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Homogeneous Kähler Manifolds and T-Algebras in $N=2$ Supergravity and Superstrings

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Abstract. Motivated by the problem of the moduli space of superconformal theories, we classify all the (normal) homogeneous Kähler spaces which are allowed in the coupling of vector multiplets to $N=2$ SUGRA. Such homogeneous spaces are in one-to-one correspondence with the homogeneous quaternionic spaces ($\neq \mathbb{H}H^n$) found by Alekseevskii. There are two infinite families of homogeneous non-symmetric spaces, each labelled by two integers. We construct explicitly the corresponding supergravity models. They are described by a *cubic* function F, as in flat-potential models. They are Kähler–Einstein if and only if they are symmetric. We describe in detail the geometry of the relevant manifolds. They are Siegel (bounded) domains of the first type. We discuss the physical relevance of this class of bounded domains for string theory and the moduli geometry. Finally, we introduce the T-algebraic formalism of Vinberg to describe in an efficient way the geometry of these manifolds. The homogeneous spaces allowed in $N=2$ SUGRA are associated to rank 3 T-algebras in exactly the same way as the symmetric spaces are related to Jordan algebras. We characterize the T-algebras allowed in $N=2$ supergravity. They are those for which the *ungraded* determinant is a polynomial in the matrix entries. The Kähler potential is simply minus the logarithm of this "naive" determinant.

1. Introduction

One promising approach [1] to the geometry of the moduli space for an abstract $2d$ superconformal field theory is the study of the low-energy supergravity corresponding to the superstring model defined by this theory. Probably, the most interesting case is that of $(2,2)$ superconformal systems, which according to a well-motivated conjecture by Gepner [2] should correspond to a σ-model on a Calabi–Yau manifold. Many results on the moduli spaces for $(2,2)$ systems were obtained using this method in refs. [3,4]. Indeed, it turns out that many problems in the moduli theory were already worked out in the context of supergravity, and hence many issues can be spelled out just by looking in the right place in the SUGRA literature. For instance, the moduli space of the $(4,4)$ $c = 6$ superconformal theories is completely determined by the (unique) geometry of the scalars' σ-models consistent with $N = 4$ $4D$ supergravity [5]. The result [1] is consistent with Gepner's conjecture [2] and the moduli space for the Kähler–Einstein metrics on the $K3$ surface, as computed in the mathematical literature [6].

However, although the method is in principle very powerful, its implementation requires to work out the details of some exercises in supergravity which are not already available in the published literature. As emphasized in Refs. [1, 3, 4] the supergravity theory which is relevant for the moduli problem (of the $(2,2)$ $c = 9$ system) is $N = 2$, so the open questions are problems in the coupling of $N = 2$ SUGRA to matter.

It is the purpose of the present paper to give full details on one such technical problem, namely to classify and construct explicitly all the couplings of vector multiplets to $N = 2$ SUGRA such that the corresponding scalar manifold is a *homogeneous* Kähler manifold. The simpler case of a *symmetric* Kähler manifold was solved some time ago by Cremmer and Van Proeyen [7].

This problem is so deeply related to the geometry of string theory, that in order to solve it, we shall use ideas coming from the analysis of the string case [3] and we shall see below how the string language gives a geometrical interpretation to the matter couplings of $N = 2$ SUGRA.

The main idea from string theory is the c-map [3]. Its physical origin is the following. The low-energy theories resulting from the compactification of type IIA and IIB superstrings on the *same* $(2,2)$ superconformal system, are related by the interchange of the vector multiplets with the hypermultiplets [1, 3]. The c-map is the operation which transforms one such effective Lagrangian into the other one. Since the hypermultiplets parametrize a quaternionic manifold [8] and the vector-multiplet scalars a Kähler manifold (of a restricted type [9, 3]), the c-map is an operation which transforms a (restricted) Kähler manifold into a quaternionic manifold (with reduced curvature [10] $\nu = -2$), and vice versa. The details of this map and its relationships with string theory are discussed in Ref. [3]. Since quaternionic and hyperKähler[1] manifolds are rather interesting geometrical objects, the c-map has surprising mathematical properties. Some of these properties were sketched in ref. [3]: there it was found that the c-map is closely related to the theory of Jordan algebras [12, 13]. We shall see that this connection extends to the more general T-algebras [14].

