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Quantum BRST Charge for Quadratically Nonlinear Lie Algebras

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Abstract. We consider the construction of a nilpotent BRST charge for extensions of the Virasoro algebra of the form $\{T_a, T_b\} = f_{ab}{}^c T_c + V^{cd}_{ab} T_c T_d$, (classical algebras in terms of Poisson brackets) and $[T_a, T_b] = h_{ab} I + f_{ab}{}^c T_c + V^{cd}_{ab}(T_c T_d)$ (quantum algebras in terms of commutator brackets; normal ordering of the product $(T_c T_d)$ is understood). In both cases we assume that the set of generators $\{T_a\}$ splits into a set $\{H_i\}$ generating an ordinary Lie algebra and remaining generators $\{S_\alpha\}$, such that only the $V^{ij}_{\alpha\beta}$ are nonvanishing. In the classical case a nilpotent BRST charge can always be constructed; for the quantum case we derive a condition which is necessary and sufficient for the existence of a nilpotent BRST charge. Non-trivial examples are the spin-3 algebra with central charge $c = 100$ and the $so(N)$-extended superconformal algebras with level $S = -2(N-3)$.

1. Introduction

Over the past few years it has become clear that conformal field theories in two dimensions play an important role in string theories and in statistical systems at the critical point (a large number of relevant papers can be found in the reprint volume [1]). Each conformal field theory is built from a set of representations of the two-dimensional conformal algebra, which is the product of two copies of the Virasoro algebra. However, in actual models there is often more symmetry than just conformal invariance. In fact, all rational conformal field theories correspond to the Virasoro algebra or some extension of it ([2, 3]). In general such extended algebras are generated by a finite set of currents of definite conformal dimension. A systematic study of finitely generated conformal algebras was initiated by Zamolodchikov in ref. [4] and has been developed further by many authors.

The extended algebras that turn up in $d = 2$ conformal field theory are quantum mechanical, i.e. they describe the (anti)commutation relations of operator-valued fields. The classical versions of these algebras, where the bracket is interpreted as a Poisson or Dirac bracket, are relevant in the study of certain hierarchies of completely integrable systems generalizing the KdV-hierarchy [5, 6].

In some examples the Fourier modes of the currents of an extended conformal algebra form an ordinary Lie algebra, or Lie superalgebra. The more general case falls outside the scope of ordinary Lie (super)algebras and involves algebras that may be called nonlinear Lie algebras. In such algebras the defining brackets contain, in addition to linear terms, terms that are multilinear in generators (see also [7]). There is a crucial difference between such nonlinear algebras at the classical and the quantum level. In both cases, the Jacobi identities are satisfied, and in both cases central extensions may be present, but at the quantum level one must define a normal-ordering prescription for the nonlinear terms, which we will denote by ( ). As we shall see, this may result in the non-vanishing of the following expression:

$$[(AB), C] - (A[B, C]) - (-)^{\sigma(B)\sigma(C)}([A, C]B), \tag{1.1}$$

where $\sigma(B) = 0$ or $+1$ if B is commuting or anticommuting, respectively.

In string theories, (extended) conformal (super)algebras play a dual role, because at the classical level the generators are also constraints which the solutions of the field equations must satisfy. At the quantum level the generators correspond to restrictions on the Fock space of states. In the modern approach these restrictions are implemented in a covariant quantization scheme, where for all generators ghost and antighost fields are introduced which are used to construct a nilpotent BRST operator. The physical sector of Fock space consists of nontrivial representations of the cohomology defined by this operator.

The constructions of nilpotent BRST operators for Kac-Moody algebras, Virasoro algebras and for some extended conformal algebras that are Lie (super)algebras have been discussed in the literature [8–10]. In this paper we consider the construction of nilpotent BRST operators for a special class of quadratically nonlinear extensions of the Virasoro algebra.

At the classical level these algebras are defined by a set of generators T_a which satisfy the following brackets:

$$\{T_a, T_b\} = f_{ab}{}^c T_c + V_{ab}^{cd} T_d T_c. \tag{1.2}$$

We make the assumption that the set of generators can be divided into a set of subalgebra generators H_i and a remaining set of generators S_α, which satisfy the following brackets

$$\begin{aligned}
\{H_i, H_j\} &= f_{ij}{}^k H_k, \\
\{H_i, S_\alpha\} &= f_{i\alpha}{}^j H_j + f_{i\alpha}{}^\beta S_\beta, \\
\{S_\alpha, S_\beta) &= f_{\alpha\beta}{}^i H_i + f_{\alpha\beta}{}^\gamma S_\gamma + V_{\alpha\beta}^{ij} H_j H_i.
\end{aligned} \tag{1.3}$$

No ordering of the last term in (1.3) is needed as the generators commute at the classical level (the bracket may in that case be viewed as a Poisson or Dirac bracket). The Jacobi identities for (1.2), (1.3) read

$$\begin{aligned}
f_{[ab}{}^d f_{c]d}{}^e &= 0, \\
V_{[ab}^{de} f_{c]d}{}^f + V_{[ab}^{df} f_{c]d}{}^e + f_{[ab}{}^d V_{c]d}^{ef} &= 0.
\end{aligned} \tag{1.4}$$

Jacobi identities containing two V objects do not occur since, according to (1.3), no two V symbols can be contracted. From (1.4) it follows that $f_{ab}{}^{c}$ are the structure constants of a Lie algebra.

In order to define the corresponding quantum algebras, we have to specify the normal ordered product ( ) of operator valued currents. We observe that all our operators have a double index structure, with one index (n or r) running over all integers or half-integers, reflecting the fact that they form a representation of the Virasoro algebra, while the other index labels the various currents. The normal ordering is with respect to the first index (n or r). Given a set of operators A_m with conformal dimension j,

$$[L_m, A_n] = \{(j-1)m - n\} A_{m+n}, \tag{1.5}$$

we define

$$\begin{aligned} (A_n X) = A_n(X) \quad &\text{if} \quad n \leqq -j \\ (X)A_n \quad &\text{if} \quad n > -j. \end{aligned} \tag{1.6}$$

In this paper we will denote these cases by $A_{n\leqq}(X)$ and $(X)A_{n>}$, respectively.

The quantum nonlinear algebras are now defined by

$$[T_a, T_b] = h_{ab} I + f_{ab}{}^{c} T_c + V_{ab}^{cd}(T_c T_d), \tag{1.7}$$

where the central charge generator I commutes with all other generators T_a. Without loss of generality we can assume that V_{ab}^{cd} is symmetric in (cd) as the antisymmetric part can be removed by redefining the $f_{ab}{}^{c}$. We shall again require that only the $V_{\alpha\beta}^{ij}$ are non-vanishing.

