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Optimal Bounds for Ratios of Eigenvalues of One-Dimensional Schrödinger Operators with Dirichlet Boundary Conditions and Positive Potentials

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Abstract. Consider the Schrödinger equation $-u'' + V(x)u = \lambda u$ on the interval $I \subset \mathbb{R}$, where $V(x) \geqq 0$ for $x \in I$ and where Dirichlet boundary conditions are imposed at the endpoints of I. We prove the optimal bound

$$\frac{\lambda_n}{\lambda_1} \leqq n^2 \quad \text{for} \quad n = 2, 3, 4, \ldots$$

on the ratio of the n^{th} eigenvalue to the first eigenvalue for this problem. This leads to a complete treatment of bounds on ratios of eigenvalues for such problems. Extensions of these results to singular problems are also presented. A modified Prüfer transformation and comparison techniques are the key elements of the proof.

1. Introduction

The Schrödinger operator $H = -\Delta + V(x)$ acting on $L^2(\Omega)$ with Dirichlet boundary conditions is known to have purely discrete spectrum if Ω is a bounded connected subset of $\mathbb{R}^d$ with smooth boundary. We denote the eigenvalues listed in ascending order (with multiplicities included) by $\{\lambda_i\}_{i=1}^{\infty}$. Furthermore, $\lambda_2 > \lambda_1$ (nondegeneracy of the groundstate) and, if $V(x) \geqq 0$ for all $x \in \Omega$, $\lambda_1 > 0$. Thus

$$0 < \lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \ldots, \tag{1.1}$$

and in this context it makes sense to consider the boundedness of the ratio λ_2/λ_1 or, more generally, of λ_n/λ_1. Such questions seem to have first been addressed by Payne et al. [16, 17] who considered the case where $V \equiv 0$ and $\Omega \subset \mathbb{R}^2$ (the stretched membrane problem). Among other things they proved the bounds

$$\frac{\lambda_2}{\lambda_1} \leqq 3 \tag{1.2}$$

and, more generally,

$$\frac{\lambda_{n+1}}{\lambda_n} \leqq 3. \tag{1.3}$$

These bounds were generalized in a straightforward way to general dimension d (still taking $V \equiv 0$) by Thompson [22] who obtained

$$\frac{\lambda_2}{\lambda_1} \leqq 1 + \frac{4}{d}, \tag{1.4}$$

and the corresponding result for λ_{n+1}/λ_n. Later, these same bounds were observed to extend to Schrödinger operators with $V \geqq 0$ by Harrell [13] and by Singer et al. [21]. In addition, Payne et al. (respectively, Thompson) considered the question of obtaining optimal bounds on eigenvalue ratios for $H = -\Delta$ and, in particular, conjectured that λ_2/λ_1 is maximal for Ω a disk in $\mathbb{R}^2$ (respectively, ball in $\mathbb{R}^d$). This would result in (1.2) holding with a constant of approximately 2.539 replacing 3 on its right-hand side (and analogous improvements for other dimensions d). A number of authors subsequently improved the constant 3 in (1.2), the best value being 2.586 proved by Chiti [7] in 1983. For further references and related material stemming from the original paper of Payne, Pólya, and Weinberger we refer the reader to our earlier papers [2, 3] and to the survey articles of Protter [18, 19] and of Kuttler and Sigillito [14].

In this paper we concentrate entirely on the one-dimensional case with particular emphasis on higher eigenvalues. While when $d = 1$ Eq. (1.4) reduces to $\lambda_2/\lambda_1 \leqq 5$, if $V \equiv 0$ there is only one domain to consider (up to inessential rescalings) and it is easily seen that in that case $\lambda_2/\lambda_1 = 4$. Thus for $H = -\Delta$ the case $d = 1$ holds little interest. However, for general Schrödinger operators with $V \geqq 0$ the situation changes. In that context the result (1.4) of Singer, Wong, Yau, and Yau reduces to $\lambda_2/\lambda_1 \leqq 5$. This bound was subsequently reproved by one of us [5] using commutation. Later, we established the optimal bound

$$\frac{\lambda_2}{\lambda_1} \leqq 4 \tag{1.5}$$

in [1] by a slight modification of the approach in [5] (see also [2] for additional comments). By using commutation, the Rayleigh–Ritz inequality, and ideas from oscillation and comparison theory we were subsequently able to prove $\lambda_3/\lambda_1 \leqq 9$. Combined with other bounds on eigenvalue ratios (developed in Sect. 3 below) this led to the result

$$\frac{\lambda_n}{\lambda_1} \leqq n^2 \tag{1.6}$$

for any positive integer n having only 2's and/or 3's as factors in its prime factorization. Obviously, this suggests that (1.6) holds for any positive integer n and this is the main result which we establish in this paper. For the general result, however, we found that we had to abandon the commutation approach in favor of an approach utilizing a modified Prüfer transformation.