More generally, it was found [3] that the c-map is connected to the classification by D. V. Alekseevskii [15] of *normal homogeneous* quaternionic manifolds, i.e. quaternionic manifolds having a *solvable* transitive group of isometries. It is conjectured [15] that these are the only *non-compact* homogeneous quaternionic manifolds.

The classification of the relevant homogeneous Kähler manifolds is constructed as follows. Take one such manifold and construct the corresponding supergravity model. Using the methods of ref. [3] we can construct its c-map, i.e. a SUGRA model

[1] HyperKähler manifolds are relevant for the global case (ref. [8, 11])

coupled to $n+1$ hypermultiplets taking value on some quaternionic space which is also homogeneous. Then it is a Alekseevskii space. Therefore, all the homogeneous Kähler spaces allowed in $N=2$ SUGRA are inverse c-images of Alekseevskii spaces, and conversely any Alekseevskii space—other than $\mathbb{H}H^n$—is the c-image of a homogeneous Kähler space allowed in $N=2$ SUGRA. Thus, the classification of such homogeneous spaces is reconducted to the known classification of the normal homogeneous quaternionic spaces [15].

As it is well known [9], the coupling of n vector-multiplets to $N=2$ SUGRA is specified by a holomorphic function of $n+1$ complex variables $F(X^0, X^1, \ldots, X^n)$, with F homogeneous of degree 2. In ref. [3] it was shown that in the case of a $4D$ type IIA superstring the function F should have the general form (*neglecting non-perturbative corrections*)

$$F(X^0, X^A) = i\frac{d_{ABC}X^A X^B X^C}{X^0}, \tag{1.1}$$

where d_{ABC} are real constants. The topological meaning of these coefficients was discussed in ref. [4] (see also Sect. 4.C.1 below). The formula in Eq. (1.1) follows from the Peccei-Quinn symmetry of ref. [16]. Of course, Eq. (1.1) holds only for a specific parametrization of the fields. This parametrization is convenient for two reasons: i) these fields are simply related to the string vertices [3] and to the geometry [4] of the "internal" Calabi-Yau space; ii) models of the form (1.1) were extensively studied in the context of the so-called flat-potential models [17, 12].

We shall show below that all the homogeneous Kähler spaces allowed in $N=2$ SUGRA—except $\mathbb{C}H^n$—are of the form in Eq. (1.1).

For all the allowed homogeneous Kähler manifolds we shall give the explicit form of the function F.

There are two infinite families of homogeneous (but non-symmetric) Kähler manifolds allowed in $N=2$ SUGRA: $K(p,q)$, (p,q integers $0 \leqq p \leqq q$), and $H(p,q)$, ($p,q \geqq 1$ integers). All these spaces have rank 3. There are some exceptional elements of these families which are symmetric spaces. These are the manifolds given by the *magical square* [12, 18]. The c-map sends the spaces $K(p,q)$ and $H(p,q)$ into the Alekseevskii quaternionic spaces $W(p,q)$ and $V(p,q)$ respectively.

The classification of the (normal) homogeneous quaternionic spaces [15] is based on the corresponding classification for the Kählerian case due to Pjateckii-Sapiro [19] and Gindikin, Pjateckii-Sapiro and Vinberg [20, 21]. The strict connection with their work allows us to go more in depth in the study of the geometrical properties of our spaces $K(p,q)$ and $H(p,q)$.

All our spaces are bounded domains in $\mathbb{C}^{n2}$. From the classification theory for such domains [20, 21], we learn (Sect. 4) that all our homogeneous spaces are Siegel domains of the first kind.

However, not all such Siegel domains can be coupled to $N=2$ SUGRA. Then, we have to characterize further the geometry of our spaces. Moreover, for a Siegel domain there may be more than one metric such that the complex automorphisms

[2] By a theorem of Borel (ref. [22]) they cannot be written as G/H with G simple (or unimodular) unless they are the symmetric spaces

act by isometries. We have to characterize geometrically the *unique* metric compatible with $N=2$ supersymmetry.

This is done using the T-algebras [14] (see Sect. 4.C.1). This theory allows us to reformulate the above results in a nice way, similar to the one encountered in the *symmetric* case [12, 7]. In that case, the result was that a *symmetric* Kähler manifold is allowed in $N=2$ SUGRA if and only if it is either a hyperbolic space or it is associated to a rank 3 Jordan algebra. This statement remains true if we replace *symmetric* with *homogeneous* and *Jordan algebras* with *T-algebras*, always of rank 3. However, only the rank 3 T-algebras whose isometric map is special or degenerate can be coupled to $N=2$ SUGRA.

