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An Analogue of P.B.W. Theorem and the Universal R-Matrix for $U_h sl(N+1)$

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Abstract. One uses Drinfeld's quantum double construction and a basis à la Poincaré–Birkhoff–Witt in $U_h n_+$ to compute an explicit formula for the quantum R-matrix.

0. Introduction

1. Definition: [1, 2] $U_h sl(N+1)$ is the topologically free $C[[h]]$ algebra generated by $X_i, Y_i, H_i, 1 \leqq i \leqq N$, with the relations:

$$[H_i, H_j] = 0, \quad [H_i, X_j] = \alpha_j(H_i)X_j,$$
$$[H_i, Y_j] = -\alpha_j(H_i)Y_j, \quad 1 \leqq i, j \leqq N,$$
$$[X_i, Y_j] = \delta_{ij}\frac{\mathrm{sh}\left(\frac{h}{2}H_i\right)}{\mathrm{sh}\left(\frac{h}{2}\right)},$$

for $|i-j| = 1, X_i^2 X_j - (e^{h/2} + e^{-h/2})X_i X_j X_i + X_j X_i^2 = 0,$

$$Y_i^2 Y_j - (e^{h/2} + e^{-h/2})Y_i Y_j Y_i + Y_j Y_i^2 = 0.$$

It is a Hopf algebra for the coproduct Δ:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i) = X_i \otimes \exp\left(\frac{h}{4}H_i\right) + \exp\left(\frac{-h}{4}H_i\right) \otimes X_i$$

$$\cdot \Delta(Y_i) = Y_i \otimes \exp\left(\frac{h}{4}H_i\right) + \exp\left(\frac{-h}{4}H_i\right) \otimes Y_i.$$

The antipode S is given by: $S(H_i) = -H_i$, $S(X_i) = -e^{h/2}X_i$, $S(Y_i) = -e^{-h/2}Y_i$.

This Hopf algebra is not cocommutative; the non-cocommutativity is measured by the so-called R-matrix, which "intertwines" Δ and the opposite comultiplication Δ' [1, 2]. The images of R in tensor products of finite dimensional representations play an important role in the construction of representations of the braid group and of link invariants. Drinfeld has indicated how the existence of this R-matrix comes from the double construction for $U_h b_+$ (see below) and indicated the general form of it; for the case of $sl(2)$, he gave an explicit formula. Our aim is to find such an explicit formula for the general case of $sl(N+1)$, following the same method. We shall introduce a convenient basis in $U_h n_+$, via the definition of analogues of root vectors, thanks to which the computations are not too complicated. We first need some preliminaries.

Giving to X_i (respectively Y_i) the degree α_i (respectively $-\alpha_i$) $U_h sl(N+1)$ is naturally Q-graded where Q is the root lattice.

One defines an *adjoint representation* ad: $U_h sl(N+1) \to \mathrm{End}(U_h sl(N+1))$ by $\mathrm{ad} = (L \otimes R)(\mathrm{Id} \otimes S)\Delta$, where L (respectively R) is the left (respectively right) representation. Let $U_h b_+$, respectively $U_h b_-$, the unital subalgebra generated by the X_i's and the H_i's, respectively by the Y_i's and the H_i's. Before introducing analogues of root vectors in $U_h b_+$, it is useful to consider the new generators $E_i = X_i \exp((-h/4)H_i)$ instead of X_i. Then

$$\Delta(E_i) = E_i \otimes 1 + \exp\left(\frac{-h}{2} H_i\right) \otimes E_i, \quad S(E_i) = \exp\left(\frac{h}{2} H_i\right) E_i.$$

In terms of the new generators, the analogues of Serre's relations can be rewritten as:

for $i \neq j$, $\mathrm{ad}(E_i)^{1-a_{ij}}(E_j) = 0$. (a_{ij}) is the Cartan matrix.

Furthermore, $\mathrm{ad}(E_i)$ acts as a twisted derivation: for $\xi, \eta \in U_h sl(N+1)$ homogeneous of degree β and γ, $\mathrm{ad}(E_i)(\xi\eta) = \mathrm{ad}(E_i)(\xi)\eta + t^{2(\alpha_i, \gamma)}\xi.\ \mathrm{ad}(E_i)(\eta)$, where $t = e^{-h/4}$.

2. Quantum R-Matrix and Quantum Double Construction

Definition 1. A quasi-triangular Hopf algebra is the data of a Hopf algebra A and of an invertible element $R \in A \otimes A$ such that: $R\Delta(x)R^{-1} = \Delta'(x)\ \forall x \in A$, where Δ' is the opposite comultiplication, and $(\Delta \otimes \mathrm{id})(R) = R^{13}R^{23}$, $(\mathrm{id} \otimes \Delta)(R) = R^{13}R^{12}$.

Then R automatically satisfies the Yang–Baxter equation.

The quantum double construction is a procedure allowing to construct a quasitriangular Hopf algebra from any Hopf algebra.

Definition and Theorem 2. *Let A be a Hopf algebra and A° be the dual algebra A^* with the opposite comultiplication. Then there exists a unique quasi-triangular Hopf algebra $(D(A), R)$ such that:*

1. *$D(A)$ contains A and A° as Hopf subalgebras;*
2. *R is the image of the canonical element of $A \otimes A^\circ$ by the embedding:*

$$A \otimes A^\circ \to D(A) \otimes D(A);$$

3. *the linear map: $A \otimes A^\circ \to D(A)$ is bijective,*

$$a \otimes b \to ab.$$

So, as a linear space, $D(A)$ can be identified with $A \otimes A^\circ$ and its algebra and coalgebra structures will be completely determined as soon as one knows how to compute a product $\xi \cdot v$, for ξ in A° and v in A as a sum of products $v_i \cdot \xi_i$, $v_i \in A$,

$\xi_i \in A^\circ$. In fact, one can give an intrinsic formula for the product $A^\circ \otimes A \to D(A)$ in terms of the map $A \otimes A^\circ \to D(A)$ described in point 3) of the theorem: let $\tau: A^\circ \otimes A \to A \otimes A^\circ$ be the permutation $\xi \otimes v \to v \otimes \xi$ then the sought for product is given by the following composition:

$$A^\circ \otimes A \xrightarrow[(\mathrm{tr} \otimes \mathrm{id})(S \otimes I^{\otimes 3})\bar{\Delta}]{} A^\circ \otimes A \xrightarrow[\tau]{} A \otimes A^\circ \xrightarrow[(I \otimes \mathrm{tr})\bar{\Delta}]{} A \otimes A^\circ \longrightarrow D(A)$$

where $\bar{\Delta}$ is the usual coproduct on the tensor product of the Hopf algebras A and A°, and $\mathrm{tr}: A \otimes A^\circ \to C$ is the contraction: $\mathrm{tr}(v \otimes \xi) = \xi(v)$.

