a(z) B_\lambda v_{K,k}^{\sigma''} \rangle, \tag{3.45}$$

$$\langle \bar{A} v_{\bar{I},\bar{i}}^{\sigma'}, \varphi_{\bar{a}}(\bar{z}) \bar{B}_\lambda v_{\bar{K}\bar{k}}^{\sigma''} \rangle \tag{3.46}$$

of densely defined operators $\varphi_a(z): \mathscr{H}_{\sigma'', L} \to \mathscr{H}_{\sigma', L}$ and $\varphi_{\bar{a}}(\bar{z}): \mathscr{H}_{\sigma'', R} \to \mathscr{H}_{\sigma', R}$. We finally set $C_{a\bar{a}}(\sigma) = D(\sigma', \sigma, \sigma'')$, and the proof of Lemma 3.2 is complete. ∎

We conclude this section by exemplifying (3.36), (3.39) and Lemma 3.2 for $\mathfrak{A}$-minimal theories, with $\mathfrak{A} = \mathrm{Vir} \oplus \overline{\mathrm{Vir}}$. Let $\phi_{i\bar{i}}(z, \bar{z})$ be a primary field (as usual

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[2] Alternatively, one could use the assumption proposed in the proof of Proposition 2.1

in the sense of Eqs. (3.21)) with conformal weights h_i and $h_{\bar{i}}$. Let

$$\mathscr{H} = \bigoplus_{j\bar{j}\in\Sigma} \mathscr{H}_{j\bar{j}} \tag{3.47}$$

be the direct sum decomposition of $\mathscr{H}$ into irreducible sub-spaces for $\mathfrak{A}$. In our example $\mathscr{H}_{j\bar{j}} = \mathscr{H}_j \otimes \mathscr{H}_{\bar{j}}$, where $\mathscr{H}_j$ is the completion of an irreducible highest weight module for Vir, and similarly for $\mathscr{H}_{\bar{j}}$. Let $I = \{j : j\bar{j} \in \Sigma\}$ and $\bar{I} = \{\bar{j} : j\bar{j} \in \Sigma\}$. We define

$$\tilde{\mathscr{H}} = \bigoplus_{\substack{j\in I \\ \bar{j}\in\bar{I}}} \mathscr{H}_j \otimes \mathscr{H}_{\bar{j}}. \tag{3.48}$$

Let $v_{j\bar{j}}$ be the highest-weight vector in $\mathscr{H}_{j\bar{j}}$. A general matrix element of $P_{i\bar{i}}\phi_{j\bar{j}}(z,\bar{z})P_{k\bar{k}}$ is of the form

$$\left\langle \prod_r L_{-n_r} \prod_s \bar{L}_{-\bar{n}_s} v_{i\bar{i}},\ \phi_{j\bar{j}}(z,\bar{z}) \prod_r L_{-m_r} \prod_s \bar{L}_{-\bar{m}_s} v_{k\bar{k}} \right\rangle. \tag{3.49}$$

The generators $L^*_{-n_r} = L_{n_r}$ can be commuted through $\phi_{j\bar{j}}$ using Eq. (3.21), and through $\prod_r L_{-m_r}$, using the Virasoro algebra (3.9). We then use that $L_n v_{k\bar{k}} = 0$, for $n > 0$. The surviving L_{-m}'s acting on $v_{k\bar{k}}$ are then commuted back through $\phi_{j\bar{j}}$ using (3.21), and the ones left over kill $v_{i\bar{i}}$. The same procedure is applied to the $\bar{L}_{-\bar{n}}$'s with the result that

$$(3.49) = P\left(z,\frac{\partial}{\partial z}\right)\bar{P}\left(\bar{z},\frac{\partial}{\partial\bar{z}}\right)\langle v_{i\bar{i}}, \phi_{j\bar{j}}(z,\bar{z})v_{k\bar{k}}\rangle$$

$$= D^{\bar{i}\bar{j}\bar{k}}_{ijk} P\left(z,\frac{\partial}{\partial z}\right)\bar{P}\left(\bar{z},\frac{\partial}{\partial\bar{z}}\right)z^{h_i - h_j - h_k}\bar{z}^{h_{\bar{i}} - h_{\bar{j}} - h_{\bar{k}}},$$

for some polynomials P, depending on $i,j,k,\underline{n}$ and $\underline{m}$, and $\bar{P}$, depending on $\bar{i},\bar{j},\bar{k},\underline{\bar{n}}$ and $\underline{\bar{m}}$. Hence (3.49) factorizes, and this provides an abinitio proof of Lemma 3.2, for $\mathfrak{A} = \text{Vir} \oplus \overline{\text{Vir}}$. Very similar arguments can be used for current algebra. In more general cases, it is advantageous to use the contour integral techniques of [1], see (3.30), to prove (3.36)–(3.40).

Remark. Since $\mathfrak{A}$ contains $\text{Vir} \otimes \overline{\text{Vir}}$, the L_n's commute with the projections P_σ. Hence if $\phi^\sigma_{j\bar{j}}$ is a primary field, the chiral fields φ_a and $\varphi_{\bar{a}}$ are primary, too:

$$[L_n, \varphi_a(z)] = z^{n+1}\frac{\partial}{\partial z}\varphi_a(z) + z^n(n+1)h_a\varphi_a(z), \tag{3.50}$$

where $h_a = h^\sigma_j$.

4. Existence and Monodromy of Conformal Blocks

In Sect. 3, we have discussed the Virasoro algebra $\text{Vir} \oplus \overline{\text{Vir}}$ and then enlarged it to a symmetry algebra, $\mathfrak{A} = [\mathscr{A} \otimes \bar{\mathscr{A}}]_{\text{loc}}$, characterized by properties (3.29) $[\Omega$ is separating for $\mathfrak{A}]$, (3.33) [locality], (3.36), (3.37) and (3.40) [Ward identities]. The chiral algebras $\mathscr{A}$ and $\bar{\mathscr{A}}$ act on an enlarged Hilbert space, $\tilde{\mathscr{H}}$, containing the

physical Hilbert space, $\mathscr{H}$, as a subspace; see (3.44). The fields ψ_j in $\mathscr{A}$ need *not* be local fields, in general. $\mathscr{F}$ is the operator algebra generated by the local fields $\{\phi_{j\bar{j}}^\sigma(z,\bar{z})\}$ and their $\mathfrak{A}$-descendants. For the purposes which we have in mind it is convenient to change some notation at this point. Therefore, we summarize those properties of $(\mathscr{F}, \mathfrak{A}, \mathscr{H}, \tilde{\mathscr{H}})$ which are used in this section:

(R1) $\mathscr{H}$ splits into a *finite* direct sum,

$$\mathscr{H} = \bigoplus_{\sigma \in \Sigma} \mathscr{H}_\sigma, \tag{4.1}$$

of subspaces, $\mathscr{H}_\sigma$, which carry irreducible representations of $\mathfrak{A}$. ∎

(R2) The algebra $\mathfrak{A}$ is a *symmetry algebra* for the conformal field theory, i.e. the Ward identities (3.36), (3.37), (3.39) and (3.40) hold for the invariant fields $\phi_{j\bar{j}}^\alpha(z,\bar{z}) =: \phi_{J\bar{J}}(z,\bar{z})$, $\alpha \in \Sigma$. (Here we have introduced the notation $J = (j, \alpha)$, $\bar{J} = (\bar{j}, \alpha)$.) ∎

(R3) Given $\sigma \in \Sigma$, let $\mathscr{H}_{\sigma,L}$ and $\mathscr{H}_{\sigma,R}$ be defined as in the paragraph preceding (3.44). We abbreviate (σ, L) by i and (σ, R) by $\bar{i}$, where i ranges over a finite set Δ and $\bar{i}$ over a finite set $\bar{\Delta}$. We define now

$$\mathscr{H}_{i\bar{i}} = \mathscr{H}_\sigma, \tag{4.2}$$

with $i\bar{i} \in \Sigma \subset \Delta \times \bar{\Delta}$. By construction

$$\mathscr{H}_{i\bar{i}} \subset \mathscr{H}_i \otimes \mathscr{H}_{\bar{i}}. \tag{4.3}$$

Let $P_{i\bar{i}}$ denote the orthogonal projection onto $\mathscr{H}_{i\bar{i}}$. Then

$$P_{i\bar{i}} \phi_{J\bar{J}}(z,\bar{z}) P_{j\bar{j}} = C_{iJj}^{\bar{i}\bar{J}\bar{j}} \varphi_{iJj}(z) \otimes \varphi_{\bar{i}\bar{J}\bar{j}}(\bar{z}), \tag{4.4}$$

where $C_{iJj}^{\bar{i}\bar{J}\bar{j}}$ are complex numbers. We also abbreviate (i, J, j) by a, $(\bar{i}, \bar{J}, \bar{j})$ by $\bar{a}$ and $C_{iJj}^{\bar{i}\bar{J}\bar{j}}$ by $C_{a\bar{a}}$, as in Lemma 3.2. The field $\varphi_{iJj}(z)$ is a densely defined operator from $\mathscr{H}_j$ to $\mathscr{H}_i$, $i, j \in \Delta$; $\varphi_{\bar{i}\bar{J}\bar{j}}(\bar{z})$ is a densely defined operator from $\mathscr{H}_{\bar{j}}$ to $\mathscr{H}_{\bar{i}}$. By construction,

$$\langle v, \varphi_{iJj}(z) v' \rangle = 0, \tag{4.5}$$

unless $v \in \mathscr{H}_i$, $v' \in \mathscr{H}_j$, and similarly for $\varphi_{\bar{i}\bar{J}\bar{j}}(\bar{z})$. ∎

(R3) forms the contents of Lemma 3.2.

Vacuum expectation values of chiral fields, $\varphi_a(z)$, are called *conformal blocks*: For $\underline{z} \in M_n^>$, where

$$M_n^> = \{\underline{z} = (z_1, \ldots, z_n) : \operatorname{Re} z_n > \cdots > \operatorname{Re} z_1\}, \tag{4.6}$$

we tentatively define

$$F_{a_1 \cdots a_n}(z_1, \ldots, z_n) = \langle \Omega, \varphi_{a_1}(z_1) \cdots \varphi_{a_n}(z_n) \Omega \rangle, \tag{4.7}$$

where $a_i = (j_{i-1}, J_i, j_i)$, with $j_0 = 1, j_n = 1$, as follows from (4.5); [$\mathscr{H}_1$ is the vacuum sector containing the vacuum Ω.]