Before evaluating the Jacobi identities for the quantum algebra we introduce as in (1.1) an operator $\hat{\Xi}_{abc}$ which measures the nonassociativity of the normally-ordered product as defined in (1.6)

$$\hat{\Xi}_{abc} \equiv [(T_a T_b), T_c] - (T_a [T_b, T_c]) - ([T_a, T_c] T_b). \tag{1.8}$$

Given (1.6), one can explicitly evaluate the terms with none, one, two and three generators. One finds the general result

$$\begin{aligned} \hat{\Xi}_{abc} = {} & f_{a\leqq c}^{t} h_{t>b} - (> \leqq) + f_{a\leqq c}{}^{t} f_{t>b}{}^{p} T_p - (> \leqq) \\ & + f_{a\leqq c}{}^{t} V_{t>b}^{pq}(T_p T_q) - (> \leqq) + V_{a\leqq c}^{pt}[(T_p T_t)_>, T_b] - (> \leqq). \end{aligned} \tag{1.9}$$

The notation $A_{\leqq} B_{>} - (> \leqq)$ indicates the combination $A_{\leqq} B_{>} - A_{>} B_{\leqq}$. The identity $A_{>} B - A B_{>} = A_{>} B_{\leqq} - A_{\leqq} B_{>}$ was used several times for combining terms into commutators. Due to the index structure of $V_{\alpha\beta}^{ij}$, it is clear that $\hat{\Xi}_{abc}$ contains terms with at most two T generators. However, for the Jacobi identities we will only need the case that the indices a, b of $\hat{\Xi}$ lie in the H-sector, and in this case $\hat{\Xi}$ simplifies further to

$$\hat{\Xi}_{ijc} = \Xi_{ijc} + \Xi_{ijc}{}^{d} T_d, \tag{1.10}$$

where

$$\Xi_{ijc} = f_{i\leqq c}{}^{t} h_{t>j} - (> \leqq), \tag{1.11}$$

$$\Xi_{ijc}{}^{d} = f_{i\leqq c}{}^{t} f_{t>j}{}^{d} - (> \leqq). \tag{1.12}$$

We recall that the notation $T_{a>}$ indicates those indices whose Virasoro index n is greater than minus the conformal spin j of T_a.

The Jacobi identities for the quantum algebra yield

$$f_{ab}{}^{d}h_{dc} + V_{ab}^{ed}\Xi_{edc} + (\text{cyclic in } abc) = 0, \tag{1.13}$$

$$f_{ab}{}^{d}f_{dc}{}^{g} + 2V_{ab}^{gd}h_{dc} + V_{ab}^{ed}\Xi_{edc}{}^{g} + (\text{cyclic in } abc) = 0, \tag{1.14}$$

$$f_{ab}{}^{d}V_{dc}^{ef} + V_{ab}^{de}f_{dc}{}^{f} + V_{ab}^{df}f_{dc}{}^{e} + (\text{cyclic in } abc) = 0. \tag{1.15}$$

Let us now show an example of an algebra of the type we described above. Perhaps the simplest nontrivial example is the so-called spin-3 or W_3 algebra introduced by Zamolodchikov in ref. [4]. The Poisson brackets for its classical version read

$$\begin{aligned}\{L_m, L_n\} &= (m-n)L_{m+n},\\ \{L_m, W_n\} &= (2m-n)W_{m+n},\\ \{W_m, W_n\} &= (m-n)(LL)_{m+n}.\end{aligned} \tag{1.16}$$

In this example, the Virasoro generators L_m span the subalgebra, while the generators W_m form the set $\{S_\alpha\}$, and V_{ab}^{cd} appears indeed only in $\{W, W\} = VLL$. In fact, the classical algebras that are found in ref. [5,6] are more general since they include a central extension, which due to the non-linearity, induces some changes in the other structure constants. However, the BRST construction for classical algebras which we will discuss in our next section, works only for classical algebras with vanishing central extension.

The commutation relations for the quantum spin-3 algebra are given by

$$\begin{aligned}[L_m, L_n] &= \frac{c}{12}m(m^2-1)\delta_{m+n} + (m-n)L_{m+n},\\ [L_m, W_n] &= (2m-n)W_{m+n},\\ [W_m, W_n] &= \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}\\ &\quad + (m-n)\{\tfrac{1}{15}(m+n+3)(m+n+2) - \tfrac{1}{6}(m+2)(n+2)\}L_{m+n}\\ &\quad + \beta(m-n)\Lambda_{m+n},\end{aligned} \tag{1.17}$$

where

$$\Lambda_m = \sum_n L_{m-n}L_n - \tfrac{3}{10}(m+3)(m+2)L_m, \tag{1.18}$$

and

$$\beta = \frac{16}{22+5c}. \tag{1.19}$$

(The deviation of this expression from the expression in Zamolodchikovs original paper [4] is due to a difference in normal ordering convention).

It was shown by Thierry-Mieg in ref. [12] that for $c = 100$ a nilpotent BRST charge for the algebra (1.17) can be constructed. The major goal of our present study is to understand better the construction he gave and to extend it to all algebras in the class we described above.

Other examples that we will treat are the series of $so(N)$-extended superconformal algebras proposed by Knizhnik [13] and Bershadsky [14] and further studied in ref. [15]. Surprisingly, an analogous series of $u(N)$-extended algebras does not allow the construction of a nilpotent BRST charge.

The outline of this paper is as follows. In Sect. 2 we will construct the general classical nilpotent BRST charge Q for the algebras in (1.2) and (1.3). In Sect. 3 we deduce the conditions for the existence of a nilpotent BRST charge Q at the quantum level. Details of some calculations are given in Appendices A and B. In Sect. 4 we apply these results to the W_3 algebra and to the $so(N)$- and $u(N)$-extended superconformal algebras. In Sect. 5 we state our results and briefly discuss cohomology and possible applications to string theory.

2. The Classical BRST Charge

There exists an algorithm for the construction of a nilpotent BRST charge for general classical Lie algebras, linear or nonlinear, due to Fradkin and Fradkina [16]. We will apply this algorithm to the quadratically nonlinear algebras with only nonvanishing $V^{ij}_{\alpha\beta}$. We will treat the formalism for bosonic algebras; the formalism for superalgebras is obtained as a straightforward generalization. For a clear review of the classical BRST formalism, see ref. [17].

For each generator T_α, one introduces an anticommuting ghost c^a and antighost $\bar{c}_a$. They satisfy the bracket relations

$$\{c^a, c^b\} = 0, \quad \{c^a, \bar{c}_b\} = \delta^a_b, \quad \{\bar{c}_a, \bar{c}_b\} = 0,$$
$$\{c^a, T_b\} = \{\bar{c}_a, T_b\} = 0. \tag{2.1}$$

We do not introduce separate ghosts and antighosts for the quadratic elements $T_a T_b$ in the algebra because invariance of a theory under T_a implies invariance under $T_a T_b$ at the classical level [18].