Application to the case of $U_h sl(N+1)$. The quasi-triangular structure of $U_h sl(N+1)$ can be deduced from that of the double of $U_h b_+$ from the following facts:

1. $(U_h b_+)^\circ$ can be identified with $U_h b_-$ as a Hopf algebra.
2. So, as linear spaces, we have: $D(U_h b_+) = U_h b_+ \otimes U_h b_- = U_h sl(N+1) \otimes U\mathscr{H}$, where $\mathscr{H}$ is the Cartan subalgebra of $sl(N+1)$.
3. We shall construct an isomorphism as in 1) for which the isomorphism $D(U_h b_+) = U_h sl(N+1) \otimes U\mathscr{H}$ is an isomorphism of algebras

4. If $\varepsilon: U\mathscr{H} \to C$ is the canonical augmentation, a quasi-triangular structure on $U_h sl(N+1)$ is defined by the image of $R \in D(U_h b_+) \otimes D(U_h b_+)$ by the composition:

$$D(U_h b_+) \otimes D(U_h b_+) \to (U_h sl(N+1) \otimes U\mathscr{H}) \otimes (U_h sl(N+1) \otimes U\mathscr{H})$$
$$\to U_h sl(N+1) \otimes U_h sl(N+1)$$

(and this mapping, when restricted to $U_h b_+ \otimes 1$ or to $1 \otimes U_h b_-$ is nothing but the natural inclusion).

Here duality should be understood in the category of Quantized Universal Envelopping algebras (Q.U.E. algebras) (cf. Drinfeld [1]). We shall freely use the formalism of Q.U.E. and Q.F.S.H. (Quantized Formal Power Series) algebras.

1. An Analogue of the Poincaré–Birkhoff–Witt Theorem for $U_h n_+$

$U_h n_+$ is the unital subalgebra generated by the E_i's. Each positive root α of $sl(N+1)$ can be written: $\alpha = \varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, $1 \leqq i < j \leqq N$. One defines by induction the root vector: $E_\alpha = \mathrm{ad}\, E_i(E_{\varepsilon_{i+1} - \varepsilon_j})$.

1. Commutation Relations Between the E_α*'s*

a) Commutation of a vector of simple root E_k with $E_{\varepsilon_i - \varepsilon_{j+1}}$.

For $k = i-1$, $\mathrm{ad}\, E_k(E_{\varepsilon_i - \varepsilon_{j+1}}) = E_{\varepsilon_{i-1} - \varepsilon_{j+1}}$. For $k = j+1$, $E_k E_{\varepsilon_i - \varepsilon_{j+1}} = \mathrm{ad}\, E_i \cdots \mathrm{ad}\, E_{j-1}(E_{j+1} E_j)$

$$E_{\varepsilon_i - \varepsilon_{j+1}} E_k = \mathrm{ad}\, E_i \cdots \mathrm{ad}\, E_{j-1}(E_j E_{j+1}),$$

so,

$$E_{\varepsilon_i - \varepsilon_{j+1}} E_{j+1} - t^{2(\alpha_j, \alpha_{j+1})} E_{j+1} E_{\varepsilon_i - \varepsilon_{j+1}} = E_{\varepsilon_i - \varepsilon_{j+2}}.$$

For $k \leqq i-2$ or $k \geqq j+2$, $E_k E_{\varepsilon_i - \varepsilon_{j+1}} = E_{\varepsilon_i - \varepsilon_{j+1}} E_k$. For $k = i$, $\mathrm{ad}\, E_i(E_{\varepsilon_i - \varepsilon_{i+2}}) =$

$(\operatorname{ad} E_i)^2(E_{i+1})=0$, and more generally:

$$\begin{aligned}\operatorname{ad} E_i(E_{\varepsilon_i-\varepsilon_{j+1}}) &= (\operatorname{ad} E_i)^2 \operatorname{ad} E_{i+1}(E_{\varepsilon_{i+2}-\varepsilon_{j+1}})\\ &= ((t^2+t^{-2})\operatorname{ad} E_i \operatorname{ad} E_{i+1} \operatorname{ad} E_i\\ &\quad - \operatorname{ad} E_{i+1}(\operatorname{ad} E_i)^2)(E_{\varepsilon_{i+2}-\varepsilon_{j+1}})=0.\end{aligned}$$

For $k\in\{i+1,\dots,j-1\}$: $\operatorname{ad} E_k(E_{\varepsilon_i-\varepsilon_{j+1}})$

$$\begin{aligned}&= \operatorname{ad} E_i \cdots \operatorname{ad} E_{k-2} \operatorname{ad} E_k \operatorname{ad} E_{k-1} \operatorname{ad} E_k(E_{\varepsilon_{k+1}-\varepsilon_{j+2}})\\ &= (t^2+t^{-2})^{-1} \operatorname{ad} E_i \cdots \operatorname{ad} E_{k-2}((\operatorname{ad} E_k)^2 \operatorname{ad} E_{k-1}\\ &\quad + \operatorname{ad} E_{k-1}(\operatorname{ad} E_k)^2)(E_{k+1})=0.\end{aligned}$$