In (4.6), (4.7) we suppose that we are working in Osterwalder–Schrader quantization, ($\gamma = \gamma_+ = \{z : \operatorname{Re} z > 0\}$). It is not entirely a trivial matter to show that the definition of conformal blocks, Eq. (4.7), makes sense, since the operators $\varphi_{a_i}(z_i)$

are unbounded. A similar problem was encountered already in the proofs of Proposition 2.1 and Lemma 3.2. It was pointed out there that difficulties with domains of unbounded operators can be avoided if one assumes e.g. that

$$\phi_{J\bar{J}}(\varepsilon, \varepsilon) e^{-\varepsilon' H} \tag{4.8}$$

is a bounded operator, for arbitrary $\varepsilon > 0$, $\varepsilon' > 0$. Clearly, the Hamiltonian H belongs to $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ and hence to $\mathfrak{A}$. It therefore *commutes* with the projections $P_{J\bar{J}}$, and hence

$$P_{i\bar{i}} \phi_{J\bar{J}}(\varepsilon, \varepsilon) P_{J\bar{J}} e^{-\varepsilon' H} \tag{4.9}$$

is a bounded operator, as well. Standard arguments sketched in the proof of Proposition 2.1 then show that the Green functions

$$H_{a_1 \bar{a}_1 \cdots a_n \bar{a}_n}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = \langle \Omega, \phi_{J_1 \bar{J}_1}(z_1, \bar{z}_1) P_{J_1 \bar{J}_1} \phi_{J_2 \bar{J}_2}(z_2, \bar{z}_2) P_{J_2 \bar{J}_2}$$
$$\cdots P_{J_{n-1} \bar{J}_{n-1}} \phi_{J_n \bar{J}_n}(z_n, \bar{z}_n) \Omega \rangle \tag{4.10}$$

are holomorphic in $(z, \bar{z})$ on $M_n^> \times M_n^>$. Since $P_{J\bar{J}}$ commutes with arbitrary Möbius transformations, $U(w)$, and Ω is invariant under U, one can, as in the proof of Proposition 2.1, continue the functions $H_{a,\bar{a}}(z, \bar{z})$ to the domain $\bigcup_w \mathcal{M}_n^{1,1}(w)$, where

$$\mathcal{M}_n^{1,1}(w) = M_n^>(w) \times M_n^>(w^*), \tag{4.11}$$

and $M_n^>(w)$ is the image of $M_n^>$ under a Möbius transformation w. From Lemma 3.2 it now follows that, for $(z, \bar{z}) \in M_n^> \times M_n^>$,

$$H_{a_1 \bar{a}_1 \cdots a_n \bar{a}_n}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = \prod_{i=1}^n C_{a_i \bar{a}_i} F_{a_1 \cdots a_n}(z_1, \ldots, z_n) \bar{F}_{\bar{a}_1 \cdots \bar{a}_n}(\bar{z}_1, \ldots, \bar{z}_n), \tag{4.12}$$

with $F_a(z)$ as in (4.8), and

$$\bar{F}_{\bar{a}}(\bar{z}) = \langle \Omega, \varphi_{\bar{a}_1}(\bar{z}_1) \cdots \varphi_{\bar{a}_n}(\bar{z}_n) \Omega \rangle. \tag{4.13}$$

By (4.11), $F_a(z)$ extends to a holomorphic function on $\bigcup_w M_n^>(w)$, and $F_{\bar{a}}(\bar{z})$ extends to the domain $\bigcup_w M_n^>(w)$. Note, however, that $F_a(z)$ need not be single-valued on $\bigcup_w M_n^>(w)$, since $\bigcup_w M_n^>(w)$ is not simply connected.

Thus, we have proven the following result.

Proposition 4.1. *The conformal blocks $F_a(z)$ and $F_a(\bar{z})$ are holomorphic functions in $z \in \bigcup_w M_n^>(w)$, $\bar{z} \in \bigcup_w M_n^>(w)$, respectively, provided (4.8) is assumed.*

Remark. In radial quantization, it would be more convenient to use assumption (3.42) and work with the domains $K_n^>$, $K_n^>(w)$, where

$$K_n^> = \{z : |z_1| > \cdots > |z_n|, -\pi < \arg z_i < \pi, i = 1, \ldots, n\}. \tag{4.14}$$

Which quantization one uses is a matter of convenience. Assumptions (3.42) and (4.8) have equivalent consequences.

Next, we recall that the Green functions

$$H_{a\bar{a}}(z,\bar{z}) \equiv H_{J_1 \bar{J}_1 \cdots J_n \bar{J}_n}^{j_1 \bar{j}_1 \cdots j_{n-1} \bar{j}_{n-1}}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n), \tag{4.15}$$

are given in terms of the conformal blocks $F_{\underline{J}}^{\underline{j}}$ and $\bar{F}_{\underline{\bar{J}}}^{\bar{\underline{j}}}$, with

$$F_{\underline{J}}^{j_1 \cdots j_{n-1}}(z_1, \ldots, z_n) \equiv F_{a_1 \cdots a_n}(z_1, \ldots, z_n), \tag{4.16}$$

$a_i = (j_{i-1}, J_i, j_i)$, $i = 1, \ldots, n$, $j_0 = j_n = 1$, by

$$H_{\underline{J}\underline{\bar{J}}}^{\underline{j}\bar{\underline{j}}}(z,\bar{z}) = g_{\underline{J}\underline{\bar{J}}}^{\underline{J}\underline{\bar{J}}} F_{\underline{J}}^{\underline{j}}(z)\bar{F}_{\underline{\bar{J}}}^{\bar{\underline{j}}}(\bar{z}), \tag{4.17}$$

where

$$g_{\underline{J}\underline{\bar{J}}}^{\underline{J}\underline{\bar{J}}} = \prod_{i=1}^{n} C_{j_{i-1}J_ij_i}^{\bar{j}_{i-1}\bar{J}_i\bar{j}_i}. \tag{4.18}$$

Here $C_{j_{i-1}J_ij_i}^{\bar{j}_{i-1}\bar{J}_i\bar{j}_i} \equiv C_{a_i\bar{a}_i}$, $a_i = (j_{i-1}, J_i, j_i)$; see (4.4) and (4.12).
    Using that

$$\sum_{(i,\bar{i})\in\Sigma} P_{i\bar{i}} = 1, \tag{4.19}$$

on $\mathcal{H}$, we see that the full Green functions, studied in Proposition 2.1 of Sect. 2, are given by

$$\begin{aligned}
H_{J_1\bar{J}_1\cdots J_n\bar{J}_n}(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n) &= \sum_{j_1\bar{j}_1,\ldots,j_{n-1}\bar{j}_{n-1}} H_{\underline{J}\underline{\bar{J}}}^{\underline{j}\bar{\underline{j}}}(z,\bar{z}) \\
&= \sum_{j_1\bar{j}_1,\ldots,j_{n-1}\bar{j}_{n-1}} g_{\underline{J}\underline{\bar{J}}}^{\underline{J}\underline{\bar{J}}} F_{\underline{J}}^{\underline{j}}(z)\bar{F}_{\underline{\bar{J}}}^{\bar{\underline{j}}}(\bar{z}).
\end{aligned} \tag{4.20}$$

The functions $\{F_{\underline{J}}^{\underline{j}}(z)\}$ form a vector space, $\mathcal{B}_{\underline{J}}$, which, by assumption (R1), is *finite-dimensional*; see Eq. (4.1). Similarly, the functions

$$\bar{F}_{\underline{\bar{J}}}^{\underline{j}}(\bar{z}) = \sum_{\underline{\bar{j}}} g_{\underline{J}\underline{\bar{J}}}^{\underline{J}\underline{\bar{J}}} \bar{F}_{\underline{\bar{J}}}^{\underline{\bar{j}}}(\bar{z})$$

form a finite-dimensional vector space $\mathcal{B}_{\underline{\bar{J}}}$.
    We introduce a basis, $\{\bar{F}_{\bar{A}}(\bar{z})\}_{A=1}^{M}$, in $\mathcal{B}_{\underline{\bar{J}}}$; (each $\bar{F}_{\bar{A}}(\bar{z})$ is a linear combination of $\{\bar{F}_{\underline{\bar{J}}}^{\underline{j}}\}$) and a basis, $\{F_A(z)\}_{A=1}^{K}$, in $\mathcal{B}_{\underline{J}}$ such that

$$H_{\underline{J}\underline{\bar{J}}}(z,\bar{z}) = \sum_{A=1}^{N_1} F_A(z)\bar{F}_A(\bar{z}) \tag{4.21}$$

for some $N_1 \leqq \min(K,M)$. It is a simple exercise in tensor algebra to convince oneself that representation (4.21) can always be achieved, the point being that $F_1(z), \ldots, F_{N_1}(z)$ are *linearly independent* functions, and $\bar{F}_1(\bar{z}), \ldots, \bar{F}_{N_1}(\bar{z})$ are *linearly independent* functions. By construction of the functions $\{F_A(z)\}$, they are holomorphic in z on $\bigcup_w M_n^>(w)$, as follows from Proposition 4.1. A similar statement holds for $\{\bar{F}_A(\bar{z})\}$.
    We define

$$H_{\underline{J}\underline{\bar{J}}}^{\pi}(z,\bar{z}) = H_{J_{\pi^{-1}1}\bar{J}_{\pi^{-1}1}\cdots J_{\pi^{-1}n}\bar{J}_{\pi^{-1}n}}(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n), \tag{4.22}$$

where π is an arbitrary permutation of $\{1, \ldots, n\}$.
    By Proposition 2.1, $H_{\underline{J}\underline{\bar{J}}}^{\pi}(z,\bar{z})$ is defined and holomorphic on $\bigcup_{\pi'} \bigcup_w M_n^{>,\pi'}(w) \times$

$M_n^{>,\pi'}(w^*)$, where

$$M_n^{>,\pi}(w) = \{\underline{z} : \operatorname{Re} w(z_{\pi^{-1}n}) > \cdots > \operatorname{Re} w(z_{\pi^{-1}1})\}, \qquad (4.23)$$

see also (2.31), (2.34). For $(\underline{z}, \bar{\underline{z}}) \in \bigcup_w M_n^>(w) \times M_n^>(w^*)$, we find, by repeating the arguments leading to (4.21),

$$H_{\underline{J}\bar{\underline{J}}}^\pi(\underline{z}, \bar{\underline{z}}) = \sum_{A=1}^{N_\pi} F_A^\pi(\underline{z}) \bar{F}_A^\pi(\bar{\underline{z}}), \qquad (4.24)$$

for some linearly independent functions $F_1^\pi(\underline{z}), \ldots, F_{N_\pi}^\pi(\underline{z})$ holomorphic on $\bigcup_w M_n^>(w)$ and linearly independent functions $\bar{F}_1^\pi(\bar{\underline{z}}), \ldots, \bar{F}_{N_\pi}^\pi(\bar{\underline{z}})$ holomorphic on $\bigcup_w M_n^>(w)$. Here N_π is some finite integer.

Representations (4.21) and (4.24), together with *locality*

$$H_{\underline{J}\bar{\underline{J}}}^\pi(\pi \underline{z}, \pi \bar{\underline{z}}) = H_{\underline{J}\bar{\underline{J}}}^{\pi'}(\pi' \underline{z}, \pi' \bar{\underline{z}}) \qquad (4.25)$$

for arbitrary permutations π and π', with

$$\pi(z_1, \ldots, z_n) \equiv (z_{\pi^{-1}1}, \ldots, z_{\pi^{-1}n}), \qquad (4.26)$$

are the *key ingredients* to extend the conformal blocks $\{F_A(\underline{z})\}$ to multi-valued functions on the space

$$M_n = \{z_1, \ldots, z_n : z_i \neq z_j, \text{ for } i \neq j\}$$

corresponding to single-valued holomorphic functions on the covering space, $\tilde{M}_n$, of M_n. In the process of analytic continuation of the conformal blocks, F_A, we need some simple facts on *braid groups*. The group, S_n, of permutations of n elements acts on M_n as described in (4.26). The braid group B_n can be defined as the fundamental group, $\pi_1(M_n/S_n)$, of M_n/S_n. For alternative definitions see [4, 5].