The BRST charge Q satisfying $\{Q, Q\} = 0$ is given by

$$Q = \sum_{n=0}^{\infty} (-)^n c^{b_{n+1}} \cdots c^{b_1} U^{(n)}_{b_1 \ldots b_{n+1}}{}^{a_1 \ldots a_n} \bar{c}_{a_n} \cdots \bar{c}_{a_1}. \tag{2.2}$$

The structure functions U follow from the Jacobi identities, and identities derived from these. For each n, one first determines a function $D^{(n)}$ which is totally antisymmetric in upper indices and lower indices and which vanishes when contracted with T generators

$$D^{(n)a_1 \ldots a_n}_{b_1 \ldots b_{n+2}} T_{a_n} = 0,$$
$$D^{(n)a_1 \ldots a_n}_{b_1 \ldots b_{n+2}} = (n+1) U^{(n+1)a_1 \ldots a_{n+1}}_{b_1 \ldots b_{n+2}} T_{a_{n+1}}. \tag{2.3}$$

Given $D^{(n)}$, one obtains $D^{(n+1)}$, and hence $U^{(n+2)}$, by evaluating the bracket $\{D^{(n)} - (n+1)U^{(n+1)}T, T_{b_{n+3}}\}$ and antisymmetrizing in all b indices. In our case, one easily finds for the first three structure functions

$$\begin{aligned} U^{(0)}{}_{b_1} &\equiv T_{b_1}, \\ U^{(1)a_1}{}_{b_1 b_2} &= -\tfrac{1}{2}(f_{b_1 b_2}{}^{a_1} + V^{a_1 p}_{b_1 b_2} T_p), \\ U^{(2)a_1 a_2}{}_{b_1 b_2 b_3} &= 0. \end{aligned} \tag{2.4}$$

The only further nonvanishing structure functions is $U^{(3)}$

$$U^{(3)a_1 a_2 a_3}{}_{b_1 b_2 b_3 b_4} = -\tfrac{1}{24} V^{pa_1}_{b_1 b_2} V^{qa_2}_{b_3 b_4} f^{a_3}_{pq}. \tag{2.5}$$

This follows from the general formula for $D^{(n)}$

$$\begin{aligned} D^{(n)a_1 \ldots a_n}{}_{b_1 \ldots b_{n+2}} &= \frac{1}{2} \sum_{p=0}^{n} (-)^{np+1} \{ U^{(p+1)a_1 \ldots a_p}{}_{b_1 \ldots b_{p+1}}, U^{(n-p)a_{p+1} \ldots a_n}{}_{b_{p+2} \ldots b_{n+2}} \} \\ &+ \sum_{p=0}^{n-1} (-)^{np+n} (p+1)(n-p+1) U^{(p+1)a_1 \ldots a_p k}{}_{b_1 \ldots b_{p+2}} U^{(n-p)a_{p+1} \ldots a_n}{}_{b_{p+3} \ldots b_{n+2} k}. \end{aligned} \tag{2.6}$$

The resulting expression for Q is

$$Q = c^a T_a - \tfrac{1}{2} f_{ab}{}^c \bar{c}_c c^a c^b - \tfrac{1}{2} V^{cd}_{ab} T_c \bar{c}_d c^a c^b - \tfrac{1}{24} V^{pr}_{ab} V^{qs}_{cd} f_{pq}{}^t \bar{c}_r \bar{c}_s \bar{c}_t c^a c^b c^c c^d. \tag{2.7}$$

One may directly verify that Q is nilpotent by using the fact that

$$V^{pr}_{[ab} V^{qs}_{cd]} f_{pq}{}^t = V^{p[r}_{[ab} V^{|q|s}_{cd]} f_{pq}{}^{t]}, \tag{2.8}$$

which follows from applying (1.4) twice. [The first three terms of Q produce a 6-(anti)ghost term in Q^2 which is canceled by the contraction of the first and last term in Q. No 12-(anti)ghost terms are present in Q^2 due to the special form of V, while the eight-(anti)ghost terms cancel due to (2.7)].

We finally remark that, if a more general form of the classical algebra (1.2) is assumed, including a central extension h_{ab}, it can be shown that the condition $Q^2 = 0$ will fix the coefficient of the extension to be zero. In other words, the critical central charge of the classical algebras with central extension is zero.

3. The Quantum BRST Charge

For the construction of a nilpotent BRST charge Q, we must define a normal-ordering prescription not only for the generator T_a, see (1.6), but also for the ghost and antighost modes. Furthermore, multiple contractions are now needed, and, as it turns out, the central extensions must be nonvanishing. (The fact that the central charge must be nonvanishing at the quantum level is well-known from the Virasoro algebra where $c = 26$.) There does not seem to exist a general algorithm for the construction of a quantum BRST charge, hence we will use the classical BRST charge as a starting point and make modifications where they appear necessary. As in our previous section, we present the formalism for bosonic algebras; the adaptations required when we are dealing with a superalgebra are easily obtained.

Usually, one introduces, given a generator T_m of conformal dimension j, ghost modes c_m with conformal dimension $(1-j)$ and antighost modes $\bar{c}_m$ with conformal dimension j satisfying $[c_m, \bar{c}_n]_+ = \delta_{m+n,0}$. It is easier to work with modes $c^m \equiv c_{-m}$

which satisfy $[c^m, \bar{c}_n]_+ = \delta^m_n$. In those cases where there are several currents labeled by I, we define $c^a \equiv c^m_I = c_{-mI} \equiv c_a$. The normal ordering defined in (1.6) for c_a and $\bar{c}_b$, yields the following result for c^a and $\bar{c}_b$:

$$\begin{aligned}(c^a X) &= c^{a>}(X) + (X)c^{a\leqq}(-)^{\sigma(X)} \\ (\bar{c}_a X) &= \bar{c}_{a\leqq}(X) + (X)\bar{c}_{a>}(-)^{\sigma(X)},\end{aligned} \tag{3.1}$$

where the symbols $>$ and $\leqq$ refer again to the Virasoro index n. Explicitly, $a_>$ stands for $n > -j$ and $a_\leqq$ stands for $n \leqq -j$, *both* for c^a and $\bar{c}_a$.

