Classical and Quantum Scattering on a Spinning Cone*

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Abstract. Solutions are presented for the Klein–Gordon and Dirac equations in the $2 + 1$ dimensional space-time created by a massive point particle, with arbitrary angular momentum. A universal formula for the scattering amplitude holds when a required self-adjoint extension of the Dirac operator is specified uniquely. Various obstacles to a consistent quantum mechanical interpretation of these results are noted.

I. Introduction

There has been completed recently a quantum-mechanical analysis of two-body scattering in $2 + 1$-dimensional [planar] gravity [1, 2]. This generally covariant theory has no propagating gravitational degrees of freedom, because in the absence of sources it is solved uniquely by flat space-time. Curvature is created by sources, but only locally at their position; elsewhere space-time remains flat. Consequently there are no gravitons, and forces are not mediated by graviton exchange; rather, they are geometrical/topological in origin, arising from global properties of space-time, which is not Minkowskian in the large, even when it is locally flat. For this reason, planar quantum gravity gives us the opportunity for examining the inter-relation between geometrical and quantum concepts, without the complication of graviton propagation. Moreover, the theory is physically realized in the presence of infinite cosmic strings [3].

The quantum mechanical results that have been established concern relative motion in a two-body, spinless system [1], and equivalently [1] motion of a spinless test particle in the presence of a [heavy] spinless source [2]. In this paper, we extend the results about test-particle motion by allowing the source and/or the test particle to carry spin, i.e. we solve the Klein–Gordon and Dirac equations. In Sect. II, we recall the space-time of a spinning source [4], and discuss classical motion. Our strategy for studying quantum scattering on a cone is presented in

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Sect. III. We review the results for a spinless test particle scattering off a spinless source [2]. Also we present a qualitative analysis of classical wave diffraction to supplement the previous quantitative discussion [1,2] and to enhance physical insight. Sections IV and V are devoted to exploring the effects of spin: the Klein–Gordon equation with a spinning source is solved in Sect. IV; the Dirac equation with a spinless and spinning source in Sect. V. It is found that the effect of spin is to introduce energy and spin dependent phases into various amplitudes, which can be given a universal form. Moreover for spinning sources, unitarity of the quantum theory is problematical. Also there are mathematical intricacies: the Dirac Hamiltonian in the presence of a spinning point particle is not self-adjoint, while a self-adjoint extension introduces a new physical parameter, whose significance remains obscure. Final remarks comprise Sect. VI. Some partial results about scattering off two sources are given in the Appendix; our analysis makes use of an orbifold.

II. Spinning Source and Classical Scattering

The space-time produced by a point source with mass M and angular momentum [spin] S gives rise to the line element [4,5]

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = (cdt + c^{-1}GSd\Theta)^2 - (dl)^2, \tag{2.1}$$

$$(dl)^2 = R^{-2GM}[(dR)^2 + R^2(d\Theta)^2] = R^{-2(1-\alpha)}[(dR)^2 + R^2(d\Theta)^2]. \tag{2.2}$$

Here c is the velocity of light, G "Newton's constant," and α [$0 < \alpha \leq 1$] is $1 - GM$. [We have rescaled G by a factor of 4 relative to our previous papers, [2,4] where α is given by $1 - 4GM$.] We call this a *spinning cone*.

In (2.2), the spatial interval is presented in conformally flat spatial coordinates, with R and Θ possessing the full radial and angular range, $0 \leq R \leq \infty, 0 \leq \Theta \leq 2\pi$. Alternatively, one may use imbedded coordinates,

$$r = R^\alpha, \quad \theta = \Theta, \tag{2.3a}$$

which also extend over the complete range, $0 \leq r \leq \infty, 0 \leq \theta \leq 2\pi$, and describe a cone imbedded in flat three-space with the constraint $z = \sqrt{(\alpha^{-2} - 1)(x^2 + y^2)}$,

$$(dl)^2 = \alpha^{-2}(dr)^2 + r^2(d\theta)^2. \tag{2.3b}$$

Finally there are the spatially flat coordinates,

$$\rho = \alpha^{-1}r, \quad \varphi = \alpha(\theta - \pi) \tag{2.4a}$$

with a limitation on the angular range, *viz.* a wedge, determined by the conical angle α, is excised from the flat plane; $0 \leq \rho \leq \infty, -\pi\alpha \leq \varphi \leq \pi\alpha$,

$$(dl)^2 = (d\rho)^2 + \rho^2(d\varphi)^2. \tag{2.4b}$$

Then spin may be hidden by redefining time,

$$T = t + GS\Theta/c^2 = t + GS\theta/c^2 = t + \alpha^{-1}GS\varphi/c^2 \tag{2.5}$$

so that with the flat spatial interval (2.4), space-time is locally Minkowskian,

$$(ds)^2 = c^2(dT)^2 - (d\rho)^2 - \rho^2(d\varphi)^2. \tag{2.6}$$

Of course the attributes of the source—mass and spin—are coded in the global properties of the locally flat variables. All the information thus resides in non-trivial boundary conditions, a fact that will be crucial in quantum mechanics: φ is excluded from the angular domain $2\pi GM$; since $(t, r, \theta + 2\pi n)$ is identified with (t, r, θ) it follows from (2.4) and (2.5) that (T, ρ, φ) is equivalent to $(T + 2\pi n GS/c^2, \rho, \varphi + 2\pi n \alpha)$. Describing physical consequences of this "time-helical" structure of the locally flat coordinates [4] is one of the goals of the present work, and is given below.

The classical equations of motion are determined by the geodesic equation,

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0. \tag{2.7}$$

The overdot indicates differentiation with respect to any convenient affine variable τ that parametrizes the path x^μ; τ satisfies $ds \propto d\tau$, where the proportionality constant vanishes for massless particles. With imbedded coordinates, we learn from (2.1) and (2.3b) that the non-vanishing components of the connection are $\Gamma^0_{r\theta} = -GS/rc$, $\Gamma^r_{\theta\theta} = -\alpha^2 r$, $\Gamma^\theta_{r\theta} = 1/r$, so that (2.7) implies [5]

$$\ddot{t} - 2GS\dot{r}\dot{\theta}/rc^2 = 0, \tag{2.8a}$$

$$\ddot{r} - \alpha^2 r\dot{\theta}^2 = 0, \tag{2.8b}$$

$$\ddot{\theta} + 2\dot{r}\dot{\theta}/r = 0. \tag{2.8c}$$

Clearly the solution for r and θ in terms of τ makes no reference to spin, which affects only the relation between time t and τ. A first integral for (2.8a) is obtained with the help of (2.8c),

$$\dot{t} + \frac{GS}{c^2}\dot{\theta} = C. \tag{2.9a}$$

The affine parameter may be chosen to be T in which case $C = 1$. Together with the remaining integrals of motion, involving the constants l and v,

$$\dot{\theta} = \frac{l}{r^2}, \tag{2.9b}$$

$$\alpha^{-2}\dot{r}^2 + \frac{l^2}{r^2} = v^2, \tag{2.9c}$$

the equations may be integrated once more to provide a description of the motion,

$$t - t_0 = T - T_0 - \frac{GS}{c^2}(\theta - \theta_0), \tag{2.10a}$$

$$r^2 = \frac{l^2}{v^2} + v^2\alpha^2(T - T_0)^2, \tag{2.10b}$$

$$\tan\alpha(\theta - \theta_0) = \frac{\alpha v^2}{l}(T - T_0). \tag{2.10c}$$

It follows that the scattering angle $\pm\omega$ is the same as in the absence of spin;

$$\Delta\theta \equiv \theta_{\text{final}} - \theta_{\text{initial}} = \pm\pi\alpha^{-1},$$

$$\pm\omega = \Delta\theta \mp \pi = \pm\pi(\alpha^{-1} - 1), \tag{2.11}$$

[the sign depends on which side the source is passed] so also is the orbit,

$$r^2\cos^2\alpha(\theta - \theta_0) = \frac{l^2}{v^2}, \tag{2.12}$$

which is traversed at a rate that is influenced by the spin, according to (2.10). In terms of the coordinate T the equations are the same as in the absence of spin. The spinning source introduces a time delay Δt that is determined by Eq. (2.10a),

$$\Delta t = \Delta T - \frac{GS}{c^2}\Delta\theta, \tag{2.13a}$$

where ΔT is the delay of the variable T and $\Delta\theta$ is the angular deflection given in (2.11). It is evident from (2.10b) or (2.10c) that ΔT vanishes; whence the time delay is only non-zero when the source carries spin,

$$\Delta t = \mp\frac{GS}{c^2}\frac{\pi}{\alpha}. \tag{2.13b}$$

The scattering angle (2.11) is presented in the imbedded coordinate system, where it measures the deflection in the asymptotic motion on the cone as projected onto the $x - y$ plane of the embedding three-space. In flat coordinates, where the connections vanish, a particle is undeflected, $\Delta\varphi = \varphi_{\text{final}} - \varphi_{\text{initial}} = \pi$; to arrive at the scattering angle one must subtract the forward direction which subtends the angle $\pi\alpha$, giving the result $\pi(1 - \alpha)$, which is in agreement with (2.11) when transformed according to (2.4a).

For a test particle with mass m, Eqs. (2.8) also follow by varying the Lagrangian,

$$L = -mc\frac{ds}{dt}. \tag{2.14}$$

This has the virtues of identifying τ with s and providing a definition of the energy,

$$E = \frac{dr}{dt}\frac{\partial L}{\partial\frac{dr}{dt}} + \frac{d\theta}{dt}\frac{\partial L}{\partial\frac{d\theta}{dt}} - L$$

$$= mc^2\sqrt{1 + c^{-2}\left(\alpha^{-2}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2\right)} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}. \tag{2.15}$$

The last identity is easily verified using (2.8b) and $d/ds = (dT/ds)(d/dT) = (1 - v^2/c^2)^{-1/2}(d/dT)$. The possibility of using this suggestive special relativistic notation is due to local flatness: when everything is expressed in the coordinates (T, ρ, φ), S disappears, only α remains in the combination l/α and v can be identified with the velocity.

Note that while the metric for the spinning cone possesses closed time-like contours [e.g. $dt = 0, dr = 0, d\theta$ constant with $r < GS/c$], there are no closed time-like [or light-like] geodesics. Nevertheless the occurrence of these contours entails on the classical level causality problems close to the source when non-gravitational forces are present. In quantum mechanics, which senses all paths not just geodesics, the closed time-like contours give rise to untoward unitarity problems, as will be seen below.

III. Overview

The following strategy for determining the quantum motion on the cone has been successfully implemented in the spinless case, [2] and will be used in Sects. IV and V for the problem with spin.

Imbedded coordinates are used because the full angular range allows for conventional partial wave analysis and identification of phase shifts in the wave equations that we solve. There are no interactions—the equations are free—except that derivatives are properly covariant with respect to the imbedded metric. The relevant equations of course depend on the spin of the test particle.

For a spinless source and spinless test particle, the Schrödinger equation was analyzed [1, 2]. This involves only the covariant Laplacian, and serves equally well as the analysis of the Klein–Gordon equation, which is equivalent to the Schrödinger equation in this simple case. When the source possesses spin, the presence of off-diagonal time-space components in the metric tensor renders the Schrödinger equation inapplicable for the spinless test particle; rather the covariant Klein–Gordon equation is solved. In either case, because the background metric is time-independent, the time-dependence of the wave function may be separated as $e^{-iEt/\hbar}$, and one is led to a stationary problem at fixed energy E, Moreover, rotational invariance of the background allows separating the θ dependence with the Ansatz $e^{in\theta}$, where n is an integer because θ possesses the full 2π range and the wave function is required to be single valued, i.e. angular momentum, $\mathscr{J} = -i\hbar\partial_\theta$, is diagonalized with eigenvalue $\hbar n$. Thus it remains only to solve for the radial function u_n, and its large-r asymptote

$$u_n(r) \xrightarrow[r \to \infty]{} \sqrt{\frac{2}{\pi\kappa r}} \cos\left(\kappa r - \frac{n\pi}{2} - \frac{\pi}{4} + \delta_n\right) \tag{3.1}$$

determines the phase shift δ_n. Here κ is related to the energy; see below.