On a bounded domain there is a preferred metric, the Bergmann one, which is invariant under all complex automorphisms [23]. At first, one would expect this to be the metric chosen by $N=2$ SUGRA (within a positive factor). However, it is not so. The Bergmann metric of a homogeneous domain is Einstein [23], whereas SUSY requires the (homogeneous) Kähler metric not to be Einstein, unless the space is symmetric. Anyhow, the SUGRA metric is related in a simple way to the Bergmann one, see Sect. 4.C.2.

The present paper is organized as follows. In Sect. 2 we shall present some preliminary material. In Subsect. 2.A we discuss duality invariance. In Subsect. 2.B we give the basics of the Alekseevskii classification of normal homogeneous quaternionic spaces. In Sect. 3 the actual construction of the couplings to $N=2$ supergravity is performed. In Subset. 3.A we compute the function F for all the homogeneous spaces. In Subsect. 3.B we show that all the models can be put in the form of Eq. (1.1). In Subsect. 3.C we show that these spaces are Kähler–Einstein if and only if they are symmetric. In Sect. 4 we study the geometry of the relevant homogeneous spaces. Subsection 4.A is concerned with the physical implications of the geometry from the string and the supergravity points of view. In Subsect. 4.B we prove that all our spaces are Siegel domains of the first kind. In Subsect. 4.C, we introduce the T-algebras and use them to give a simpler characterization of the homogeneous spaces allowed in $N=2$ SUGRA and their Kähler metrics.

2. Homogeneous Kähler Spaces in $N=2$ SUGRA: Preliminaries

In this section we review some results we need in order to construct explicitly the coupling of n vector multiplets to $N=2$ supergravity such that the resulting σ-models are *homogeneous.*

First of all we have to explain the idea underlying the construction of these homogeneous couplings to $N=2$ SUGRA. We have to find for each Alekseevskii space (except the quaternionic hyperbolic ones) a holomorphic function F describing the corresponding coupling to $N=2$ supergravity. The function F is found using the theorem of Sect. 2.A of ref. [3] (see also ref. [24]): if in a supergravity theory we have a group of symmetries which acts transitively on the scalars' manifold and acts on the vectors by duality transformations, then all the couplings are uniquely determined by the group itself and the representation to which the field-strengths belong. In our case all the isometries of the vector-multiplet scalar manifold act by duality transformations. Therefore, in the case of a *homogeneous*

Kähler manifold, the function F is fully determined by its algebra and by its realization on the field-strengths. As we shall see, the classification of the homogeneous quaternionic spaces also specifies this realization.

Given the crucial importance of duality transformations, we begin this section by reviewing their geometry (in the $N = 2$ case). Then we introduce the relevant results by Alekseevskii and we show how from them we can get the realization of the isometry group on the field-strengths.

2.A. Duality in $N = 2$ Supergravity. As it is well known [9] the coupling of n vector multiplets to $N = 2$ supergravity is specified by a holomorphic function $F(X^0, X^1, \ldots, X^n)$ homogeneous of degree 2. However, it is more convenient to consider the function $S(X^I) = iF(X^I)/2$. In ref. [3] it was shown that $S(X)$ can be interpreted as the generating function for a canonical (holomorphic) transformation. The momenta are $P_I = \partial_I S(X)$. In the complex phase space there are three different symplectic structures. The transformation defined by S leaves invariant two of them. The equation $P_I = \partial_I S(X)$ defines a complex submanifold $\mathscr{F}[S]$ embedded in the phase space.

The real physical fields are the "Cartesian" coordinates $Z^A = X^A/X^0$ $(A = 1, \ldots, n)$. The corresponding physical Kähler potential is [9]

$$G = -\ln[z^I N_{IJ}(z, \bar{z})\bar{z}^J]. \tag{2.1}$$

For more details see the Appendix of ref. [3], as well as the standard reference on the $N = 2$ tensor calculus, ref. [9].