In later sections we discuss optimal bounds for λ_m/λ_n for any integers m, n and show how our results can be extended to singular cases such as Schrödinger operators on the line with $V(x)$ going to infinity as $x \to \pm\infty$.

2. The Optimal Bound $\lambda_n/\lambda_1 \leqq n^2$

In this section we prove the following optimal result for the ratio λ_n/λ_1.

Theorem 2.1. *Let $H = -d^2/dx^2 + V(x)$ be a Schrödinger operator acting on $L^2(I)$ where $I \subset \mathbb{R}$ is a finite closed interval and where Dirichlet boundary conditions are imposed at both endpoints of I. Assume also that $V \in L^1(I)$ and $V \geqq 0$ a.e. on I. Then the ratio λ_n/λ_1 of the n^{th} eigenvalue of H to the first eigenvalue of H satisfies the bound*

$$\frac{\lambda_n}{\lambda_1} \leqq n^2. \tag{2.1}$$

Furthermore, this bound is optimal and, for $V \in L^1(I)$ and $n > 1$, equality obtains if and only if $V \equiv 0$ a.e. on I.

Proof. We begin by restricting consideration to continuous potentials and to the interval $I = [0, 1]$. Since any interval $[a, b]$ can be transformed into the interval $[0, 1]$ by a translation and a rescaling and since λ_n/λ_1 will be unaffected by such a transformation it is clear that the latter of these restrictions causes no loss of generality. The former restriction will be removed near the end of the proof.

The proof will be by contradiction. Fix n and suppose that $\lambda_n/\lambda_1 > n^2$ for some potential $V \in C(I)$. Let M be a bound for V on I and consider all $\varepsilon > 0$ for which $\lambda_n/\lambda_1 > (n+\varepsilon)^2$. We show that we can modify V while respecting the bound M so that $\lambda_n/\lambda_1 = (n+\varepsilon)^2$ for this new V. This follows easily from consideration of the family of operators $H(\eta) = -d^2/dx^2 + \eta V(x)$ for $\eta \in [0, 1]$. Clearly λ_n/λ_1 is greater than $(n+\varepsilon)^2$ when $\eta = 1$ and is n^2 when $\eta = 0$. By continuity of eigenvalues with respect to η it follows that there is an $\eta_0 \in (0, 1)$, where $\lambda_n/\lambda_1 = (n+\varepsilon)^2$. Thus, if V is replaced by $\eta_0 V$ we have the desired result and since $\eta_0 < 1$ the maximum of the potential will have been decreased, and therefore M will remain an upper bound. In the following we shall continue to call the potential V. A specific choice for ε will be made later in the proof.

For the main part of the argument we shall need to introduce modified Prüfer variables $r(x)$ and $\theta(x)$. For the Schrödinger equation

$$-u'' + V(x)u = \lambda u \tag{2.2}$$

these will be defined by the transformation

$$u(x) = r(x)\sin(\sqrt{\lambda}\theta(x)), \tag{2.3}$$

$$u'(x) = \sqrt{\lambda}r(x)\cos(\sqrt{\lambda}\theta(x)). \tag{2.4}$$

These imply that the angular variable θ must satisfy the differential equation

$$\frac{d\theta}{dx}(x) = 1 - \frac{V(x)}{\lambda}\sin^2(\sqrt{\lambda}\theta(x)) \equiv F(x, \theta; \lambda). \tag{2.5}$$

We define θ_1 (respectively, θ_n) to be the solution to this equation with $\lambda = \lambda_1$ (respectively, $\lambda = \lambda_n$) and satisfying the initial condition $\theta(0) = 0$. Since these solutions correspond to the eigenfunctions u_1 and u_n and since the corresponding functions $r(x)$ will never vanish we must have

$$\theta_1(1) = \frac{\pi}{\sqrt{\lambda_1}} \tag{2.6}$$

and

$$\theta_n(1) = \frac{n\pi}{\sqrt{\lambda_n}}. \tag{2.7}$$

These results follow in exact analogy to the argument for the standard Prüfer angle where it is shown that $\theta' > 0$ whenever the argument of the sine function causes it to vanish (see, for example, [6]).

