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Non-Holonomicity of the S Matrix and Green Functions in Quantum Field Theory: A Direct Algebraic Proof in Some Simple Situations

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Abstract. A criterion of non-holonomicity of the S matrix and Green functions in some basic simple situations of relativistic quantum field theory, previously established in an indirect way or with supplementary assumptions, is reobtained from unitarity equations by a direct and general algebraic argument.

1. Introduction

Holonomicity in the sense of M. Sato [SKK, KK] is an important notion in the analysis of singularities of distributions (or hyperfunctions). Its meaning in situations of interest in this note is recalled below. It was conjectured in [S] that the momentum-space S matrix and Green functions of relativistic quantum field theory should satisfy holonomicity properties at their Landau singularities. As a matter of fact, holonomicity is satisfied in some simple well-known situations, where singularities are poles, logarithms or, at 2-particle thresholds, square-roots, and also [KS1] for a particular class of singularities which includes previous ones. On the other hand, Feynman integrals are always holonomic (see [KK] and references therein). However, the further analysis from various viewpoints [BI1,2, BP, KS2] (perturbative or non-perturbative field theory, S-matrix theory) indicates that the S matrix and Green functions are probably non-holonomic in general and leads one to consider formulations of the idea of [S] involving infinite convergent expansions in terms of holonomic contributions. For related investigations and results in particular in constructive field theory, see [I1, IM, I2].

Actual proofs of non-holonomicity have been given in [BI1, BP] in some basic simple situations. The purpose of this note is to present a new more direct proof, providing a better understanding of mechanisms that generate non-holonomicity in these situations: at the 2-particle threshold $s=4\mu^2$ itself if the dimension d of space-time is odd and, more generally, in a simplified theory with no subchannel interaction, at the m-particle threshold $s=(m\mu)^2$ in a $m\to m$ process if

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$\beta = [(m-1)d - m - 1]/2$ is integer, e.g. $m = 3$, d arbitrary; $\mu > 0$ is the mass and s is the squared center-of-mass energy of the channel. We denote below by F the (momentum-space) connected, amputated Green function and by T its on-mass-shell restriction (corresponding to the S matrix). Either in approaches based on the Bethe–Salpeter (BS) or a BS-type equation, as in constructive field theory, or in axiomatic approaches, F and T are shown to be analytic or meromorphic, in the complex variable $\sigma = s - (m\mu)^2$, in a multisheeted domain around $\sigma = 0$ and to satisfy, at β integer, the unitarity-type equations

$$F_0 - F_r = rF_0 * F_r, \quad \forall r, \tag{1}$$

$$T_0 - T_r = rT_0 * T_r, \quad \forall r, \tag{2}$$

where F_r, or T_r, is the determination obtained at $\sigma > 0$ after r turns (in the anticlockwise sense) around $\sigma = 0$, and $*$ denotes on-mass-shell convolution. A new simpler algebraic derivation of Eqs. (1), (2) as also of related results, will be given in Appendix 1.

Holonomicity, if it holds, means (essentially) either that the number of sheets is finite with $F_r = F_0$ (or $T_r = T_0$) for some r, or more generally that it is infinite but that the vector space generated by successive determinations is finite dimensional: "finite-determination property"; equivalently F_r (or T_r) is, for some r, a linear combination of $F_0, \ldots, F_{r-1}$ (or $T_0, \ldots, T_{r-1}$). The non-holonomicity of T, which implies that of F, is derived in [BI1, Sect. 5] from the further condition, usual at $m = 2$, of hermitian analyticity. However, the latter appears to be unessential and the following more general criterion has been established in [BP]. (A weaker result is first derived there from the BS equation).

Criterion. T satisfies the finite-determination property if and only if $T_0^{*(p)} \equiv 0$ for some positive integer p ($T_0^{*(p)} = T_0 * \cdots * T_0, p$ factors).