A *duality transformation* is defined [3] to be a transformation of the phase space which leaves invariant the symplectic structures and the submanifold $\mathscr{F}[S]$. These maps are linear transformations belonging to the group $Sp(2n + 2, \mathbb{R})$, so the duality group must be a subgroup of the real symplectic group [24]. Let Σ

$$\Sigma = \begin{pmatrix} B & D \\ C & -B^T \end{pmatrix} \tag{2.2}$$

be a generator of the duality group, acting on the (complex) phase space as

$$\begin{aligned} \delta X^I &= B^I{}_J X^J + D^{IJ} P_J, \\ \delta P_I &= -B^J{}_I P_J + C_{IJ} X^J. \end{aligned} \tag{2.3}$$

The corresponding transformation on the vector-field strengths $F^I{}_{\mu\nu}$ $(I = 0, \ldots, n;\ F^0{}_{\mu\nu}$ is the graviphoton field strength) is

$$\begin{aligned} \delta F^I_{\mu\nu} &= B^I{}_J F^J_{\mu\nu} + D^{IJ} G_{J\mu\nu}, \\ \delta G_{I\mu\nu} &= -B^J{}_I G_{J\mu\nu} + C_{IJ} F^J_{\mu\nu}, \end{aligned} \tag{2.4}$$

where

$$\tilde{G}_{I\mu\nu} = -\frac{\partial L}{\partial F^{I\mu\nu}}. \tag{2.5}$$

The requirement that Σ leaves invariant $\mathscr{F}[S]$ gives [9, 3]

$$C_{IJ} X^J - B^J{}_I S_J = S_{IJ}(B^J{}_K X^K + D^{JK} S_K). \tag{2.6}$$

It is elementary to check that this transformation is an isometry of the Kähler metric in Eq. (2.2) (see also ref. [9]).

In principle, to determine the function S we have to solve Eq. (2.6), which is a rather complicated equation. However, the interpretation of Σ as the generator of a canonical transformation allows for a simplification of the analysis.

Lemma. *If S is homogeneous of degree 2, Eq. (2.6) is equivalent to first order scalar equation*

$$\frac{1}{2}\frac{\partial S}{\partial X^I}D^{IJ}\frac{\partial S}{\partial X^J}+\frac{\partial S}{\partial X^I}B^I{}_J X^J-\frac{1}{2}C_{IJ}X^I X^J=0. \tag{2.7}$$

Proof. That Eq. (2.6)$\Rightarrow$Eq. (2.7) follows by multiplying Eq. (2.6) by X^I and using the homogeneity condition $S_I = S_{IJ}X^J$. Instead, that Eq. (2.7)$\Rightarrow$Eq. (2.6) is seen by derivation of Eq. (2.8) with respect X^I.

Equation (2.7) is much easier to analyze than Eq. (2.6). Equation (2.7) is easily seen to be the analytic continuation to complex values of the stationary Hamilton–Jacobi equation for the "Hamiltonian" H_Σ

$$H_\Sigma=\tfrac{1}{2}P_I D^{IJ}P_J+P_I B^I{}_J X^J-\tfrac{1}{2}C_{IJ}X^I X^J. \tag{2.8}$$

In fact $H_\Sigma(p,q)$ is the generator of the canonical (symplectic) transformation in Eq. (2.3),

$$\delta P_I=[H_\Sigma,P_I],\quad \delta X^I=[H_\Sigma,X^I]. \tag{2.9}$$

Our models have a *solvable* algebra of duality transformations with generators Σ^a $(a=1,\dots,2n)$

$$[\Sigma^a,\Sigma^b]=f^{ab}{}_c\Sigma^c. \tag{2.10}$$

Let $H^a{}_\Sigma(p,q)$ be the corresponding Hamilton functions. The condition of invariance for $\mathscr{F}[S]$ under the corresponding duality transformation reads

$$H^a{}_\Sigma(\partial_1 S,X^I)=0. \tag{2.11}$$

The Poisson bracket of two Hamilton functions reproduces the original duality algebra

$$[H^a_\Sigma,H^b_\Sigma]=f^{ab}{}_c H^c_\Sigma. \tag{2.12}$$

This equation is the integrability condition of Eq. (2.11); it is just the closure condition for the Lie algebra of duality transformations [25].

2B. Basics of Normal Kähler and Quaternionic Algebras. In this subsection we shall briefly review the results of Alekseevskii [15] which are relevant for our construction. In particular, we shall need the classification of quaternionic solvable (metric) Lie algebras and the corresponding results for the Kählerian algebras [19, 20].