For $k=j$: $E_jE_{\varepsilon_i-\varepsilon_{j+1}} = \operatorname{ad} E_i \cdots \operatorname{ad} E_{j-2}(E_jE_{\varepsilon_{j-1}-\varepsilon_{j+1}})$

$$E_{\varepsilon_i-\varepsilon_{j+1}}E_j = \operatorname{ad} E_i \cdots \operatorname{ad} E_{j-2}(E_{\varepsilon_{j-1}-\varepsilon_{j+1}}E_j).$$

But, $E_{\varepsilon_{j-1}-\varepsilon_{j+1}} = E_{j-1}E_j - t^{-2}E_jE_{j-1}$, so

$$\begin{aligned}E_jE_{\varepsilon_{j-1}-\varepsilon_{j+1}} - t^{-2}E_{\varepsilon_{j-1}-\varepsilon_{j+1}}E_j &= -t^{-2}(E_j^2E_{j-1}+E_{j-1}E_j^2)+E_jE_{j-1}E_j(1+t^{-4})\\ &= 0 \text{ according to Serre's relations.}\end{aligned}$$

b) Commutation of E_α and E_β, $\alpha=\varepsilon_i-\varepsilon_{p+1}, \beta=\varepsilon_j-\varepsilon_{k+1}$.
For $j\geqq p+2$: $E_\alpha E_\beta = E_\beta E_\alpha$. For $j=p+1$: put $\alpha'=\varepsilon_i-\varepsilon_p$, one has: $E_\alpha = E_{\alpha'}E_p - t^{-2}E_pE_{\alpha'}$ and $E_{\alpha'}E_\beta = E_\beta E_{\alpha'}$. So

$$\begin{aligned}E_\alpha E_\beta &= E_{\alpha'}E_pE_\beta - t^{-2}E_pE_\beta E_{\alpha'},\\ E_\beta E_\alpha &= E_{\alpha'}E_\beta E_p - t^{-2}E_\beta E_pE_{\alpha'},\end{aligned}$$

so

$$\begin{aligned}E_\alpha E_\beta - t^{-2}E_\beta E_\alpha &= E_{\alpha'}E_{\alpha_p+\beta} - t^{-2}E_{\alpha_p+\beta}E_{\alpha'}\\ \text{go on} &=\\ &= E_iE_{\varepsilon_{i+1}-\varepsilon_{k+1}} - t^{-2}E_{\varepsilon_{i+1}-\varepsilon_{k+1}}E_i\\ &= E_{\alpha+\beta}.\end{aligned}$$

For $j\leqq p$: Up to exchanging the roles of α and β, one may suppose that i is the smallest index which appears. Put $\gamma=\alpha_j+\cdots+\alpha_p$; $\alpha=\alpha_i+\cdots+\alpha_{j-1}+\gamma$ and $\beta=\gamma+\alpha_{p+1}+\cdots\alpha_k$. Then: $E_\alpha E_\beta = \operatorname{ad} E_i \cdots \operatorname{ad} E_{j-2}(E_{\alpha_{j-1}+\gamma}E_\beta)$,

$$\begin{aligned}E_\beta E_\alpha &= \operatorname{ad} E_i \cdots \operatorname{ad} E_{j-2}(E_\beta E_{\alpha_{j-1}+\gamma})\\ E_{\alpha_{j-1}+\gamma}E_\beta &= \operatorname{ad} E_{j-1}(E_\gamma E_\beta) - t^{2(\alpha_{j-1},\gamma)}E_\gamma E_{\alpha_{j-1}+\beta}\\ E_\beta E_{\alpha_{j-1}-\gamma} &= t^{-2(\alpha_{j-1},\beta)}(\operatorname{ad} E_{j-1}(E_\beta E_\gamma) - E_{\alpha_{j-1}+\beta}E_\gamma).\end{aligned}$$

But $E_j, E_{j+1},\dots,E_p$ commute with $E_{\alpha_{j-1}+\beta}$, so $E_\gamma E_{\alpha_{j-1}+\beta} = E_{\alpha_{j-1}+\beta}E_\gamma$. Put $\gamma=\alpha_j+\gamma'$: one has in the same way: $E_{\gamma'}E_\beta = E_\beta E_{\gamma'}$.

Then: $E_\gamma E_\beta = \operatorname{ad} E_j(E_\gamma, E_\beta)$ and $E_\beta E_\gamma = t^{-2(\alpha_j,\beta)}\operatorname{ad} E_j(E_\beta E_{\gamma'})$ so

$$\begin{aligned}&E_\gamma E_\beta - t^{-2(\gamma,\beta)}E_\beta E_\gamma = 0,\\ &E_{\alpha_{j-1}+\gamma}E_\beta - t^{-2(\alpha_{j-1}+\gamma,\beta)}E_\beta E_{\alpha_{j-1}+\gamma} = (t^2-t^{-2})E_\gamma E_{\alpha_{j-1}+\beta},\end{aligned}$$

and

$$E_\alpha E_\beta - t^{2(\alpha,\beta)} E_\beta E_\alpha = (t^2 - t^{-2}) E_\gamma E_{\alpha_i + \cdots + \alpha_k}.$$

Remark. This supposes (with notations as above) that $i \leqq j-1$ and $k \geqq p+1$. But for $i=j$, i.e. $\alpha=\gamma$, we saw in the course of the proof that: $E_\gamma E_\beta - t^{-2(\gamma,\beta)} E_\beta E_\gamma = 0$ for $k=p$, i.e. $\beta=\gamma$, $E_{\alpha_{j-1}+\gamma} E_\gamma = \operatorname{ad} E_j \cdots \operatorname{ad} E_{p-1}(E_{\alpha_{j-1}+\gamma} E_p)$,

$$E_\gamma E_{\alpha_{j-1}+\gamma} = \operatorname{ad} E_j \cdots \operatorname{ad} E_{p-1}(E_p E_{\alpha_{j-1}+\gamma}),$$

$E_p E_{\alpha_{j-1}+\gamma} - t^{-2(\alpha,\alpha_{j-1}+\gamma)} E_{\alpha_{j-1}+\gamma} E_p = 0$ so $E_\gamma E_{\alpha_{j-1}+\gamma} - t^{-2(\gamma,\alpha_{j-1}+\gamma)}) E_\gamma E_{\alpha_{j-1}+\gamma} = 0$, and $E_\gamma E_\alpha - t^{-2(\gamma,\alpha)} E_\alpha E_\gamma = 0$.