The scattering solution is constructed as a superposition of partial waves, appropriate to the planar problem,

$$\psi(\mathbf{r}) = \sum_n e^{i(\delta_n + n\pi/2)} u_n(r) e^{in\theta}. \tag{3.2}$$

This may be decomposed into an incoming wave and a scattered wave,

$$\psi = \psi_{\text{in}} + \psi_{\text{sc}}, \tag{3.3}$$

the latter defining the scattering amplitude f as

$$\psi_{\mathrm{sc}}(\mathbf{r}) \xrightarrow[r\to\infty]{} \sqrt{\frac{i}{r}}\, f(\theta)e^{i\kappa r}. \tag{3.4}$$

Ordinarily, the incoming wave is taken to be a conventional plane wave,

$$\psi_{\mathrm{in}}(\mathbf{r}) = e^{i\kappa r\cos\theta}. \tag{3.5}$$

It then follows that the scattering amplitude is given by

$$f(\theta) = \frac{1}{\sqrt{-2\pi\kappa}}\sum_n (e^{2i\delta_n}-1)e^{in\theta}. \tag{3.6}$$

For scattering on a cone, δ_n increases with n, owing to the persistence of the "interaction" at large distances—the test particle is never out of range of the source because space is not asymptotically Euclidean. Consequently, (3.6) must be regulated, for example by multiplying the summand by a convergence factor $e^{-|n|\varepsilon}$ and setting ε to zero at the end of the calculation. This results in a well-defined. expression for f, which however contains angular delta functions [2].

An alternative procedure yields a scattering amplitude without the delta functions. It is recalled that the large-r asymptote of a plane wave is a radial wave, times a "scattering amplitude" that is an angular delta function in the forward and backward directions. Therefore, we can consider that portion of ψ_{sc} leading to delta functions in f as belonging to the asymptote of a plane wave, and move its contribution to the incoming wave. In other words, we can present a decomposition alternative to (3.3), (3.4) and (3.5), [2]

$$\psi = \tilde{\psi}_{\mathrm{in}} + \tilde{\psi}_{\mathrm{sc}}, \tag{3.7}$$

such that $\tilde{\psi}_{\mathrm{in}}$ is a *superposition* of plane waves, $\tilde{\psi}_{\mathrm{sc}}$ still tends to a radial wave

$$\tilde{\psi}_{\mathrm{sc}}(\mathbf{r}) \xrightarrow[r\to\infty]{} \sqrt{\frac{i}{r}}\, \tilde{f}(\theta)e^{i\kappa r}, \tag{3.8}$$

and $\tilde{f}$ coincides with f, except for the delta functions.

To find the alternative decomposition, (3.7) and (3.8), we construct the scattering solution (3.2) by evaluating the partial wave sum with the help of a contour integral representation for u_n. This results in a contour integral representation for ψ, from which $\tilde{\psi}_{\mathrm{in}}$ and $\tilde{\psi}_{\mathrm{sc}}$ are easily determined [1].

For a spinless particle of mass m moving on a spinless cone, the Schrödinger/Klein–Gordon equation in embedded coordinates reads [2] [henceforth we set c, G and $\hbar$ to unity]

$$-\left(\frac{\alpha^2}{r}\partial_r r\partial_r + \frac{1}{r^2}\partial_\theta^2\right)u_n(r)e^{in\theta} = \alpha^2\kappa^2 u_n(r)e^{in\theta},$$

$$n\in\mathbf{Z}, \quad \alpha^2\kappa^2 = \begin{cases} 2mE & \text{Schrödinger} \\ E^2-m^2 & \text{Klein–Gordon} \end{cases}. \tag{3.9}$$

The radial wave function is a Bessel function,

$$u_n(r) = (-1)^{(n-|n|)/2} J_{|n|/\alpha}(\kappa r) \tag{3.10}$$

whose asymptotic form determines the phase shift,

$$\delta_n = -\frac{|n|\pi}{2}(\alpha^{-1} - 1) = -\tfrac{1}{2}|n|\omega. \tag{3.11}$$

From (3.5) the scattering amplitude is found [2],

$$f(\theta) = -\frac{i\pi}{\sqrt{2\pi\kappa}} \sum_n (\delta(\theta + \omega - 2\pi n) + \delta(\theta - \omega - 2\pi n) - 2\delta(\theta - 2\pi n)) + \tilde{f}(\theta), \tag{3.12a}$$

$$\tilde{f}(\theta) = \frac{i}{\sqrt{2\pi\kappa}} \left(\frac{1}{1 - e^{i(\theta+\omega)}} - \frac{1}{1 - e^{i(\theta-\omega)}} \right) = \frac{1}{\sqrt{2\pi\kappa}} \frac{\sin\omega}{\cos\omega - \cos\theta}. \tag{3.12b}$$

The singularity in $\tilde{f}$ at $\theta = \pm\omega$ [plus periodic repetitions] is a principal value, while the delta functions emerge when the regulator ε is set to zero.

To evaluate the sum over angular momentum eigenstates in the scattering solution (3.2), the Bessel function is represented by the Schläfli contour integral. The summation is then elementary and one finds, after some adjustment of contours [2], the following expression for the scattering solution [1],

$$\psi^f(\rho, \varphi) = \int \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\boldsymbol{\rho}} \frac{1}{1 - e^{iz/\alpha}}. \tag{3.13}$$

The superscript f denotes the wave function transformed by (2.4a) to flat coordinates $\boldsymbol{\rho} = (-\rho\cos\varphi, -\rho\sin\varphi)$, $k = \alpha\kappa$ so that $k\rho = \kappa r$. The vector $\mathbf{k}(z)$ is rotated by complex angle z, $\mathbf{k}(z) = (k\cos z, k\sin z)$ and the contour over which z ranges is depicted in Fig. 1a and equivalently in Fig. 1b.

It is clear that ψ^f satisfies the free, flat equation $-\nabla_\rho^2 \psi^f = k^2\psi^f$, while some further manipulation with the contour shows that $\psi^f(\rho, \varphi + 2\pi\alpha) = \psi^f(\rho, \varphi)$.

The equivalent contours in Fig. 1b give the decomposition of ψ^f into $\tilde{\psi}^f_{in}$ and $\tilde{\psi}^f_{sc}$. The former is the contribution from the [negative] Cauchy contour around the poles of $(1 - e^{iz/\alpha})^{-1}$ at $z = z_n = 2\pi n\alpha$ and leaves a superposition of plane waves, with variously rotated incoming momenta,

$$\tilde{\psi}^f_{in}(\rho, \varphi) = \alpha \sum_n{}' e^{i\mathbf{k}(z_n)\cdot\boldsymbol{\rho}} = \alpha \sum_n{}' e^{-ik\rho\cos(\varphi - 2\pi n\alpha)}. \tag{3.14}$$

The prime on the sum indicates that z_n must lie in the internal $[-\pi + \varphi, \pi + \varphi]$. The integration over the vertical contours gives $\tilde{\psi}^f_{sc}$. The integral has not been evaluated in closed form, but its large distance asymptote is found to give a radial wave, with scattering amplitude $\tilde{f}$ of (3.12b), which in flat coordinates reads [1]

$$\tilde{f}^f(\varphi) = \frac{1}{2\sqrt{2\pi k}} \left[\left(\mathrm{ctn}\frac{\varphi - \pi}{2\alpha} - i \right) - \left(\mathrm{ctn}\frac{\varphi + \pi}{2\alpha} - i \right) \right]. \tag{3.15}$$

The above formulations will be used in Sect. IV, where we solve the Klein–Gordon equation on a spinning cone. An obvious generalization to a spinor wave function is utilized in Sect. V for the Dirac equation.
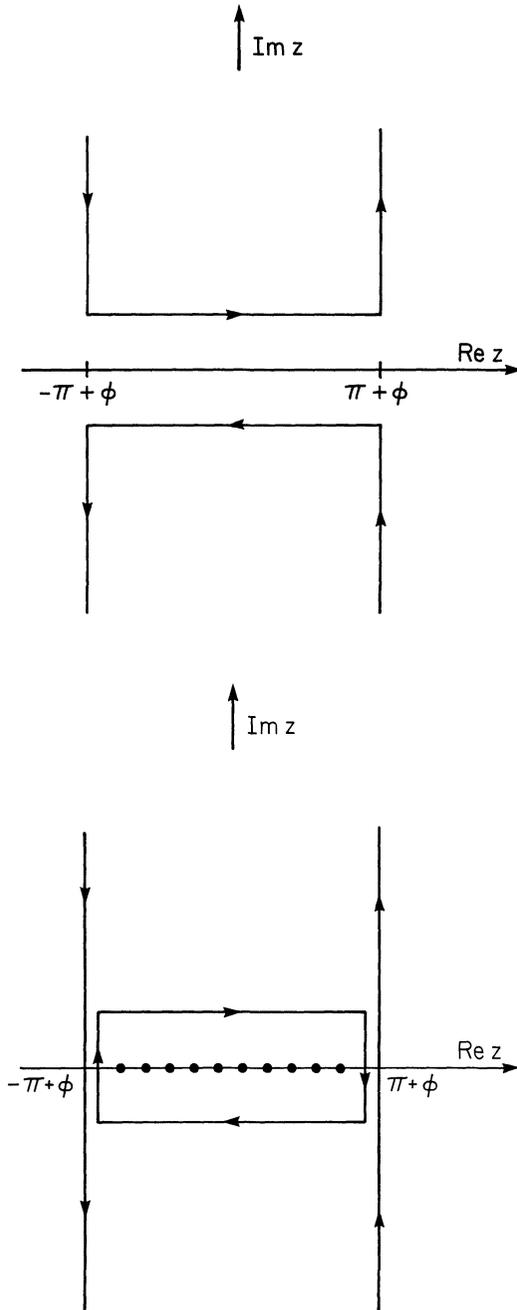
Fig. 1. a—Integration contour for the representation of $\psi^f(\rho, \varphi)$. **b**—Integration contour for the representation of $\psi^f(\rho, \varphi)$ equivalent to that in Fig. 1a but giving rise to the alternative decomposition $\psi^f = \tilde{\psi}_{\text{in}}^f + \tilde{\psi}_{\text{sc}}^f$. The incoming wave $\tilde{\psi}_{\text{in}}^f$, is given by the [negative] Cauchy contour around the poles at $z = 2\pi n\alpha$, indicated by heavy dots. The integrals along the left and right vertical contours determine the scattered wave $\tilde{\psi}_{\text{sc}}^f$, whose large distance asymptote defines the scattering amplitude $\tilde{f}^f$
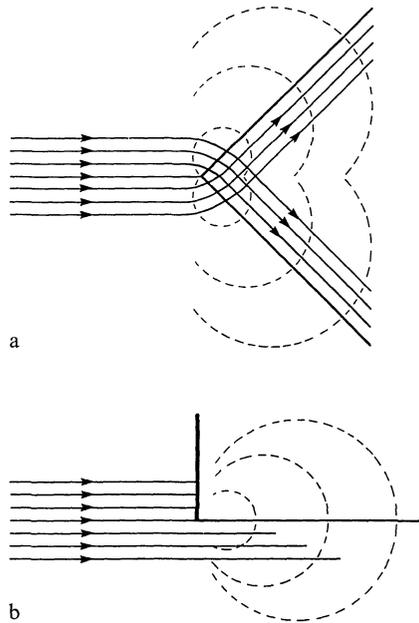
Fig. 2. a—Qualitative pictorialization for scattering of waves on an obstacle at the origin. The sharp lines are classical trajectories with scattering angle $\pm \omega$, the sign depending on which side the trajectory passes the source. The envelope to the right of the source, formed by heavy diagonal lines, is the sharp geometrical shadow. Broken lines represent diffraction on two sharp edges, even though no edge is actually present—the source [conical defect] provides the "edges." **b**—Fraunhofer diffraction around an edge—heavy vertical line. The horizontal lines are the light rays of geometrical optics. Half of them are cut off by the edge, thus forming the geometrical shadow—medium heavy horizontal line. The diffraction is indicated by broken lines

Next we present a qualitative discussion of wave diffraction around a sharp edge, which gives an intuitive, physical understanding of the above formulas. For the remainder of this section we restrict ourselves to the case $S = 0$, i.e. $T = t$, but we shall later comment on the general situation. In optics, whose terminology we shall often use, the classical geodesic equation corresponds to the approximation of geometrical optics. These classical trajectories, obtained from (2.10), are shown as lines in Fig. 2a. They are a function only of the impact parameter, but not of the energy, *viz.* they only depend on the initial y-coordinate, which is l/v. The scattering angle $\pm \omega$ does not vary with the impact parameter, except in its sign that depends on which side the test particle trajectory passes the source.

From this picture it is now easy to understand the quantum scattering [Fig. 2a]. Instead of a sharp geometrical shadow [indicated by heavy diagonal lines], each half of the incoming wave-packet [approximated by plane waves] gives rise to a diffraction pattern [indicated by broken lines] after passing the source at the right or at the left. For the scattering amplitude we thus expect two parts:

1. The contribution that is already present classically, i.e. two wave packets with

half the flux in the directions $\pm \omega$. In the scattering amplitude (3.12a) these correspond to the first two delta functions. The third delta function arises from the forward wave, which is present in conventional scattering theory [hidden in $\exp(ikr\cos\theta)$] but absent in our problem and thus has to be subtracted.