We begin by recalling some definitions [15]. A *metric* Lie algebra is simply a Lie algebra endowed with a Euclidean metric $\langle\,,\,\rangle$. To every metric Lie algebra G there corresponds a homogeneous Riemannian space $\mathscr{G}=\exp G$, i.e. the corresponding group space equipped with the metric $\langle\,,\,\rangle$ on $T_e\mathscr{G}\approx G$. If the metric algebra G is *completely solvable*, G is called a normal algebra and the corresponding Riemannian space is a normal homogeneous space.

Let $(V, \langle\ ,\ \rangle)$ be a metric Lie algebra with V the underlying algebra. For each $x \in V$ we have a skew-symmetric endomorphism of V called the *Nomizu operator* L_x defined by

$$2\langle L_x y, z\rangle = \langle [x,y], z\rangle - \langle x, [y,z]\rangle - \langle y, [x,z]\rangle. \tag{2.13}$$

In terms of these operators the curvature endomorphisms are $(x, y \in V)$

$$\mathrm{Riem}\,(x,y) = [L_x, L_y] - L_{[x,y]}, \tag{2.14}$$

$$\mathrm{ric}\,(x,y) = -\sum_i \langle \mathrm{Riem}\,(x, e_i)y, e_i\rangle, \tag{2.15}$$

where e_i is an orthonormal basis for V.

The *holonomy algebra* is defined as the linear Lie algebra Γ generated by the curvature operators $\mathrm{Riem}\,(x,y)$ and their commutators with the Nomizu operators $[L_w, \ldots, [L_z, \mathrm{Riem}\,(x,y)]\ldots]$.

A *complex structure* of V is a skew-symmetric endomorphism J, with $J^2 = -1$. Its centralizer in the Lie algebra $\Lambda^2(V)$ is denoted by $C(J)$. $C(J)$ is isomorphic to the Lie algebra of $U(n)$.

A *quaternionic structure* of V is a linear Lie algebra Q generated by two anticommuting complex structures. In a natural basis J_1, J_2, J_3 for Q, we have

$$J_\alpha^2 = -1 \quad J_\alpha J_\beta = J_\gamma \quad (\alpha, \beta, \gamma \text{ cyclic permutation of } 1,2,3). \tag{2.16}$$

The centralizer and the normalizer of Q in $\Lambda^2(V)$ are denoted by $C(Q)$ and $N(Q)$; we have $N(Q) = Q + C(Q)$. Obviously, Q and $C(Q)$ are the Lie algebras of $Sp(1)$ and $Sp(n)$, respectively.

A normal metric Lie algebra is called *Kählerian* if on V there is a complex structure J whose centralizer $C(J)$ contains the holonomy algebra Γ; it is called *quaternionic* if there is a quaternionic structure Q such that $N(Q)$ contains Γ, and the endomorphisms from Q map the curvature Riem to zero (this second condition is automatically satisfied in a dimension larger than four [10]).

There is a natural one-to-one correspondence between Kählerian (respectively quaternionic) normal Lie algebras and Kähler (respectively quaternionic) simply connected normal homogeneous spaces [15, 19–21]. So the classification of the corresponding spaces is equivalent to the algebraic problem of the classification of the relevant normal Lie algebras.

Exploiting the solvability, it is easy to see [15] that any (normal) quaternionic algebra should contain a one-dimensional[3] quaternionic subalgebra E—called the *canonical quaternionic subalgebra*. This subalgebra is totally geodesic (i.e. $L_E E \subset E$). There are only two one-dimensional quaternionic algebras [15], namely ${A^1}_1$ and ${C^1}_1$. The corresponding four dimensional quaternionic spaces are symmetric

$$\begin{aligned} A_1^1&: \quad SU(2,1)/[SU(2) \otimes U(1)], \\ C_1^1&: \quad Sp(1,1)/[Sp(1) \otimes Sp(1)]. \end{aligned} \tag{2.17}$$

[3] In the quaternionic sense (i.e. one quarter of the real dimension)

For a given dimension, there is a unique quaternionic algebra whose canonical subalgebra is isomorphic to $C^1{}_1$. The associated quaternionic spaces are the hyperbolic spaces $\mathbb{H}H^n$. All the other normal quaternionic algebras have a canonical subalgebra isomorphic to $A^1{}_1$.