Proposition 4.2. *The braid group B_n acts freely on $\tilde{M}_n$. A fundamental domain for this action is $\Delta_1 = M_n^>$.[3] Let Δ_b be the image of Δ_1 under $b \in B_n$. Then $\Delta_b \cap \Delta_{b'} = \varnothing$ if $b \neq b'$.*

Proof. Fix a point $P \in M_n$. The covering space $\tilde{M}_n$ can be described as the set of pairs $(\underline{z}, [\gamma])$ where $\underline{z} \in M_n$, and $[\gamma]$ is a homotopy class of paths from P to $\underline{z}$. Similarly, B_n can be described as the set of pairs $(\pi, [\gamma])$, where $\pi \in S_n$ and $[\gamma]$ is a homotopy class of paths, γ, from P to $\pi(P)$. The multiplication law in B_n is given by

$$(\pi, [\gamma])(\pi', [\gamma']) = (\pi \circ \pi', [\gamma \circ \pi(\gamma')]).$$

One verifies that

$$(\pi, [\gamma])(\underline{z}, [\gamma']) = (\pi(\underline{z}), [\gamma \circ \pi(\gamma')])$$

defines a free action of B_n on $\tilde{M}_n$. Let $(\underline{z}, [\gamma])$ be some point in $\tilde{M}_n$, and let $\pi \in S_n$

---

[3] More precisely, Δ_1 is the lift of $M_n^>$ in $\tilde{M}_n$

be such that $\mathrm{Re}\, z_{\pi(1)} < \cdots < \mathrm{Re}\, z_{\pi(n)}$. Choose P to be $(1, 2, \ldots, n-1, n)$. Then

$$(z, [\gamma]) = b(\pi^{-1}(z), [\gamma_0]),$$

where $b \in B_n$ is given by

$$b = (\pi, [\gamma \circ \pi(\gamma_0)^{-1}]),$$

and $[\gamma_0]$ is the unique homotopy class of paths, joining P to $\pi^{-1}(z)$ without leaving $M_n^>$. Thus the set of pairs $(z, [\gamma_0])$, where $\mathrm{Re}\, z_1 < \cdots < \mathrm{Re}\, z_n$ and γ_0 does not leave $M_n^>$, is a fundamental domain, denoted by Δ_1, for the action of B_n on $\tilde{M}_n$. To prove the second part of the proposition, it is enough to consider the case where $b' = 1$. Then if $b(z, [\gamma_0]) = (z', [\gamma_0'])$, with $(z, [\gamma_0])$ and $(z', [\gamma_0'])$ in Δ_1, it follows that $z = z'$ and $[\gamma_0] = [\gamma_0']$, by the uniqueness of $[\gamma_0]$. Thus $b = 1$. ∎

From now on we denote a point, $(z, [\gamma])$, in $\tilde{M}_n$ by Z, and identify a function $f(z)$ defined on $M_n^>$ with the function $f(Z)$ defined on Δ_1 by setting $f(Z) \equiv f(z, [\gamma_0]) = f(z)$. We also note that Möbius transformations, w, (in particular rotations), act on $\tilde{M}_n$ in the obvious way $w: (z, [\gamma]) \mapsto (w(z), [w(\gamma)])$. The image of Δ_b under w is denoted by $\Delta_b(w)$.

We now return to representations (4.21), (4.24) of the Green functions $H_{J\bar{J}}(z, \bar{z})$:

$$H_{J\bar{J}}(z, \bar{z}) = \sum_{A=1}^{N_1} F_A(Z) \bar{F}_A(\bar{Z}), \tag{4.27}$$

for $Z \in \Delta_1(w)$, $\bar{Z} \in \Delta_1(w^*)$. The right-hand side of (4.27) can be viewed as a parametric representation of an n-dimensional surface, S, with surface parameters $\bar{Z}$, in the N_1-dimensional vector space, $\mathcal{R}$, spanned by $\{F_A(Z)\}_{A=1}^{N_1}$. The linear independence of $\{\bar{F}_A(\bar{Z})\}_{A=1}^{N_1}$ implies that this surface is not contained in any hyperplane of $\mathcal{R}$ of positive co-dimension. To prove this claim, represent a function $F(Z) \in \mathcal{R}$ as a vector $(\lambda_1, \ldots, \lambda_{N_1}) \in \mathbb{E}^{N_1}$, given by $F(Z) = \sum_{A=1}^{N_1} \lambda_A F_A(Z)$. If our claim were false, there would exist a vector $(\lambda_1^0, \ldots, \lambda_{N_1}^0) \neq 0$ orthogonal to $(\bar{F}_1(\bar{Z}), \ldots, \bar{F}_{N_1}(\bar{Z}))$, for all $\bar{Z}$. Hence

$$\sum_{A=1}^{N_1} \lambda_A^0 \bar{F}_A(\bar{Z}) = 0, \quad \text{for all } \bar{Z}.$$

This contradicts the linear independence of $\{\bar{F}_A(\bar{Z})\}_{A=1}^{N_1}$. By analyticity, our claim is true whenever $\bar{Z}$ ranges over an arbitrarily small, non-empty open subset, K, of $\bigcup_w \Delta_1(w)$.

We conclude that, given any such K, we can find points $\bar{Z}_1, \ldots, \bar{Z}_{N_1}$ in K and complex numbers $\mu_C^1, \ldots, \mu_C^{N_1}$ such that

$$F_C(Z) = \sum_{B=1}^{N_1} \mu_C^B \sum_{A=1}^{N_1} F_A(Z) \bar{F}_A(\bar{Z}_B) = \sum_{B=1}^{N_1} \mu_C^B H_{J\bar{J}}(Z, \bar{Z}_B), \tag{4.28}$$

for every $C \in \{1, \ldots, N_1\}$. [Here $H_{J\bar{J}}(Z, \bar{Z}) = H_{J\bar{J}}(z, \bar{z})$, for $Z \in \Delta_1(w)$, $\bar{Z} \in \Delta_1(w^*)$.] Similarly, given some non-empty, open subset, K^π, of $\bigcup_w \Delta_1(w)$, one can find points

$\bar{Z}_1, \ldots, \bar{Z}_{N_\pi}$ in K^π and complex numbers $\kappa_C^1, \ldots, \kappa_C^{N_\pi}$ such that

$$F_C^\pi(Z) = \sum_{B=1}^{N_\pi} \kappa_C^B \sum_{A=1}^{N_\pi} F_A^\pi(Z) \bar{F}_A^\pi(\bar{Z}_B) = \sum_{B=1}^{N_\pi} \kappa_C^B H_{\underline{I},\underline{\bar{I}}}^\pi(\bar{Z}, \bar{Z}_B), \qquad (4.29)$$

for every $C \in \{1, \ldots, N_\pi\}$.

Suppose that, for some $b = (\pi, [\gamma]) \in B_n$ and a Möbius transformation w, $\Delta_{b^{-1}} \cap \Delta_1(w) \neq \varnothing$, and $\Delta_{(b^*)^{-1}} \cap \Delta_1(w^*) \neq \varnothing$, where $b^* = (\pi, [\gamma^*])$, and γ^* is the path complex conjugate to γ. Choosing suitable points $\bar{Z}_1, \ldots, \bar{Z}_{N_1}$ in $\Delta_{(b^*)^{-1}} \cap \Delta_1(w^*)$ and complex numbers $\mu_C^1, \ldots, \mu_C^{N_1}$, we have from (4.28), locality (see (4.25)) and (4.24) that

$$F_C(Z) = \sum_{B=1}^{N_1} \mu_C^B H_{\underline{I},\underline{\bar{I}}}(Z, \bar{Z}_B) = \sum_{B=1}^{N_1} \mu_C^B H_{\underline{I},\underline{\bar{I}}}^\pi(bZ, b^* \bar{Z}_B) = \sum_{A=1}^{N_\pi} R_C^A(b) F_A^\pi(bZ) \qquad (4.30)$$

for $Z \in \Delta_{b^{-1}} \cap \Delta_1(w)$, where

$$R_C^A(b) \equiv \sum_{B=1}^{N_1} \mu_C^B \bar{F}_A^\pi(b^* \bar{Z}_B). \qquad (4.31)$$

Similarly, using (4.29) and locality we find

$$F_C^\pi(Z) = \sum_{A=1}^{N_1} \tilde{R}_C^A(b^{-1}) F_A(b^{-1} Z), \qquad (4.32)$$

for $b^{-1} Z \in \Delta_{b^{-1}} \cap \Delta_1(w)$, and a suitable choice of $\bar{Z}_1, \ldots, \bar{Z}_{N_\pi}$ in $\Delta_{(b^*)^{-1}} \cap \Delta_1(w^*)$, with

$$\tilde{R}_C^A(b^{-1}) \equiv \sum_{B=1}^{N_\pi} \kappa_C^B \bar{F}_A(\bar{Z}_B). \qquad (4.33)$$

Comparison of (4.30) and (4.32) shows that

$$N_1 = N_\pi, \quad \text{and} \quad \tilde{R}(b^{-1}) = R(b)^{-1}. \qquad (4.34)$$

Next, observe that the right-hand side of (4.30) is holomorphic in $Z \in \bigcup_w \Delta_{b^{-1}}(w)$, i.e. $bZ \in \bigcup_w \Delta_1(w)$. Hence $F_C(Z)$ extends to a function holomorphic on $\bigcup_w \Delta_1(w) \cup \bigcup_{w'} \Delta_{b^{-1}}(w')$, provided $\Delta_{b^{-1}} \cap \Delta_1(w) \neq \varnothing$, for some w. This procedure can be repeated and will yield an extension of $F_A(Z)$ to a holomorphic function on $\tilde{M}_n$, thanks to the following fact.