2. $\tilde{f}$ represents true quantum mechanical scattering. Each summand in Eq. (3.12b) comes from the diffraction pattern of the upper or lower part of the wave. This is analogous to Fraunhofer diffraction on a sharp edge [heavy vertical line in Fig. 2b]. Thus one can use the standard approximation methods based on Huygen's principle to obtain the leading singularities of $\tilde{f}$. Superposing the well-known $1/\sin\theta$ result in the directions $\pm \omega$ yields an approximation that reproduces correctly the leading singularity of $\tilde{f}$ close to the $\pm \omega$ directions,

$$\tilde{f}_{app.}(\theta) \propto \frac{1}{\sin(\theta + \omega)} - \frac{1}{\sin(\theta - \omega)}.$$

In fact Kirchhoff's formulae and thus Huygen's principle, on which the argument is based, yield only the correct leading amplitude close to the geometrical shadow region [6]. This is insensitive to the [unspecified] orientation of the edge that serves to cut off half the wave. Removing this arbitrariness gives rise to subleading corrections, which assemble into the complete scattering amplitude $\tilde{f}$.

The above picture also allows us to understand the absence of diffraction for $\alpha^{-1} = $ integer [2]. In this case the classical scattering angle ω is an integral multiple of π and the two shadow lines in Fig. 2a precisely overlap, combining into a dispersion free scattered wave. Thus the quantum mechanical scattering amplitude $\tilde{f}$ vanishes, leaving the classical result. For $\alpha^{-1} = $ even integer [classical back-scattering] our time-independent formulation of scattering results in a standing wave [2].

Having thus interpreted the main features of the scattering amplitude (3.12), let us formulate what one would expect for a test particle with spin: A rotation of the wave function of a particle with spin s by $\pm \omega$ introduces the additional phase $\exp(\mp is\omega)$. We thus expect this factor to multiply the delta function and diffraction contributions of the two respective "half-waves" propagating in the directions $\pm \omega$. In the case of a spin $\frac{1}{2}$-particle this will be demonstrated to be the only modification. An "exchange" argument will also allow us to understand the results for a spinning source.

In conclusion, let us remark that there is an approximation involved in the above qualitative analysis. In the embedded coordinate system that we employ, plane waves do not [locally] solve the wave equation, even far away from the scatterer; $\exp(i\kappa r\cos\theta/\alpha)$ should be used to represent monochromatic waves. Asymptotically and close to the direction of propagation [which is the region relevant for scattering] the two however are identical. Note that our exact quantitative analysis results in an incoming wave that superposes precisely such monochromatic waves; see (3.14). Alternatively one can present an analogous discussion in locally flat coordinates (2.4) where $\exp(i\kappa r\cos\theta/\alpha)$ becomes the plane wave $\exp(ik\rho\cos\varphi)$, but pictures lose their simplicity due to identifications of angles depending on which side the source is passed. The shape of the scattering amplitude

$\tilde{f}$ is insensitive to these coordinate changes, because radial waves are not distorted by a conical metric, and one just multiplies all angles by α.

IV. Klein–Gordon Equation on a Spinning Cone

We solve the equation

$$(\Box + m^2)\psi = 0, \tag{4.1}$$

where $\Box$ is the covariant d'Alembertian in imbedded coordinates,

$$\Box = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu = \partial_t^2 - \frac{\alpha^2}{r} \partial_r r \partial_r - \frac{1}{r^2} (S\partial_t - \partial_\theta)^2. \tag{4.2}$$

The *Ansatz*

$$\psi(t; \mathbf{r}) = e^{-iEt} \psi(\mathbf{r}) \tag{4.3}$$

reduces (4.1) to

$$\left[-\frac{\alpha^2}{r} \partial_r r \partial_r + \frac{(ES - i\partial_\theta)^2}{r^2} \right] \psi(\mathbf{r}) = (E^2 - m^2)\psi(\mathbf{r}). \tag{4.4}$$

Observe that a solution to (4.4) is

$$\psi(\mathbf{r}) = e^{-iES\theta} \psi_{S=0}(\mathbf{r}),$$

$$\psi(t; \mathbf{r}) = e^{-iE(t + S\theta)} \psi_{S=0}(\mathbf{r}) = e^{-iET} \psi_{S=0}(\mathbf{r}),$$

where $\psi_{S=0}$ is a solution in the absence of spin, e.g. any of the solutions considered in the previous section: the partial wave with definite angular momentum $u_n(r)e^{in\theta}$ of (3.10) or the scattering wave function in (3.2). However, such solutions are not acceptable because they do not satisfy proper boundary conditions: Since $\psi_{S=0}$ is periodic in 2π, the above is not periodic but obeys $\psi(t; r, \theta + 2\pi) = \psi(t + 2\pi S; r, \theta)$ [ES is not necessarily an integer [7]]; equivalently in locally flat coordinates, $\psi^f(T; \rho, \varphi + 2\pi\alpha) = \psi^f(T; \rho, \varphi)$. Moreover, since $\psi_{S=0}(0, \theta) = 1$, the above is ill-defined at the origin.

On the contrary, according to our discussion following Eq. (2.6), we impose the boundary conditions

$$\psi(t; r, \theta + 2\pi) = \psi(t; r, \theta), \tag{4.5a}$$

$$\psi^f(T + 2\pi S; \rho, \varphi + 2\pi\alpha) = \psi^f(T; \rho, \varphi). \tag{4.5b}$$

Thus the correct solution is periodic in θ and the time coordinate T, not t, jumps. This produces a non-trivial dependence on S, in contrast to the solution with improper boundary conditions whose spin dependence is merely a phase, which is unobservable in a closed system.

To construct a periodic solution of (4.4) we proceed, as in Sect. III, by making an *Ansatz* with angular momentum eigenvalue n,

$$\psi(\mathbf{r}) = u_n(r)e^{in\theta}, \quad n \in \mathbf{Z}. \tag{4.6}$$

The radial function satisfies

$$\left(-\frac{1}{r}\partial_r r \partial_r + \frac{(ES+n)^2}{\alpha^2 r^2}\right)u_n(r) = \kappa^2 u_n(r), \quad \alpha^2 \kappa^2 = E^2 - m^2, \tag{4.7}$$

with the regular solution given by a Bessel function,

$$\begin{aligned}
u_n(r) &= J_{|n+ES|/\alpha}(\kappa r), & n &\geqq -[ES], \\
&= (-1)^n J_{|n+ES|/\alpha}(\kappa r), & n &< -[ES],
\end{aligned} \tag{4.8}$$

whose phase shift is

$$\delta_n = \frac{\pi}{2}\left(n(1-\alpha^{-1}) - \frac{ES}{\alpha}\right) = -\frac{1}{2}\left(n\omega + \frac{\pi}{\alpha}ES\right) \quad n \geqq -[ES]$$

$$= -\frac{\pi}{2}\left(n(1-\alpha^{-1}) - \frac{ES}{\alpha}\right) = \frac{1}{2}\left(n\omega + \frac{\pi}{\alpha}ES\right) \quad n < -[ES]. \tag{4.9}$$

Here $[x]$ is the largest integer less than or equal to x. The scattering amplitude is found from (3.6), which after regularization yields

$$f(\theta) = -\frac{i\pi}{\sqrt{2\pi\kappa}}\sum_n (e^{i\pi ES/\alpha}\delta(\theta + \omega - 2\pi n)$$

$$+ e^{-i\pi ES/\alpha}\delta(\theta - \omega - 2\pi n) - 2\delta(\theta - 2\pi n)) + \tilde{f}(\theta). \tag{4.10a}$$

$$\tilde{f}(\theta) = \frac{i}{\sqrt{2\pi\kappa}}\left(e^{i\pi ES/\alpha}\frac{e^{-i[ES](\theta+\omega)}}{1-e^{i(\theta+\omega)}} - e^{-i\pi ES/\alpha}\frac{e^{-i[ES](\theta-\omega)}}{1-e^{i(\theta-\omega)}}\right). \tag{4.10b}$$

The result (4.10) is similar to the $S = 0$ expression (3.12), except that each of the two terms in $\tilde{f}$ is multiplied by different S-dependent phase factors, which are discontinuous in energy [for fixed S]. However, the full $\tilde{f}$ is continuous, as is the delta function contribution to f.

Two comments are in place here:
First, we observe that for a massless spinning target particle ($\alpha = 1$, $S \neq 0$) Eq. (4.7) coincides with the Schrödinger equation governing the motion of a charged (e) particle moving in the field of an infinitely thin solenoid, which carries a magnetic flux Φ, when the replacement

$$ES \leftrightarrow \frac{e\Phi}{2\pi} \tag{4.11}$$

is made. When $\alpha \neq 1$, (4.7) together with the replacement (4.11) describes the combined gravitational and Aharonov–Bohn interaction. For further elaborations on this analogy to the Aharonov–Bohm effect and on the corresponding "gauge-invariance" in the gravitational case see ref. [7].

Second, one correctly obtains the time delay (2.13b) [with $c = 1$, $G = 1$] in the presence of a spinning target from formula (4.9) by using Wigner's result [8]

$$\Delta t = 2\frac{\partial \delta_n(E)}{\partial E}. \tag{4.12}$$

The sign of the time delay is determined by [and is opposite to] the sign of the angular momentum in intrinsic coordinates, which describes the side on which the classical trajectory passes the source.

Next we construct the scattering wave function from (3.2). The Bessel function is represented by the Schläffi contour integral, the sum is evaluated, and the contours are shifted in the same manner as in the spinless case [2], to produce the final result, which we present in locally flat coordinates,

$$\psi^f(\rho, \varphi) = e^{-iES\varphi/\alpha} \int \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\boldsymbol{\rho}} \frac{e^{i\{ES\}z/\alpha}}{1 - e^{iz/\alpha}}. \tag{4.13}$$

The contour is the same as for (3.13) [Fig. 1], as is the notation $\mathbf{k}(z)$ and $\boldsymbol{\rho}$; $\{x\}$ is the fractional part of x: $\{x\} \equiv x - [x]$. Finally, the complete time-dependent solution

$$\psi^f(T; \rho, \varphi) = e^{-iEt}\psi^f(\rho, \varphi) = e^{-iET} \int \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\boldsymbol{\rho}} \frac{e^{i\{ES\}z/\alpha}}{1 - e^{iz/\alpha}} \tag{4.14}$$

clearly satisfies the flat Klein–Gordon equation and depends on the characteristics of the source—mass and spin—through the weight function in the integrand. This is as it must be, so that the boundary conditions (4.5) are satisfied. That they indeed hold, can be easily verified from (4.13) and (4.14). [For the verification, it is convenient to change the contour of Fig. 1a, by shifting its φ dependence into the integrand.]

The integral $\int (dz/2\pi)(e^{i\{ES\}z/\alpha}/(1 - e^{iz/\alpha}))$ over our contour is zero when ES is not an integer. Then the wave function (4.13) or (4.14) vanishes at the origin—a centrifugal-like barrier is responsible for this. An exception occurs when ES happens to be an integer, because then $\int (dz/2\pi)(1/(1 - e^{iz/\alpha})) = 1$ and $\psi^f(0, \varphi) = e^{-iES\varphi/\alpha}$. [This follows also, but less reliably, from the series representation (3.2): each partial wave (4.8) vanishes at the origin, except for integral ES, where the nth partial wave, with $n = -ES$, leads to $J_0(0) = 1$.] Consequently the φ-dependent phase in (4.13) causes no problem at the origin in contrast to $\psi_{S=0}$. The S-dependent phase factors in the integrand of (4.13) and (4.14) are discontinuous in ES: $\{ES\}$ vanishes when ES approaches an integer from above, but equals unity when the same integer is approached from below. Nevertheless, the wave function is continuous, because the potential discontinuity, proportional to $\int (dz/2\pi)e^{i\mathbf{k}(z)\cdot\boldsymbol{\rho}}(1 - e^{iz/\alpha})/(1 - e^{iz/\alpha}) = \int (dz/2\pi)e^{i\mathbf{k}(z)\cdot\boldsymbol{\rho}}$, integrates to zero on our contour.