The crucial part of the proof is a comparison between θ_1 and θ_n. We use a standard result from the theory of ordinary differential equations to do this. Specifically, set $F_1(x,\theta) = F(x,\theta;\lambda_1)$ and $F_n(x,\theta) = F(x,\theta;\lambda_n)$ and consider the two differential equations $\theta_1' = F_1(x,\theta_1)$ and $\theta_n' = F_n(x,\theta_n)$ with initial conditions $\theta_1(0) = 0 = \theta_n(0)$ on the interval $[0,X]$, where $X\in(0,1)$ is any point where $\theta_1(X) = \pi/2\sqrt{\lambda_1}$. Since $0 < \theta_1 < \pi/\sqrt{\lambda_1}$ for $x\in(0,1)$ and θ_1 is continuous we can be sure that there is a $\delta > 0$ such that $\sqrt{\lambda_1}\,\theta_1 < \pi - \delta$ for all $x\in[0,X]$. Indeed, the maximum of θ_1 on the interval $[0,X]$ either must be $\theta_1(X) = \pi/2\sqrt{\lambda_1}$ or must occur at a point x where $0 = \theta_1'(x) = 1 - \lambda_1^{-1} V(x)\sin^2[\sqrt{\lambda_1}\,\theta_1(x)]$. This shows that θ_1 cannot come too close to $\pi/\sqrt{\lambda_1}$ since if it did θ_1' would be positive, and thus θ_1 would continue to increase until it passed beyond $\pi/\sqrt{\lambda_1}$ rather than eventually arriving at the value $\pi/2\sqrt{\lambda_1}$ when x reaches X. In particular, since $V(x) \leqq M$ for all $x\in I$ and $\lambda_1 \geqq \pi^2$, θ_1 cannot get so large that $\sin^2[\sqrt{\lambda_1}\,\theta_1] < \pi^2/M$. The crucial observation to be made here is that the bound δ on how close $\sqrt{\lambda_1}\,\theta_1$ can come to π is controlled solely by the bound M. Thus, given the bound M a bound δ can be found and this bound is independent of the value of the parameter ε discussed earlier. We are therefore free to choose ε arbitrarily (subject, of course, to the previous restrictions) and, in particular, we impose the added condition

$$\varepsilon < \frac{n\delta}{\pi - \delta}. \tag{2.8}$$

We are now ready to complete the comparison argument. For $x\in[0,X]$, $\theta_1(x)$ remains in the interval $[0,(\pi-\delta)/\sqrt{\lambda_1}]$ and it will follow that

$$\theta_n(x) \geqq \theta_1(x) \quad \text{for} \quad x\in[0,X] \tag{2.9}$$

if we can show that

$$F_n(x,\theta) \geqq F_1(x,\theta) \quad \text{for} \quad (x,\theta)\in[0,X]\times[0,(\pi-\delta)/\sqrt{\lambda_1}] \tag{2.10}$$

(see, for example, Birkhoff and Rota [6], pp. 26–28). Since $V(x) \geqq 0$ this amounts to showing that

$$\frac{\sin^2[\sqrt{\lambda_n}\,\theta]}{\lambda_n} \leqq \frac{\sin^2[\sqrt{\lambda_1}\,\theta]}{\lambda_1} \quad \text{for} \quad \theta\in[0,(\pi-\delta)/\sqrt{\lambda_1}], \tag{2.11}$$

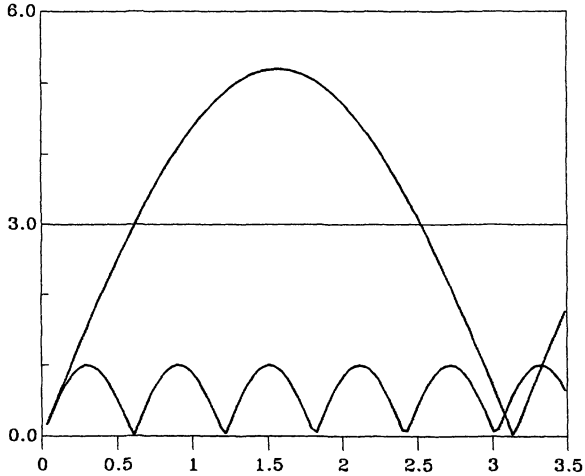

Fig. 1. Graph illustrating the inequality (2.12). Here $n = 5$, and $\varepsilon = 0.2$

or equivalently

$$|\sin(n+\varepsilon)t| \leqq (n+\varepsilon)\sin t \quad \text{for} \quad t \in [0, \pi - \delta]. \tag{2.12}$$

This latter inequality follows easily from the fact that $\sin(n+\varepsilon)t$ and $(n+\varepsilon)\sin t$ have equal values and equal derivatives at $t = 0$ and $\sin(n+\varepsilon)t$ has a smaller second derivative on $(0, \pi/2(n+\varepsilon)]$ than $(n+\varepsilon)\sin t$. This situation is depicted in Fig. 1.