We shall put on the set of positive roots R_+ a total order $\beta(1) < \beta(2) < \cdots < \beta(n)$, such that the ordered monomials $E_{\beta(1)}{}^{m_1} \cdots E_{\beta(n)}{}^{m_n}$, $(m_1, \ldots, m_n) \in N^n$ form a basis of $U_h n_+$. We record the following computational lemma, which is easily checked by induction, and which will be useful next.

Lemma

$$\Delta(E_{\varepsilon_i - \varepsilon_{j+1}}) = E_{\varepsilon_i - \varepsilon_{j+1}} \otimes 1 + (1 - e^h) \sum E_{\varepsilon_i - \varepsilon_{k+1}} \exp \frac{-h}{2} H_{\varepsilon_{k+1} - \varepsilon_{j+1}} \otimes E_{\varepsilon_{k+1} - \varepsilon_{j+1}}$$
$$+ \exp \frac{-h}{2} H_{\varepsilon_i - \varepsilon_{j+1}} \otimes E_{\varepsilon_i - \varepsilon_{j+1}}. \quad \text{(Sum from } k=i \text{ to } k=j-1).$$

Put

$$u_1 = E_{\varepsilon_i - \varepsilon_{j+1}} \otimes 1,$$
$$u_2 = E_{\varepsilon_i - \varepsilon_j} \exp(-h/2) H_j \otimes E_j,$$
$$u_3 = E_{\varepsilon_i - \varepsilon_{j-1}} \exp \frac{-h}{2} H_{\varepsilon_{j-1} - \varepsilon_{j+1}} \otimes E_{\varepsilon_{j-1} - \varepsilon_{j+1}},$$
$$u_{j-i+1} = E_i \exp \frac{-h}{2} H_{\varepsilon_{i+1} - \varepsilon_{j+1}} \otimes E_{\varepsilon_{i+1} - \varepsilon_{j+1}},$$
$$u_{j-i+2} = \exp \frac{-h}{2} H_{\varepsilon_i - \varepsilon_{j+1}} \otimes E_{\varepsilon_i - \varepsilon_{j+1}}.$$

Then $u_k u_l = e^{-h} u_l u_k$ for $k > l$. As $\Delta(E_\alpha) = u_1 + (1 - e^h)(u_2 + \cdots + u_{j-1+1}) + u_{j-i+2}$, one can compute $\Delta(E_\alpha)^n$ by the q-multinomial formula:

$$\Delta(E_\alpha)^n = \sum_{n_1 + \cdots + n_{j-1+2} = n} \frac{\phi_n(e^{-h})}{\phi_{n_i}(e^{-h}) \cdots \phi_{n_{j-i+2}}(e^{-h})}$$
$$\cdot (1 - e^{-h})^{n_2 + \cdots + n_{j-i+1}} u_1^{n_1} \cdots u_{j-i+2}^{n_{j+i-2}}$$

where

$$\phi_n(q) = (1-q)(1-q^2) \cdots (1-q^n),$$
$$u_{k+1}{}^r = e^{(h/4)r(r-1)} (E_{\varepsilon_i - \varepsilon_{j-k+1}})^r \exp \frac{-rh}{2} H_{\varepsilon_{j-k+1} - \varepsilon_{j+1}} \otimes (E_{\varepsilon_{j-k+1} - \varepsilon_{j+1}})^r.$$

2. A Basis à la Poincaré–Birkhoff–Witt for $U_h n_+$

Definition. Let R_+ be the set of positive roots, $\alpha_1, \ldots, \alpha_N$ the simple roots. Let's consider the following total order on R_+:

$$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \ldots, \alpha_1 + \cdots + \alpha_N, \alpha_2,$$
$$\alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \ldots, \alpha_2 + \cdots + \alpha_N, \alpha_3, \alpha_3 + \alpha_4,$$
$$\alpha_3 + \alpha_4 + \alpha_5, \ldots, \alpha_3 + \cdots + \alpha_N, \alpha_4, \ldots, \alpha_{N-1} + \alpha_N, \alpha_N,$$

and let's note $\beta(1) < \beta(2) < \cdots < \beta(n)$ the inverse total order (i.e. $\beta(1) = \alpha_N, \beta(2) = \alpha_{N-1} + \alpha_N, \ldots,$).

Theorem. *The set of elements* $E_{\beta(1)}m_1 \cdots E_{\beta(n)}m_n$, $(m_1, \ldots, m_n) \in N^n$ *from a basis of* $U_h n_+$.

Proof.

a) One knows that the monomials in $E_1, \ldots, E_N$, generate $U_h n_+$, so a fortiori the (non)-ordered monomials in E_α's. To see that the set above is generator, it is enough to prove that each element $E_{\beta(i_1)} \cdots E_{\beta(i_k)}$, is a linear combination of ordered elements as above, with $m_1 + \cdots + m_n \leqq k$.

We shall make a double induction: first on k, and for k fixed, on i_1.
—The case $k = 1$ is clear.
—Suppose the assertion true for k: we now prove by induction on i_1 that it holds for $k+1$. So, let's consider an element $E_{\beta(i_1)} \cdots E_{\beta(i_{k+1})}$.

i) For $i_1 = 1$, apply the induction hypothesis on k to $E_{\beta(i_2)} \cdots E_{\beta(i_{k+1})}$.
ii) If $i_1 > 1$, applying again the induction hypothesis on k to $E_{\beta(i_2)} \cdots E_{\beta(i_{k+1})}$, one sees that $E_{\beta(i_1)} E_{\beta(i_2)} \cdots E_{\beta(i_{k+1})}$ is a linear combination of elements $E_{\beta(i_1)} E_{\beta(j)}{}^{m_j} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n}$ with $m_j + \cdots m_n \leqq k$.

If $i_1 \leqq j$: we are O.K.