The alternative contour of Fig. 1b gives the alternative decomposition (3.7). The poles of the integrand are at their spinless locations $z = z_n = 2\pi n\alpha$; the [negative] Cauchy contour around the poles determines $\tilde{\psi}_{in}$,

$$\tilde{\psi}_{in}^f(\rho, \varphi) = \alpha e^{-iES\varphi/\alpha} \sum_n{}' e^{i\mathbf{k}(z_n)\cdot\boldsymbol{\rho}} e^{i\{ES\}2\pi n}$$

$$= \alpha \sum_n{}' e^{-ik\rho \cos(\varphi - 2\pi n\alpha)} e^{-i(ES/\alpha)(\varphi - 2\pi n\alpha)}. \tag{4.15}$$

As in the spinless case, the sum is restricted so that the poles lie in the interval $[-\pi + \varphi, \pi + \varphi]$, and the incoming wave is a superposition of plane waves, with variously rotated incoming wave vectors and spin dependent relative phase factors.

Note that $\tilde{\psi}_{in}^{f}$ is continuous in energy, but discontinuous in φ, as in the spinless case. The latter discontinuity is a consequence of the separation in (3.7): $\tilde{\psi}_{sc}^{f}$ is similarly discontinuous, while ψ^{f} of (4.13) varies continuously with φ. The contribution from the vertical contours defines $\tilde{\psi}_{sc}^{f}$, whose asymptote is determined as in the spinless case [1, 2]. We find (3.8), with scattering amplitude $\tilde{f}$ of (4.10), which in locally flat coordinates reads

$$\tilde{f}^{f}(\varphi) = \frac{e^{-i[ES]\varphi/\alpha}}{2\sqrt{2\pi k}} \left[e^{-i\pi\{ES\}/\alpha} \left(\operatorname{ctn} \frac{\varphi - \pi}{2\alpha} - i \right) - e^{i\pi\{ES\}/\alpha} \left(\operatorname{ctn} \frac{\varphi + \pi}{2\alpha} - i \right) \right]. \quad (4.16)$$

Even though the various phase factors are discontinuous in energy, $\tilde{f}$ is continuous, and the description of scattering by the Klein–Gordon equation on a spinning cone shows no untoward features, other than the divergence of the total cross section—the angular integral of $|\tilde{f}|^{2}$—owing to the singularities in $\tilde{f}$. This was already observed in the spinless case, and is attributed to the long range nature of the "interaction." As a consequence, the optical theorem is inapplicable [2].

Nevertheless, it must be appreciated that a complete quantum mechanical framework does not naturally accompany solutions of a relativistic spinless wave equation that is second order in time. Moreover, an attempt to construct such a framework for our problem runs afoul of the closed time-like paths mentioned at the end of Sect. II. Let us observe that the Klein–Gordon equation (4.1), (4.2) may also be presented as

$$-\left((1 - S^{2}/r^{2})^{1/2}\partial_{t} + (1 - S^{2}/r^{2})^{-1/2}\frac{S}{r^{2}}\partial_{\theta} \right)^{2} \psi(t; \mathbf{r})$$

$$= \left(-\frac{\alpha^{2}}{r}\partial_{r}r\partial_{r} - (r^{2} - S^{2})^{-1}\partial_{\theta}^{2} + m^{2} \right)\psi(t; \mathbf{r}). \quad (4.17)$$

A quantum mechanical equation should be first order in time, so one is led to consider the "square-root" of (4.17),

$$\left((1 - S^{2}/r^{2})^{1/2}i\partial_{t} + (1 - S^{2}/r^{2})^{-1/2}\frac{iS}{r^{2}}\partial_{\theta} \right)\psi(t; \mathbf{r})$$

$$= \sqrt{-\frac{\alpha^{2}}{r}\partial_{r}r\partial_{r} - (r^{2} - S^{2})^{-1}\partial_{\theta}^{2} + m^{2}}\,\psi(t; \mathbf{r}), \quad (4.18a)$$

which may also be written as

$$i\partial_{t}\psi = H\psi,$$
$$H = (1 - S^{2}/r^{2})^{-1/2}(-\partial_{i}g^{(2)ij}\partial_{j} + m^{2})^{1/2} - (r^{2} - S^{2})^{-1}iS\partial_{\theta}. \quad (4.18b)$$

In (4.18b) the metric with superscript "2" refer to the 2-space metric tensor $-g_{ij}$ of (2.1) and (2.3b),

$$g_{ij}^{(2)} = (1 - S^{2}/r^{2})(\partial^{ij} - \hat{r}^{i}\hat{r}^{j}) + \alpha^{-2}\hat{r}^{i}\hat{r}^{j}, \quad (4.19a)$$

$$g^{(2)ij} = (1 - S^{2}/r^{2})^{-1}(\delta^{ij} - \hat{r}^{i}\hat{r}^{j}) + \alpha^{2}\hat{r}^{i}\hat{r}^{j}, \quad (4.19b)$$

$$g^{(2)} \equiv \det g_{ij}^{(2)} = \alpha^{-2}(1 - S^{2}/r^{2}). \quad (4.19c)$$

Equation (4.18) is not mathematically equivalent to (4.17), because the differential operator on the left-hand side of (4.18a) does not commute with the operator on the right-hand side. Nevertheless, (4.18) may be independently motivated by constructing the Hamiltonian corresponding to the Lagrangian (2.14). One finds

$$H = (1 - S^2/r^2)^{-1/2}(p_i g^{(2)ij} p_j + m^2)^{1/2} + (r^2 - S^2)^{-1} S\varepsilon^{ij} r^i p_j, \qquad (4.20)$$

where

$$p_i \equiv \frac{\partial L}{\partial \dfrac{dr^i}{dt}}. \qquad (4.21)$$

Clearly the Hamiltonian operator in (4.18b) corresponds to a particular ordering of (4.20).

The natural measure for computing the inner product between solutions to (4.18) is $d^2\mathbf{r}\sqrt{g^{(2)}} = d^2\mathbf{r}\alpha^{-1}(1 - S^2/r^2)^{1/2}$. With this inner product, H of (4.18b) would be self-adjoint, except that in the region $r < S$ the square root of $1 - S^2/r^2$ is imaginary, spoiling hermiticity of H and reality of the norm. The closed time-like contours of the metric (2.1) also inhabit this domain of r.

We shall see that for the first order Dirac equation on a spinning cone, whose quantum mechanical interpretation is much more immediate than for the second order Klein–Gordon equation, similar problems in the region $r < S$ interfere with a consistent quantum mechanical framework.

V. Dirac Equation on a Spinning Cone

1. Preliminaries. To begin, we review the $2 + 1$-dimensional free Dirac theory and describe the appropriate modification to the conical scattering formalism outlined in Sect. III [9].

We take 2×2 γ-matrices appropriate for 3-dimensional spacetime: $\gamma^0 \equiv \beta = \sigma^3$, $\gamma^1 = i\sigma^2$ and $\gamma^2 = -i\sigma^1$, obeying the relation

$$\gamma^a\gamma^b = \eta^{ab} - i\varepsilon^{abc}\gamma_c, \qquad (5.1)$$

where η^{ab} is the Minkowski metric tensor and ε^{abc} is the 3-dimensional Levi–Civita symbol ($\varepsilon^{012} = 1$). In our index conventions, Roman letters from the beginning of the alphabet denote Minkowski vectors, from the middle they represent spatial [coordinate] two-vectors and Greek letters denote space-time vectors.

Solutions to the free massive Dirac equation in Minkowski space,

$$(i\gamma^a\partial_a - m)\psi^o = 0, \qquad (5.2)$$

can be expanded in plane waves. For positive energy these are

$$\psi_{\mathbf{k}}^o(\mathbf{r})e^{-iEt} = \begin{pmatrix} \sqrt{E+m} \\ \sqrt{E-m}\ \ e^{i\theta_0} \end{pmatrix} e^{-i(Et - \mathbf{kr})}, \qquad (5.3)$$

where the normalization is fixed by $\psi^{o\dagger}\psi^o = 2E$. Here $k_x + ik_y =$

$ke^{i\theta_0} = \sqrt{E^2 - m^2}\, e^{i\theta_0}$ and the associated current is

$$j^a = \bar{\psi}\gamma^a\psi = 2(E, \mathbf{k}). \tag{5.4}$$

[The Dirac adjoint of spinors is defined by $\bar{\psi} \equiv \psi^\dagger\gamma^{a=0}$.] Alternatively one can diagonalize the angular momentum operator,

$$\mathscr{J} = -i\partial_\theta + \tfrac{1}{2}\sigma^3, \tag{5.5}$$

yielding partial waves with angular momentum $n + \tfrac{1}{2}$,

$$u_n^o(\mathbf{r})e^{-iEt} = \exp\left\{i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\}u_n^o(r)e^{-iEt}, \tag{5.6a}$$

$$u_n^o(r) = \begin{pmatrix} \sqrt{E + m}\; J_n(kr) \\ i\sqrt{E - m}\; J_{n+1}(kr) \end{pmatrix}, \quad n \in \mathbf{Z}. \tag{5.6b}$$

The expansion of the plane wave (5.3) in this basis is

$$\psi_{\mathbf{k}}^o(\mathbf{r}) = \sum_n i^n e^{-in\theta_0} u_n^o(\mathbf{r}). \tag{5.7}$$

We emphasize that the positive energy, two-component solution of the massive Dirac equation has only one degree of freedom, whose spin is 1/2 [as seen in the rest frame] and the theory is parity violating [10]. Therefore, in the presence of interactions we shall find only a single set of phase shifts δ_n and one scattering amplitude f. To determine it, eigenfunctions of $\mathscr{J}$ are constructed, as in the scalar case,

$$u_n(\mathbf{r}) = \exp\left\{i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\}\begin{pmatrix} \sqrt{E + m}\; u_n^{(1)}(\kappa r) \\ i\sqrt{E - m}\; u_n^{(2)}(\kappa r) \end{pmatrix} \tag{5.8}$$

whose asymptote

$$u_n^{(1)}(\kappa r) \xrightarrow[r\to\infty]{} \sqrt{\frac{2}{\pi\kappa r}}\cos\left(\kappa r - \frac{n\pi}{2} - \frac{\pi}{4} + \delta_n^{(1)}\right),$$

$$u_n^{(2)}(\kappa r) \xrightarrow[r\to\infty]{} \sqrt{\frac{2}{\pi\kappa r}}\cos\left(\kappa r - \frac{n\pi}{2} + \frac{\pi}{4} + \delta_n^{(2)}\right) \tag{5.9}$$

[with κ a function of energy E] determines the phase shifts $\delta_n^{(i)}$, which will be shown to obey the relation $\delta_n^{(1)} = \delta_n^{(2)} \equiv \delta_n$. The scattering solution [with $\theta_0 = 0$]

$$\psi(\mathbf{r}) = \sum_n e^{i(\delta_n + n\pi/2)}u_n(\mathbf{r}) \tag{5.10}$$

defines the scattering amplitude *via* the decomposition

$$\psi = \psi_{\text{in}} + \psi_{\text{sc}}, \tag{5.11}$$

and

$$\psi_{\text{sc}}(\mathbf{r}) \xrightarrow[r\to\infty]{} \sqrt{\frac{i}{r}}f(\theta)\begin{pmatrix} \sqrt{E + m} \\ \sqrt{E - m}\; e^{i\theta} \end{pmatrix}e^{ikr}, \tag{5.12}$$

where the above spinor gives rise to a purely radial current. Taking the incoming

wave to be a plane wave in the x-direction [(5.3) with $k \to \kappa(E)$ and $\theta_0 = 0$] gives the previous scalar formula (3.6) for f in terms of the phase shifts δ_n.

2. Solutions of the Dirac Equation on the Cone. It would seem that a way to obtain solutions of the Dirac equation on the spinning cone is to solve first the flat equation in the coordinates (T, ρ, φ) and then to impose the proper boundary conditions (4.5). It turns out however, that for $S \neq 0$ the behavior at the origin is non-trivial. Requiring the wave function to be regular does not lead to a self-adjoint Hamiltonian. In order to study this question further we have to construct the Hamiltonian and therefore the Dirac equation in the coordinates (t, r, θ).