The inequality continues to hold out to $\pi - \delta$ because of our choice of ε. In particular, $\sin(n+\varepsilon)t$ has a zero at $t_1 = n\pi/(n+\varepsilon) < \pi$ so that the inequality certainly holds on $[0, t_1]$ and

$$t_1 = \frac{n\pi}{n+\varepsilon} > \frac{n\pi}{n + n\delta/(\pi - \delta)} = \pi - \delta.$$

Thus $\theta_n(x) \geqq \theta_1(x)$ on $[0, X]$ and, specifically,

$$\theta_n(X) \geqq \theta_1(X). \tag{2.13}$$

(It might be remarked that θ_n can leave the interval $[0, (\pi - \delta)/\sqrt{\lambda_1}]$ without harm since it could only leave by becoming too large and then the desired inequality would continue to hold trivially; if θ_n later reentered the given interval the comparison argument would take effect again.)

To complete the proof we observe that the same argument can be applied to compare Prüfer angles starting from the right endpoint of the interval and proceeding over to $x = X$. Specifically, one compares $\tilde{\theta}_1(x) \equiv (\pi/\sqrt{\lambda_1}) - \theta_1(x)$ and $\tilde{\theta}_n(x) \equiv (n\pi/\sqrt{\lambda_n}) - \theta_n(x)$ and obtains

$$\frac{n\pi}{\sqrt{\lambda_n}} - \theta_n(x) \geqq \frac{\pi}{\sqrt{\lambda_1}} - \theta_1(x) \quad \text{for} \quad x \in [X, 1]. \tag{2.14}$$

In particular, this shows that

$$\frac{n\pi}{\sqrt{\lambda_n}} - \theta_n(X) \geqq \frac{\pi}{\sqrt{\lambda_1}} - \theta_1(X) \tag{2.15}$$

and, when added to (2.13) this yields

$$\frac{n\pi}{\sqrt{\lambda_n}} \geqq \frac{\pi}{\sqrt{\lambda_1}}. \tag{2.16}$$

But this is equivalent to

$$\frac{\lambda_n}{\lambda_1} \leqq n^2,$$

contradicting the fact that $\lambda_n/\lambda_1 = (n+\varepsilon)^2 > n^2$. This proves the inequality (2.1).

To see that the result continues to hold for all $V \in L^1(I)$ we observe that any L^1 function can be approximated arbitrarily closely in L^1 by a continuous function on I and that the eigenvalues of H are continuous functionals of V with respect to the L^1-norm.

Finally, to see that $\lambda_n/\lambda_1 = n^2$ implies that V must be identically zero we observe that we can make the same comparison arguments in this case (in fact, one argument suffices for the whole interval since Fig. 1 now applies with $\varepsilon = 0$) and the inequality $\theta_n(x) \geqq \theta_1(x)$ will become strict as soon as V is positive on a set of positive measure. But then we would obtain $\lambda_n/\lambda_1 < n^2$ by evaluating the inequality at $x = 1$ which gives a contradiction.

Remark. A modified Prüfer angle similar to the one employed here was used earlier in a somewhat different context by Crandall and Reno [10].

3. Other Eigenvalue Ratios

In this section we develop optimal bounds for the ratios of any two eigenvalues for the class of Schrödinger operators treated in Sect. 2. Specifically we prove

Theorem 3.1. *Let $H = -d^2/dx^2 + V(x)$ be as in Theorem 2.1 Then the ratio of any two eigenvalues, λ_m and λ_n, obeys*

$$\frac{\lambda_m}{\lambda_n} \leqq \left[\frac{m}{n}\right]^2. \tag{3.1}$$

Furthermore this bound is optimal for the class of operators considered and if n divides m and $n \neq m$ it is saturated if and only if $V = 0$ a.e. *If n does not divide m then the bound is never saturated* (i.e. "$\leqq$" *may be replaced by* "$<$") *but multiple-well examples can be constructed which come arbitrary near to saturating the bound.*

Remark. The expression $[x]$ (the "ceiling" function of x) denotes the least integer greater than or equal to x.

The key element of the proof of Theorem 3.1 is the result for λ_m/λ_n when n

divides m. Since this argument is completely self-contained and perhaps of independent interest we present it as a separate proposition.

Proposition 3.2. *Let H be as above. If k and n are positive integers*

$$\frac{\lambda_{kn}}{\lambda_n} \leqq k^2 \tag{3.2}$$

with equality for $k > 1$ if and only if $V = 0$ a.e.