The case $T_0^{*(p)} \equiv 0$ is "pathological" and is excluded physically (e.g. from hermitian analyticity: $T_0^{*(p)} \equiv 0$ then implies $T_0 = 0$). The proof of this criterion in [BP] makes recourse to the kernel U defined from T in [BI1, Sect. 6] (up to modifications in some cases) via the equation

$$T = U + \left(\frac{i}{2\pi} \ln \sigma\right) T * U, \tag{3}$$

and shown from Eq. (2) at $r = 1$ to be locally analytic at $\sigma = 0$ or uniform around it. However, this proof is indirect and requires various technicalities, and the question arises whether a simpler proof, directly from Eqs. (2) is possible. Such a proof is given in [BI1, Sect. 3], but under conditions on β, T_0 that play there an important role, so that the feasibility of a general proof remains at that stage unclear.

This note presents a simple purely algebraic result which shows (Sect. 3) that the above criterion does follow directly from Eqs. (2) without conditions. This result is stated and proved in Sect. 2 in a way directly suited for the application. A more refined analysis, including further results, is given in Appendix 2 and in a more general mathematical framework in [L].

Appendix 3 presents for completion a simple alternative proof of non-holonomicity using the kernel U, when U is analytic at $\sigma = 0$. This proof relies more on analyticity and less on algebraic properties than that of Sect. 2.

2. Algebraic Result

We consider algebras $\mathscr{A}$ generated by the elements of a family $\{T_r\}, r \in \mathbb{N}$, and their products, and satisfying the following properties:

(P1) $$T_r = T_0 + r T_0 T_r, \quad \forall r \in \mathbb{N}. \tag{4}$$

(P2) Translation invariance of linear relations: if there exists a linear relation $\sum a_r T_r = 0$ (involving a finite number of terms with non-zero coefficients a_r), then all translated relations $\sum a_r T_{r+\gamma} = 0$, where γ is a positive or negative integer (such that $r + \gamma \geqq 0$ if $a_r \neq 0$), also hold.

The existence of linear relations is (as in the physical context) equivalent, in view of (P2), to the fact that the vector space generated by the family $\{T_r\}$ is finite-dimensional. Algebras $\mathscr{A}$ under consideration include those for which (P2) is empty: no linear relation. The question we wish to investigate is whether the existence of linear relations is compatible with (P1) (P2). The theorem below shows that the answer is positive but only for a particular class of algebras: linear relations are then well determined and these algebras are characterized in terms of T_0 by the condition $T_0^p = 0$ for some $p \in \mathbb{N}$.

Theorem. *Let $\mathscr{A}$ be an algebra satisfying* (P1) (P2). *The following properties, if they hold, are equivalent*:

(i) *The vector space generated by the family $\{T_r\}$ is finite-dimensional, with dimension R.*

(ii) $$\sum_{r=0}^{R} (-1)^r \binom{R}{r} T_r = 0, \tag{5}$$

and there is no linear relation $\sum_{r=0}^{R'} a_r T_r = 0$ for $R' < R$.

(iii) $$T_0^{R+1} = 0, \tag{6}$$

and $T_0^{R'+1}$ non-zero for $R' < R$.

Algebras satisfying (P1) (P2) *and* (i)–(iii) *exist for each R, are commutative and all their elements are well determined linear combinations of $T_0, T_1, \ldots, T_{R-1}$, or $T_0, T_0^2, \ldots, T_0^R$. In particular*:

$$T_r = \sum_{n=1}^{R} r^{n-1} T_0^n, \quad \forall r, \tag{7}$$

$$T_0^n = \sum_{r=1}^{R} a_r^{(n)} T_r, \quad \forall n, \tag{8}$$

where the coefficients $a_r^{(n)}$ are the (unique) solutions of the equations

$$\sum_{r=1}^{R} r^\alpha a_r^{(n)} = \delta_{\alpha, n-1}, \quad \alpha = 0, 1, \ldots, R-1. \tag{9}$$

Equation (8) reduces at $n=1$ to Eq. (5). At $n>R$, $\delta_{\alpha,n-1}=0$, $\forall\alpha=0,1,\ldots,R-1$ so that $a_r^{(n)}=0$, $\forall r=1,\ldots,R$. Hence $T_0^n=0$.