We are now ready to tackle the construction of the quantum BRST charge. The minimal change in the classical BRST charge would occur if only the constants $f_{ab}{}^c$ and V^{cd}_{ab} were to be replaced by new constants $F_{ab}{}^c$ and W^{cd}_{ab} which depend again on $f_{ab}{}^c$, V^{cd}_{ab} and h_{ab}. (Recall that Ξ_{abc} is not an independent operator, see (1.10)–(1.12).) Due to the fact that $f_{ab}{}^c$ and h_{ab} always increase the difference between the number of lower and upper indices, it follows that $f_{ab}{}^c$ can at most be modified, except for overall constants, by an fV term, while V^{cd}_{ab} cannot be modified at all, since any contraction between two V symbols vanishes. This leads to the following ansatz for the quantum BRST charge:

$$\begin{aligned}Q &= c^a T_a - \tfrac{1}{2}F_{bc}{}^a(\bar{c}_a c^b c^c) - \tfrac{1}{2}V^{cd}_{ab}T_c(\bar{c}_d c^a c^b) \\ &\quad - \tfrac{1}{24}V^{kp}_{tu}V^{lq}_{vw}F_{kl}{}^r(\bar{c}_p\bar{c}_q\bar{c}_r c^t c^u c^v c^w).\end{aligned} \tag{3.2}$$

For the evaluation of the anticommutator $[Q, Q]_+$ we derive two lemmas which follow from (3.1):

Lemma I.

$$\begin{aligned}[(\bar{c}_a c^b c^c), (\bar{c}_p c^t c^u)]_+ &= 2\delta_p{}^{[c}(\bar{c}_a c^{b]} c^t c^u) - 2(\bar{c}_p c^b c^c c^{[u})\delta_a{}^{t]} \\ &\quad + 2(\delta_{a_\leqq}{}^t\delta_{p>}{}^{[c} - (> \leqq))(c^{b]}c^u) - 2(\delta_{a_\leqq}{}^u\delta_{p>}{}^{[c} - (> \leqq))(c^{b]}c^t).\end{aligned} \tag{3.3}$$

Lemma II.

$$\begin{aligned}[(\bar{c}_a c^b c^c), (\bar{c}_p\bar{c}_q\bar{c}_r c^t c^u c^v c^w)]_+ &= 6\delta_p{}^c(\bar{c}_a\bar{c}_q\bar{c}_r c^b c^t c^u c^v c^w) - 4\delta_a{}^t(\bar{c}_p\bar{c}_q\bar{c}_r c^b c^c c^u c^v c^w) \\ &\quad + 6(\delta_{p\leqq}{}^c\delta_{q\leqq}{}^b - (> >))(\bar{c}_a\bar{c}_r c^t c^u c^v c^w) \\ &\quad + 24(\delta_{a\leqq}{}^t\delta_{p>}{}^c - (> \leqq))(\bar{c}_q\bar{c}_r c^b c^u c^v c^w) \\ &\quad + 24(\delta_{a\leqq}{}^t\delta_{q>}{}^b\delta_{r>}{}^c + > \leqq \leqq)(\bar{c}_p c^u c^v c^w),\end{aligned} \tag{3.4}$$

where the right-hand side of (3.4) is to be antisymmeterized in bc, pqr and $tuvw$. These identities follow straightforwardly from Wick's theorem applied to each of the two products of normal-ordered operators, where one uses that $\langle 0|\bar{c}_a c^b|0\rangle = \delta_{a>}{}^b$ and $\langle 0|c^b\bar{c}_a|0\rangle = \delta_{a\leqq}{}^b$.

We shall now analyze the contributions to $[Q, Q]_+$ order by order in the (anti)ghosts. We begin with the 2-ghost terms, and end with the 8-(anti)ghost terms. All higher terms vanish since c^t, c^u, c^v, c^w in the last term of Q in (3.2) all have coset indices, while $\bar{c}_p, \bar{c}_q, \bar{c}_r$ all have subalgebra indices.

Terms in Q^2 with two ghosts arise from single contractions of the first term in Q with itself and with the second and third terms, and double contractions of the second and third terms among themselves. The result is proportional to

$$\begin{aligned}&[h_{bd}+(F_{bc>}{}^{a}F_{a\leqq d}{}^{c}-(\leqq >))]c^{b}c^{d}\\&\quad+[f_{bd}{}^{g}-F_{bd}{}^{g}+2(f_{bc>}{}^{\alpha}V_{a\leqq d}{}^{cg}-(\leqq >))]T_{g}c^{b}c^{d},\end{aligned}\tag{3.5}$$

where we used that $T_cT_d-(T_cT_d)=[T_{c>},T_d]$. These terms cancel if and only if

$$F_{ab}{}^{c}=f_{ab}{}^{c}+\delta f_{ab}{}^{c},\tag{3.6}$$

where

$$\delta f_{ab}{}^{c}=(f_{ae>}{}^{d}V^{ec}_{d\leqq b}-(\leqq >))-a\leftrightarrow b,\tag{3.7}$$

and

$$h_{ab}=[-F^{d}_{ac>}F_{d\leqq b}{}^{c}+(\leqq >)].\tag{3.8}$$

Note that the right-hand side of (3.8) is antisymmetric in (ab).

The first of these equations determines the modification of the structure constants while the second fixes the central extension. Due to the fact that only $\delta f_{\alpha\beta}{}^{i}$ is non-vanishing, the F factor in the last term of Q reduces to f. It remains to be checked that the condition (3.8) is compatible with the Jacobi identities (1.13)–(1.15). In fact, one can show that (3.8) together with the last two Jacobi identities in (1.14) and (1.15) implies the first Jacobi identity (1.13), so that (3.8) takes the place of (1.13).

Through Eqs. (3.2), (3.6), (3.7) the expression for Q in terms of the tensors h, f and V has been fixed completely. We will now show that also the higher order terms in ghosts and antighosts in Q^2 vanish if the condition (3.8) is satisfied. This shows that the condition (3.8), together with the Jacobi identities (1.14) and (1.15), is sufficient for the existence of a nilpotent BRST charge. We shall analyze the contributions to Q^2 in increasing order of ghosts and antighosts.

The terms proportional to $\bar{c}ccc$ arise from single contractions of the first with the third term, the second and third terms in Q among themselves and triple contractions of the second and third terms with the fourth term. The result reads

$$[V_{ab}{}^{de}h_{ce}+F_{ae}{}^{d}F_{bc}{}^{e}]\bar{c}_{d}c^{a}c^{b}c^{c}.\tag{3.9}$$

Double contractions of the type $TT\bar{c}c$ vanish identically due to the fact that $V^{ij}_{\alpha\beta}$ is symmetric in ij. With the Jacobi identity (1.14) Eq. (3.9) is rewritten as

$$[V_{ab}{}^{de}h_{ec}+V_{ab}{}^{ef}\Xi_{efc}{}^{d}+\delta f_{ae}{}^{d}f_{bc}{}^{e}+f_{ae}{}^{d}\delta f_{bc}{}^{e}]\bar{c}_{d}c^{a}c^{b}c^{c}.\tag{3.10}$$

The strategy is now to substitute the result (1.12) for $\Xi_{efc}{}^{d}$ into the second term in (3.10), and then to rewrite this term such that it cancels all other terms in (3.10). The details of the calculations are involved and relegated to Appendix A.