Proposition 4.3. *For each $b \in B_n$, there exists a finite sequence $1 = b_0, b_1, \ldots, b_k = b$ of elements of B_n and angles $\varphi_1, \ldots, \varphi_{k-1}, -\pi < \varphi_j < \pi, j = 1, \ldots, k-1$, such that*

$$\Delta_{b_i}(r_{\varphi_i}) \cap \Delta_{b_{i+1}}$$

contains a non-empty open set. [*Here r_φ denotes a rotation through an angle φ around $z = 0$.*]

Proof. We introduce the standard generators $\tau_1, \ldots, \tau_{n-1}$ of the braid group B_n: $\tau_i = (t_i, [\gamma_i])$, where t_i denotes the transposition of i with $i+1$ and γ_i is the path leaving $1, \ldots, i-1, i+2, \ldots, n$ constant and exchanging i with $i+1$ along a positively oriented path; see Fig. 1; (we have set $P = (1, \ldots, n) \in M_n^>$).
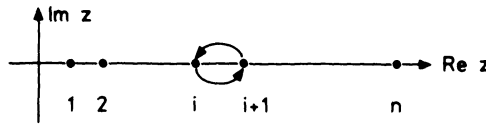
Fig. 1

Every $b \in B_n$ can be written as a word in the generators $\tau_1, \ldots, \tau_{n-1}$:

$$b = \tau_{i_1}^{\varepsilon_1} \cdots \tau_{i_k}^{\varepsilon_k},$$

with $\varepsilon_j = \pm 1$, $j = 1, \ldots, k$, $k = 1, 2, \ldots$. We define

$$b_j = \tau_{i_1}^{\varepsilon_1} \cdots \tau_{i_j}^{\varepsilon_j}.$$

To complete the proof of the proposition, it is enough to show that, for all i and $\varepsilon = \pm 1$ there exists an angle φ, with $-\pi < \varphi < \pi$, such that $\Delta_1(r_\varphi) \cap \Delta_{\tau_i^\varepsilon}$ contains a non-empty open subset. This is obvious geometrically. ∎

Equations (4.30), (4.32), (4.34) and Proposition 4.3 show that the conformal blocks $F_A(Z)$ extend to holomorphic functions on $\tilde{M}_n$, with

$$F_A(b^{-1}Z) = \sum_{B=1}^{N} R_A^B(b) F_B^\pi(Z) \tag{4.35}$$

for some representation R of B_n; ($N = N_1 = N_\pi$, for all $\pi \in S_n$, $b = (\pi, [\gamma]) \in B_n$). A similar analysis can be performed for the functions $\bar{F}_A(\bar{Z})$, with the result that every $\bar{F}_A(\bar{Z})$ extends to a function holomorphic on $\tilde{M}_n$, with

$$\bar{F}_A(b^{-1}\bar{Z}) = \sum_{B=1}^{N} \bar{R}_A^B(b) \bar{F}_B^\pi(\bar{Z}). \tag{4.36}$$

From *locality* (4.25) it then follows that

$$\sum_A F_A(b^{-1}Z) \bar{F}_A((b^*)^{-1}\bar{Z}) = \sum_{A,B,C} F_B^\pi(Z) R_B^A(b) \bar{R}_C^A(b^*) \bar{F}_C^\pi(\bar{Z}) = \sum_B F_B^\pi(Z) \bar{F}_B^\pi(\bar{Z}),$$
$$\tag{4.37}$$

for arbitrary $Z, \bar{Z}$ and all $b \in B_n$. Using the linear independence of the functions $\{F_A\}$ and $\{\bar{F}_A\}$, we conclude that

$$R(b) \bar{R}^T(b^*) = \mathbb{1}, \quad \text{i.e.}$$
$$\bar{R}(b) = R^T(b^*)^{-1}, \quad \text{for all} \quad b \in B_n. \tag{4.38}$$

We summarize our findings in the following theorem.

Theorem 4.4. *Let* $\{G_{J_1 \bar{J}_1 \cdots J_n \bar{J}_n}(z_1, z_1^*, \ldots, z_n, z_n^*)\}_{n=0}^{\infty}$ *be the Euclidean Green functions of a two-dimensional conformal field theory satisfying properties* (R1), (R2), (R3) *and assumption* (4.8). *Then these Green functions are the restrictions of functions* $H_{J\bar{J}}(Z, \bar{Z})$ *holomorphic in* $Z = (z_1, \ldots, z_n, [\gamma])$ *and* $\bar{Z} = (\bar{z}_1, \ldots, \bar{z}_n, [\bar{\gamma}])$ *on the space* $\tilde{M}_n \times \tilde{M}_n$ *to the Euclidean domain* $\{\bar{Z} : \bar{Z} = (z_1^*, \ldots, z_n^*, [\gamma^*])\}$. *Moreover,* $H_{J\bar{J}}(Z, \bar{Z})$ *is a sum of products of conformal blocks only depending on* $Z, \bar{Z}$, *respectively, which transform according to matrix representations of the braid group* B_n *under the action of* B_n *on* $\tilde{M}_n$.

The next problem we propose to tackle is to describe the representations R and $\bar{R}$ of the braid groups B_n appearing in (4.35) and (4.36) more explicitly. Progress in this direction can be made under an additional assumption that we shall formulate, now. The key idea is to study the monodromy of *four-point conformal blocks*. More precisely, we consider the following expectation values:

$$K^{k\bar{k}}_{J_1\bar{J}_1 J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2) = \langle A \otimes \bar{A} v^{\alpha}_{i\bar{i}}, \ \phi_{J_1\bar{J}_1}(z_1,\bar{z}_1) P_{k\bar{k}} \phi_{J_2\bar{J}_2}(z_2,\bar{z}_2) B \otimes \bar{B} v^{\beta}_{j\bar{j}} \rangle, \quad (4.39)$$

where $v^{\alpha}_{i\bar{i}}$ and $v^{\beta}_{j\bar{j}}$ are invariant states for $\mathfrak{A} = [\mathscr{A} \otimes \bar{\mathscr{A}}]_{\text{loc}}$,

$$A = \psi_{j_1,a_1} \cdots \psi_{j_r,a_r} \in \mathscr{A},$$
$$\bar{A} = \psi_{\bar{j}_1,b_1} \cdots \psi_{\bar{j}_s,b_s} \in \bar{\mathscr{A}}, \quad (4.40)$$

where

$$[L_0 - \bar{L}_0, A \otimes \bar{A}] = n A \otimes \bar{A}, \quad n \in \mathbb{Z}, \quad (4.41)$$

and

$$\psi_{j,a} = \oint_{|z|=1} z^{a+h_J-1} \psi_j(z) dz. \quad (4.42)$$

The generators $\psi_{\bar{j},b}$ are defined similarly, and $B \in \mathscr{A}$, $\bar{B} \in \bar{\mathscr{A}}$ have the same properties, (4.40), (4.41), as A and $\bar{A}$, respectively. See also (3.30)–(3.33). We recall that $v^{\alpha}_{i\bar{i}}$ is an eigenvector of $(L_0, \bar{L}_0)$ with eigenvalues $(h_I, h_{\bar{I}})$, where $h_I \equiv h^{\alpha}_i$, $h_{\bar{I}} \equiv \bar{h}^{\alpha}_i$, for $I = (i, \alpha)$, $\bar{I} = (\bar{i}, \alpha)$. Next, we recall that

$$\phi_{J\bar{J}}(z,\bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0} \phi_{J\bar{J}}(1,1) z^{-L_0-h_J} \bar{z}^{-\bar{L}_0-h_{\bar{J}}}; \quad (4.43)$$

see (3.38). For $A, \bar{A}$ as in (4.40), (4.41), $A \otimes \bar{A} v^{\alpha}_{i\bar{i}}$ is an entire vector for L_0 and for $\bar{L}_0$. The same is true for $B \otimes \bar{B} v^{\beta}_{j\bar{j}}$, $U(t_a) A \otimes \bar{A} v^{\alpha}_{i\bar{i}}$, $U(t_a) B \otimes \bar{B} v^{\beta}_{j\bar{j}}$, where t_a is translation by $a \in \mathbb{E}^2$. Since $P_{k\bar{k}}$ commutes with $z^{L_0} \bar{z}^{\bar{L}_0}$, we now conclude, using (3.42), that the function $K^{kk}_{J_1\bar{J}_1 J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2)$ extends to a holomorphic function on $\bigcup_a K^{>}_2(a) \times K^{>}_2(a^*)$, where

$$K^{>}_2 = \{z_1,z_2 : |z_1| > |z_2|, -\pi < \arg z_i < \pi, i = 1,2\},$$

and $K^{>}_2(a)$ is the image of $K^{>}_2$ under t_a.

Next, we note that, by locality of the fields $\phi_{J\bar{J}} \in \mathscr{F}$, the function

$$K_{J_1\bar{J}_1 J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2) \equiv \sum_{k\bar{k} \in \Sigma} K^{k\bar{k}}_{J_1\bar{J}_1 J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2) \quad (4.44)$$

is symmetric under interchanging 1 and 2. Hence $K_{J_1\bar{J}_1 J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2)$ extends to a holomorphic function on

$$\left[\bigcup_a K^{>}_2(a) \times K^{>}_2(a^*)\right] \cup \left[\bigcup_a K^{<}_2(a) \times K^{<}_2(a^*)\right], \quad (4.45)$$

where

$$K^{<}_2 = \{z_1,z_2 : |z_1| < |z_2|, -\pi < \arg z_i < \pi, i = 1,2\}.$$

By (4.4), (see also Lemma 3.2, (3.41)),

$$K^{k\bar{k}}_{J_1\bar{J}_1 J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2) = g_{k\bar{k}} F^k_{J_1 J_2}(z_1,z_2) \bar{F}^{\bar{k}}_{\bar{J}_1\bar{J}_2}(\bar{z}_1,\bar{z}_2), \quad (4.46)$$

where ($\mathscr{H}_{m\bar{m}} = \mathscr{H}_\alpha$, $\mathscr{H}_{n\bar{n}} = \mathscr{H}_\beta$),

$$g_{k\bar{k}} = C_{mJ_1k}^{\bar{m}\bar{J}_1\bar{k}} C_{kJ_2n}^{\bar{k}\bar{J}_2\bar{n}}, \tag{4.47}$$

$$F_{J_1J_2}^k(z_1,z_2) = \langle Av_i^\alpha,\ \varphi_{mJ_1k}(z_1)\varphi_{kJ_2n}(z_2)Bv_j^\beta\rangle, \tag{4.48}$$

and $\bar{F}_{\bar{J}_1\bar{J}_2}^{\bar{k}}$ is defined similarly.

From (4.43)–(4.46) we conclude that Theorem 4.4 applies to the functions $K_{J_1\bar{J}_1J_2\bar{J}_2}(z_1,\bar{z}_1,z_2,\bar{z}_2)$. Our purpose is then to analyze the properties of the representations, R and $\bar{R}$, of B_2 determined by the conformal blocks $\{F_{J_1J_2}^k, \bar{F}_{\bar{J}_1\bar{J}_2}^{\bar{k}}: k\bar{k}\in\Sigma\}$. In order to avoid too much empty generality, we require some additional assumptions typical of what might be called "rational" conformal field theories, [22].

Definition 4.5. A conformal field theory on $\mathscr{H}$ is called *of order* $(p,\bar{p})$, if there exist positive integers $p\geq 1$, $\bar{p}\geq 1$ such that

$$\text{spec}(L_0P_{j\bar{j}}) \subseteq \{H_J + p^{-1}\mathbb{Z}\}\cap(0,\infty),$$

$$\text{spec}(\bar{L}_0P_{j\bar{j}}) \subseteq \{H_{\bar{J}} + \bar{p}^{-1}\mathbb{Z}\}\cap(0,\infty). \tag{4.49}$$

[The fact that $\text{spec}(L_0P_{j\bar{j}}) \subseteq (0,\infty)$ follows from the positivity of L_0; see Proposition 2.2.]

Given $m\bar{m}\in\Sigma$, $n\bar{n}\in\Sigma$ and $J_1\bar{J}_1$, $J_2\bar{J}_2$, with $\phi_{J_1\bar{J}_1}\neq 0$, $\phi_{J_2\bar{J}_2}\neq 0$, we define an index set

$$I(m,J_1,J_2,n) \equiv \{k:\varphi_{mJ_1k}(z_1)\varphi_{kJ_2n}(z_2)\not\equiv 0\}$$

and a complex vector space

$$W(m,J_1,J_2,n) \equiv \{(\lambda_k\in C):k\in I(m,J_1,J_2,n)\};$$

$I(\bar{m},\bar{J}_1,\bar{J}_2,\bar{n})$ and $W(\bar{m},\bar{J}_1,\bar{J}_2,n)$ are defined similarly. Then the numbers $g_{k\bar{k}}$, defined in (4.47), can be interpreted as the matrix elements of a linear map

$$\left.\begin{array}{l} g:W(\bar{m},\bar{J}_1,\bar{J}_2,\bar{n})\to W(m,J_1,J_2,n)\\ (\lambda_{\bar{k}}\mapsto \sum_{\bar{k}\in I(\bar{m},\bar{J}_1,\bar{J}_2,\bar{n})} g_{k\bar{k}}\lambda_{\bar{k}}) \end{array}\right\}. \tag{4.50}$$

We are now prepared to state a key assumption.