In the string context [3] this canonical subalgebra has the physical interpretation of the *universal* sector, i.e. that sector of the massless theory whose vertices do not contain the fields of the internal superconformal theory. This sector is the same for all type II superstring compactified on *any* (2,2) superconformal theory. Physically, the four generators of the $A^1{}_1$ subalgebra are: the dilatation corresponding to a rescaling of the dilaton field ϕ, the Peccei-Quinn symmetry associated to the "axion" $B_{\mu\nu}$ and two Peccei-Quinn symmetries corresponding to the two R–R scalars whose vertices are bilinear in the two space-time SUSY generators. It follows from the analysis of ref. [3] that only the second class of quaternionic spaces is relevant for our purposes.

In this last case one can show [15] that $V = U + U'$, where U and U' are isomorphic, as vector spaces. One can choose the basis for Q such that the isomorphism is simply J_2. Then $J_1 U = U$ and [15]

$$[U,U] \subset U, \quad [U,U'] \subset U', \quad [U',U'] \subset \{e_1\}. \tag{2.18}$$

It is easily checked that U is a totally geodesic subalgebra of V. Hence, U is a Kählerian algebra with respect to the (integrable) complex structure J_1. U is called the *principal Kählerian subalgebra* of the quaternionic algebra V. Let F_0 be the intersection of U with the canonical subalgebra E. In an orthonormal basis for $F_0, \{e_0, e_1\}$ (with $e_1 = J_1 e_0$), the Lie algebra F_0 reads $[e_0, e_1] = e_1$. Of course, $E = F_0 + F'_0$, where F'_0 is the image of F_0 under the isomorphism J_2. It turns out that the principal subalgebra U is always a *direct* sum $F_0 + W$, with W some normal Kähler algebra. Consequently, the corresponding homogeneous Kähler manifold is always a direct product

$$\mathscr{U} = \frac{SU(1,1)}{U(1)} \otimes \mathscr{W}, \quad \mathscr{W} = \exp W. \tag{2.19}$$

Comparing the above results with the analysis of ref. [3], we get the explicit expression for the c-map of the homogeneous Kähler spaces $\mathscr{W}$

$$c\colon \mathscr{W} = \exp W \to \mathscr{V} = \exp V, \tag{2.20}$$

It is easy to check that this map has all the physical and mathematical properties of the c-map, as defined in ref. [3].

Since the spaces of the form $\mathscr{V}$ exhaust all the relevant homogeneous quaternionic spaces, from the discussion in the introduction we conclude that *all the homogeneous Kähler manifolds allowed in* $N = 2$ SUGRA *arise in this way, namely as the non-trivial factor space* $\mathscr{W}$ *of the principal Kählerian geodesic submanifold* $\mathscr{U}$ *of a homogeneous quaternionic manifold.*

Therefore, the classification of the relevant Kähler spaces is reduced to the algebraic problem of finding what Kähler algebras W lead to homogeneous quaternionic spaces. This is the problem solved for us by Alekseevskii [15].

From Eq. (2.18) we see that there is a representation $u \to T_u$ of the Kähler algebra U in U' induced by the adjoint representation of V, $T_u u' = [u, u'] \in U'$. In order for the algebra V to be quaternionic this representation T should have a number of properties ($Q1, \ldots, Q8$ of Lemma 5.5 of ref. [15]) which fully characterize the quaternionic algebra V. Such a representation is called a Q-representation. Then the homogeneous Kähler manifolds allowed in $N=2$ supergravity are in one-to-one correspondence with Q-representations of Kählerian normal algebras of the form $F_0 + W$. For lack of space we shall not comment upon the properties defining a Q-representation. However, the condition $Q7$ is physically so crucial that we should mention it. On U' there is a complex structure J (related to J_1) and hence a skew-Hermitian form $\langle J \ldots, \ldots \rangle$. The representation T of W (but *not* of all U) is *symplectic* with respect this form. If W has complex dimension n, U and U' have (real) dimension $2n+2$. Therefore, on U' the group $\mathscr{W}$ acts as a (totally solvable) subgroup of $Sp(2n+2, \mathbb{R})$. This is not a surprise, since $\mathscr{W}$ is a duality group and hence it should be a subgroup of $Sp(2n+2, \mathbb{R})$. In fact, we have more: the representation T is nothing else than the realization of the duality algebra W on the vector fields, Eq. (2.4).