The terms proportional to $\bar{c}\bar{c}cccc$ in Q^2 come from the following terms in Q: the simple contractions between term 1 and 4, and between term 3 with itself, and further the double contractions between term 2 and term 3 with term 4. The resulting expression is

$$\begin{aligned}&[\tfrac{1}{4}(f^{a}{}_{p\leqq q\leqq}V^{pe}_{tu}f_{ef}{}^{q}V^{fr}_{vw}-(\gg))-(f_{tc>}{}^{q}V_{q\leqq u}{}^{ce}f_{ef}{}^{a}V^{fr}_{vw}-(\leqq >))\\&\quad+\tfrac{1}{4}V_{tu}{}^{ae}V^{rf}_{vw}h_{ef}]\bar{c}_{a}\bar{c}_{r}c^{t}c^{u}c^{v}c^{w}.\end{aligned}\tag{3.11}$$

This result looks simple, but it requires involved algebra to show that the sum of these terms indeed vanishes. The details of this calculation are relegated to Appendix B.

The remaining terms to be analyzed are the terms proportional to $\bar{c}\bar{c}\bar{c}ccccc$. They arise from simple contractions between the second and third terms with the last term in Q. Due to the index restrictions on V only the f term in F survives the contractions. For the same reason the contractions between term 3 and the last term vanish. The remaining contributions, due to term 2, are thus proportional to f times VVf, and are exactly the same as in the classical computation. To show that they cancel one only needs the second and third Jacobi identities. The Jacobi identities are, of course, modified at the quantum level, but in this case the extra terms in the Jacobi identities do not contribute due to the index structure of V. This concludes the proof that the quantum BRST charge Q in (3.2), with F as in (3.6)–(3.7), is nilpotent if the condition (3.8) is satisfied.

4. Examples

In this section we apply our results to the examples that we already announced in Sect. 1.

4.1. The Quantum Spin-3 Algebra. The structure constants h, f, V for this algebra read in explicit form (we write $T_{(0m)} = L_m, T_{(1m)} = W_m$)

$$h_{(0m)(0n)} = \frac{c}{12} m(m^2 - 1)\delta_{m+n},$$

$$h_{(1m)(1n)} = \frac{c}{360} m(m^2 - 1)(m^2 - 4)\delta_{m+n},$$

$$f^{(0p)}_{(0m)(0n)} = (m - n)\delta^p_{m+n},$$

$$f^{(1p)}_{(0m)(1n)} = (2m - n)\delta^p_{m+n},$$

$$f^{(0p)}_{(1m)(1n)} = \left(\left(\frac{1}{15} - \frac{3\beta}{10}\right)P_1(m, n) - \tfrac{1}{6}P_2(m, n)\right)\delta^p_{m+n},$$

$$V^{(0p)(0q)}_{(1m)(1n)} = \beta(m - n)\delta^{p+q}_{m+n}, \tag{4.1}$$

where β was defined in (Eq. 1.19) P_1 and P_2 are polynomials given by

$$P_1(m, n) = (m - n)(m + n + 3)(m + n + 2),$$
$$P_2(m, n) = (m - n)(m + 2)(n + 2). \tag{4.2}$$

A straightforward calculation shows that the modified structure constants $F^{(0p)}_{(1m)(1n)}$ are given by

$$F^{(0p)}_{(1m)(1n)} = \left(\frac{1 - 17\beta}{15} P_1(m, n) - \frac{1 - 17\beta}{6} P_2(m, n)\right)\delta^p_{m+n}. \tag{4.3}$$

Working out the condition (3.8) for this algebra leads to two independent conditions on the central charge c, corresponding to the components $h_{(0m)(0n)}$ and $h_{(1m)(1n)}$, respectively,

$$c = 26 + 74 = 100, \quad c = \tfrac{1044}{5}(1 - 17\beta) = 1044\frac{c-50}{5c+22}. \tag{4.4}$$

These conditions are simultaneously satisfied for $c = 100$. The nilpotent BRST operator takes the following form:

$$\begin{aligned} Q = {} & c^{(0m)}T_{(0m)} + c^{(1m)}T_{(1m)} \\ & - \tfrac{1}{2}(m-n)\bar{c}_{(0(m+n))}c^{(0m)}c^{(0n)} \\ & - \tfrac{1}{2}(2m-n)\bar{c}_{(1(m+n))}c^{(0m)}c^{(1n)} \\ & - \tfrac{125}{522}(\tfrac{1}{15}P_1(m,n) - \tfrac{1}{6}P_2(m,n))\bar{c}_{(0(m+n))}c^{(1m)}c^{(1n)} \\ & - \tfrac{4}{261}(m-n)T_{(0(m+n-r))}\bar{c}_{(1r)}c^{(0m)}c^{(1n)}. \end{aligned} \tag{4.5}$$

The $\bar{c}\bar{c}\bar{c}cccc$ terms, which in general occur in the expression for Q, are identically zero in this case. Our result for Q agrees with the expression obtained in ref. [12].

4.2. Quantum $so(N)$-Extended Superconformal Algebras. These algebras, which were first written down by Knizhnik [13] and by Bershadsky [14], are generated by Virasoro generators L_n, supercurrents $G^i_r, i = 1, 2, \ldots, N$ and $\widehat{so}(N)$ generators $J^a_n, a = 1, 2, \ldots, N(N-1)/2$, where $r \in \mathbf{Z} + 1/2$ (Neveu–Schwarz sector) or $r \in \mathbf{Z}$ (Ramond sector). The (anti)commutation relations read

$$\begin{aligned} [L_m, L_n] &= \frac{c}{12}(m^3 - m)\delta_{m+n} + (m-n)L_{m+n}, \\ [L_m, G_{ir}] &= \left(\frac{m}{2} - r\right)G_{i(m+r)}, \\ [L_m, J_{an}] &= -nJ_{a(m+n)}, \\ [G_{ir}, G_{js}]_+ &= \frac{B}{2}(r^2 - \tfrac{1}{4})\delta^{ij}\delta_{r+s} + 2\delta^{ij}L_{r+s} \\ & \quad + 1/2K(r-s)t^a_{ij}J_{a(r+s)} + \gamma\Pi^{ab}_{ij}(J_aJ_b)_{r+s}, \\ [J_{am}, G_{ir}] &= t^a_{ji}G_{j(m+r)}, \\ [J_{am}, J_{bn}] &= -Sm\delta^{ab}\delta_{m+n} + f^{abc}J_{c(m+n)}, \end{aligned} \tag{4.6}$$

where t^a_{ij} and f^{abc} satisfy

$$\begin{aligned} & [t^a, t^b] = f^{abc}t^c, \quad tr(t^at^b) = -2\delta^{ab}, \\ & t^a_{ij}t^a_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \quad f^{abc}f^{abd} = 2(N-2)\delta^{cd}, \end{aligned} \tag{4.7}$$

and the tensor Π^{ab}_{ij} is given by

$$\Pi^{ab}_{ij} = t^a_{im}t^b_{mj} + t^b_{im}t^a_{mj} + 2\delta^{ab}\delta_{ij}. \tag{4.8}$$

This algebra is associative if and only if the constants c, B, γ are chosen as follows:

$$c = 1/2S\frac{6S+N^2-10}{S+N-3}, \quad B = KS,$$

$$K = \frac{2S+N-4}{S+N-3}, \quad \gamma = 1/2\frac{1}{S+N-3}, \tag{4.9}$$

leaving S as a freely adjustable parameter.