Proof. The proof is by induction on n. For $n = 1$ this is the content of Theorem 2.1 above. Suppose now that $\lambda_{kn}/\lambda_n \leqq k^2$ and that we want to prove the corresponding result for $n + 1$. We let u_{n+1} and $u_{k(n+1)}$ denote the eigenfunctions for λ_{n+1} and $\lambda_{k(n+1)}$ respectively, and let w_1 be the first zero of u_{n+1} in the interior of the interval I and w_2 the k^{th} zero of $u_{k(n+1)}$. In addition, with $u(x, \lambda)$ defined as the solution to

$$-u'' + V(x)u = \lambda u \quad x \in I \tag{3.3}$$

obeying the initial conditions

$$u(a, \lambda) = 0, \quad u'(a, \lambda) = 1, \tag{3.4}$$

we let $z_l(\lambda)$ denote the l^{th} zero of $u(x, \lambda)$ in $(a, b]$ (we have taken $I = [a, b]$ here). It is a well known fact (see, for example, Courant–Hilbert, vol. I [9, pp. 454–455]) that $z_l(\lambda)$, for each $l = 1, 2, 3, \ldots$, is a monotone decreasing function of λ for $\lambda \geqq \lambda_l$. Clearly, $w_1 = z_1(\lambda_{n+1})$ and $w_2 = z_k(\lambda_{k(n+1)})$.

Now suppose that V is a potential for which the bound (3.2) is violated for $n + 1$ replacing n. We consider the consequences in the two cases $w_2 \geqq w_1$ and $w_2 < w_1$ in turn. If $w_2 \geqq w_1$ we consider the Dirichlet problem on the interval $[a, w_1]$. Let its eigenvalues be denoted $\tilde{\lambda}_n$ $(n = 1, 2, 3, \ldots)$. Since $z_k(\lambda)$ is decreasing in λ and $z_k(\lambda_{k(n+1)}) = w_2 \geqq w_1$ it follows that $\tilde{\lambda}_k \geqq \lambda_{k(n+1)}$. Because $\tilde{\lambda}_1 = \lambda_{n+1}$ this yields a problem for which $\tilde{\lambda}_k/\tilde{\lambda}_1 \geqq \lambda_{k(n+1)}/\lambda_{n+1} > k^2$, contradicting Theorem 2.1.

If $w_2 < w_1$ we consider the other part of the interval, $[w_1, b]$. By considerations similar to those given above (working now from the initial point $x = b$) we can arrive at a problem where $\tilde{\lambda}_{kn}/\tilde{\lambda}_n > k^2$, contradicting our induction hypothesis.

Thus we have proved the bound (3.2). To characterize the cases where equality occurs one can make the analogous arguments to those above (now assuming $\lambda_{k(n+1)}/\lambda_{n+1} = k^2$) to see inductively that $V = 0$ a.e. both in $[a, w_1]$ and $[w_1, b]$ and hence in $I = [a, b]$. This induction is based upon the characterization of the case of equality given in Theorem 2.1.

Remarks.

1. One can also prove the bound (3.2) using the method of Barnsley [4] for estimating eigenvalues. Specifically, to prove $\lambda_{kn}/\lambda_n \leqq k^2$ one would let $z_0 = a$, $z_1, z_2, \ldots, z_{n-1}, z_n = b$ denote the zeros of u_n and consider Dirichlet subproblems on each of the intervals $[z_{i-1}, z_i]$ for $i = 1, 2, \ldots, n$. If $\tilde{u}_k^i$ denotes the k^{th} eigenfunction for subproblem i one can take

$$u(x) = (-1)^{i-1} \tilde{u}_k^i(x) \quad \text{for} \quad x \in [z_{i-1}, z_i] \quad \text{and} \quad 1 \leqq i \leqq n \tag{3.5}$$

as a trial function for estimating λ_{kn} in the Barnsley method. One then obtains

$$\frac{\lambda_{kn}}{\lambda_n} \leqq \frac{\left[\max\limits_{1 \leqq i \leqq n} \tilde{\lambda}_k^i\right]}{\lambda_n} = \max_{1 \leqq i \leqq n} \frac{\tilde{\lambda}_k^i}{\tilde{\lambda}_1^i} \leqq k^2, \tag{3.6}$$

where $\tilde{\lambda}_1^i$ denotes the first eigenvalue for subproblem i, and we have used the fact that $\lambda_n = \tilde{\lambda}_1^i$ for $1 \leqq i \leqq n$ and that $\lambda_k/\lambda_1 \leqq k^2$ is known for each subproblem. Unfortunately, characterization of the cases of equality seems difficult from this point of view.

2. Mahar and Willner [15] proved a similar result for the eigenvalues of the equation of a vibrating string with variable density.