In other words, the matrices Σ^a of Subsect. 2.A are just the linear transformations T_w written in a canonical basis for the symplectic structure of U'. The couplings of the vector multiplets are determined once we know the representation T. But T is induced by the adjoint rep. of V, and so it is known from the classification of the quaternionic normal algebras.

The physical picture emerging from the work of Alekseevskii is quite appealing. The elements of the Kählerian Lie algebra W are identified with the physical scalar fields of the vector-multiplets. Then the isomorphism J_2 is nothing else than the supersymmetry transformation mapping the scalars into the vector fields of the corresponding multiplet, F'_0 being related to the graviphoton field strengths. Thus, the condition that the relevant isometries have a Q-representation is, essentially, the same as the request that these symmetries commute with local supersymmetry.

It remains to describe the classification of the possible transitive duality algebras W and their representations. To do this we need some more definitions [15]. We agree that all the Lie algebras below are written in an orthonormal basis.

A *key* algebra is a two-dimensional Kähler algebra $F = \{h, g\}$, $g = Jh$, with $[h, g] = \mu g$. The positive number μ is the *root* of the key algebra. An *elementary* Kähler algebra is an algebra of the form $F + X$, where F is a key subalgebra with root μ, X is the orthogonal complement to F and we have $\mathrm{ad}_h | X = \mu/2$, $\mathrm{ad}_g | X = 0$ and $[x, y] = \langle Jx, y \rangle g$, for $x, y \in X$. The basic result in the classification of the Kählerian algebras [19] is that every normal non-degenerate Kählerian algebra is the *semidirect* sum of elementary algebras, $U = \sum_i U_i$, with $U_i = F_i + X_i$. The explicit structure of this semidirect sum is described in Proposition 6.2 of ref. [15]. The number k of elementary algebras in U is called the *rank* of the algebra U.

The normal Kählerian algebras having Q-representations are classified according to their rank and type. (The type is the largest value of $(\mu_i)^{-2}$.) One shows [15] that the relevant algebras W have rank $k \leqq 3$, and the allowed values for the roots are $1, 1/\sqrt{2}$ and $1\sqrt{3}$, so that we can have type 1, 2 and 3. All the algebras W with rank strictly less than 3 lead to *symmetric* Kähler (and quaternionic) spaces. It is elementary to see that the function F can be put in the *stringy* (cubic) form if $M \equiv \sum_i (\mu_i)^{-2} = 3$.

There is a unique Kähler algebra of type 3 which admits a Q-representation [15]. It has the form $U = F_0 + F$, where F is a key algebra with root $1/\sqrt{3}$. The associated quaternionic space is the symmetric coset $G_{2(+2)}/[SU(2) \otimes SU(2)]$. The symmetric Kähler space associated to a *key* Kähler algebra is the coset $SU(1,1)/U(1)$, with the metric normalized as

$$R_{\alpha\bar{\beta}} = -2\mu^2 g_{\alpha\bar{\beta}}. \tag{2.21}$$

For this type 3 algebra, the representation T is the one induced under the embedding $F \to SU(1,1)$ by the representation 4 of $SU(1,1)$. From this fact, we infer that this is the model described by the function $F = i(X^1)^3/X^0$. The associated quaternionic manifold $G_{2(+2)}/[SU(2) \otimes SU(2)]$ is what we got as the c-image of $SU(1,1)/U(1)$ in ref. [3] by more elementary means.

There is also a unique Kähler algebra of type 2 which admits a Q-representation [15]. It has the form $U = F_0 + W$ with $W = F + F'$ the *direct* sum of two Kähler algebras with roots $1/\sqrt{2}$ and 1. Hence the corresponding Kähler space is reducible in the product of two cosets $SU(1,1)/U(1)$ with curvature -1 and -2 respectively. Since $M = 3$ it has the cubic form and it corresponds to the function $F = iX^1(X^2)^2/X^0$. This is easily checked since T is induced by the representation $(3, 2)$ of $SU(1,1) \otimes SU(1,1)$. Here we find the connection of ref. [3] between curvatures (*quantization* of the Newton constant) and the quantum numbers of the field-strengths under the $SU(1,1)$ group. The associated quaternionic space (i.e. its c-map) is $SO(3,4)/[SO(3) \otimes SO(4)]$.