We will use the notation $T_{(0m)} = L_m, T_{(ir)} = G_{ir}, T_{(am)} = J_{am}$. When applying the results of Sect. 3 to this algebra, we have to keep in mind that the present algebra is a superalgebra, which implies that the signs in various expressions are different.

The structure constants $f^{(am)}_{(ir)(js)}$ are renormalized according to the expression (3.6). We find

$$F^{(am)}_{(ir)(js)} = (r-s)t^a_{ij}\delta^m_{r+s}. \tag{4.10}$$

Observe that in this example the renormalization $f \to F$ boils down to a scaling of the structure constants $f^{(am)}_{(ir)(js)}$ with a factor $(\frac{1}{2}K)^{-1}$.

If we now inspect the condition (3.8) we find that three conditions have to be satisfied

• (from the components $h_{(0m)(0n)}$)

$$c = 2(13 - \tfrac{11}{2}N + 1/2(N^2-N)) = N^2 - 12N + 26. \tag{4.11}$$

This critical charge is obtained as the sum of contributions $2(-1)^{2\lambda}(6\lambda^2 - 6\lambda + 1)$ from each generator of conformal dimension λ.

• (from the components $h_{(ir)(js)}$)

$$B = 16 - 6N \tag{4.12}$$

• (from the components $h_{(am)(bn)}$)

$$S = -2(N-3). \tag{4.13}$$

These conditions are compatible with the relations (2.8) that were needed for the associativity of the algebra. We may therefore conclude that the following charge, constructed from currents in a representation having $S = -2(N-3)$, is nilpotent (the ghost/antighost for the generators L_m, G^i_r and J^a_m are denoted by $c^m, b_m, \gamma^{ir}, \beta_{ir}, C^{am}$ and B_{am}, respectively)

$$\begin{aligned} Q = {} & c^m L_m + \gamma^{ir} G_{ir} + C^{am} J_{am} - 1/2(m-n) b_{m+n} c^m c^n + \left(\frac{m}{2} - r\right) \beta_{j(m+r)} c^m \gamma^{ir} \\ & + n B_{a(m+n)} c^m C^{an} - b_{r+s} \gamma^{ir} \gamma^{is} - 1/2(r-s) t^a_{ij} B_{a(r+s)} \gamma^{ir} \gamma^{js} + t^a_{ij} \beta_{j(m+r)} C^{am} \gamma^{ir} \\ & - 1/2 f^{abc} B_{c(m+n)} C^{am} C^{bn} - 1/2 \gamma \Pi^{ab}_{ij} J_{a(r+s-m)} B_{bm} \gamma^{ir} \gamma^{js} \\ & - \tfrac{1}{24}\gamma^2 \Pi^{ab}_{ij} \Pi^{cd}_{kl} f^{ace} \delta^{m+n+p}_{r+s+t+u} B_{bm} B_{dn} B_{ep} \gamma^{ir} \gamma^{js} \gamma^{kt} \gamma^{lu}. \end{aligned} \tag{4.14}$$

The existence of a nilpotent charge at $S = -2(N-3)$ was already announced in ref. [15]. We notice that the 7-(anti)ghost term in Q is non-vanishing in this example. In this respect this example is more generic than the example of the spin-3 algebra.

4.3. Quantum u(N)-Extended Superconformal Algebras. These algebras have a bosonic part which is generated by Virasoro generators L_n, and generators $J_{Am} = \{J_{am}, J_{0m}\}$, $a = 1, 2, \ldots, N^2 - 1$, generating an $\hat{u}(N)$ Kac-Moody algebra. There are $2N$ supercurrents $G_{\alpha r}, \bar{G}_{\alpha r}$, $\alpha = 1, 2, \ldots, N$. The (anti-)commutation relations read

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n},\\
[L_m, G_{\alpha r}] &= \left(\frac{m}{2} - r\right)G_{\alpha(m+r)},\\
[L_m, \bar{G}_{\alpha r}] &= \left(\frac{m}{2} - r\right)\bar{G}_{\alpha(m+r)},\\
[L_m, J_{An}] &= -nJ_{A(m+n)},\\
[G_{\alpha r}, G_{\beta s}]_+ &= 0,\\
[\bar{G}_{\alpha r}, \bar{G}_{\beta s}]_+ &= 0,\\
[G_{\alpha r}, \bar{G}_{\beta s}]_+ &= \frac{B}{2}(r^2 - \tfrac{1}{4})\delta_{\alpha\beta}\delta_{r+s} + 2\delta_{\alpha\beta}L_{r+s} + 1/2K(r-s)\lambda^A_{\alpha\beta}J_{A(r+s)}\\
&\quad + 1/2Q(r-s)\lambda^0_{\alpha\beta}J_{0(r+s)} + \gamma P^{AB}_{\alpha\beta}(J_A J_B)_{r+s},\\
[J_{Am}, G_{\alpha r}] &= -\lambda^A_{\alpha\beta}G_{\beta(m+r)},\\
[J_{Am}, \bar{G}_{\alpha r}] &= \lambda^A_{\beta\alpha}\bar{G}_{\beta(m+r)},\\
[J_{am}, J_{bn}] &= -Sm\delta^{ab}\delta_{m+n} + f^{abc}J_{c(m+n)},\\
[J_{0m}, J_{0n}] &= -\tilde{S}m\delta_{m+n},
\end{aligned} \tag{4.15}$$

where the generators $\lambda^a_{\alpha\beta}$ of the group $SU(N)$ in the fundamental representation and the $U(1)$ generator $\lambda^0_{\alpha\beta} = i\delta_{\alpha\beta}\sqrt{(N-2)/N}$ are antihermitian. In the summation over repeated indices $A, B, \ldots$ the zeroth components are taken with a minus sign; $\delta^{00} = -1$. We have

$$\begin{aligned}
&[\lambda^a, \lambda^b] = f^{abc}\lambda^c, \quad tr(\lambda^a\lambda^b) = -2\delta^{ab},\\
&f^{abc}f^{abd} = 4N\delta^{cd}, \quad \lambda^A_{\alpha\beta}\lambda^A_{\gamma\delta} = -2\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\beta}\delta_{\gamma\delta}.
\end{aligned} \tag{4.16}$$

The tensor $P^{AB}_{\alpha\beta}$ is given by

$$P^{AB}_{\alpha\beta} = \lambda^A_{\alpha\gamma}\lambda^B_{\gamma\beta} + \lambda^B_{\alpha\gamma}\lambda^A_{\gamma\beta} + 2\delta^{AB}\delta_{\alpha\beta}. \tag{4.17}$$

The algebras are associative if and only if the constants c, B, K, Q, γ and $\tilde{S}$ are chosen as follows ($N \neq 2$):

$$c = \frac{3S(N+S)+(N-1)(2+(N-1)S)}{S+2(N-1)}, \quad B = KS,$$

$$K = 4(N+S)\gamma, \quad Q = 4N\gamma,$$

$$\gamma = 1/2\frac{1}{S+2(N-1)}, \quad \tilde{S} = -N-S. \tag{4.18}$$

If we now inspect the condition (3.8) for the components of h corresponding to J_{am} and J_{0m}, respectively, we obtain the following conditions on $S, \tilde{S}$:

$$S = -4(N-1), \quad \tilde{S} = 2(N-2). \tag{4.19}$$

The combination of these conditions is in conflict with the associativity condition $\tilde{S} = -N-S$. We therefore conclude that there is no choice for S possible such that the BRST operator (3.2) is nilpotent.