We take the *dreibein* in (t, r, θ) coordinates to be [11]

$$E_a{}^\mu = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{S}{r}\sin\theta & \alpha\cos\theta & -\dfrac{1}{r}\sin\theta \\ -\dfrac{S}{r}\cos\theta & \alpha\sin\theta & \dfrac{1}{r}\cos\theta \end{pmatrix} \tag{5.13a}$$

with the inverse

$$e_\nu{}^b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\alpha}\cos\theta & \dfrac{1}{\alpha}\sin\theta \\ S & -r\sin\theta & r\cos\theta \end{pmatrix}. \tag{5.13b}$$

The usual relations are obeyed,

$$E_a{}^\mu g_{\mu\nu} E_b{}^\nu = \eta_{ab}, \tag{5.14a}$$

$$E_a{}^\mu \eta^{ab} E_b{}^\nu = g^{\mu\nu}. \tag{5.14b}$$

The spin connection $\omega_{\mu;ab} = -\omega_{\mu;ba}$, which forms the covariant derivative for a spinor, $D_\mu \equiv \partial_\mu + \frac{1}{2}\omega_{\mu;ab}\sigma^{ab}$, $\sigma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]$, may be written in three dimensions as $\omega_{\mu;ab} = \varepsilon_{abc}\omega_\mu{}^c$. Also from (5.1), $\sigma^{ab} = -\dfrac{i}{2}\varepsilon^{abc}\gamma_c$. Hence $D_\mu = \partial_\mu - \dfrac{i}{2}\omega_\mu{}^a \gamma_a$. The form of $\omega_\mu{}^a$ is determined by

$$\varepsilon^{\lambda\mu\nu}\partial_\mu e_\nu{}^a = \varepsilon^{\lambda\mu\nu}\varepsilon^a{}_{bc}\omega_\mu{}^b e_\nu{}^c. \tag{5.15}$$

With (5.13) the only non-vanishing component is

$$\omega_\theta{}^0 = \alpha - 1. \tag{5.16}$$

The Dirac equation

$$(i\gamma^a E_a{}^\mu D_\mu - m)\psi = 0 \tag{5.17a}$$

on the spinning cone reads

$$\left\{ i\left[\gamma^0\partial_t + \gamma^r\left(\alpha\partial_r - \frac{1-\alpha}{2r}\right) + \gamma^\theta\left(-\frac{S}{r}\partial_t + \frac{1}{r}\partial_\theta\right)\right] - m \right\}\psi = 0, \tag{5.17b}$$

where $\gamma^r = \cos\theta\gamma^1 + \sin\theta\gamma^2$, $\gamma^\theta = -\sin\theta\gamma^1 + \cos\theta\gamma^2$.

The rotational invariance of (5.17b) enables us to choose positive energy solutions that are simultaneously angular momentum eigenfunctions, with eigenvalue $n + \frac{1}{2}$,

$$u_n(\mathbf{r})e^{-iEt} = \exp\left\{i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\}\begin{pmatrix}\sqrt{E+m} & u_n^{(1)}(r)\\ i\sqrt{E-m} & u_n^{(2)}(r)\end{pmatrix}e^{-iEt}, \quad n\in\mathbf{Z}, \quad (5.18)$$

and (5.17b) reduces to the following system of coupled equations.

$$\begin{pmatrix} (E-m) & i\left[\alpha\left(\partial_r + \dfrac{1}{2r}\right) + \dfrac{ES + n + \frac{1}{2}}{r}\right] \\ i\left[-\alpha\left(\partial_r + \dfrac{1}{2r}\right) + \dfrac{ES + n + \frac{1}{2}}{r}\right] & -(E+m) \end{pmatrix} \times$$

$$\times \begin{pmatrix}\sqrt{E+m} & u_n^{(1)}(r)\\ i\sqrt{E-m} & u_n^{(2)}(r)\end{pmatrix} = 0. \quad (5.19)$$

The solutions for $E^2 > m^2$ are

$$u_n^{(1)}(r) = (\varepsilon_n)^n J_{\varepsilon_n(\nu/\alpha)}(\kappa r),$$
$$u_n^{(2)}(r) = (\varepsilon_n)^{n+1} J_{\varepsilon_n(\nu/\alpha + 1)}(\kappa r). \quad (5.20)$$

Here $\nu \equiv n + ES + (1-\alpha)/2$, $\kappa = 1/\alpha\sqrt{E^2 - m^2}$, $\varepsilon_n = \pm 1$ and the same sign has to be chosen for the upper and lower component. The asymptotic form of the Bessel functions determines the phase shifts; as claimed they are identical for the upper and lower components and are given by

$$\delta_n = -\varepsilon_n \frac{\pi}{2}\left(\frac{\nu}{\alpha} - n\right). \quad (5.21)$$

The choice of the sign ε_n for arbitrary S will require some discussion. We shall therefore first describe the results for $S = 0$, which are quite interesting in themselves, and on that basis proceed to discuss arbitrary S.

Let us remark that just as in the scalar case the replacement (4.11) converts (5.19) into the Dirac equation in an Aharonov-Bohm field, without gravity for $\alpha = 1$, with gravity for $\alpha < 1$.

2.a Results for a Spinless source. For a spinless source ($S = 0$) the orders of the Bessel functions appearing in Eq. (5.20) are $(\varepsilon_n/\alpha)(n + (1 \mp \alpha)/2)$. Since $0 < \alpha \leq 1$, we must choose $\varepsilon_n = \mathrm{sign}(n + (1-\alpha)/2) = \mathrm{sign}\,n$, (sign $0 \equiv 1$) to have both components regular at the origin.

We now rewrite our formulae for the phase shifts in a way that allows deriving results for the present case from the calculations of Sect. IV. At the same time, this suggests a conjecture for the case in which both the test particle and the source carry spin.

The phase shifts δ_n from (5.21) can be written as

$$\delta_n = - \varepsilon_n \frac{\pi}{2\alpha}((1-\alpha)n + (1-\alpha)/2),$$

$$\varepsilon_n = \text{sign}(n + (1-\alpha)/2). \tag{5.22}$$

On the other hand, the phase shifts in Eq. (4.9) can be given by

$$\delta_n = - \varepsilon_n \frac{\pi}{2\alpha}((1-\alpha)n + ES),$$

$$\varepsilon_n = \text{sign}(n + ES). \tag{5.23}$$

Comparing these formulae we see that the present result is related to the corresponding expression for a spinless particle on a spinning cone by the replacement

$$ES \to (1-\alpha)/2 = \frac{\alpha\omega}{2\pi}. \tag{5.24}$$

Since the integral part $[(1-\alpha)/2]$ vanishes, the scattering amplitude for the Dirac particle in the field of a spinless source is [from (4.10)]

$$f(\theta) = \frac{-i\pi}{\sqrt{2\pi\kappa}} \sum_n \left(e^{i\omega/2}\delta(\theta + \omega - 2\pi n) + e^{-i\omega/2}\delta(\theta - \omega + 2\pi n) - 2\delta(\theta - 2\pi n)\right) + \tilde{f}(\theta),$$

$$\tilde{f}(\theta) = \frac{i}{\sqrt{2\pi\kappa}}\left(e^{i\omega/2}\frac{1}{1 - e^{i(\theta + \omega)}} - e^{-i\omega/2}\frac{1}{1 - e^{i(\theta - \omega)}}\right). \tag{5.25}$$

[We postpone the contour representation for the wave function and the alternative decomposition $\psi = \tilde{\psi}_{\text{in}} + \tilde{\psi}_{\text{sc}}$, until the next sub-section.]

These results can be described by the wave picture, presented at the end of Sect. III. As anticipated in that discussion, the diffraction is given by the same amplitude as for a scalar test particle [Eq. (3.12)]. except that the parts stemming from the two half waves continuing in the $\pm \omega$-directions are multiplied by a phase factor $\exp(\mp (i/2)\omega)$ appropriate for a spin 1/2 particle rotated by that angle. This produces of course some appreciable physical changes, e.g. while the scattering amplitude $\tilde{f}$ still vanishes for $\alpha^{-1} = $ odd integer, it no longer vanishes for $\alpha^{-1} = $ even integer [classical backscattering], because the two half waves differ by a sign and thus no longer combine to a dissipation-free wave.

The phase shift formulae (5.22) and (5.23) can both be represented by a universal expression,

$$\delta_n = - \varepsilon_n \frac{\pi}{2\alpha}((1-\alpha)J^t + E^t J^s) = - \varepsilon_n \frac{\pi}{2\alpha}(E^s J^t + E^t J^s),$$

$$\varepsilon_n = \text{sign}(\alpha n + E^s J^t + E^t J^s). \tag{5.26}$$

Here E and J denote energy and total [i.e. orbital and spin] angular momentum, s and t label source and test particle respectively and we use the relation $E^s = M = 1 - \alpha$ between source mass and deficit angle. Thus the above wave picture also allows us to relate to the results of the spinless test particle scattering off a spinning source by interpreting Eq. (5.24) as an "exchange" of the "source" and "test" labels $ES = E^t S^s \to E^s S^t = M/2 = (1-\alpha)/2$, where S denotes the respective spin.

The above leads to the conjecture that Eqs. (5.26) remain valid if source and test particles carry spin, i.e. these formulae provide a universal description for the scattering of spinning particles. Let us now see how this compares with our results for a Dirac particle on a spinning background.

2b. Results for a Spinning Source. We rewrite the orders of the Bessel functions in (5.20) as $(\varepsilon_n/\alpha)(n + ES + \frac{1}{2}(1 - \alpha))$ and $(\varepsilon_n/\alpha)(n + ES + \frac{1}{2}(1 + \alpha))$, giving the phase shifts

$$\delta_n = -\varepsilon_n \frac{\pi}{2\alpha}((1 - \alpha)(n + \tfrac{1}{2}) + ES). \tag{5.27}$$

Apart from the rule that fixes ε_n, which is still to be determined, (5.27) is of the form (5.26), thus partially confirming the conjectured universal description. However specification of ε_n is complicated by the fact that for

$$n + \tfrac{1}{2}(1 - \alpha) < -ES < n + \tfrac{1}{2}(1 + \alpha), \tag{5.28a}$$

or equivalently

$$ES = -(n + \tfrac{1}{2}(1 - \alpha) + \eta\alpha), \quad 0 < \eta < 1, \tag{5.28b}$$

there is no choice for ε_n which renders both components of the wave function regular at the origin. [For non-integer $\lambda, J_\lambda(z)$ behaves as z^λ at the origin.] The singular solutions must not be rejected altogether, because as we now show this would lead to a Hamiltonian that is not self-adjoint. Indeed in order to preserve self-adjointness of the Hamiltonian the regularity requirement at the origin must be relaxed [12].

First we derive the "measure" for spatial integrals of our spinors. The current

$$j^a E_a{}^\mu = \bar{\psi}\gamma^a\psi E_a{}^\mu = j^\mu \tag{5.29}$$

is covariantly conserved,

$$D_\mu j^\mu = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}j^\mu = 0. \tag{5.30}$$

Hence the spatial integral over the density ρ [here and in the following $\sigma^r = \cos\theta\sigma^1 + \sin\theta\sigma^2; \sigma^\theta = -\sin\theta\sigma^1 + \cos\theta\sigma^2$]

$$\rho = j^{\mu=0} = \psi^\dagger\gamma^{a=0}\gamma^{\mu=0}\psi = \psi^\dagger\left(1 - \frac{S}{r}\sigma^\theta\right)\psi \tag{5.31}$$

is time independent and gives a suitable measure. The eigenvalues of $(1 - (S/r)\sigma^\theta)$ are $(1 \pm (S/r))$; thus for $r < S$ the measure is not positive definite. This difficulty, which arises from the region of r with closed timelike contours, prevents a consistent quantum mechanical interpretation and points towards second quantization where ρ is interpreted as the charge density.

From the Dirac equation (5.17b) we can construct the Hamiltonian by isolating the time derivative,

$$i\partial_t\psi = H\psi = \left(\gamma^0 - \frac{S}{r}\gamma^\theta\right)^{-1}\left\{(-i)\left[\gamma^r\left(\alpha\partial_r - \frac{1-\alpha}{2r}\right) + \gamma^\theta\frac{1}{r}\partial_\theta\right] + m\right\}\psi$$

$$= \left(1 - \frac{S}{r}\sigma^\theta\right)^{-1}\left\{(-i)\left[\sigma^r\left(\alpha\partial_r - \frac{1-\alpha}{2r}\right) + \sigma^\theta\frac{1}{r}\partial_\theta\right] + \beta m\right\}\psi. \quad (5.32)$$

Time is separated in the usual way and the angular dependence is removed by working in an eigenspace of the angular momentum $\mathscr{J}$ (5.5) with eigenvalue $n + 1/2$,

$$\psi(t;\mathbf{r}) = \exp\left\{i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\}\chi(r)e^{-iEt}. \quad (5.33)$$

In this eigenspace the Hamiltonian can be represented by a radial operator h acting or radial spinors $\chi(r) = \begin{pmatrix} \chi^{(1)}(r) \\ \chi^{(2)}(r) \end{pmatrix}$,

$$H\psi(t;\mathbf{r}) = \exp\left\{i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\}h\chi(r)e^{-iEt},$$

$$h = \left(1 - \frac{S}{r}\sigma^2\right)^{-1}\left\{-i\sigma^1\left(\alpha\partial_r + \frac{\alpha}{2r}\right) + \sigma^2\frac{n + 1/2}{r} + \beta m\right\}. \quad (5.34)$$

The measure for the r integral of radial spinors is

$$\exp\left\{-i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\}\left(1 - \frac{S}{r}\sigma^\theta\right)\exp\left\{i(n + \tfrac{1}{2} - \tfrac{1}{2}\sigma^3)\theta\right\} = \left(1 - \frac{S}{r}\sigma^2\right) \equiv \mathscr{M}.$$

$$(5.35)$$

Because $\mathscr{M}^{-1}$ appears as a factor, the radial Hamiltonian is singular at $r = S$. Nevertheless h is unambiguously defined for $r < S$ and the eigenfunctions (5.20) do not reflect the singularity at this point.