(R4) *Non-Degeneracy Condition*: We assume that the conformal field theory on the Hilbert space $\mathscr{H}$ is of order $(p,\bar{p})$. If the distance $\text{dist}(\text{spec}(L_0P_{k\bar{k}}),\text{spec}(L_0P_{k'\bar{k}'})) \equiv d(k\bar{k},k'\bar{k}')$, for $k\bar{k}\in\Sigma$ and $k'\bar{k}'\in\Sigma$, $k\bar{k}\neq k'\bar{k}'$, obeys

$$d(k\bar{k},k'\bar{k}') = 0, \tag{4.51}$$

and if $\varphi_{kJn}(z)\not\equiv 0$, for some n and J, then

$$\varphi_{k'Jn}(z) \equiv 0. \tag{4.52}$$

An analogous assumption is made for the fields $\varphi_{\bar{k}\bar{J}\bar{n}}(\bar{z})$, $\varphi_{\bar{k}'\bar{J}\bar{n}}(\bar{z})$ if $\bar{d}(k\bar{k},k'\bar{k}') = 0$.

Finally, we assume that the matrix g defined in (4.50) is a *regular* matrix, for arbitrary $m\bar{m}$ and $n\bar{n}$ in Σ and arbitrary $J_1\bar{J}_1$, $J_2\bar{J}_2$. ∎

Definition 4.6 A conformal field theory, consisting of $(\mathscr{F},\mathfrak{A},\mathscr{H},\bar{\mathscr{H}})$, is called *rational* if it has properties (R1)–(R4).

The simplest rational theories are the *minimal models*; see (3.23), [1]. Other examples are provided by theories for which $\mathfrak{A} = \mathscr{A} \otimes \bar{\mathscr{A}}$, with $\mathscr{A}$ and $\bar{\mathscr{A}}$ some spin-1 current algebras, such as the Wess–Zumino–Witten models [17]. These models are examples of theories of order $(1,1)$. However, theories with parafermions [21] are of order $(p \geq 1, \bar{p} \geq 1)$, with $(p,\bar{p}) \neq (1,1)$.

Our non-degeneracy condition has the following important consequence.

Lemma 4.7. *We assume that* $(\mathscr{F},\mathfrak{A},\mathscr{H},\bar{\mathscr{H}})$ *has properties* (R1)–(R4). *Then, for fixed* $J_1,\ldots,J_n$, *the conformal blocks,*

$$F_{J_1\cdots J_n}^{j_1\cdots j_{n-1}}(z_1,\ldots,z_n) = F_{a_1\cdots a_n}(z_1,\ldots,z_n),\qquad (4.53)$$

defined in (4.7), (4.16), *are linearly independent functions. Furthermore, for fixed* m,J_1,J_2 *and* n, *the functions* $F_{J_1J_2}^k(z_1,z_2)$, $k \in I(m,J_1,J_2,n)$, *defined in* (4.48), *are linearly independent.*

Proof. We claim that, for all $k = 1,\ldots,n-1$,

$$\exp\left(\tau\sum_{r=1}^{k} h_{J_r}\right)F_{J_1\cdots J_n}^{j_1\cdots j_{n-1}}(e^\tau z_1,\ldots,e^\tau z_k, z_{k+1},\ldots,z_n)$$

$$= \exp(-\tau H(k))[\text{const.} + O(e^{-(\tau/p)})],\qquad (4.54)$$

where h_J is the conformal weight of $\varphi_{jJk}(z)$, see (3.50), for some $H(k) \in \mathrm{spec}\,(L_0 P_{j_kj_k})$. Equation (4.54) follows from (4.49), since intermediate states contributing to $F_{\tilde{J}}^l$ between the kth and the $k+1^{\text{st}}$ argument of $F_{\tilde{J}}^l$ are in the range of the projection $P_{j_kj_k}$; (see (4.10), (4.12)). By (4.54), a conformal block, $F_{\tilde{J}}^l$ can be a linear combination of non-zero conformal blocks, $F_{\tilde{J}}^{l^l}$, $l = 1,\ldots,N$, with non-zero coefficients $\lambda_1,\ldots,\lambda_N$, for some $N \geq 1$, only if $d(j_k\bar{j}_k, j_k^l\bar{j}_k^l) = 0$, for all $k = 1,\ldots,n-1$, for some l. Clearly $j_n = j_n^l = 1$, hence, by the non-degeneracy assumption (4.51), (4.52), $j_{n-1} = j_{n-1}^l$. Using (4.51) and (4.52) again, we then conclude that $j_{n-2} = j_{n-2}^l$, and so on. Thus $\underline{j} = \underline{j}^l$, and hence $F_{\tilde{J}}^l = F_{\tilde{J}}^{l^l}$, for some l. This proves the first part of Lemma 4.7. To prove the linear independence of the functions $F_{J_1J_2}^k(z_1,z_2)$, we note that by (4.48), (4.43) and (4.40), (4.42),

$$\exp[\tau(h_{J_1} - H_A(m))]F_{J_1J_2}^k(e^\tau z_1, z_2) = \exp(-\tau H(k))[\text{const.} + O(e^{-(\tau/p)})],\qquad (4.55)$$

for some $H_A(m)$ (independent of k), and some $H(k)$. The remaining part of the proof is similar to the one before. ∎

Equations (4.35), (4.36), Lemma 3.7 and the regularity of the matrix $g = (g_{k\bar{k}})$, see (R4), have an important consequence: For $b = (t,[\gamma]) \in B_2$,

$$F_{J_1J_2}^k(b^{-1}(Z_1,Z_2)) = \sum_{k'} R_{k'}^k(b) F_{J_{t(1)}J_{t(2)}}^{k'}(Z_1,Z_2),\qquad (4.56)$$

where $R(b) = R(A,m,J_1,J_2,B,n;b)$ is a regular matrix on the vector space

$W(m, J_1, J_2, n)$. Similarly,

$$\bar{F}^{\bar{k}}_{\bar{J}_1 \bar{J}_2}(b^{-1}(\bar{Z}_1, \bar{Z}_2)) = \sum_{\bar{k}'} \bar{R}^{\bar{k}}_{\bar{k}'} \bar{F}^{\bar{k}'}_{\bar{J}_{t(1)} \bar{J}_{t(2)}}(\bar{Z}_1, \bar{Z}_2). \tag{4.57}$$

[Taking into account Lemma 3.7 and the regularity of g, one sees that the details of the proofs of (4.56) and (4.57) are very similar to the arguments (4.27)–(4.34) used to prove (4.35) and (4.36).] Equations (4.56), (4.57) together with *locality* imply that

$$R(b)^T g \bar{R}(b^*) = g. \tag{4.58}$$

The proof of (4.58) is analogous to the one of (4.38).

Next, we note that B_2 is generated by a single element $\tau_1 \equiv \tau$; see Fig. 1. We define a matrix R by setting

$$R = R(\tau). \tag{4.59}$$

Then, for $b = \tau^m \in B_2$, $m \in \mathbb{Z}$,

$$R(b) = R^m. \tag{4.60}$$

Hence, the representations of B_2 determined by the conformal blocks $\{F^k_{J_1 J_2}\}_{k \in I(m, J_1, J_2, n)}$, $\{F^{\bar{k}}_{\bar{J}_1 \bar{J}_2}\}$ are completely described by two matrices, R and $\bar{R}$. Since $\tau^* = \tau^{-1}$, (4.58) is equivalent to

$$R^T g \bar{R}^{-1} = g. \tag{4.61}$$

We now claim that, under reasonable assumptions on the chiral algebras $\mathscr{A}$ and $\bar{\mathscr{A}}$ discussed below,

$$R = R(A, m, J_1, J_2, B, n) \equiv R(m, J_1, J_2, n) \tag{4.62}$$

is *independent* of A and B, and

$$\bar{R} = \bar{R}(\bar{A}, \bar{m}, \bar{J}_1, \bar{J}_2, \bar{B}, \bar{n}) \equiv \bar{R}(\bar{m}, \bar{J}_1, \bar{J}_2, \bar{n}) \tag{4.63}$$

is *independent* of $\bar{A}$ and $\bar{B}$.

This has remarkable consequences: It says that the representation of B_2 determined by

$$\langle A v_i^\alpha, \varphi_{m J_1 k}(z_1) \varphi_{k J_2 n}(z_2) B v_j^\beta \rangle$$

(see (4.48)) under interchanging 1 and 2 and exchanging z_1 and z_2 along the path τ shown in Fig. 2 is *independent* of admissible operators $A, B \in \mathscr{A}$.

Hence (4.62) says that

$$\varphi_{m J_1 k}(z_1) \varphi_{k J_2 n}(z_2) = \sum_{k'} R(m, J_1, J_2, n)^{k'}_k \varphi_{m J_2 k'}(z_2) \varphi_{k' J_1 n}(z_1), \tag{4.64}$$

for $|z_2| > |z_1|$, where the left-hand side of (4.64) is interpreted as the analytic continuation of $\varphi_{m J_1 k}(z) \varphi_{k J_2 n}(z')$ from $z = z_2$, $z' = z_1$ to $z = z_1$, $z' = z_2$ along a path τ as in Fig. 2 *not* enclosing any other operators. Equation (4.64) permits us to describe the monodromy of an arbitrary conformal block, $F^{j_1 \cdots j_n - 1}_{J_1 \cdots J_n}(Z)$, (see (4.16)
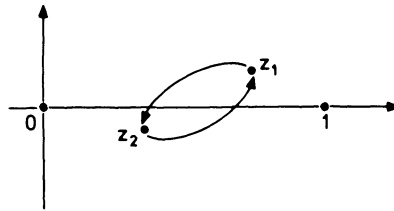
Fig. 2

and (4.7)) in terms of the matrix R introduced in (4.59), (4.62): Let

$$I \equiv \{(iJj): \varphi_{iJj}(z) \not\equiv 0\}.$$

Let V be the complex vector space given by

$$\{(\lambda_a \in \mathbb{C}): a = (iJj) \in I\}. \tag{4.65}$$

We define an endomorphism, $R = (R_{ab}^{cd})$, from $V \otimes V$ to $V \otimes V$ by setting

$$R_{ab}^{cd} = R_{(iJk)(kJ'j)}^{(iJ'k')(k'Jj)} \equiv R(i, J, J', j)_k^{k'}, \left.\right\}$$

and

$$R_{ab}^{cd} = 0, \quad \text{otherwise.} \tag{4.66}$$

Let $V_1, \ldots, V_n$ be isomorphic copies of V and define

$$R_i \equiv \mathbb{1}|_{V_1} \otimes \cdots \otimes \mathbb{1}|_{V_{i-1}} \otimes R|_{V_i \otimes V_{i+1}} \otimes \mathbb{1}|_{V_{i+2}} \otimes \cdots \otimes \mathbb{1}|_{V_n}. \tag{4.67}$$

The associativity of the algebra generated by the chiral fields $\varphi_{iJk}(z)$ and Eqs. (4.64), (4.65) imply [5] that