If $i_1 > j$: $E_{\beta(i_1)} E_{\beta(j)}{}^{m_j} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n} = E_{\beta(i_1)} E_{\beta(j)} E_{\beta(j)}{}^{m_j - 1} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n}$.
But we have computed the "commutation relations" between the E_α's and there are essentially three possibilities: for some non-zero coefficients λ and μ:

$$E_{\beta(i_1)} E_{\beta(j)} - \lambda E_{\beta(j)} E_{\beta(i_1)} = \begin{cases} 0 \\ \mu E_{\beta(i_1) + \beta(j)} \\ \mu E_\gamma E_{\alpha' + \gamma + \beta'} \quad \text{where } \beta(i_1) = \alpha' + \gamma, \beta(j) = \gamma + \beta' \end{cases},$$

and so: $\beta(i_1) > \gamma > \beta(j)$.

Then:

$$E_{\beta(i_1)} E_{\beta(j)} E_{\beta(j)}{}^{m_j - 1} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n}$$
$$= \begin{cases} \lambda E_{\beta(j)} E_{\beta(i_1)} E_{\beta(j)}{}^{m_j - 1} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n} \\ \lambda E_{\beta(j)} E_{\beta(i_1)} E_{\beta(j)}{}^{m_j - 1} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n} + \mu E_{\beta(i_1) + \beta(j)} E_{\beta(j)}{}^{m_j - 1} \cdots E_{\beta(n)}{}^{m_n} \\ \lambda E_{\beta(j)} E_{\beta(i_1)} E_{\beta(j)}{}^{m_j - 1} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n} + \mu E_\gamma E_{\alpha' + \gamma + \beta'} E_{\beta(j)}{}^{m_j - 1} \cdots E_{\beta(n)}{}^{m_n}. \end{cases}$$

In the first case, induction on k allows to reorder $E_{\beta(i_1)} E_{\beta(j)}{}^{m_j - 1} E_{\beta(j+1)}{}^{m_{j+1}} \cdots E_{\beta(n)}{}^{m_n}$ as a linear combination of monomials with at most k terms, then as $j < i_1$, one uses induction on i_1.

In the second case: the first term is treated in the same way, and the second one comes from induction on k.

In the third case: proceed for the two terms as in the first case, as $\gamma < \beta(i_1)$.

b) Let us prove now that the $E_{\beta(1)}{}^{m_1}\cdots E_{\beta(n)}{}^{m_n}$ are linearly independent. Let Q be the root lattice; $U_h n_+, U_h sl(N+1)$, $U_h sl(N+1)\otimes U_h sl(N+1)$ are $Q\times Q$-graded. For Q-degree reasons, the $E_{\beta(j)}$ are independent.

$\Delta: U_h n_+ \to U_h b_+ \otimes U_h n_+$ preserves the Q-degree and $\Delta(E_\beta)$ has a component of bidegree $(\alpha_i, \beta-\alpha_i)$ if and only if β is of the form $\beta=\alpha_i+\cdots$. Then the component of bidegree $(n\alpha_i, n(\beta-\alpha_i))$ of $\Delta(E_\beta)^n$ is proportional to $E_i{}^n \exp(-nh/2)H_{\beta-\alpha_i}\otimes (E_{\beta-\alpha_i})^n$. (See lemma).

In the same way, the component of bidegree $((m+m_1+\cdots+m_r)\alpha_i,\dots)$ of $\Delta((E_{\alpha_i+\cdots+\alpha_{i+r}})^{m_r}(E_{\alpha_i+\cdots+\alpha_{i+r-1}})^{m_{r-1}}\cdots(E_{\alpha_i+\alpha_{i+1}})^{m_1}E_{\alpha_i}{}^m)$ is proportional to:

$$(E_i)^{m+m_1+\cdots+m_r}\exp\frac{-h}{2}H\otimes(E_{\alpha_{i+1}+\cdots+\alpha_{i+r}})^{m_r}\cdots(E_{\alpha_{i+1}})^{m_1}$$

and the monomial on the right of $\otimes$ is already well ordered. More generally, consider the component of bidegree $(p\alpha_i,\dots)$ of $\Delta(E_{\beta(1)}{}^{m_1}\cdots E_{\beta(n)}{}^{m_n})$, with p maximal: it is proportional to $E_i{}^p\exp(-h/2)H\otimes E_{\beta(1)'}{}^{m_1}\cdots E_{\beta(n)'}{}^{m_n}$, with $\beta(k)'=\beta(k)-\alpha_i$ if $\beta(k)=\alpha_i+\cdots$ and $\beta(k)'=\beta(k)$ if not. When reordering $E_{\beta(1)'}{}^{m_1}\cdots E_{\beta(n)'}{}^{m_n}$, the only commutations than one has to do are between two vectors of the form $E_{\alpha_{i+1}+\cdots}$, these commutations are of the type: $E_\gamma E_\delta=\lambda E_\delta E_\gamma$ for a non-zero λ. So, the sought for component is proportional to a monomial

$$E_i{}^p\exp\frac{-h}{2}H\otimes E_{\beta(1)}{}^{m'_1}\cdots E_{\beta(n)}{}^{m'_n}.$$

Consider now a linear relation between the $E_{\beta(1)}{}^{m_1}\cdots E_{\beta(n)}{}^{m_n}$: one can assume they all have the same Q-degree: $\Sigma m_i\beta(i)$ is fixed. We prove by induction on this degree that the relation is trivial.

—we saw that if this Q-degree is $\beta(i)$.