Proof of Theorem 3.1. Letting $k = [m/n]$ we have, by Proposition 3.2,

$$\lambda_m \leqq \lambda_{kn} \leqq k^2 \lambda_n \tag{3.7}$$

(since $k \geqq m/n$ and this implies $m = (m/n)n \leqq kn$). Obviously, if n does not divide m, then $m < kn$ and the first inequality above becomes strict whereas if m divides n then $m = kn$ and the bound will saturate if and only if $V = 0$ a.e. This proves Theorem 3.1 except for the claim of optimality in the cases where n does not divide m. For these results we must introduce multiple-well examples. Roughly speaking, if V has l identical wells separated by large barriers, then the first set of l eigenvalues of V will be nearly $\tilde{\lambda}_1$, the first eigenvalue of a single well, the second set of l eigenvalues of V will be nearly $\tilde{\lambda}_2$, the second eigenvalue of a single well, etc. Thus, if $n \leqq l < m/(k-1)$, where $k = [m/n]$ an l-well example will have $\lambda_m/\lambda_n \approx \tilde{\lambda}_k/\tilde{\lambda}_1$. In particular, an n-well example should suffce. These remarks can be made precise through the notion of Dirichlet decoupling [8]. Specifically, we consider the potential $v(x)$ on $[0,1]$ defined by

$$v(x) = \begin{cases} 0, & \text{for } |x - \frac{1}{2}| < \frac{1}{2} - \varepsilon \\ M, & \text{for } \frac{1}{2} - \varepsilon \leqq |x - \frac{1}{2}| \leqq \frac{1}{2}, \end{cases}$$

where $0 < \varepsilon < 1/2$ and where $M > 0$. Let $h = -d^2/dx^2 + v(x)$ with Dirichlet boundary conditions at $x = 0$ and $x = 1$. Also, define the potential $V(x)$ on $[0, n]$ by repeating $v(x)$ n times across this interval and let $H = -d^2/dx^2 + V(x)$ be defined with Dirichlet boundary conditions at $x = 0$ and $x = n$. Finally, let H^D be H but with additional Dirichlet conditions imposed at the points $1, 2, \ldots, n-1$. Then H^D has the same spectrum has h but each eigenvalue now has multiplicity n. By results of [8] (see Propositions 6 and 7, pp. 263–264), if M is sufficiently large the resolvents $(H+1)^{-1}$ and $(H^D+1)^{-1}$ may be made arbitrarily close in operator norm. This in turn implies that the first j eigenvalues of H can be made arbitrarily close to the corresponding eigenvalues of H^D for any finite j by taking M sufficiently large [20, pp. 27–29]. Since the eigenvalues of H^D are precisely those of h with multiplicity n this gives an exact result concerning the clustering into groups of n for the eigenvalues of H alluded to above. Finally, since the eigenvalues of h approach those of the Dirichlet problem on an interval of length 1–2ε as M goes to infinity we see that this suffices to establish the remaining statements of the theorem.

Remark. When $k=1$ $m \leqq n$ and the result can be translated to a lower-bound result for λ_m/λ_n for $m>n$. It yields $\lambda_m/\lambda_n>1$ for $m>n$ which is uninteresting except for the observation that there is no better bound.

4. Singular Problems

We now give a brief discussion of problems which are singular either in the sense of occurring on infinite intervals or of V being singular near finite endpoints (or both). We assume throughout that V is in $L^1_{\text{loc}}(\Omega)$, where Ω denotes the open interval under consideration. We begin with the case of an infinite interval, which may by convention be taken as either $\mathbb{R}$ or $(0, \infty)$.

Theorem 4.1. *Suppose $H=-d^2/dx+V(x)$ on $L^2(\Omega)$, where $\Omega=\mathbb{R}$ or $(0,\infty)$. If $\Omega=\mathbb{R}$ suppose that $V \in L^1(-a,a)$ for any $a>0$ and if $\Omega=(0,\infty)$ suppose that $V \in L^1(0,a)$ and let a Dirichlet boundary condition be imposed at 0. Suppose further that $V(x) \geqq 0$ for all $x \in \Omega$ and that*

$$\lim_{|x| \to \infty} V(x)=\infty. \tag{4.1}$$

Then the bound

$$\frac{\lambda_m}{\lambda_n} \leqq \left[\frac{m}{n}\right]^2 \tag{4.2}$$

holds for all eigenvalues λ_m and λ_n of H.