This result is rather surprising. In all previous examples, including for example also pure Virasoro and Kac–Moody algebras, the condition (3.8) could always be satisfied by tuning the central charge parameter to an appropriate critical value. Here we have an example where the condition is violated for all values of the central charge parameter, such that our construction of Q breaks down. It would be very interesting to understand better the reason why the BRST construction works for the $so(N)$-extended superconformal algebras, but fails for the $u(N)$-extended series.

5. Summary and Conclusions

The main result of this paper is the explicit construction of a nilpotent BRST-operator for quadratically non-linear Lie algebras of the form (1.7), that are characterized by the tensors $h_{ab}, f_{ab}{}^{c}$, and V_{ab}^{cd}, with the special index-structure of V assumed. Associativity of this algebra is equivalent with the Jacobi identities (1.13), (1.14) and (1.15). We showed that a nilpotent BRST-charge can be constructed if the conditions (3.8), (1.14) and (1.15), which together imply also the first Jacobi identity (1.13), are satisfied. We showed some examples where (3.8), (1.14) and (1.15) are satisfied if the central charge is tuned to a critical value. We also showed the example of the $u(N)$-extended superconformal algebras, where the associativity conditions (1.13)–(1.15) are satisfied, but the condition (3.8) is violated for all choices of the free parameter.

For those cases where the conditions (3.8), (1.14) and (1.15) hold we can construct the cohomology of the operator Q in the Fock space, possibly restricted to a well defined ghost number. We expect that this restricted space corresponds to the "physical Hilbert-space" for some model where the symmetry algebra is realized locally (as a gauged symmetry). As the Virasoro algebra is part of the algebra, we expect an interpretation of such models as string theories. For the case of the $so(N)$-extended superconformal algebras these would be a new type of N-extended superstrings.

Let us mention some comments concerning these ideas. For the spin-3 algebra the critical value for c is in the unitary domain. However, we do not know how to obtain a representation actually realizing this value. The spin-3 algebra is realized as a symmetry algebra of the $SU(3)$, $k=1$ Wess–Zumino–Witten model. However one has then $c=2$. The tensor product of $c=2$ matter multiplets is not possible due to the nonlinearity of the algebra; here the analogy with ordinary bosonic strings breaks down. Furthermore, an interpretation in terms of string-geometry of the higher-spin symmetries is still lacking. For the $so(N)$-extended superconformal algebras the level for a critical representation is negative, which implies that such representations can never be unitary. It is not clear to us how far this is an obstruction for attaching a physical interpretation to the BRST-cohomology. Also here a problem is that explicit realizations at the critical level have not been constructed.

Clearly, a lot of work on the BRST-construction for quantum nonlinear Lie algebras remains to be done. The class of algebras we chose to analyze in the present paper contains some interesting examples, but not all of them. In particular, all the so-called Casimir algebras [19], except the simplest ones associated with $su(2)$ and $su(3)$, fall outside this class. (In general, in the product of a dimension-s operator $Q^{(s)}$ with itself a term T^{s-1} is expected. Therefore, if operators of spin $s>3$ are present, the algebra is no longer quadratic.) Still, it seems likely that also for these algebras a nilpotent BRST-charge can be constructed along the lines developed in this paper. For the Casimir algebras associated to simply laced classical Lie algebras an explicit expression for the critical central charge was proposed in [9].

It will be clear that a direct generalization of the method we used in the present work to more general nonlinear Lie algebras will lead to enormous calculational complications. Possibly, a scheme where a BRST current $Q(z)$ is constructed directly from information contained in the operator product algebra of the currents $T_a(z)$, without passing to a commutator algebra in terms of Fourier modes, will be easier to work with. We leave these matters for future investigations.

A. Appendix A

In this appendix, we bring $\Xi^{d}_{\ ef[a} V^{ef}_{bc]}$ in a form which is such that the cancellation of the $\bar{c}ccc$ terms in Q^2, Eq. (3.10), becomes obvious. We start by substituting Eq. (1.12) for $\Xi^{d}_{\ efc}$,

$$V^{ef}_{ab}\,\Xi^{d}_{\ efc} = -\,V^{ef}_{ab}\big(f_{ce\leqq}{}^{g} f_{g>f}{}^{d} - (\leqq\ >)\big). \tag{A.1}$$

Here and for the rest of this appendix, we assume that all formulas are antisymmetrized in abc.

Applying the third Jacobi identity to this yields

$$\begin{aligned} V^{ef}_{ab}\,\Xi^{d}_{\ efc} = {} & 2V^{dq}_{bt>} f_{aq}{}^{p} f_{cp\leqq}{}^{t} - (\leqq\ >) + 2V^{dq}_{bt} f_{aq\leqq}{}^{p} f_{cp>}{}^{t} - (\leqq\ >) \\ & + V^{dp}_{t>q} f_{cp\leqq}{}^{t} f_{ab}{}^{q} - (\leqq\ >) + 2V^{pq}_{bt>} f_{aq}{}^{d} f_{cp\leqq}{}^{t} - (\leqq\ >) \\ & + V^{dq}_{ab} f_{t>q}{}^{p} f_{cp\leqq}{}^{t} - (\leqq\ >). \end{aligned} \tag{A.2}$$