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \tag{4.68}$$

and, by definition (4.67),

$$R_i R_j = R_j R_i, \quad \text{for} \quad |j - i| \geq 2. \tag{4.69}$$

Equation (4.68) is a special case of the Yang–Baxter equation [23]. We claim that the matrices R_i define a representation

$$R^{(n)}: B_n \mapsto \text{End}(V^{\otimes n}). \tag{4.70}$$

To see this, write an element $b \in B_n$ as a word in the generators τ_i, $i = 1, \ldots, n-1$, as in the proof of Proposition 4.3:

$$b = \tau_{i_1}^{\varepsilon_1} \cdots \tau_{i_k}^{\varepsilon_k}, \quad \varepsilon_j = \pm 1.$$

One then defines

$$R^{(n)}(b) = \prod_{j=1}^{k} R_{i_j}^{\varepsilon_j}. \tag{4.71}$$

Equations (4.68) and (4.69) ensure that (4.71) defines a representation of the braid group, B_n, on the plane. Because of (4.64), this representation describes the transformation properties of the conformal blocks, $F_{a_1 \cdots a_n}(\mathcal{Z})$, under the action of

B_n on $\tilde{M}_n$:

$$F_{a_1\cdots a_i a_{i+1}\cdots a_n}(\tau_i^{-1}\underline{Z}) = \sum_{a_i', a_{i+1}'} R_{a_i a_{i+1}}^{a_i' a_{i+1}'} F_{a_1\cdots a_i' a_{i+1}'\cdots a_n}(\underline{Z}). \tag{4.72}$$

This follows from (4.64) by rewriting this equation as

$$\varphi_a(z_1)\varphi_b(z_2) = \sum_{cd} R_{ab}^{cd}\varphi_c(z_2)\varphi_d(z_1), \tag{4.73}$$

with $a = (iJj), \dots$. More generally,

$$F_{\underline{a}}(b^{-1}\underline{Z}) = \sum_{\underline{a}'} R_{\underline{a}}^{\underline{a}'}(b) F_{\underline{a}'}(\underline{Z}), \tag{4.74}$$

where $R(b)$ is given by (4.71). Analogous results hold for the conformal blocks $\bar{F}_{\underline{a}}(\bar{Z})$ and the matrix $\bar{R}$.

It remains to discuss the basic properties (4.62) and (4.63) from which the results derived above follow. In order to "derive" (4.62) and (4.63), we return to definitions (4.39) and (4.48) of the functions $K_{J_1\bar{J}_1 J_2\bar{J}_2}^{k\bar{k}}(z_1, \bar{z}_1, z_2, \bar{z}_2)$, $F_{J_1 J_2}^k(z_1, z_2)$. We introduce so-called $\mathfrak{A}$-*descendant fields* $\phi_{J\bar{J}}^{[\underline{a}, \underline{b}]}(z, \bar{z})$:

$$\phi_{J\bar{J}}^{[\underline{a}, \underline{b}]}(z, \bar{z}) = \prod_{k=1}^{N} \oint_{C_k} dz_k (z_k - z)^{a_k + h_{J_k} - 1} \psi_{j_k}(z_k)$$

$$\prod_{l=1}^{M} \oint_{\bar{C}_k} d\bar{z}_l (\bar{z}_l - \bar{z})^{b_l + h_{\bar{l}_l} - 1} \psi_{\bar{j}_l}(\bar{z}_l) \phi_{J\bar{J}}(z, \bar{z}), \tag{4.75}$$

$$\underline{a} = (a_1, \dots, a_N), \quad a_i \in \mathbb{R}, \quad \underline{b} = (b_1, \dots, b_M), \quad b_i \in \mathbb{R},$$

$N, M = 0, 1, 2, \dots$. Here C_k is a circle of radius ε/k, centered at z, $\bar{C}_k$ is a circle of radius $\bar{\varepsilon}/k$, centered at $\bar{z}$, and (4.75) holds if both sides are inserted into a Green function, and $\varepsilon > 0$, $\bar{\varepsilon} > 0$ are chosen to be so small that the contours C_1 and $\bar{C}_1$ do not enclose any fields not appearing on the right-hand side of (4.75). Using (4.40)–(4.42) and the fact that $v_{ii}^\alpha, v_{jj}^\beta$ are invariant states, one now shows that

$$K_{J_1\bar{J}_1 J_2\bar{J}_2}^{k\bar{k}}(z_1, \bar{z}_1, z_2, \bar{z}_2) \equiv \langle A \otimes \bar{A} v_{ii}^\alpha, \phi_{J_1\bar{J}_1}(z_1, \bar{z}_1) P_{k\bar{k}} \phi_{J_2\bar{J}_2}(z_2, \bar{z}_2) B \otimes \bar{B} v_{jj}^\beta \rangle$$

$$= \sum_{\underline{a}_1, \underline{b}_1, \underline{a}_2, \underline{b}_2} \lambda(\underline{a}_1, \underline{a}_2)\bar{\lambda}(\underline{b}_1, \underline{b}_2) \tag{4.76}$$

$$\cdot \langle v_{ii}^\alpha, \phi_{J_1\bar{J}_1}^{[\underline{a}_1, \underline{b}_1]}(z_1, \bar{z}_1) \phi^{[\underline{a}_2, \underline{b}_2]}(z_2, \bar{z}_2) v_{jj}^\beta \rangle,$$

for some complex coefficients $\lambda(\underline{a}_1, \underline{a}_2)$ and $\bar{\lambda}(\underline{b}_1, \underline{b}_2)$ completely determined by the algebraic relations in $\mathscr{A}$, $\bar{\mathscr{A}}$ respectively. In (4.76) we have used that $A \otimes \bar{A}$ and $B \otimes \bar{B}$ commute with $P_{k\bar{k}}$.

From (4.76) and (4.46)–(4.48) we conclude that

$$F_{J_1 J_2}^k(z_1, z_2) = \langle A v_i^\alpha, \varphi_{mJ_1 k}(z_1) \varphi_{kJ_2 n}(z_2) B v_j^\beta \rangle$$

$$= \sum_{\underline{a}_1, \underline{a}_2} \lambda(\underline{a}_1, \underline{a}_2) \langle v_i^\alpha, \varphi_{mJ_1 k}^{[\underline{a}_1]}(z_1) \varphi_{kJ_2 n}^{[\underline{a}_2]}(z_2) v_j^\beta \rangle, \tag{4.77}$$

where $\varphi_{mJk}^{[\underline{a}]}$ is defined as in (4.75) but with $M = 0$. An equation similar to (4.77) holds for the functions $\bar{F}_{\bar{J}_1\bar{J}_2}^{\bar{k}}(\bar{z}_1, \bar{z}_2)$.

The point is now that using Ward identities, (see (3.36), (3.37), (3.40)) one can

usually prove that the representation of B_2 determined by

$$\langle v_i^\alpha, \varphi_{mJ_1k}^{[a_1]}(z_1)\varphi_{kJ_2n}^{[a_2]}(z_2)v_j^\beta\rangle \tag{4.78}$$

is *independent* of $[\underline{a}_1]$, $[\underline{a}_2]$, [i.e. Eq. (4.56) also holds for the function (4.78), with the *same* matrix $R_{k\cdot}^k(b)$].

In the arguments outlined above, Eq. (4.76) and (4.78) have not been derived from our basic assumptions. It will be studied elsewhere how to derive these statements from a general definition of chiral algebras. The reader familiar with refs. [1, 17, 18, 20 and 21] may, however, verify without too much trouble the following result.

Theorem 4.8. *If $\mathscr{A}$, $\bar{\mathscr{A}}$ are Virasoro-, or current algebras then (4.76)–(4.78) and hence (4.62)–(4.64) hold. For Neveu–Schwarz [20] and parafermion [21] algebras (4.62)–(4.64) hold.*

Remark. Theorem 4.8 and the analysis preceding it show that, for rational conformal field theories based on chiral algebras $\mathscr{A}, \bar{\mathscr{A}}$ which are algebras of currents and parafermions, the monodromy of the "four-point functions"

$$\langle v_i^\alpha, \varphi_{mJ_1k}(z_1)\varphi_{kJ_2n}(z_2)v_j^\beta\rangle$$

is completely described by a Yang–Baxter matrix, R, and determines the monodromy of arbitrary conformal blocks, $F_{J_1\ldots J_n}^{j_1\ldots j_{n-1}}(z_1,\ldots,z_n)$. These results can be coded into the commutation relations (4.64) for the chiral fields $\varphi_{iJk}(z)$.

5. Conformal Field Theory as Representation Theory of Chiral Algebras

The purpose of this last section is to describe the general mathematical structure of two-dimensional conformal field theory that has emerged from our analysis in Sects. 2–4. It will be the subject of our next paper to initiate a systematic analysis of that structure. The upshot of our analysis is that two-dimensional conformal field theory is, in essence, a chapter in the representation theory of an infinite dimensional "symmetry algebra" $\mathfrak{A} = [\mathscr{A}\otimes\bar{\mathscr{A}}]_{\text{loc.}}$, where $\mathscr{A}$ and $\bar{\mathscr{A}}$ are chiral algebras. Here we shall attempt to make this statement more precise. For the sake of simplicity we shall limit our analysis to the case where the algebras $\mathscr{A}$ and $\bar{\mathscr{A}}$ are generated by local currents, so that $\mathfrak{A} = \mathscr{A}\otimes\bar{\mathscr{A}}$. But the general case is not substantially more complicated.

Let us first clarify what is meant by the notion of a (local) chiral algebra. An abstract chiral algebra, $\mathscr{A}$, is an algebra generated by (unbounded) operator-valued distributions $\psi_i(x)$, $x\in\mathbb{R}$, and i ranges over a finite or countably infinite index set I. There is an involution $*$, $i\in I \to i^*\in I$, on I such that, for all $i\in I$,

$$\psi_i(x)^* \equiv \psi_{i^*}(x) \quad \text{is a generator of } \mathscr{A}. \tag{5.1}$$

General elements of $\mathscr{A}$ are polynomials in the generators $\psi_i(x)$, $i\in I$, smeared out with arbitrary test functions of compact support.

The generators $\psi_i(x)$, $i\in I$, satisfy quadratic relations of the form

$$\psi_i(x)\psi_j(x) = [R^\varepsilon]_{ij}^{kl}\psi_k(x)\psi_l(x), \tag{5.2}$$

where $R = (R_{ij}^{kl})$ is a matrix on $\mathbb{C}^{|I|} \otimes \mathbb{C}^{|I|}$ which is a solution of the Yang–Baxter equation (with spectral parameter $\rightarrow i\infty$). Furthermore,

$$\varepsilon = \operatorname{sig}(x - x') = \pm 1. \tag{5.3}$$

One assumes that there are no further quadratic relations between the generators of $\mathscr{A}$, for $x \neq x'$.