The loss of positivity of the measure means that we cannot apply the usual apparatus of von Neumann deficiency subspaces in order to check the self-adjointness of h and/or construct its self-adjoint extensions, because this theory relies on positive definiteness of the inner product of a Hilbert space. But we can examine these questions directly and construct an appropriate domain for h to be self-adjoint.

Let us restate the general problem: In order to establish *symmetry* of h we have to prove for two arbitrary radial spinors χ and $\tilde{\chi}$ that

$$\int_0^\infty rdr\tilde{\chi}^\dagger(r)\mathscr{M}\overrightarrow{h}\chi(r) = \int_0^\infty rdr\,\tilde{\chi}^\dagger(r)\overleftarrow{h}^\dagger\mathscr{M}\chi(r) = \int_0^\infty rdr\tilde{\chi}^\dagger(r)\overleftarrow{h}\mathscr{M}\chi(r), \quad (5.36)$$

where the first equality defines $h^\dagger$ and the second is true for symmetric h [arrows indicate the direction of derivatives, which act only on the spinors]. This equality can easily be established, as long as the boundary term from the radial partial integration vanishes,

$$\lim_{r\to 0} r(\tilde{\chi}^\dagger(r)\sigma^1\chi(r)) = \lim_{r\to 0} r(\tilde{\chi}^{(1)*}(r)\chi^{(2)}(r) + \tilde{\chi}^{(2)*}(r)\chi^{(1)}(r)) = 0. \quad (5.37)$$

[We always assume everything to be well-behaved at ∞.]

A symmetric Hamiltonian is *self-adjoint* when its domain coincides with the

domain of its conjugate. It is seen from Eq. (5.37) that if the domain of h consists of spinors $\chi(r)$ regular at the origin [and square integrable with the measure $\mathcal{M}$] the domain of $h^\dagger$ is larger, since the spinors $\tilde{\chi}(r)$ could diverge as $r^{\varepsilon-1}$, $\varepsilon > 0$, yet still satisfy Eq. (5.37) and remain square integrable. We thus have to posit a condition that assures (5.37) and is *symmetric* in χ and $\tilde{\chi}$. An acceptable criterion involves two parameters $0 \leq \eta \leq 1$ and $-(\pi/2) < \lambda \leq (\pi/2)$; we demand that both χ and $\tilde{\chi}$ satisfy

$$\lim_{r \to 0} (mr)^n \chi_1(r) \cos \lambda = i \lim_{r \to 0} (mr)^{1-n} \chi_2(r) \sin \lambda \qquad (5.38)$$

[m is inserted in order to assure proper dimensionality].

For fixed values of λ and η the radial energy eigenfunctions [see Eq. (5.18) and (5.20)] are

$$u_n(r) = u_n^+(r) \equiv \begin{pmatrix} \sqrt{E+m} & J_{\nu/\alpha} & (\kappa r) \\ i\sqrt{E-m} & J_{((\nu/\alpha)+1)} & (\kappa r) \end{pmatrix} \qquad \text{for} \quad \nu > -\eta\alpha, \quad (5.39a)$$

$$u_n(r) = u_n^-(r) \equiv \begin{pmatrix} \sqrt{E+m} & (-1)^n J_{-\nu/\alpha} & (\kappa r) \\ i\sqrt{E-m} & (-1)^{n+1} J_{-((\nu/\alpha)+1)} & (\kappa r) \end{pmatrix} \qquad \text{for} \quad \nu < -\eta\alpha, \quad (5.39b)$$

$$u_n(r) = \sin \mu u_n^+(r) + \cos \mu u_n^-(r) \qquad \text{for} \quad \nu = -\eta\alpha. \quad (5.39c)$$

Here again $\nu = n + ES + (1-\alpha)/2$. As n is an integer, the last case only occurs for special values of E. For $\eta = 0, 1$ the parameter μ is irrelevant, as $u_n^+ = u_n^-$ in (5.39c). Otherwise μ is fixed by (5.38), when we recall that $J_n(z) \xrightarrow[z \to 0]{} (z/2)^n 1/\Gamma(1+\eta)$,

$$\tan \mu = (-1)^n \sqrt{\frac{E-m}{E+m}} \left(\frac{\kappa}{2m}\right)^{2\eta-1} \frac{\Gamma(1-\eta)}{\Gamma(\eta)} \tan \lambda. \qquad (5.40)$$

The solutions (5.39) remain square integrable [at the origin], even in the limiting case $\eta = 0, 1$, since $J_{-1} = -J_1$ is regular there.

The corresponding phase shifts are given by

$$\delta_n = \delta_n^+ \equiv -\frac{\pi}{2\alpha}((1-\alpha)(n+\tfrac{1}{2}) + ES) \quad n > -(ES + \tfrac{1}{2}(1-\alpha) + \eta\alpha), \quad (5.41a)$$

$$\delta_n = \delta_n^- \equiv \frac{\pi}{2\alpha}((1-\alpha)(n+\tfrac{1}{2}) + ES) \qquad n < -(ES + \tfrac{1}{2}(1-\alpha) + \eta\alpha). \quad (5.41b)$$

If E is such that an equality sign can arise in the above definition of the domain of the solutions, the remaining phase shift is determined from Eq. (5.39c),

$$\tan \delta_n = \frac{\tan \mu - 1}{\tan \mu + 1} \tan \delta_n^+ \quad n = -(ES + \tfrac{1}{2}(1-\alpha) + \eta\alpha), \qquad (5.41c)$$

where $\delta_n^\pm$ are defined in (5.41a, b).

The scattering amplitude is

$$f(\theta) = \frac{-i\pi}{\sqrt{2\pi\kappa}} \sum_n (e^{i(\pi ES/\alpha + \omega/2)}\delta(\theta + \omega - 2\pi n) + e^{-i(\pi ES/\alpha + \omega/2)}\delta(\theta - \omega - 2\pi n)$$

$$- 2\delta(\theta - 2\pi n)) + \tilde{f}(\theta). \tag{5.42}$$

For energies E different from the special values in (5.41c) one has

$$\tilde{f}(\theta) = \frac{i}{\sqrt{2\pi\kappa}} \left(e^{i(\pi ES/\alpha + \omega/2)} \frac{e^{-i[ES + (1-\alpha)/2 + \eta\alpha](\theta + \omega)}}{1 - e^{i(\theta + \omega)}} \right.$$

$$\left. - e^{-i(\pi ES/\alpha + \omega/2)} \frac{e^{-i[ES + (1-\alpha)/2 + \eta\alpha](\theta - \omega)}}{1 - e^{i(\theta - \omega)}} \right). \tag{5.43a}$$

On the other hand if there is an $n^* \in \mathbf{Z}$ such that $E = E^* \equiv -1/S(n^* + (1-\alpha)/2 + \eta\alpha)$, then $\tilde{f}$ reads

$$\tilde{f}(\theta) = \frac{i}{\sqrt{2\pi\kappa}} \left(e^{i(\pi ES/\alpha + \omega/2)} \frac{e^{-i(ES + (1-\alpha)/2 + \eta\alpha)(\theta + \omega)}}{1 - e^{i(\theta + \omega)}} \right.$$

$$\left. + e^{-i(\pi ES/\alpha + \omega/2)} \frac{e^{-i(ES + (1-\alpha)/2 + \eta\alpha)(\theta - \omega)}}{1 - e^{-i(\theta - \omega)}} - e^{2i\delta_{n^*}}e^{-i(ES + (1-\alpha)/2 + \eta\alpha)\theta} \right), \tag{5.43b}$$

where an explicit expression for the last summand is obtained from Eq. (5.41c) with the help of the identity $\exp(2i\delta_{n^*}) = (1 + i\tan\delta_{n^*})^2/(1 + \tan^2\delta_{n^*})$.

We observe that for generic η the scattering amplitude is discontinuous in energy at such a special value E^*. Upon defining $f_{E^*}^\pm(\theta) = \lim_{\varepsilon \to 0} f_{E^* \pm \varepsilon}(\theta)$, it follows from Eq. (3.6) [or with more effort from Eq. (5.43a)],

$$f_{E^*}^+(\theta) - f_{E^*}^-(\theta) = \frac{e^{in^*\theta}}{\sqrt{-2\pi\kappa}}(\exp(2i\delta_{n^*}^+) - \exp(2i\delta_{n^*}^-))$$

$$= \frac{e^{in^*\theta}}{\sqrt{-2\pi\kappa}}(e^{i\pi(n^* + \eta)} - e^{-i\pi(n^* + \eta)}) = \frac{2ie^{in^*}(\theta + \pi)}{\sqrt{-2\pi\kappa}} \sin\eta\pi. \tag{5.44}$$

This vanishes only for

$$\eta = 0, 1. \tag{5.45a}$$

With Eq. (5.41) the last equation can be restated as

$$\exp 2i\delta_{n^*}^+ = \exp 2i\delta_{n^*}^- = \exp 2i\delta_{n^*} = \pm 1. \tag{5.45b}$$

Thus when Eq. (5.41a) is satisfied we have indeed $f_{E^*}^+(\theta) = f_{E^*}^-(\theta) = f_{E^*}(\theta)$, i.e. Eq. (5.43b) coincides with the upper and lower limit in (5.43a) and the scattering amplitude is continuous in energy.

The wave function depends on the parameter λ via Eqs. (5.39c) and (5.40), but for $\eta = 0$ or 1, $u_n^\pm(r)$ in Eq. (5.39) coincide [possibly modulo a sign] at $v = -\eta\alpha = 0$ or $-\alpha$ respectively and thus the scattering amplitude is in fact independent of λ.

The value $\eta = 0$ is further preferred, because then the present phase shifts fit the universal form (5.26) in agreement with the conjecture following that equation.

From now on we take $\eta = 0, \lambda = 0$. With this particular choice for the self-adjoint

extension, the upper ["large"] component always is regular at the origin, while the lower ["small"] component can diverge, but remains square integrable and satisfies (5.38) with $\eta = 0$, $\lambda = 0$.

The contour representation for the wave function is derived in the same way as in Sect. IV. Formulae (5.20) and (5.22) together with the Schläfli representation for the Bessel function yield the following expression [for $\eta = 0$, $\lambda = 0$], which is defined on the same contour as (3.10) and (4.11) [Fig. 1 with $\varphi \equiv \alpha(\theta - \pi)$],

$$\psi(t; r, \theta) = e^{-iEt} e^{-iES(\theta - \pi)} \int \frac{dr}{2\pi} e^{-i\kappa r \cos(z - \alpha(\theta - \pi))}$$

$$\cdot \frac{e^{i\{ES + (1 - \alpha)/2\}z\alpha}}{1 - e^{iz/\alpha}} e^{-(i\omega/2\pi)\alpha(\theta - \pi)\sigma^3} \begin{pmatrix} \sqrt{E + m} \\ \sqrt{E - m} \quad e^{iz} \end{pmatrix}. \qquad (5.46)$$

It is not hard to see that this indeed solves the Dirac equation (5.17b) and to check that the proper boundary conditions (4.5) are obeyed. $\psi^f(T; \rho, \varphi)$ is obtained via the coordinate transformations (2.4) and (2.5),

$$\psi^f(T; \rho, \varphi) = e^{-iET} \int \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\rho} \frac{e^{i\{ES + (1 - \alpha)/2\}z/\alpha}}{1 - e^{iz/\alpha}} e^{-(i\omega/2\pi)\varphi\sigma^3} \begin{pmatrix} \sqrt{E + m} \\ \sqrt{E - m} \quad e^{iz} \end{pmatrix}. \qquad (5.47)$$

This is not a solution $\psi^o(T; \rho, \varphi)$ of the free Dirac equation in coordinates $(T; \rho, \varphi)$ that one would have naively presented. The relation between these two is

$$\psi^f(T; \rho, \varphi) = R\psi^o(T; \rho, \varphi), \qquad (5.48a)$$

$$R = e^{i(\pi/2)\sigma^3} e^{-(i\omega/2\pi)\varphi\sigma^3}. \qquad (5.48b)$$

[The first factor in the right-hand side of (5.48b) just takes care of the π in the definition $\varphi = \alpha(\theta - \pi)$.] This can be checked by performing the coordinate transformations (2.4) and (2.5) on Eq. (5.17b) and observing that the resulting equation is indeed related to the conventional free Dirac equation in radial coordinates by the relation R.

$$R^{-1} \left\{ i \left[\gamma^0 \partial_T + \gamma^r \left(\partial_\rho - \frac{1}{\rho} \frac{\omega}{2\pi} \right) + \gamma^\theta \left(\frac{1}{\rho} \partial_\varphi \right) \right] - m \right\} R$$

$$= i \left[\gamma^0 \partial_T + \gamma^\rho \partial_\rho + \gamma^\varphi \frac{1}{\rho} \partial_\varphi \right] - m. \qquad (5.49)$$

Here $\gamma^\rho = \cos \varphi \gamma^1 + \sin \varphi \gamma^2$; $\gamma^\varphi = -\sin \varphi \gamma^1 + \cos \varphi \gamma^2$.