Now one uses the resummation identity

$$(>\text{all}\leqq)-(\leqq\text{all}>)+(\text{all}\leqq\,>)-(\text{all}>\,\leqq)=(>\,\leqq\text{all})-(\leqq\,>\text{all}), \tag{A.3}$$

to rewrite the first three terms of (A.2) as

$$(\delta f)^{d}_{ea} f^{e}_{bc}. \tag{A.4}$$

Here the defining relation (3.7) and the relation

$$V^{pa}_{qr} f_{a[b}{}^{d} f_{c]d}{}^{r} = \tfrac{1}{2} V^{pa}_{qr} f_{bc}{}^{d} f_{da}{}^{r}, \tag{A.5}$$

where used. Equation (A.5) follows from the second Jacobi identity (1.14), the explicit form of Ξ^{g}_{edc} and the properties of V. The fourth term in (A.2) is recognized as

$$\delta f_{cb}{}^{e} f_{ae}{}^{d}, \tag{A.6}$$

while from Eq. (3.8) it follows that the least term becomes

$$V^{de}_{ab} h_{ce}. \tag{A.7}$$

Putting all this together gives

$$V^{ef}_{ab} \Xi^{d}_{efc} = -V^{de}_{ab} h_{ec} - \delta f_{ae}{}^{d} f_{bc}{}^{e} - f^{d}_{ae} \delta f_{bc}{}^{e} \tag{A.8}$$

which combined with Eq. (3.10) shows that the $\bar{c}ccc$ terms in Q^2 vanish. Notice that Eq. (A.8) implies that the second Jacobi identity (1.14) can be rewritten in the simple form

$$F_{[ab}{}^{d} F_{c]d}{}^{g} + V^{gd}_{[ab} h_{c]d} = 0. \tag{A.9}$$

B. Appendix B

In this appendix we prove the cancellation of the $\bar{c}\bar{c}cccc$ terms in Q^2. This amounts to showing that

$$\begin{aligned} &\tfrac{1}{2}\big(f_{p\leqq q\leqq}{}^{a} V^{pe}_{tu} f_{ef}{}^{q} V^{fr}_{vw} - (>\,>)\big) - \big(f_{tc>}{}^{q} V^{ce}_{q\leqq u} f_{ef}{}^{a} V^{fr}_{vw} - (\leqq\,>)\big) \\ &\quad - \big(f_{tc>}{}^{q} V^{ce}_{vw} f_{ef}{}^{a} V^{fr}_{q\leqq u} - (\leqq\,>\big) - \tfrac{1}{2}\big(V^{ae}_{tu} V^{rf}_{vw} f_{ep>}{}^{q} f_{q\leqq f}{}^{p} - (\leqq\,>)\big) \end{aligned} \tag{B.1}$$

vanishes. Here and in the next we assume that all formulae are antisymmetrized in *ar* and *tuvw*. Equation (B.1) follows from Eq. (3.11) where we substituted the value of the central extension (3.8). Similar to Eq. (A.3) we have another resummation identity

$$(\leqq\,\leqq\text{all})-(>\,>\text{all})=(\leqq\text{all}\leqq)-(>\text{all}>)-(\text{all}>\,\leqq)+(\text{all}\leqq\,>). \tag{B.2}$$

Applying this to the first term of (B.1) yields

$$\tfrac{1}{2} f_{p\leqq q}{}^{a} f_{ef}{}^{q} V^{pe}_{tu} V^{fr}_{vw} - \tfrac{1}{2} f_{pq}{}^{a} f_{e>f}{}^{q} V^{pe}_{tu} V^{fr}_{vw} - \tfrac{1}{2}\big(f_{pq>}{}^{a} f_{e\leqq f}{}^{q} V^{pe}_{tu} V^{fr}_{vw} - (\leqq\,>)\big), \tag{B.3}$$

where we used the trivial resummation identity

$$(\leqq\,\leqq)-(>\,>)=(\leqq\text{all})-(\text{all}>). \tag{B.4}$$

The last term of (B.1) together with the last term of (B.3) give

$$-\tfrac{1}{2} f_{e\leqq f}{}^{q}\big(f_{pq>}{}^{a} V^{pe}_{tu} + f_{pq>}{}^{e} V^{pa}_{tu}\big) V^{rf}_{vw} - (>\,\leqq). \tag{B.5}$$

On the terms between brackets we apply the third Jacobi identity (1.15), giving

$$-(f_{e\leqq f}{}^{q}f_{pt}{}^{e}V^{pa}_{q>u}V^{rf}_{vw}-(>\leqq))+(f_{e\leqq f}{}^{q}f_{q>t}{}^{p}V^{ae}_{pu}V^{rf}_{vw}-(>\leqq)). \tag{B.6}$$

We now concentrate on the second term of (B.1). We apply the third Jacobi identity on $V^{ce}_{q\leqq u}f_{ef}{}^{a}$, which yields

$$(f_{e\leqq f}{}^{p}f_{q>t}{}^{e}V^{aq}_{pu}V^{rf}_{vw}-(>\leqq))-(f_{ef}{}^{q}f_{q>t}{}^{p}V^{ae}_{p\leqq u}V^{rf}_{vw}-(\leqq>)). \tag{B.7}$$

Adding (B.6) and (B.7) results in

$$f_{ef}{}^{q}f_{qt}{}^{p}V^{ae}_{p>u}V^{rf}_{vw}-f_{e>f}{}^{q}f_{qt}{}^{p}V^{ae}_{pu}V^{rb}_{vw}-f_{ef}{}^{p}f_{q\leqq t}{}^{e}V^{aq}_{pu}V^{rf}_{vw}+f_{ef}{}^{p}f_{qt}{}^{e}V^{aq}_{p\leqq u}V^{rf}_{vw}. \tag{B.8}$$

If one now rewrites the ff-terms in the last two terms of (B.8) using the second Jacobi identities, one finds that (B.8) vanishes. We also used the fact that $f_{t[a}{}^{p}V^{qt}_{bc]}$ vanishes by the third Jacobi identity when p does not belong to the linear Lie subalgebra. To resume, the remaining terms are term three in (B.1) and the first two terms of (B.3).

Term three in (B.1) reduces to

$$-f_{t>c}{}^{q}V^{fr}_{qu}f_{ef}{}^{a}V^{ce}_{vw}, \tag{B.9}$$

and we apply the third Jacobi identity on $f_{tc}{}^{q}V^{fr}_{qu}$, giving

$$(\tfrac{1}{2}V^{qf}_{tu}f_{c>q}{}^{r}+\tfrac{1}{2}V^{qr}_{tu}f_{c>q}{}^{f})f_{ef}{}^{a}V^{ce}_{vw}. \tag{B.10}$$

Now, one observes that these terms vanish if there would have been no restriction on the summation, this because it would be symmetric in ar. From this one sees that (B.10) can be replaced by

$$-(\tfrac{1}{2}V^{qf}_{tu}f_{c\leqq q}{}^{r}+\tfrac{1}{2}V^{qr}_{tu}f_{c\leqq q}{}^{f})f_{ef}{}^{a}V^{ce}_{vw}. \tag{B.11}$$

Applying the third Jacobi identity on $f_{cq}{}^{r}V^{af}_{tu}$, one rewrites the first term of (B.11) to a form which is precisely minus term one in (B.3). A similar procedure makes that term two in (B.11) cancels against term two in (B.3). This completes our proof that (B.1) and as such all the $\bar{c}\bar{c}cccc$ terms in Q^2 vanish.

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