It is assumed, furthermore, that $\mathscr{A}$ carries a representation of the subgroup, $\cong PSL(2, \mathbb{R})$, of the Möbius group, $\cong PSL(2, \mathbb{C})$, which leaves the real axis invariant as a group of *automorphisms of $\mathscr{A}$: For $w \in PSL(2, \mathbb{R})$, there exists a linear operator, τ_w, on $\mathscr{A}$ such that $\tau_w \circ \tau_{w'} = \tau_{w \circ w'}$, and

$$\tau_w(A \cdot B) = \tau_w(A) \cdot \tau_w(B), \quad \tau_w(A^*) = \tau_w(A)^*,$$

for all A, B in $\mathscr{A}$; moreover

$$\tau_w(\psi_i(x)) = \left(\frac{dw}{dx}\right)^{h_i}(x)\psi_i(w(x)), \tag{5.4}$$

for some $h_i \in \mathbb{R}$, called conformal weight of ψ_i. We assume that the generators $\psi_i(x)$ are "analytic vectors" for τ_w, in the sense that (5.4) has an analytic continuation in w to a complex neighborhood of $SL(2, \mathbb{R})$ in $SL(2, \mathbb{C})$. Clearly, it must be assumed that (5.2) and (5.4) are compatible which puts restrictions on the possible Yang–Baxter matrices, R, appearing in (5.2): E.g.

$$R_{ij}^{kl} = 0, \quad \text{unless } h_i = h_l, \quad \text{and} \quad h_j = h_k. \tag{5.5}$$

See [5] for more details.

We shall assume that, among the generators $\psi_i(x)$, $i \in I$, there is an identity operator, $\psi_0(x) \equiv 1 = 1^*$, independent of x, (i.e. $h_0 = 0$). We do not exclude further polynomial relations between the generators $\psi_i(x)$ of $\mathscr{A}$—in addition to (5.2)—of degree higher than two.

A state, ω, on $\mathscr{A}$ is a linear functional on $\mathscr{A}$ with the property that

$$\omega(A^*A) \geqq 0, \quad \text{for all} \quad A \in \mathscr{A}. \tag{5.6}$$

Every state, ω, on $\mathscr{A}$ determines a representation, π_ω, of $\mathscr{A}$ on some Hilbert space. This is the contents of the so-called Gel'fand–Naimark–Segal construction. We assume that $(\mathscr{A}, \tau_w)$ is such that there is precisely one state ω_0 on $\mathscr{A}$ such that

1. $$\omega_0(\tau_w(A)) = \omega_0(A), \tag{5.7}$$

for all $A \in \mathscr{A}$ and all $w \in PSL(2, \mathbb{R})$, i.e. ω_0 is $PSL(2, \mathbb{R})$-invariant; and

2. $$\omega_0(A^*A) = 0 \quad \text{implies} \quad A = 0, \tag{5.8}$$

for all $A \in \mathscr{A}$.

Remark. One might envisage requiring that there be a representation, τ, of a central extension of $\operatorname{Diff}(S^1)$ as a *automorphism group of $\mathscr{A}$, where $\operatorname{Diff}(S^1)$ is the group of diffeomorphisms of the real line conjugated to diffeomorphisms of the

circle which is the image of $\mathbb{R}$ under the Möbius transformation

$$x \to -\frac{x-i}{x+i}. \tag{5.9}$$

In that case one might require that (5.4) holds for all diffeomorphisms, w, of the kind just introduced, i.e. that the generators $\psi_i(x)$, $i \in I$, are *primary fields*. However, it is *inconsistent* to assume that (5.7) hold for all $w \in \text{Diff}(S^1)$. In this paper, we do not pursue this line of thought.

It will be discussed in a separate publication how to classify general chiral algebras $(\mathcal{A}, \tau_w)$ characterized by properties (5.1), (5.2), (5.4), (5.7) and (5.8). It turns out that such algebras are algebras of currents of arbitrary spin $s = 1, 2, 3, \ldots$ and of fields representing a slight generalization of parafermions. One simple consequence of a general classification is that if

$$R_{ij}^{kl} = \delta_i^l \delta_j^k, \tag{5.10}$$

for all i, j, k and l in I, then

$$h_i \in \mathbb{Z}_+, \quad \text{for all} \quad i \in I, \tag{5.11}$$

i.e. $\mathcal{A}$ is an algebra of *local* currents of spin $s_i = h_i = 1, 2, 3, \ldots, i \in I$.

Henceforth we shall focus our attention on chiral algebras $(\mathcal{A}, \tau_w)$ satisfying (5.10). More general algebras will be studied elsewhere. Most of the concepts discussed below can be introduced in the general case, but the analysis and notations would become more cumbersome.

If $\mathcal{A}$ is an algebra of local currents we can, alternatively, work with generators, $\psi_i(e^{i\sigma})$, $-\pi < \sigma \leq \pi$, $i \in I$, defined on the unit circle in $\mathbb{C}$. This follows from (5.4), using the Möbius transformation (5.9). In that case we can trade the generators $\psi_j(e^{i\sigma})$, $j \in I$, for their Fourier-Laurent coefficients,

$$\psi_{j,n} = \int_{-\pi}^{\pi} \psi_j(e^{i\sigma}) e^{i(n+h_j)\sigma} d\sigma. \tag{5.12}$$

By (5.4), (5.9), (5.1) we have that

$$\psi_{j,n}^* = \psi_{j^*,-n}; \tag{5.13}$$

see also (3.30), (3.31). The algebra generated by $\{\psi_{j,n} : j \in I, n \in \mathbb{Z}\}$ is denoted by $\mathcal{A}$.

Let w_σ be a rotation of the unit circle through an angle σ. Then it follows from (5.4) and (5.12) that

$$\tau_{w_\sigma}(\psi_{j,n}) = e^{-in\sigma} \psi_{j,n}.$$

We define $\mathcal{A}_n$ to be the linear subspace of $\mathcal{A}$ of all elements $A \in \mathcal{A}$ for which

$$\tau_{w_\sigma}(A) = e^{-in\sigma} A.$$

Then $\mathcal{A}$ is $\mathbb{Z}$-graded, with

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n. \tag{5.14}$$

We also define $\mathcal{A}_< = \bigoplus_{n>0} \mathcal{A}_n$, $\mathcal{A}_> = \bigoplus_{n<0} \mathcal{A}_n$.

If $|I| \equiv \operatorname{card} I < \infty$, then $\mathscr{A}_0$ is often a finite-dimensional algebra. We shall assume this henceforth.

For the purpose of constructing two-dimensional conformal field theories, one must study the representation theory of $(\mathscr{A}, \tau_w)$. Since we are interested in unitary theories, we are only interested in representations, π, of $(\mathscr{A}, \tau_w)$ with the following properties:

(a) π is a representation of $(\mathscr{A}, \tau_w)$ on a separable Hilbert space, $\mathscr{H}_\pi$, which is *unitary*, i.e.

$$\pi(A^*) = \pi(A)^*, \quad \text{for all} \quad A \in \mathscr{A}, \tag{5.15}$$

where A^* is defined through (5.1); and π is *covariant*, i.e. there exists a unitary representation, U_π, of $SL(2, \mathbb{R})$ (now defined as those Möbius transformations which map the unit circle onto itself) on $\mathscr{H}_\pi$ such that, for all $A \in \mathscr{A}$ and $w \in SL(2, \mathbb{R})$,

$$\pi(\tau_w(A)) = U_\pi(w)\pi(A)U_\pi(w)^{-1}, \tag{5.16}$$

as an operator equation on a dense domain, $\mathscr{D}_\pi$, in $\mathscr{H}_\pi$ which is invariant under $\pi(\mathscr{A})$.

(b) We assume that $\mathscr{D}_\pi$ can be chosen such that it consists of analytic vectors for $U_\pi(w)$, i.e., for $\Phi \in \mathscr{D}_\pi$, $U_\pi(w)\Phi$ has an analytic continuation in w to some complex neighborhood of $SL(2, \mathbb{R})$ in $SL(2, \mathbb{C})$.

(c) Let w_σ denote the rotation of the unit circle through an angle σ. Then

$$U_\pi(w_\sigma) = e^{i\sigma L_0^\pi}, \tag{5.17}$$

for some selfadjoint operator L_0^π on $\mathscr{H}_\pi$. We require that

$$L_0^\pi \geqq 0, \tag{5.18}$$

and that $h_\pi = \inf \operatorname{spec}(L_0^\pi)$ is an eigenvalue of L_0^π of finite multiplicity.

Note that by (5.15), $\pi(\mathscr{A}_0)$ commutes with L_0^π. For the last part of assumption (c) to hold it is therefore commonly necessary that $\mathscr{A}_0$ be finite-dimensional.

Representations, π, of $(\mathscr{A}, \tau_w)$ satisfying properties (a)–(c) are called "*positive-energy representations*," [19]. Let L be a list of all positive-energy representations of $(\mathscr{A}, \tau_w)$. By assumption (5.7), L contains precisely one representation, $\pi_1 \equiv \pi_{\omega_0}$, $(1 \in L)$, on a Hilbert space $\mathscr{H}_1 \equiv \mathscr{H}_{\pi_1}$ containing a vector $\Omega \in \mathscr{D}_{\pi_1}$ that is *invariant* under $U_1 \equiv U_{\pi_1}$, with $L_0^1 \Omega = 0$. By (5.8), Ω is separating for $\mathscr{A}$, i.e.

$$\text{if } \pi_1(A)\Omega = 0 \quad \text{then} \quad A = 0, \tag{5.19}$$

for all $A \in \mathscr{A}$. [Note that this does *not* imply that Ω is separating for $\mathscr{A}$.]

By assumption (b), every positive-energy representation, π, of $(\mathscr{A}, \tau_w)$ determines operators $L_{\pm 1}$ such that

$$(L_1^\pi)^* = L_{-1}^\pi, \quad \text{and} \quad [L_1^\pi, L_{-1}^\pi] = 2L_0^\pi. \tag{5.20}$$

Moreover, L_1^π generates Möbius transformations of the form $z \to z/1 + \tau z, L_{-1}^\pi$ generates translations $z \to z + \tau$. Vectors in $\mathscr{D}_\pi$ are analytic vectors for $L_0^\pi, L_{\pm 1}^\pi$.

Remarks. (1) Note that the fact that $L_0^1 \Omega = 0$ follows from the invariance of $\Omega \in \mathscr{D}_{\pi_1}$ under U_1 and from (5.20).

(2) If $\mathscr{A}$ were a C^* algebra then (5.18) would imply that the spectral projections of L_0 are contained in $\pi(\mathscr{A})''$, the weak closure of $\pi(\mathscr{A})$. When working with unbounded operators—as we do—one might *assume* that there is a generator $T(e^{i\sigma})\in\mathscr{A}$, with conformal weight $h_T = 2$, such that, for $n = -1, 0, 1$,

$$L_n^\pi = \int_{-\pi}^{\pi} d\sigma\, e^{i(n+2)\sigma} \pi(T(e^{i\sigma})), \qquad (5.21)$$

in every positive-energy representation π of $(\mathscr{A}, \tau)$. Then a variant of the Lüscher–Mack theorem would imply that (5.21) defines Virasoro generators, for $n\in\mathbb{Z}$, so that $\mathscr{A}$ contains Vir.

We define a linear deformation map, δ_z, depending on a complex number z, with $0 < |z| < \infty$, on the linear space, $G(\mathscr{A})$, spanned by the generators $\{\psi_{j,n}\}_{j\in\mathscr{L}, n\in\mathbb{Z}}$, of $\mathscr{A}$,

$$\delta_z(\psi_{j,n}) = \sum_{k=-h_j+1}^{\infty} \binom{n+h_j-1}{k+h_j-1} z^{n-k} \psi_{j,k}, \qquad (5.22)$$

where $\binom{n}{m}$ is the usual binomial coefficient defined to vanish, for $m > n \geq 0$, with $\binom{0}{0} = 1$. (Our definition of δ_z is motivated by the contour integral formalism of [1]). Note that δ_z is *not* a $*$ endomorphism of $\mathscr{A}$, and that

$$\delta_z(1) = 0, \qquad (5.23)$$

(so that δ_z 'kills" the central charge).