Proof.

a) (i)$\leftrightarrow$(ii)

We show below that (i)$\rightarrow$(ii) (The converse is trivial). Property (i) yields the existence of a linear relation

$$\sum_{r=0}^{R} a_r T_r = 0 \tag{10}$$

between the $R+1$ vectors $T_0,\ldots,T_R$, and the absence of linear relations between $T_0,\ldots,T_{R-1}$ (which would entail, in view of (P2), applied with $\gamma=1,2,\ldots$, dimension $<R$). Hence, $a_R\neq 0$ in Eq. (10), as also $a_0\neq 0$ (otherwise, (P2), with $\gamma=-1$, would yield a relation between $T_0,\ldots,T_{R-1}$), and Eq. (10) is unique up to multiplication of all coefficients a_r by a common scalar. Putting $a_0=-1$, we show below that these coefficients are necessarily those of Eq. (5).

Equation (10) yields, in view of (P2):

$$\sum_{r=0}^{R} a_r T_{r+1} = 0. \tag{11}$$

After multiplication, on the left, of Eq. (11) by T_0 and use of Eq. (4) ($T_0T_j=(T_j-T_0)/j$), one also obtains:

$$\sum_{r=0}^{R} a_r \frac{T_{r+1}}{r+1} - T_0 C = 0; \quad C=\sum_{j=0}^{R}\frac{a_j}{j+1}. \tag{12}$$

Elimination of T_{R+1} between Eqs. (11) and (12) yields:

$$\sum_{r=1}^{R} T_r\left\{a_{r-1}\left(\frac{1}{r}-\frac{1}{R+1}\right)\right\} - T_0 C = 0. \tag{13}$$

The unicity of Eq. (10) then shows that

$$C a_r = a_{r-1}\left(\frac{1}{r}-\frac{1}{R+1}\right) \tag{14}$$

and therefore $C\neq 0$ and:

$$a_r = -\frac{1}{[C(R+1)]^r}\binom{R}{r}, \quad r=1,2,\ldots,R. \tag{15}$$

Finally, Eqs. (15) and the definition of C (see Eq. (12)) then give (with $r'=r+1$):

$$C\left[\sum_{r'=1}^{R+1}\{C(R+1)\}^{-r'}\binom{R+1}{r'}+1\right]=0. \tag{16}$$

The left-hand side of Eq. (16) is identical to $C[1+(1/C(R+1))]^{R+1}$, so that $C(R+1)=-1$. Equation (5) thus follows from Eq. (15). Q.E.D.

b) *Equation* (5)$\rightarrow$*Eqs.* (6) *and* (8)

Equation (8) reduces at $n=1$ to Eq. (5). Assuming it holds for some n, one obtains by multiplication on the left of all terms by T_0, use of Eq. (4) $(T_0 T_r = (T_r - T_0)/r)$ and replacement of T_0 by its expression provided by Eq. (8) at $n=1$:

$$T_0^{n+1} = \sum_{r=1}^{R} a_r^{(n+1)} T_r, \tag{17}$$

where

$$a_r^{(n+1)} = \frac{a_r^{(n)}}{r} - a_r^{(1)} \left(\sum_{j=1}^{R} \frac{a_j^{(n)}}{j} \right). \tag{18}$$

Since $\sum_{r=1}^{R} a_r^{(1)} = 1$, one checks that $\sum_{r=1}^{R} a_r^{(n+1)} = 0$. On the other hand, since $\sum r^\alpha a_r^{(1)} = 0$, $\alpha \geqq 1$, one has:

$$\sum_{r=1}^{R} r^\alpha a_r^{(n+1)} = \sum_{r=1}^{R} r^{\alpha-1} a_r^{(n)}. \quad \alpha \geqq 1. \tag{19}$$