—Among the monomials of the relation, consider the biggest integer i such that there appears a $E_{\alpha_i+\cdots}$ with a non-zero exponent. Let n be the biggest total exponent at which all $E_{\alpha_i+\cdots}$ appear. Only the monomials in which this total exponent is exactly n will have a component of degree $(n\alpha_i,\dots)$ after applying Δ. From the relation we started with, we deduce a relation between the monomials $E_{\beta(1)}{}^{m'_1}\cdots E_{\beta(n)}{}^{m'_n}$ which appeared on the right of $\otimes$, and if the $E_{\beta(1)}{}^{m_1}\cdots E_{\beta(n)}{}^{m_n}$ are two by two distinct, it is the same for the $E_{\beta(1)}{}^{m'_1}\cdots E_{\beta(n)}{}^{m'_n}$. (The $\beta(k)$ of the form $\alpha_i+\cdots$ are replaced by $\alpha_{i+1}+\cdots$, or 1 if $\beta(k)=\alpha_i$, and the others are unchanged.) As the $E_{\beta(1)}{}^{m'_1}\cdots E_{\beta(n)}{}^{m'_n}$ have a Q-degree strictly smaller than the one we started with, this new relation is trivial. But the coefficients of this relation are non-zero multiples of the coefficients of the initial relation (see remarks made above): so the coefficients of all the monomials in which $E_{\alpha_i+\cdots}$ appears with exponent n should be zero; this contradicts the choice of n.

Remark. $U_h n_+$ is a (non-homogeneous) quadratic algebra, generated by the E_α's, and the existence of the P.B.W. basis implies that it is a Koszul algebra in the sense of Priddy. [3,4].

The choice has the following interest:

Proposition. *Let* $(U_h n_+)_j$ *be the subspace of* $U_h n_+$ *with basis* $E_{\beta(1)}{}^{m_1}\cdots E_{\beta(j)}{}^{m_j}$ *with* $(m_1,\dots,m_j)\in N^j$. *Then:*

a) $(U_h n_+)_j$ *is a subalgebra of* $U_h n_+$.
b) $(U_h n_+)_j$ *is a sub-* $U_h b_+$ *right-comodule of* $U_h n_+$, i.e.

$$\Delta((U_h n_+)_j) \subset U_h b_+ \otimes (U_h n_+)_j$$

II. Computation of the Universal R-Matrix

1. Construction of a Basis of the Q.F.S.H. *Dual to* $U_h b_+$

From the previous theorem, we deduce that $E_{\beta(1)}{}^{m_1} \cdots E_{\beta(n)}{}^{m_n} H_1{}^{r_1} \cdots H_N{}^{r^N}$, is a basis of $U_h b_+$. Let's introduce linear forms $\xi_1, \dots, \xi_N$ and $\eta_\gamma, \gamma \in R_+$, defined by: $\xi_i(H_i) = 1$; zero on the other monomials; $\eta_\gamma(E_\gamma) = 1$, zero on the other monomials.

Lemma 1.

i) $\langle \xi_i^n, H_i^n \rangle = n!$ *and* ξ_i^n *is zero on the other monomials.*

ii) $\langle \eta_\gamma{}^n, E_\gamma{}^{n'} \rangle = \prod_{k=1}^{n} \left(\frac{1 - e^{-kh}}{1 - e^{-h}} \right) \delta_{n,n'}$.

iii) $\langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n} \xi_1{}^{r_1} \cdots \xi_N{}^{r_N}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n)}{}^{m'_n} H_1{}^{r'_1} \cdots H_N{}^{r'_N} \rangle$

$$= \prod_{i=1}^{n} \delta_{m_i, m'_i} \prod_{i=1}^{N} \delta_{r_i, r'_i} \frac{\Phi_{m_1}(e^{-h})}{(1 - e^{-h})^{m_1}} \cdots \frac{\Phi_{m_n}(e^{-h})}{(1 - e^{-h})^{m_n}} r_1! \cdots {}_N!$$

Proof.

i) is immediate.

ii) $\langle \eta_\gamma{}^n, E_\gamma{}^{n'} \rangle = \langle \eta_\gamma{}^{n-1} \otimes \eta_\gamma, \Delta(E_\gamma{}^{n'}) \rangle$

$$= \left\langle \eta_\gamma{}^{n-1} \otimes \eta_\gamma, \frac{\Phi_{n'}(e^{-h})}{\Phi_{n'-1}(e^{-h}) \Phi_1(e^{-h})} E_\gamma{}^{n'-1} \exp \frac{-h}{2} H_\gamma \otimes E_\gamma \right\rangle$$

$= \frac{1 - e^{-n'h}}{1 - e^{-h}} \langle \eta_\gamma{}^{n-1}, E_\gamma{}^{n'-1} \rangle$ and the result follows by induction on n.

iii) One checks immediately:

$$\langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n} \xi_1{}^{r_1} \cdots \xi_N{}^{r_N}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n)}{}^{m'_n} H_1{}^{r'_1} \cdots H_N{}^{r'_N} \rangle$$
$$= \langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n)}{}^{m'_n} \rangle \prod \delta_{r_i, r'_i} r_1! \cdots r_N!$$

Put $X = \langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n)}{}^{m'_n} \rangle$

$$= \langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n - 1} \otimes \eta_{\beta(n)}, \Delta(E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n-1)}{}^{m'_{n-1}}) \Delta(E_{\beta(n)}{}^{m'_n}) \rangle.$$

But $\Delta(E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n-1)}{}^{m'_{n-1}}) \in U_h b_+ \otimes (U_h n_+)_{n-1}$ and $\eta_{\beta(n)}$ is zero on $(U_h n_+)_{n-1}$.

So, $X = \langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n - 1} \otimes \eta_{\beta(n)}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n-1)}{}^{m'_{n-1}} \otimes 1 \Delta(E_{\beta(n)}{}^{m'_n}) \rangle$

$$= \frac{1 - e^{-m'_n h}}{1 - e^{-h}} \langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n - 1}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n)}{}^{m'_n - 1} \rangle$$

$$= \frac{\Phi_{m_r}(e^{-h})}{(1 - e^{-h})^{m_r}} \delta_{m_n, m_n'} \langle \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n-1)}{}^{m_n - 1}, E_{\beta(1)}{}^{m'_1} \cdots E_{\beta(n-1)}{}^{m'_{n-1}} \rangle,$$

and applying the same argument, one gets the result.

2. Commutation Relations and Coproduct in $(U_h b_+)^*$

We note $\eta_i = \eta_{\alpha_i}$ for α_i a simple root.