Next, we introduce an analogue of the notion of tensor operators in group theory: Let $i_1, \ldots, i_n, j$ and k be positive-energy representations of $(\mathscr{A}, \tau_w)$. Let v_l be a vector in $\mathscr{H}_{i_l}$. A *generalized vertex* $V_{jk}(v_1, \ldots, v_n; Z_1, \ldots, Z_n)$ is an operator-valued function of $(Z_1, \ldots, Z_n)\in\tilde{M}_n$, (with $|z_i| < 1$, for $i = 1, \ldots, n$) mapping a dense domain in $\mathscr{H}_k$ to a dense domain in $\mathscr{H}_j$, with the following properties:

(a) $V_{jk}(v_1, \ldots, v_n; Z_1, \ldots, Z_n)$ is multi-linear in the arguments $v_1, \ldots, v_n$;

(b) $\pi_j(A) V_{jk}(v_1, \ldots, v_n; Z_1, \ldots, Z_n) = \sum_{l=1}^{n} V_{jk}(v_1, \ldots, \pi_{i_l}(\delta_{z_l}(A))v_l, \ldots, v_n; Z_1, \ldots, Z_n)$

$$+ V_{jk}(v_1, \ldots, v_n; Z_1, \ldots, Z_k)\pi_k(A), \qquad (5.24)$$

for every generator $A = \int_{-\pi}^{\pi} e^{i(n+h_j)\sigma} \psi_j(\sigma) d\sigma$ of $\mathscr{A}$; and

(c) If v_l is a highest-weight vector for Vir then, for all $m\in\mathbb{Z}$,

$$\left[z_l^{m+1} \frac{\partial}{\partial z_l} + z_l^m (m+1) h_l \right] V_{jk}(v_1, \ldots, v_l, \ldots, v_n; Z_1, \ldots, Z_l, \ldots, Z_n)$$
$$= V_{jk}(v_1, \ldots, L_m^{i_l}(z_l)v_l, \ldots, v_n; Z_1, \ldots, Z_l, \ldots, Z_n), \qquad (5.25)$$

where

$$L_m^i(z) = \pi_i(\delta_z(L_m)).$$

A special case of (5.25) is

$$\frac{\partial}{\partial Z_l} V_{jk}(v_1,\ldots,v_l,\ldots,v_n;Z_1,\ldots,Z_l,\ldots,Z_n)$$

$$= V_{jk}(v_1,\ldots,L^{i_l}_{-1}v_l,\ldots,v_n;Z_1,\ldots,Z_l,\ldots,Z_n). \tag{5.26}$$

This defines a flat connection on the space of generalized vertices. The holonomy matrices of this connection associated with paths corresponding to elements of the braid group B_n determine a representation, $R^{(n)}$, of B_n.

A special example of a generalized vertex is

$$\varphi_{jvk}(z) = V_{jk}(v,z), \tag{5.27}$$

for $v \in \mathcal{H}_i$, $|z| < 1$. The operators $\varphi_{jvk}(z)$ correspond to the ones constructed in Lemma 3.2 from a sequence of Green functions of a two-dimensional conformal field theory.

In principle, the notion of generalized vertices is purely *representation theoretic*. One might hope that, given some $(\mathscr{A},\tau_w)$ for which all positive-energy representations are known, one could construct all generalized vertices satisfying (a)–(c), above, in particular, one could construct the operators $\varphi_{jvk}(z)$, on the basis of purely representation theoretic arguments. This hope has materialized for the case where $\mathscr{A}$ is the Virasoro algebra with central charge $c = 1 - 6/p(p+1), p \geq 3$; see [24]. In this case, a basis of generalized vertices is obtained from the operators

$$V_{jk}^{j_1\cdots j_{n-1}}(v_1,\ldots,v_n;z_1,\ldots,z_n) = \prod_{l=1}^{n} \varphi_{j_{l-1}v_lj_l}(z_l), \tag{5.28}$$

with $j_0 = j$, $j_n = k$, and $(z_1,\ldots,z_n) \in K_n^>$, where

$$K_n^> \equiv \{z:|z_1| > \cdots > |z_n|, -\pi < \arg z_i < \pi, i = 1,\ldots,n\},$$

by analytic continuation in $(z_1,\ldots,z_n)$ to $\tilde{M}_n$. We now assume that this property holds for all chiral algebras considered henceforth. In this case, the representation $R^{(n)}$ of B_n can be determined from the representation R of B_2 obtained from the generalized vertices $V_{jk}^i(v_1,v_2;Z_1,Z_2)$ which are analytic continuations of the product $\varphi_{jv_1i}(z_1)\varphi_{iv_2k}(z_2)$.

We assume that the representation of B_2 determined by the operators $\{V_{jk}^i(v_1,v_2;Z_1,Z_2)\}_{i \in L}$ only depends on j,i_1,i_2 and k, but not on the choice of $v_1 \in \mathcal{H}_{i_1}$, $v_2 \in \mathcal{H}_{i_2}$, and has the form

$$V_{jk}^i(v_1,v_2;Z_1,Z_2) = \sum_m R(j,i_1,i_2,k)_m^i V_{jk}^m(v_2,v_1;\tau(Z_1,Z_2)), \tag{5.29}$$

where the matrix $R(j,i_1,i_2,k)$ is a solution of the Yang–Baxter equation; see (4.62), (4.64); τ is the generator of B_2. [Part of this assumption follows from (5.25)–(5.26).] In more informal notation, (5.29) says that

$$\varphi_{jv_1i}(z_1)\varphi_{iv_2k}(z_2) = \sum_m R(j,i_1,i_2,k)_m^i \varphi_{jv_2m}(z_2)\varphi_{mv_1k}(z_1), \tag{5.30}$$

if z_1 and z_2 are exchanged along a positively oriented path. If the vertices introduced in (5.28) form a basis of generalized vertices then (5.29) determines the representations $R^{(n)}$ of B_n, for all n.

How much of the structure described here can be derived from the representation theory of $(\mathscr{A}, \tau_w)$ remains to be investigated. The only example that is essentially completely understood is the example of the Virasoro algebra with central charge $c = 1 - 6/p(p + 1)$, $p \geq 3$; see [24]. In more general cases, we have succeeded in deriving a list of constraints on the matrices $R(j, i_1, i_2, k)$ that follow from the structure described above and to derive the chiral fusion rules [25]. The fusion rules permit us, in principle, to calculate matrices $R(j, i_1, i_2, k)$ from a few basic R-matrices. These results will be presented in paper II [26] of this series. Assuming that the matrices $R(j, i_1, i_2, k)$ are all given, the construction of generalized vertices $V_{jk}(v_1, \ldots, v_n, Z_1, \ldots, Z_n)$ satisfying properties (a)–(c) can be viewed as a generalization of the Riemann–Hilbert problem. We have essentially no results to report on its solution, but the subject is under investigation.

Let us now suppose that we are given a pair of chiral algebras $(\mathscr{A}, \tau_w)$, $(\bar{\mathscr{A}}, \bar{\tau}_w)$ with all the properties described above. We propose to sketch how one may reconstruct a local, unitary conformal field theory from these data. More details will appear in papers II and III of this series.

With $(\mathscr{A}, \tau_w)$ we associate chiral operators $\varphi_{jvk}(z)$ having all the properties described above. Similarly, the operators $\varphi_{\bar{j}\bar{v}\bar{k}}(\bar{z})$ correspond to $(\bar{\mathscr{A}}, \bar{\tau}_w)$. We define an index set

$$\Delta \equiv \{ jik : \varphi_{jvk}(z) \neq 0, \text{ for some } v \in \mathscr{H}_i \}$$

and a complex vector space

$$V \equiv \{ (\lambda_{jik} \in \mathbb{C}) : jik \in \Delta \}. \tag{5.31}$$

The objects $\bar{\Delta}, \bar{V}$ are defined similarly. Let $R : V \otimes V \to V \otimes V$ and $\bar{R} : \bar{V} \otimes \bar{V} \to \bar{V} \otimes \bar{V}$ be the Yang–Baxter matrices generating the representations $R^{(n)}$ of B_n on the space of generalized vertices V_{jk}, the representations $\bar{R}^{(n)}$ on $\bar{V}_{\bar{j}\bar{k}}$, respectively; see (4.66)–(4.71).

We now look for coefficients, $C_{jik}^{\bar{j}\bar{i}\bar{k}}$, such that the fields

$$\phi_{v \otimes \bar{v}}(z, \bar{z}) = \sum_{j, k, \bar{j}, \bar{k}} C_{jik}^{\bar{j}\bar{i}\bar{k}} \varphi_{jvk}(z) \otimes \varphi_{\bar{j}\bar{v}\bar{k}}(\bar{z}) \tag{5.32}$$

with $v \in \mathscr{H}_i$, $\bar{v} \in \mathscr{H}_{\bar{i}}$, are *local* fields, (in particular, their vacuum expectation values are all symmetric). We may interpret the coefficients $C_{jik}^{\bar{j}\bar{i}\bar{k}}$ as the matrix elements of a linear map C from $\bar{V}$ to V. It is easy to show that the fields $\phi_{v \otimes \bar{v}}(z, \bar{z})$ defined in (5.32) are local iff

$$R(\tau)^T [C \otimes C] \bar{R}(\tau^{-1}) = C \otimes C. \tag{5.33}$$

This is an overdetermined system of equations for the matrix elements $C_{jik}^{\bar{j}\bar{i}\bar{k}}$; see [27] and papers II and III. These equations have solutions, provided the matrices R and $\bar{R}$ satisfy certain polynomial constraints derived and analyzed in Papers II and III. These constraints have been vertified for the example where $\mathscr{A}$ and $\bar{\mathscr{A}}$ are isomorphic to the Virasoro algebra with central charge $c = \bar{c} = 1 - 6/p(p + 1), p \geq 3$. This leads to the minimal models.

For more general classes of models, the basic problem is to construct the chiral fields $\varphi_{jvk}(z)$, $\varphi_{\bar{j}\bar{v}\bar{k}}(\bar{z})$ and to calculate their vacuum expectation values. The problem

of solving (5.33) is then comparatively easy. Paper II is intended to represent a first step towards a general theory of chiral fields $\varphi_{jvk}(z)$. We analyze the properties of the set of matrices $\{A_i\}_{i \in L}$, where $(A_i)_{jk}$ is the number of chiral fields $\varphi_{jvk}(z) \not\equiv 0$, where v is an *invariant* vector in $\mathscr{H}_i$. We also analyze chiral fusion, i.e. we derive equations for the coefficients in the operator product expansions of products of chiral fields. These results are similar to some recent results of Moore and Seiberg [25].

Our results, in particular the notion of generalized vertices and their properties, chiral fusion, etc., provide a convenient starting point for constructing local, conformal field theories on Riemann surfaces of arbitrary genus. We hope to present results on this problem in a future publication.

Ideas somewhat related to the ones developed in this paper have recently appeared in [28] and in [25]. We thank the authors of these papers for sending us their preprints prior to publication.

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