Proof. Consider first the case $\Omega=\mathbb{R}$. Then if for $b>0$ we define $H^D(b)$ to be the operator obtained from H by imposing Dirichlet boundary conditions at $x=\pm b$ a result of Combes, Duclos, and Seiler [8, Proposition A.I.8, p. 296] shows that $H^D(b)$ converges to H in norm resolvent sense as $b \to \infty$. This implies that any finite initial set of eigenvalues of $H^D(b)$ can be made arbitrarily near to the corresponding eigenvalues of H by taking b large enough. Now $H^D(b)=H_1(b) \oplus H_2(b)$, where $H_1(b)=H^D(b) \restriction L^2(-b,b)$ and $H_2(b)=H^D(b) \restriction (L^2(-\infty,-b) \cup L^2(b,\infty))$, and since $V(x) \to \infty$ as $|x| \to \infty$ we can pick b so large that $\sigma(H_2(b))$ lies above any initial set of eigenvalues of H. Thus for any $\varepsilon>0$ and positive integer N we can find b sufficiently large that

$$|\lambda_i(H)-\lambda_i(H_1(b))|<\varepsilon \quad \text{for} \quad 1 \leqq i \leqq N.$$

Since the eigenvalues $\lambda_i(H_1(b))$ obey all the bounds proved previously for finite intervals and $\lambda_1(H)>0$, this estimate shows that the same bounds apply to the eigenvalues of H. Finally, the proof for the case $\Omega=(0,\infty)$ follows from that for $\Omega=\mathbb{R}$ by extending V to $\mathbb{R}$ by reflection and then considering the even-order eigenvalues of the operator so obtained.

To handle the other cases mentioned above we work with conditions allowing us to approximate eigenvalues in terms of $C_0^\infty(\Omega)$ trial functions. These are, of course very natural for problems with Dirichlet boundary conditions.

Theorem 4.2. *Let $H=-d^2/dx^2+V(x)$ on $L^2(\Omega)$ where $V=0$ on Ω and $B \in L^1_{\text{loc}}(\Omega)$*

and suppose that $C_0^\infty(\Omega)$ is a form-core for H. Then if $\lambda_1(H) > 0$,

$$\frac{\lambda_m}{\lambda_n} \leqq \left[\frac{m}{n}\right]^2, \tag{4.3}$$

where λ_m and λ_n are any two eigenvalues of H. (If necessary, the "eigenvalues" of H should be taken as those deriving from the min-max principle [9, 20].)

Proof. Since $C_0^\infty(\Omega)$ is a form-core for H and H is bounded below, given $\varepsilon > 0$ and a positive integer N we can find orthonormal C_0^∞ functions $\varphi_i(x)$, $i = 1, 2, \ldots, N$, such that the $N \times N$ matrix A with elements $(\varphi_i, H\varphi_j)$ for $1 \leqq i, j \leqq N$ has eigenvalues $\lambda_i(A)$ satisfying

$$0 < \lambda_i(A) - \lambda_i(H) < \varepsilon \quad \text{for} \quad 1 \leqq i \leqq N. \tag{4.4}$$

Since the φ_i's have compact support we can assume that they are all supported in a closed interval $[c, d] \subset \Omega$. Next we consider the operator $H' = -d^2/dx^2 + V(x)$ on the interval $[c, d]$ with Dirichlet boundary conditions imposed at both endpoints. By the min-max principle the first N eigenvalues of H' lie between those of H and A. It therefore follows that

$$\lambda_i(H) < \lambda_i(H') \leqq \lambda_i(A) < \lambda_i(H) + \varepsilon \quad \text{for} \quad 1 \leqq i \leqq N, \tag{4.5}$$

so that

$$|\lambda_i(H) - \lambda_i(H')| < \varepsilon \quad \text{for} \quad 1 \leqq i \leqq N. \tag{4.6}$$

The proof can now be completed as in the proof of Theorem 4.1 since H' is an operator to which all our previous results apply and $\lambda_1(H) > 0$.

Remarks. (1) It would suffice here for H to have a form-core of compactly-supported functions. (2) This result covers finite-interval problems where V becomes singular near one or both endpoints of the interval, and thus it applies to operators obtained from H by commutation [11, 12]. It also covers problems on $(0, \infty)$, where V has a singularity at 0 as well as problems on $(0, \infty)$ and $\mathbb{R}$ where V has a variety of behaviors as $|x| \to \infty$. As such, it extends the result of Theorem 4.1. (3) As in Sect. 3 one could show that the results of Theorems 4.1 and 4.2 are optimal for their respective classes of potentials by providing examples that come arbitrarily near to saturating the bounds. In addition, we expect that within the context of nonnegative potentials in $L^1_{\text{loc}}(\Omega)$ one has equality for n dividing m and $m \neq n$ if and only if $V = 0$ a.e. for finite-interval problems and that for infinite-interval problems the bounds are never saturated except when $m = n$ or when one considers λ_m/λ_n with $m < n$ and both λ_m and λ_n sit at the bottom of the essential spectrum of H. However, these latter expectations have not been proved, and they are perhaps of only technical interest.