Lemma 2.

i) $\xi_i \xi_j = \xi_j \xi_i$,

ii) $[\xi_i, \eta_j] = -\frac{h}{2}\eta_j \delta_{ij}$,

iii) $\eta_i \eta_{i+1} - e^{h/2}\eta_{i+1}\eta_i = (1-e^h)\eta_{\alpha_i+\alpha_{i+1}}$
$[\eta_i, \eta_j] = 0$ if $|i-j| \geqq 2$.

iv) For $\alpha_i > \alpha$, one has: $\eta_i \eta_\alpha - e^{(h/2)(\alpha,\alpha_i)}\eta_\alpha \eta_i = (1-e^h)\eta_{\alpha_i+\alpha}$ if $\alpha_i + \alpha \in R_+$ and 0 if not.

Proof.

i) is immediate.
ii) $\eta_j \xi_i$ is non-zero only on $E_j H_i$ where its value is 1. $\xi_i \eta_j$ is non-zero only on $E_j H_i$ where its value is 1, and if $i=j$, on E_j where its values is $\langle \xi_i \otimes \eta_i, \exp -(h/2) H_i \otimes E_i \rangle = -h/2$.
iii) For our order, $\alpha_{i+1} < \alpha_i$, so $\eta_{i+1}\eta_i$ is non-zero only on $E_{i+1}E_i$; where its value is 1. On the contrary, $\eta_i \eta_{i+1}$ may be non-zero also on $E_{\alpha_i+\alpha_{i+1}}$:

$$\langle \eta_i \eta_{i+1}, E_{i+1}E_i \rangle = \left\langle \eta_i \otimes \eta_{i+1}, \exp -\frac{h}{2} H_{i+1} E_i \otimes E_{i+1} \right\rangle = e^{h/2},$$

$$\langle \eta_i \eta_{i+1}, E_{\alpha_i+\alpha_{i+1}} \rangle = \left\langle \eta_i \otimes \eta_{i+1}, (1-e^h) E_i \exp -\frac{h}{2} H_{i+1} \otimes E_{i+1} \right\rangle = (1-e^h).$$

iv) $\eta_\alpha \eta_i$ is non-zero only on $E_\alpha E_i$ where it is 1.

One shows then that $\eta_i \eta_\alpha$ is zero on each monomial of degree $\geqq 3$ in the E_γ's. The only monomial of degree 2 on which it is not zero is $E_\alpha E_i$ and its value on it is $e^{(h/2)(\alpha,\alpha_i)}$. It can be non-zero on E_γ only if $\alpha = \alpha_{i+1} + \dots + \alpha_i$, $\gamma = \alpha_i + \alpha$ and then $\langle \eta_\alpha \eta_i, E_\gamma \rangle = 1$.

Corollary. *As an algebra;* $(U_h b_+)^*$ *is generated by the* ξ_i*'s and* η_i*'s. Furthermore, one has from* iii) *and* iv) *the analogues of Serre's relations*:

$$\eta_i \eta_j = \eta_j \eta_i \quad \text{if} \quad |i-j| \geqq 2,$$
$$\eta_i^2 \eta_{i\pm 1} - (e^{h/2} + e^{-h/2})\eta_i \eta_{i\pm 1}\eta_i + \eta_{i\pm 1}\eta_i^2 = 0.$$

Lemma 3.

i) $\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i$,
ii) $\Delta(\eta_i) = \eta_i \otimes 1 + \exp(-\xi_{i-1} + 2\xi_i - \xi_{i+1}) \otimes \eta_i$ *with the evident modification for* $i=1$ *or* $i=N$.

Proof.

i) is immediate.
ii) As the operation of "commuting" two root vectors can never give a simple root vector, a priori $\Delta(\eta_i)$ will be non-zero only on: $E_i \otimes 1$ (where it is 1) and

$H_1^{r_1}\cdots H_N^{r_N}\otimes E_i$.

$$\langle \Delta(\eta_i), H_1^{r_1}\cdots H_N^{r_N}\otimes E_i\rangle = \langle \eta_i, H_1^{r_1}\cdots H_N^{r_N}E_i\rangle$$
$$= \langle \eta_i, E_i(H_1+\alpha_i(H_1))^{r_1}\cdots(H_N+\alpha_i(H_N))^{r_N}\rangle$$

so; we must have $r_j = 0$ for $j \notin \{i-1, i, i+1\}$.

$$\langle \Delta(\eta_i), H_{i-1}{}^{r_{i-1}}H_i^{r_i}H_{i+1}{}^{r_{i+1}}\otimes E_i\rangle = \langle \eta_i, E_i(H_{i-1}-1)^{r_{i-1}}(H_i+2)^{r_i}(H_{i+1}-1)^{r_{i+1}}\rangle$$
$$=(-1)^{r_{i-1}}2^{r_i}(-1)^{r_{i+1}}.$$

$$\Delta(\eta_i) = \eta_i\otimes 1 + \sum_{p,q,r}(-1)^p\frac{\xi_{i-1}^p}{p!}2^q\frac{\xi_i^q}{q!}(-1)^r\frac{\xi_{i+1}^r}{r!}\otimes\eta_i, \quad \text{i.e.}$$
$$\Delta(\eta_i) = \eta_i\otimes 1 + \exp(-\xi_{i-1}+2\xi_i-\xi_{i+1})\otimes\eta_i.$$

For the identification with the Q.F.S.H. associated with the Q.U.E. algebra $(U_hb_+)^\circ$, it is useful to introduce: $\zeta_1 = \xi_1 - \frac{1}{2}\xi_2$,

$$\zeta_i = \xi_i - \tfrac{1}{2}(\xi_{i-1}+\xi_{i+1}) \quad 2\leqq i\leqq N-1,$$
$$\zeta_N = \xi_N - \tfrac{1}{2}\xi_{N-1}.$$

Then: $\Delta(\zeta_i) = \zeta_i\otimes 1 + 1\otimes\zeta_i$

$$[\zeta_i,\zeta_j]=0,$$
$$[\zeta_i,\eta_j]=0 \quad \text{if} \quad j\in\{i-1,i,i+1\},$$
$$[\zeta_i,\eta_{i\pm 1}] = \frac{h}{4}\eta_{i\pm 1},$$
$$[\zeta_i,\eta_i] = -\frac{h}{2}\eta_i,$$

and $\Delta(\eta_i) = \eta_i\otimes 1 + \exp(2\zeta_i)\otimes\eta_i$.