Equations (9) follow by induction on n. As already remarked, Eq. (6) is a particular case of Eq. (8) $(n = R+1)$. Q.E.D.

c) *Equation* (6) $\to$ *Eqs.* (5) *and* (7)

Equation (4), when applied successively N times, always yields:

$$T_r = \sum_{n=1}^{N} r^{n-1} T_0^n + r^N T_0^N T_r, \quad \forall r, \quad \forall N = 1, 2, \ldots. \tag{20}$$

If $T_0^{R+1} = 0$, Eqs. (7) follow (with e.g. $N = R+1$). Hence

$$\sum_{r=0}^{R} a_r T_r = \sum_{l=1}^{R} T_0^l \left[\sum_{r=0}^{R} a_r r^{l-1} \right]. \tag{21}$$

Coefficients of all factors T_0^l in the right-hand side of Eq. (21) vanish when the coefficients a_r are the solutions of Eqs. (9) at $n=1$. Equation (5) follows.

d) Any relation $T_0^{R'+1} = 0$, $R' < R$, would yield by the same method as above the relation $\sum_{r=1}^{R'} (-1)^r \binom{R'}{r} T_r = 0$. This completes the proof that (ii) $\to$ (iii). The proof that (iii) $\to$ (ii) is completed similarly.

Remaining points of the theorem are easily established. One checks that all relations obtained for each given R are consistent. (E.g. Eqs. (7) and (6) yield Property (P1)).

3. Application to Non-Holonomicity

The product $T_i T_j$ in the application is the convolution $T_i * T_j$. Property (P1) is satisfied in the case β integer (Eq. (2)). Property (P2) holds by analytic continuation around $\sigma = 0$ (action of the monodromy group). The criterion of non-holonomicity presented in Sect. 1 thus follows from the theorem of Sect. 2.

Remark. Let T be expressed in terms of σ and of incoming and outgoing angular variables Ω', Ω''. The absence of p such that $T_0^{*(p)} \equiv 0$ (as σ varies) entails not only the absence of the linear relation $\sum a_i T_i(\sigma, \Omega', \Omega'') = 0$, but also the absence of relations of the form $\sum a_i(\sigma) T_i(\sigma, \Omega', \Omega'') = 0$ where the sum is finite and with functions a_i locally analytic or uniform in the neighborhood of $\sigma = 0$. (Property (P2) applies equally, by analytic condition, to such relations. The analysis of Sect. 2 can then be applied for each given σ, apart possibly from zeroes of some of the functions involved. Arguments of analyticity allow one to conclude.)

Appendices

Details and proofs of various results, in particular in Appendices 2 and 3, have been omitted for conciseness. They will be found in [IL].

1. Bethe–Salpeter and Unitarity Equations: New Algebraic Derivation. The following lemma will provide a new, simplified and unified algebraic derivation of (i) the algebraic equivalence (due originally to J. Bros) of the analyticity or uniformity of the Bethe–Salpeter kernel G ($G_0 = G_1$) and of the 2-particle asymptotic completeness relation $F_0 - F_1 = F_0 * F_1$, (ii) Eqs. (1), hence Eqs. (2) by restriction to the mass-shell, at β integer (or $F_2 = F_0$ at β half-integer), and (iii) results on the kernel U.

Lemma 1. *The relations* $A_k = A_{k+1} + A_k o^{(k)} A_{k+1}$, $k = 1, 2, \dots, n$ *where* $o^{(1)}, o^{(2)}, \dots$ *are linear operations with associativity properties, yield* $A_1 = A_{n+1} + A_1(o^{(1)} + o^{(2)} + \dots + o^{(n)}) A_{n+1}$.