Examples

(a) For $V(x) = x^2$ on $\mathbb{R}$ one has $\lambda_n = 2n - 1$ $(n = 1, 2, 3, \ldots)$ and then $\lambda_m/\lambda_n = (2m-1)/(2n-1)$ which behaves roughly like m/n rather than like $(m/n)^2$. Even for λ_2/λ_1 we get only $\lambda_2/\lambda_1 = 3$. Thus the harmonic oscillator does not come near to saturating the bounds presented here.

(b) For $V(x)=p(p+1)\tanh^2 x\,(p>0)$ the eigenvalues are given by $\lambda_n=p(p+1)-(p-n+1)^2$ for indices n satisfying $0<n<p+1$. One obtains, for example, $\lambda_2/\lambda_1=(3p-1)/p=3-1/p$, and again the bound appears rather far from being saturated.

The results of these examples are not unexpected since on infinite intervals Dirichlet boundary conditions are most closely approximated by fast-growing potentials. Thus, potentials like $V(x)=|x|^\nu$ for ν large would be expected to come closer to saturating the bounds. Indeed, the bounds will be saturated in the limit $\nu\to\infty$, which gives the finite square-well.

5. Concluding Remarks

In this paper complete results for bounds on eigenvalue ratios of one-dimensional Schrödinger operators with Dirichlet boundary conditions and nonnegative potentials are given. Some problems that remain open here are the proof of bounds such as

$$\lambda_3 \leqq \tfrac{7}{3}\lambda_1 + \tfrac{5}{3}\lambda_2, \tag{5.1}$$

and similar inequalities for higher eigenvalues and the proof of bounds like

$$\frac{\lambda_m}{\lambda_n} \leqq \left(\frac{m}{n}\right)^2 \quad \text{for} \quad m>n\geqq 1 \tag{5.2}$$

under the additional constraint that V be convex. The convexity hypothesis might also be presumed to yield a better lower bound than 1 in the case of λ_m/λ_n for $m>n$. These seems like reasonable conjectures since convexity of V rules out all multiple-well examples. Also inequality (5.1) is reasonable in that it yields $\lambda_3\leqq 9\lambda_1$ and $\lambda_3<4\lambda_2$ when combined with $\lambda_2\leqq 4\lambda_1$ and $\lambda_1<\lambda_2$, respectively (indeed, this explains where we obtained the coefficients $\frac{7}{3}$ and $\frac{5}{3}$).

In addition, there is a possibility that the techniques of Sect. 2 can be extended to higher-dimensional problems with spherical symmetry. This problem is currently under investigation.

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Note added in proof. We have recently realized that the proof of our main result, Theorem 2.1, can be simplified considerably. Specifically, one can avoid having to work from both ends of the interval to the intermediate point X and, in addition, the somewhat delicate arguments involving ε, δ, and the bound M can all be dispensed with. The key to this simplification is the observation that we need only show that $F_n(x,\theta) \geqq F_1(x,\theta)$ (eq. (2.10)) holds for $(x,\theta) \in [0,1] \times [0, n\pi/\sqrt{\lambda_n}]$ to arrive at a contradiction to the assumption that $\lambda_n/\lambda_1 > n^2$. This is equivalent to showing $\sin^2[\sqrt{\lambda_n}\theta]/\lambda_n \leqq \sin^2[\sqrt{\lambda_1}\theta]/\lambda_1$ for $\theta \in [0, n\pi/\sqrt{\lambda_n}]$ or, more simply, to $|\sin ns| \leqq \sqrt{\lambda_n/\lambda_1}\, \sin[ns/\sqrt{\lambda_n/\lambda_1}]$ for $s \in [0,\pi]$ and this is easily seen to hold since it amounts to showing that the appropriate inequality holds from 0 out to the n^{th} positive zero of $|\sin ns|$ (see Fig. 1). Since $\theta_n(t) \in [0, n\pi/\sqrt{\lambda_n}]$ for all $t \in [0,1]$ it follows from the comparison theorem [6] that $\theta_1(t) \leqq \theta_n(t)$ for all $t \in [0,1]$ and this contradicts the fact that $\theta_1(1) = \pi/\sqrt{\lambda_1} > n\pi/\sqrt{\lambda_n} = \theta_n(1)$, concluding the proof.

In connection with our introductory remarks concerning the extension of the results of Payne, Pólya, and Weinberger and of Thompson to Schroedinger operators with positive potentials it has recently come to our attention that Allegretto was aware of this prior to either Harrell or Singer, Wong, Yau, and Yau. Allegretto's results appear in the article "Lower bounds on the number of points in the lower spectrum of elliptic operators," Can. J. Math. **31**, 419–426 (1979).