Remark. $2\langle\zeta_i, H_j\rangle = (\alpha_i,\alpha_j)$.

3. The Identification of $(U_hb_+)^*$ *with the* Q.F.S.H *Associated with* U_hb_- *with the Opposite comultiplication.* Let Δ' this opposite comultiplication and S' the related antipode. With these, one can define another adjoint representation $\mathrm{ad}' = (L\otimes R)(I\otimes S')\Delta'$. In U_hb_-, one introduces the new generators $F_i = Y_i\exp(h/4)H_i$ and $\mathrm{ad}'(F_i)$ has the same properties with respect to U_hb_- as $\mathrm{ad}(E_i)$ had with respect to U_hb_+.

In particular, for each positive root $\alpha = \alpha_i + \cdots + \alpha_j$, one defines the analogue of the root vector F_α as: $F_\alpha = \mathrm{ad}'(F_i)(F_{\alpha_{i+1}} + \cdots + \alpha_j)$. From the computations made above, one easily gets:

Proposition. *For every* $\lambda_1(h),\ldots,\lambda_N(h)\in C[[h]]$, *with h-valuation* 1, *the map*

$$\phi_\lambda:(U_hb_+)^\circ\to \text{Q.F.S.H.}(U_hb_-),$$
$$\zeta_i\to\frac{h}{4}H_i,$$
$$\eta_i\to\lambda_i(h)F_i$$

defines an isomorphism of Hopf algebras.

We shall see that there is a unique choice of λ_i's such that the Hopf algebra structure of $D(U_h b_+)$ induces the one of $U_h sl(N+1)$.

An easy computation gives that, in $U_h sl(N+1)$,

$$[E_i, F_j] = \delta_{ij} \frac{\operatorname{sh}\left(\frac{h}{2} H_i\right)}{\operatorname{sh}\left(\frac{h}{2}\right)} e^{h/2}.$$

We compare it with $[E_i, \eta_j]$ computed in $D(U_h b_+)$ thanks to the intrinsic formula given in the introduction. One has: $(\operatorname{tr}\otimes \operatorname{id})(S\otimes I^{\otimes 3})\Delta(\eta_j \otimes E_i) = -\delta_{ij}\exp(2\zeta_i)\otimes 1 + \eta_j \otimes E_i$, so:

$$[E_i, \eta_j] = \delta_{ij}\left(\exp(2\zeta_i) - \exp -\frac{h}{2} H_i\right).$$

The image by ϕ_λ: $[E_i, F_j]\lambda_j(h) = \delta_{ij} 2\operatorname{sh}((h/2)H_i)$.

So, $\lambda_j(h) = (1-e^{-h})$. One also checks that, in $D(U_h b_+)$, $[H_i, \eta_j] = -(\alpha_j, \alpha_i)\eta_j$.

Corollary. *The map*: $D(U_h b_+) \to U_h sl(N+1)$

$$E_i \to E_i$$
$$\eta_i \to (1-e^{-h})F_i$$
$$H_i \to H_i$$
$$\zeta_i \to \frac{h}{4} H_i$$

defines a (surjective) morphism of Hopf algebras. So, the image of the canonical element of $D(U_h b_+)\otimes D(U_h b_+)$ *defines a quasi-triangular structure on* $U_h sl(N+1)$.

4. The Canonical Elements of $D(U_h b_+)\otimes D(U_h b_+)$ *and the Universal R-Matrix of* $U_h sl(N+1)$. In terms of the P.B.W. basis of $U_h b_+$ and of its dual basis, the canonical element is given by:

$$R = \sum_{\substack{(r_1,\ldots,r_N)\in N^N \\ (m,\ldots,m_n)\in N^n}} \frac{(1-e^{-h})^{m_1}}{r_1!\cdots r_N!\,\Phi_{m_1}(e^{-h})} \cdots \frac{(1-e^{-h})^{m_n}}{\Phi_{m_n}(e^{-h})} E_{\beta(1)}{}^{m_1} \cdots E_{\beta(n)}{}^{m_n} \cdot H_1{}^{r_1} \cdots H_N{}^{r_N} \otimes \eta_{\beta(1)}{}^{m_1} \cdots \eta_{\beta(n)}{}^{m_n} \xi_1{}^{r_1} \cdots \xi_N{}^{r_N}.$$

This can be written in a more compact way by using the q-exponential:

$$e(u;q) = \sum \frac{u^n}{\Phi_n(q)}.$$

With these notations, one has:

$$R = \prod e((1-e^{-h})E_{\beta(i)} \otimes \eta_{\beta(i)}; e^{-h}) \cdot \exp\left(\sum H_j \otimes \xi_j\right),$$

where the product is made in the order $1 < 2 < \cdots < n$.

Now:—the image of $\sum H_j \otimes \xi_j$ in $U_h sl(N+1) \otimes U_h sl(N+1)$ is $(h/2)t_0$, where to $t_0 \in \mathscr{H} \otimes \mathscr{H}$ corresponds to the scalar product (,).

—from $\eta_i\eta_{i+1} - e^{h/2}\eta_{i+1}\eta_i = (1-e^h)\eta_{\alpha_i+\alpha_{i+1}}$, one has $\eta_{\alpha_i+\alpha_{i+1}} = -e^{-h}(1-e^{-h}) F_{\alpha_i+\alpha_{i+1}}$, and by induction on the length $l(\alpha)$ of the root α

$$\eta_\alpha = (-e^{-h})^{l(\alpha)-1}(1-e^{-h})F_\alpha.$$

Theorem. *The universal R-matrix of $U_h sl(N+1)$ is given by:*

$$R = \prod e((-e^{-h})^{l(\beta(i))-1}(1-e^{-h})^2 E_{\beta(i)} \otimes F_{\beta(i)}; e^{-h}) \cdot \exp\left(\frac{h}{2} t_0\right).$$

with the same convention as above for ordering the product.

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