The correct $\psi^f(T; \rho, \varphi)$ is not a zero mode of the free Dirac operator (5.49) [but instead is related to it via the transformation (5.48)], since these zero modes $\psi^o(T; \sigma, \varphi)$ can be only periodic in φ with the conventional period 2π—as indeed the operator itself has this period—whereas $\psi^f(T; \rho, \varphi)$ has to obey the boundary condition (4.5).

The remainder of this section is completely analogous to the scalar case in Sect. IV. Again the alternative contour of Fig. 1b gives a decomposition that can be interpreted as an incoming and a scattered wave. Equation (5.47) shows that the poles inside the Cauchy contour are at the same locations $z = z_n = 2\pi n\alpha$ as in

the spinless case. They determine the alternative incoming wave,

$$\tilde{\psi}_{\text{in}}^{f}(\rho, \varphi) = \alpha e^{-iES\varphi/\alpha} \sideset{}{'}\sum_{n} e^{\mathbf{k}(z_n)\cdot\rho} e^{i\{ES + (1-\alpha)/2\}2\pi n} e^{-(i\omega/2\pi)\varphi\sigma^3} \begin{pmatrix} \sqrt{E+m} \\ \sqrt{E-m} \quad e^{i2\pi n\alpha} \end{pmatrix}$$

$$= \alpha \sideset{}{'}\sum_{n} e^{-ik\rho\cos(\varphi - 2\pi n\alpha)} e^{-i(ES/\alpha + \omega\sigma^3/2\pi)(\varphi - 2\pi n\alpha)} \begin{pmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{pmatrix}. \quad (5.50)$$

The prime on the sum again restricts the summation so the poles lie in the interval $[-\pi + \varphi, \pi + \varphi]$. The integral along the vertical contours gives the scattered wave. Asymptotically we find the form (5.12), with a scattering amplitude that agrees with (5.41), and in flat coordinates reads

$$\tilde{f}^{f}(\varphi) = \frac{e^{-i[ES + (1-\alpha)/2]\varphi/\alpha}}{2\sqrt{2\pi k}} \left[e^{-i\pi\{ES + (1-\alpha)/2\}/\alpha} \left(\operatorname{ctn} \frac{\varphi - \pi}{2\alpha} - i \right) \right.$$

$$\left. - e^{i\pi\{ES + (1-\alpha)/2\}/\alpha} \left(\operatorname{ctn} \frac{\varphi + \pi}{2\alpha} - i \right) \right]. \quad (5.51)$$

In conclusion, we observe that for $\eta = -\nu/\alpha$ non-integer and $\tan \lambda < 0$ the hamiltonian h of Eq. (5.34) also has bound states, with eigenfunctions

$$B_\nu \sim \begin{pmatrix} \sqrt{m + E} & K_{\nu/\alpha}(|\kappa|r) \\ i\sqrt{m - E} & K_{(\nu/\alpha) + 1}(|\kappa|r) \end{pmatrix}, \quad (5.52)$$

where K_η is the modified Bessel function. The energy of these states is implicitly determined by

$$\frac{(1 + E/m)^{(\nu/\alpha) + 1}}{(1 - E/m)^{-\nu/\alpha}} = -\alpha 2^{2(\nu/\alpha) + 1} \frac{\Gamma((\nu/\alpha) + 1)}{\Gamma(-\nu/\alpha)} \tan \lambda. \quad (5.53)$$

See also the comments at the end of Sect. VI.

VI. Summary

Scattering amplitudes that we have found in various cases can be summarized with one universal formula,

$$\tilde{f}^{f}(\varphi) = \frac{e^{-i[E^s S^t + E^t S^s]\varphi/\alpha}}{2\sqrt{2\pi k}} \left[e^{-i\pi\{E^s S^t + E^t S^s\}/\alpha} \left(\operatorname{ctn} \frac{\varphi - \pi}{2\alpha} - i \right) \right.$$

$$\left. - e^{i\pi\{E^s S^t + E^t S^s\}/\alpha} \left(\operatorname{ctn} \frac{\varphi + \pi}{2\alpha} - i \right) \right]. \quad (6.1)$$

This possesses [principal value] singularities at $\varphi = \pm \pi + 2\pi n\alpha$, which arise from the long-range "interaction" and interfere with the optical theorem. Furter unitarity difficulties are present when the source carries spin $S^s \neq 0$, and closed time-like contours are possible. Then the Klein–Gordon "Hamiltonian" and the natural measure become complex and hermiticity is lost. For the Dirac Hamiltonian the pathology is less severe. The measure loses positivity and the Hamiltonian must

be extended into a one-parameter family to achieve a self-adjoint operator. Formula (6.1) applies for one value of the parameter; the physical meaning of other values is unclear.

It remains to be seen whether second quantization eliminates these pathologies. It not, perhaps one should reject as physically unrealistic a spinning point particle [spinning string!] or at least the portion of its spacetime $r < S$ that contains closed time-like contours. [The Dirac equation in a spinless conical background does not encounter these problems. Therefore, by exchanging test particle and source in our universal formula (6.1), the special case of a scalar test particle and $S^s = 1/2$ may be understood without encountering causal difficulties.]

Second quantization of the Dirac field raises new questions. It is known that fermion loops induce a gravitational Chern–Simons term [13], which then liberates the graviton, rendering it a propagating, massive excitation [14]. At present, it is a mystery to us how this new degree of freedom is reconciled with the first quantized results presented here.

Quantum mechanical puzzles remain even with spinless particles. For example 't Hooft has argued [1] that when the source is treated dynamically, the Hamiltonian for relative motion is $H = \sqrt{p^2 + M^2} + \sqrt{p^2 + m^2}$ [where p is the relative momentum] and H operates on a space with deficit angle given by its eigenvalue, i.e. $\alpha = 1 - H$. Thus states of different energy inhabit different spaces, and this truly "Machian" phenomenon destroys quantum mechanical linearity and superposition.

Finally we remark that when we accept energy-dependent boundary conditions the most appropriate choice for η in (5.38) is $\eta = -v/\alpha$ [whenever the latter can lie between 0 and 1—otherwise there is no need for a self-adjoint extension]. The scattering amplitude will then depend continuously on the energy for all values of λ and bound states arise for $\tan \lambda < 0$.

Appendix

In this Appendix we study a quantum mechanical particle in the presence of two sources, and solve the relevant wave equation for the simplest example. As will be seen, our analysis makes use of an orbifold.

The line element

$$(ds)^2 = (dt)^2 - (dl)^2 \tag{A.1}$$

in the presence of N static point particles with masses M_i, located at positions $\mathbf{R}_i$, is given in conformally flat spatial coordinates by [4]

$$(dl)^2 = \prod_{i=1}^{N} \frac{d\mathbf{R} \cdot d\mathbf{R}}{|\mathbf{R} - \mathbf{R}_i|^{2M_i}} = \prod_{i=1}^{N} \frac{dZ \, dZ^*}{[Z - Z_i]^{M_i}[Z^* - Z_i^*]^{M_i}}. \tag{A.2}$$

In the second equation we use complex coordinates $Z = X + iY$, $Z^* = X - iY$, etc. That equation also indicates how passage to localy flat coordinates is effected. Define

$$z(Z) = \int^Z \frac{d\mathscr{Z}}{\prod\limits_{i=1}^{N} (\mathscr{Z} - Z_i)^{M_i}}, \tag{A.3}$$

then

$$(dl)^2 = dz\,dz^*, \tag{A.4}$$

and information about the sources is coded in the boundary conditions on the locally flat variables z and z^*. Near the ith source, z is approximately given by $\dfrac{(Z - Z_i)^{1 - M_i}}{1 - M_i} \prod_{j \neq i} (Z_i - Z_j)^{-M_j}$. Thus when the ith source is encircled, z acquires the phase $e^{-i2\pi M_i}$, and in the vicinity of the ith source z has to be identified with $e^{-i2\pi M_i} z$.

Henceforth, we specialize to two point particles; without loss of generality we may assume that they are placed on the X axis, equidistant from the Y axis, separated by a distance $2a$. Thus, the integral in (A.3) becomes

$$z = \int^z \frac{d\mathscr{Z}}{(\mathscr{Z} + a)^{M_1}(\mathscr{Z} - a)^{M_2}}. \tag{A.5}$$

For arbitrary M_i, this cannot be expressed in terms of elementary functions. But the geometry of the variable z is easily described pictorially, in Fig. 3. The two particles are denoted by numbered circles. The cut-out region is the shaded domain to the left of the solid line, and the line segments marked by $\times$ are identified, as are those marked by $\otimes$. [The first particle appears in two locations, but they are identified.] Thus the excision consists of a wedge with opening angle $2\pi(M_1 + M_2)$ plus a further region between the two particles.

The integral (A.5) is elementary only when the M_i's are rational and sum to one. Setting $M_2 = p/q = 1 - M_1$ and changing variables $(\mathscr{Z} + a/\mathscr{Z} - a)^{1/q} = \bar{\mathscr{Z}}$ transforms (A.5) into

$$z = - \sum_{n=1}^{q} \omega^n(q) \int^{(Z + a/Z - a)^{1/q}} d\mathscr{Z} \, \frac{\bar{Z}^{p-1}}{\mathscr{Z} - \omega^n(q)}, \tag{A.6}$$

where the $\omega^n(q)$ comprise the q roots of unity, $e^{i2\pi n/q}$. These satisfy, for any integer $\mathcal{N}$ that is not an integral multiple of q,

$$\sum_{n=1}^{q} \omega^{n\mathcal{N}}(q) = 0. \tag{A.7a}$$

It also follows that

$$\frac{q}{\mathscr{Z}^q - 1} = \sum_{n=1}^{q} \frac{\omega^n(q)}{\mathscr{Z} - \omega^n(q)}, \tag{A.7b}$$

which was used to arrive at (A.6).

While the integral (A.6) is elementary for any p, we further restrict the discussion to the simplest case $p = 1$, $q = 2$, i.e., $M_1 = M_2 = 1/2$,

$$z = \ln\left[\left(\frac{Z + a}{Z - a}\right)^{1/2} + 1 \right] - \ln\left[\left(\frac{Z + a}{Z - a}\right)^{1/2} - 1 \right] = \ln\left[\frac{Z}{a} + \left(\frac{Z^2}{a^2} - 1\right)^{1/2} \right]. \tag{A.8}$$

This maps the first particle, at $Z = -a$, into $z = \pm i\pi \bmod 2\pi i$, while the second