Proof. At $n=2$, write e.g. $A_1 = A_2 + A_1 o^{(1)}(A_3 + A_2 o^{(2)} A_3)$ and use $A_1 o^{(1)}(A_2 o^{(2)} A_3) = (A_1 o^{(1)} A_2) o^{(2)} A_3 = A_1 o^{(2)} A_3 - A_2 o^{(2)} A_3$. The induction on n is trivial.

Applications

(i) The BS equation reads $F = G + FoG = G + GoF$, where the Feynman-type convolution o satisfies the relation $o_0 - o_1 = *$. Starting from $G_0 = G_1$, write e.g. $F_0 = G_0 + F_0 o_0 G_0$, $G_0 = G_1, G_1 = F_1 - G_1 o_1 F_1$. Conversely, write $G_0 = F_0 - G_0 o_0 F_0$, $F_0 = F_1 + F_0 * F_1$, $F_1 = G_1 + F_1 o_1 G_1$.
(ii) From the BS equation, with $G_0 = G_1$, the same derivation as above for $r \geqq 1$, with the index 1 replaced by r, noting that at β integer $\underline{*}_0 = \underline{*}_1$ so that $o_0 - o_r = r*$ ($\underline{*}$ is analytic continuation of $*, \underline{*}_0 \equiv *$). In the axiomatic framework that starts from the relation $F_0 - F_1 = F_0 * F_1$, direct derivation from the successive relations $F_1 - F_2 = F_1 * F_2, \dots$ obtained by analytic continuation. (For β half-integer, $\underline{*}_1 = -\underline{*}_0$, so that $F_1 - F_2 = -F_1 * F_2$ and $F_2 = F_0$.)
(iii) See [IL].

2. Mathematical Complement to Sect. 2. The following results hold [IL] if property (P.2) is not assumed:

(i) The existence of linear relations entails the existence of R and of $c_1, \dots, c_R$ such that $T_0, T_0^2, \dots, T_0^R$ are independent and

$$T_0^{R+1} = \sum_{i=1}^{R} c_i T_0^i. \tag{22}$$

(ii) Conversely, Eq. (22) (with $T_0, \ldots, T_0^R$ independent) characterizes a well defined algebra all elements of which are well defined linear combinations of $T_0, \ldots, T_0^R$, or $T_0, T_1, \ldots, T_{R-1}$, with "distorted" coefficients depending on $c_1, \ldots, c_R$. The existence of two "translated" linear relations implies $c_1 = c_2 = \cdots c_R = 0$ and results of Sect. 2 are reobtained.

3. Proof of Non-Holonomicity Using the Kernel U. The kernel U defined either [BI1] in terms of T via Eq. (3) or [I3] in terms of the BS kernel G is shown in either case to be uniform around $\sigma = 0$. We assume below it is analytic at $\sigma = 0$, as shown in [BI3] in weakly coupled models of constructive theory at $m = 2$, $d = 3$.

For $\beta > 0$, T is then locally equal [KS2, BI1] to the sum of the convergent series $\sum_n U^{\hat{*}(n+1)}[(1/2i\pi)\sigma^\beta \ln \sigma]^n$, where $* = \sigma^\beta \hat{*}$ and where coefficients $U^{\hat{*}(n+1)}$ are locally analytic (and bounded in modulus by cst^n). Each term in $(\ln \sigma)^n$ satisfies a finite-determination property of the form (5) with $R \geqq n+1$. Since its minimal degree $(n+1)$ increases with n, holonomicity is not expected for the sum unless $U^{*(p)} \equiv 0$ for some >0 integer p (or equivalently [BP] $T_0^{*(p)} \equiv 0$) in which case the sum stops at $n = p - 1$, or unless rearrangements of terms occur in the infinite sum as in the case of the series $\sum (1/n!) (\ln \sigma)^n$, equal to σ. Such rearrangements can be excluded in the present situation and non-holonomicity is established: see [IL].

At $\beta = 0$, as in models at $m = 2$, $d = 3$, the series above is divergent, but the argument can be adapted by using a decomposition of U.

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