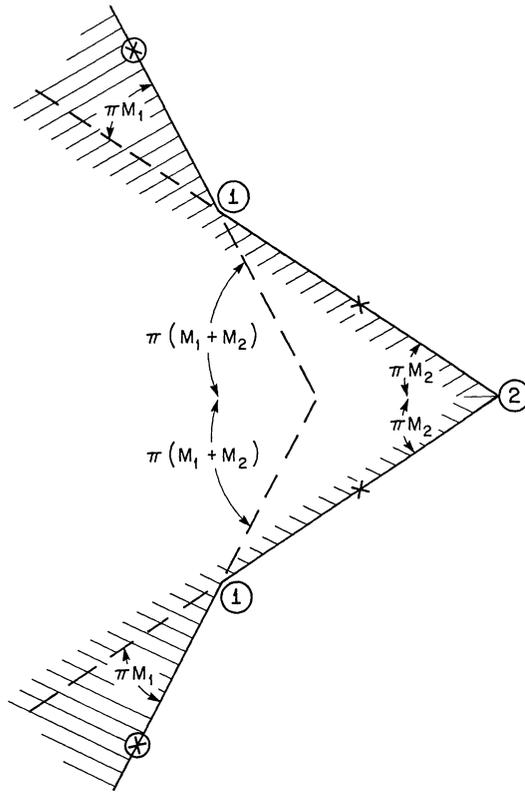**Fig. 3.** Geometry of space for two static particles with mass M_1 and M_2. The cut out region at M_2 has opening angle $2\pi M_2$, and the edges of the excised wedge are identified [$\times$]. Particle M_1 requires a cut out region with opening angle $2\pi M_1$, which is divided into two by the [doubled but identified] axis proceeding from M_2. The edges are identified again [$\otimes$] and the total cut out region, indicated by shading, is described by a cut out wedge appropriate to a single particle with mass $M_1 + M_2$ located at an intermediate point, plus a further excision between the two particles

particle at $Z = a$ is now at $z = 0 \bmod 2\pi i$. The inverse formula

$$Z = a \cosh z,$$
$$X = a \cosh x \cos y, \quad Y = a \sinh x \sin y \tag{A.9}$$

shows that y is a 2π-periodic variable, whose range may be taken in the interval $[-\pi, \pi]$, while x extends from 0 to ∞. In other words, the line $y = -\pi$ is identified with $y = \pi$; also at $x = 0$, y is identified with $-y$; see Fig. 4. The space is a semiinfinite cylinder, with one end pinched together. When the z variable is extended over the entire complex plane, points z and $-z = ze^{-i2\pi M_i}$ are identified, and also points $z \bmod 2\pi i$ are identified with each other. Our space is an orbifold. Its underlying manifold is an infinite cylinder, which is divided by the group $\mathbf{Z}_2$ of rotations by π. The conical singularities with deficit angle π, introduced at the two fixed points 0 and $\pm i\pi$ of this operation, correspond to the two particles.
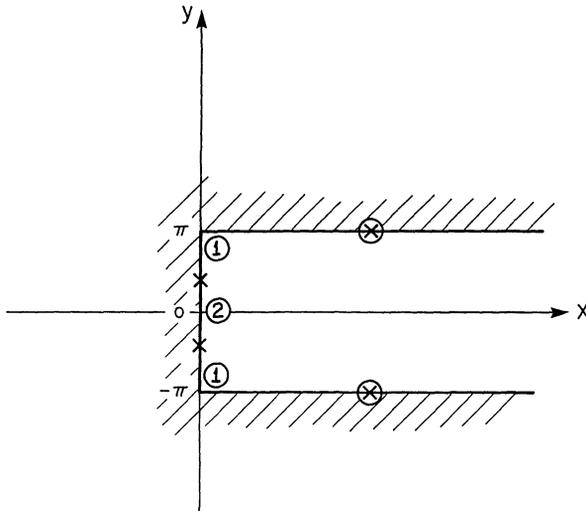
Fig. 4. Geometry of space for two static particles with equal masses, $M_i = 1/2$

It is instructive to compare this space with that of a single particle of mass $M = M_1 + M_2 = 1$, located at the origin. In conformally flat spatial coordinates, the appropriate line element is given by (2.2) with $\alpha = 0$. Changing variables $x = \ln R$, $y = \Theta$, gives $(dl)^2 = (dx)^2 + (dy)^2$. The variable y is 2π-periodic as in (A.9), but x extends from $-\infty$, where now the particle is locted, to ∞.

To describe a quantum mechanical test particle interacting with the double source we must diagonalize the Hamiltonian $H = \sqrt{p^2 + m^2}$. This is achieved by diagonalizing p^2—the Klein–Gordon equation is equivalent to the Schrödinger equation. We thus solve

$$-\nabla^2 \psi(x, y) = k^2 \psi(x, y) \tag{A.10}$$

on the plane with a restriction on the wave function that incorporates the orbifold identification,

$$\psi(-x, -y) = \psi(x, y), \tag{A.11a}$$

$$\psi(x, y + 2\pi) = \psi(x, y). \tag{A.11b}$$

Note it would be insufficient to require equality merely on the identified line segments: $\psi(x, -\pi) = \psi(x, \pi)$, $\psi(0, -y) = \psi(0, y)$. Of course, these conditions are implied by (A.11), but they are weaker, since they merely define ψ on the boundary of the region, and do not encode the necessary identification. The unique, orthonormal eigensolutions are

$$u_n(x, y) = \frac{1}{\pi}\left(\frac{k}{k_n}\right)^{1/2} \cos(k_n x + ny),$$

$$n = 0, \pm 1, \pm 2, \ldots, \pm [k], \quad k_n = \sqrt{k^2 - n^2}. \tag{A.12}$$

There is no scattering, the waves are standing waves. They are not angular momentum eigenfunctions, because diagonalizing $(1/i)\partial_y$ is incompatible with the requirement (A.11a). However angular momentum squared may be diagonalized, as indeed it is in (A.12). [Recall that for a single particle with mass $M = 1/2$, there too is no scattering [2]. The partial wave (3.10) in locally flat spatial coordinates is $J_{2n}(k\rho)e^{i2n\varphi}$ and the scattering solution (3.2) sums to $\cos \mathbf{k}\cdot\rho$, when $\mathbf{k}$ points in an arbitrary direction.]

One may also construct the heat kernel for the Laplacian on our orbifold,

$$\langle x'y'|e^{\tau(\nabla^2/2)}|xy\rangle = \sum_n \int_{|n|}^{\infty} \frac{dk}{\pi^2} e^{-\tau(k^2/2)} \left(\frac{k}{k_n}\right) \cos(k_n x + ny) \cos(k_n x' + ny')$$

$$= \frac{1}{\sqrt{2\pi\tau}} e^{-(x-x')^2/2\tau} \frac{1}{2\pi} \Theta_3\left(\frac{y-y'}{2}|e^{-\tau/2}\right)$$

$$+ \frac{1}{\sqrt{2\pi\tau}} e^{-(x+x')^2/2\tau} \frac{1}{2\pi} \Theta_3\left(\frac{y+y'}{2}|e^{-\tau/2}\right), \qquad (A.13)$$

where Θ_3 is the theta function of the third kind.

$$\Theta_3\left(\frac{y}{2}|e^{-\tau/2}\right) = \sum_n e^{-n^2\tau/2}e^{iny} = \sqrt{\frac{2\pi}{\tau}} \sum_n e^{-(y-2\pi n)^2/2\tau}. \qquad (A.14)$$

Finally, we extend the above results to the case when the two particles also carry spin S_i. The multi-particle line element [5] reduces in our case to

$$(ds)^2 = \left(dt + S_1 \frac{(\mathbf{R}+\mathbf{a})}{|\mathbf{R}+\mathbf{a}|^2} \times d\mathbf{R} + S_2 \frac{(\mathbf{R}-\mathbf{a})}{|\mathbf{R}-\mathbf{a}|^2} \times d\mathbf{R}\right)^2$$

$$- \frac{d\mathbf{R}\cdot d\mathbf{R}}{|\mathbf{R}+\mathbf{a}| \, |\mathbf{R}-\mathbf{a}|} \qquad (\mathbf{a} = (a,0))$$

$$= \left(dt - i\frac{S_1}{2} d\ln\frac{Z+a}{Z^*+a} - i\frac{S_2}{2} d\ln\frac{Z-a}{Z^*-a}\right)^2 - \frac{dZ \, dZ^*}{\sqrt{(Z^2-a^2)(Z^{*2}-a^2)}}. \qquad (A.15)$$

Upon changing variables as in (A.3), (A.8) and (A.9), this becomes

$$(ds)^2 = (dT)^2 - dz \, dz^*,$$

$$T = t - i\frac{S_1}{2}\ln\frac{\cosh z + 1}{\cosh z^* + 1} - i\frac{S_2}{2}\ln\frac{\cosh z - 1}{\cosh z^* - 1}. \qquad (A.16)$$

Solutions to the Klein–Gordon equation satisfying the conditions (A.11) are

$$\psi_n(t; x, y) = e^{-iEt} \left[\frac{\cosh\dfrac{z}{2}}{\cosh\dfrac{z^*}{2}}\right]^{-ES_1/4} \left[\frac{\sinh\dfrac{z}{2}}{\sinh\dfrac{z^*}{2}}\right]^{-ES_2/4} \qquad \times$$

$$\cdot \frac{1}{\pi} \left(\frac{K_n + K_n^*}{|K_n|} \right)^{-1/2} \cos \tfrac{1}{2}(K_n^* z + K_n z^*)$$

$$= e^{-iET} \frac{1}{\pi} \left(\frac{K_n + K_n^*}{|K_n|} \right)^{-1/2} \cos \tfrac{1}{2}(K_n^* z + K_n z^*),$$

$$K_n = k_n + in; \quad k_n = \sqrt{E^2 - m^2 - n^2}. \tag{A.17}$$

This is just [the time-dependent version of] (A.12) with an additional phase $-((ES_1/2)\arg\cosh(z/2) + (ES_2/2)\arg\sinh(z/2))$. In the second equality above the phase is hidden in T.

References

1. 't Hooft, G.: Commun. Math. Phys. **117**, 685 (1988)
2. Deser, S., Jackiw, R.: Commun. Math. Phys. **118**, 495 (1988)
3. Gott, J.: Ap. J. **288**, 422 (1985); for a review see Vilenkin, A.: Phys. Rep. **121**, 263 (1985). The spinning cosmic string was considered by Mazur, P.: Phys. Rev. **D34**, 1925 (1986), and simultaneously to our work by Harari, D. and Polychronakos, A.: Phys. Rev. **D38**, 3320 (1988)
4. Deser, S., Jackiw, R., 't Hooft, G.: Ann Phys. (NY) **152**, 220 (1984)
5. The generalization to arbitrary numbers of spinning point particles is given by Clément, G.: Int. J. Theor. Phys. **24**, 281 (1985), who also studies classical motion
6. Sommerfeld, A.: Math. Ann, **47**, 317 (1896) and Optics. New York, NY: Academic Press 1954
7. We see no reason for energy to be quantized or time t to be periodic as has been claimed by Mazur, P.: Phys. Rev. Lett. **57**, 929 (1986). Our objections to the reasoning in that paper coincide with those raised by Samuel, J., Iyer, B.: Phys. Rev. Lett. **59**, 2379 (1987) and are apparently accepted by Mazur, P.: Phys. Rev. Lett. **59**, 2380 (1987)
8. Wigner, E.: Phys. Rev. **98**, 145 (1955)
9. A mathematical investigation of Dirac operators on spaces with conical singularities is by Chou, A.: Trans. Am. Math. Soc. **289** 1 (1985)
10. Deser, S., Jackiw, R., Templeton, S.: Ann Phys. (NY) **140**, 372 (1982) The normalization in (5.4) is chosen so that a well-defined zero mass limit may be taken. But recall that the interpretation of the massless theory is quite different, as might be expected since spin is discontinuous when m passes through zero. A redefinition of the fields in the massless Dirac theory

$$\phi(\mathbf{k})e^{i\mathbf{k}\mathbf{r}} = \left(\begin{matrix} \sqrt{k_x - ik_y} & 0 \\ 0 & \sqrt{k_x + ik_y} \end{matrix} \right) \exp\left\{ \frac{\pi}{4} \hat{k}^i \sigma^i \right\} \psi_{\mathbf{k}}^o(\mathbf{r})$$

exhibits their almost scalar nature, with discontinuous behaviour under rotation R_θ,

$$\phi(\mathbf{k}) \xrightarrow[R_\theta]{} \begin{cases} \phi(R_\theta^{-1}\mathbf{k}) & \text{if } 0 \leq \theta \leq 2\pi \\ -\phi(R_\theta^{-1}\mathbf{k}) & \text{if } 2\pi < \theta < 4\pi \end{cases}$$

because the functions $\sqrt{k_x \mp ik_y}$ are double-valued; see Binegar, B.: J. Math. Phys. **23**, 1511 (1982). Parity is not violated in this case

11. One has to choose the *dreibein* that yields the proper flat space limit $[S = 0, \alpha = 1]$, with solutions to the Dirac equation that are single-valued and periodic in θ. This would not be the case if in (5.13) $\theta = 0$

12. Similar problems have been studied in monopole physics by Goldhaber, A.: Phys. Rev. **D16**, 1815 (1977) and Callias, C.: Phys. Rev. **D16**, 3068 (1977), who found that the s-wave radial Dirac Hamiltonian is not self-adjoint. Chou in ref. 9 recognized the possible loss of self-adjointness of the Dirac operator on conical spaces, but did not study extensions

13. Alvarez-Gaumé, L., Della Pietra, S., Moore, G.: Ann. Phys. (NY) **163**, 288 (1985); Goni, M., Valle, M.: Phys. Rev. **D34**, 648 (1986); Vuorio, I.: Phys. Lett. **B175**, 176 (1986)

14. Deser, S., Jackiw, R., Templeton, S.: Phys. Rev. Lett. **48**, 975 (1982) and Ann. Phys. (NY) **140**, 372 (1982