There remains to discuss the algebras W of type 1. It can be shown [15] that in this case W has rank 1 or 3. By our previous remarks, the first case leads to symmetric Kähler manifolds and the second one has the stringy (cubic) form. The type 1, rank 1 case corresponds to the complex hyperbolic spaces $\mathbb{C}H^n = SU(1,n)/[S(U(1) \otimes U(n))]$ (the so-called *minimal* coupling [9]). This can be easily checked from Proposition 9.1 of ref. [15]. The corresponding quaternionic space is $SU(2,n)/[S(U(2) \otimes U(n))]$ in agreement with the more elementary arguments of ref. [3]. Again, in the case $n = 1$ we get the coset $SU(1,1)/U(1)$ but this time with curvature -2.

The last case, type 1 rank 3 Kähler algebras, is the really interesting one. W has the form $W = (F_1 + X_1) + (F_2 + X_2) + F_3$ (F_i are key algebras with root 1, and $F_i + X_i$ are elementary Kähler algebras). It is convenient to set $X = X_2$ and $X_1 = Y + Z$, where $[F_2, Y] = 0$ and $[F_2, Z] = Z$. We decompose the vector spaces X, Y into the eigenspaces (eigenvalues $\pm 1/2$) of the adjoint action of h_3 and for the space Z of h_2. The corresponding eigenspaces are denoted as X_+ and $X_- = JX_+$, etc.

With these notations the *general* type 1 rank 3 normal Kähler algebra W, such that $F_0 + W$ possesses a Q-representation has the form [15] (we do not rewrite the commutation relations which define the elementary Kähler algebras)

$$[F_i, F_j] = 0 \quad i \neq j, \tag{2.22a}$$

$$[h_3, Y_\pm] = \pm\tfrac{1}{2} Y_\pm, \tag{2.22b}$$

$$[h_3, X_\pm] = \pm\tfrac{1}{2} X_\pm, \tag{2.22c}$$

$$[h_2, Z_\pm] = \pm\tfrac{1}{2}Z_\pm, \tag{2.22d}$$

$$[g_3, Y_+] = [g_2, Z_+] = [g_3, X_+] = 0, \tag{2.22e}$$

$$[g_3, Y_-] = Y_+ \quad [g_2, Z_-] = Z_+ \quad [g_3, X_-] = X_+, \tag{2.22f}$$

$$[F_1, X] = [F_2, Y] = [F_3, Z] = 0, \tag{2.22g}$$

$$[Y, Z] = [X, Z_+] = [X_-, Y_-] = [X_+, Y_+] = 0, \tag{2.22h}$$

$$[X_-, Y_+] \subset Z_+, \tag{2.22i}$$

$$[x_-, z_-] = \frac{1}{\sqrt{2}}\psi(x_-, z_-), \tag{2.22j}$$

$$\langle [x_-, y_+], z_+ \rangle = -\frac{1}{\sqrt{2}}\langle Jy_+, \psi(x_-, Jz_+)\rangle, \tag{2.22k}$$

$$[Jx, z_-] = J[x, z_-], \quad [x_+, Jy_+] = -[Jx_+, y_+], \tag{2.22l}$$

$$(x_- \in X_-, x \in X, z_- \in Z_-, x_+ \in X_+ \text{ and } y_+ \in Y_+), \tag{2.22m}$$

where $\psi: X_- \otimes Z_- \to Y_-$ is an *isometric* mapping, i.e. a map linear in its two arguments such that

$$\langle \psi(X_-, Z), \psi(X_-, Z_-)\rangle = \langle X_-, X_-\rangle\langle Z_-, Z_-\rangle. \tag{2.23}$$

Given three vector spaces X_-, Z_- and Y_- and an isometric map ψ, we satisfy all the conditions defining a Q-representation T_w except the closure of the algebra,

$$[T_w, T_{w'}] = T_{[w,w']}. \tag{2.24}$$

This last condition gives us two classes of solutions [15],

i) $X = 0$, or
ii) $X, Y, Z \neq 0$ and $\dim Y_- = \dim Z_-$. Such isometric maps are called *special.*

The theory of special isometric mappings is equivalent to those of the Clifford modules [15, 26], i.e. to that of the Dirac matrices.

Finally, we give the explicit form of the matrices representing the algebra W, Eqs. (2.22), on U'. These correspond to the duality generators of Eq. (2.3). We denote by a $\sim$ the element of U' obtained by a given element of U under the isomorphism J_2.

The three h generators are represented by

$$\begin{aligned} T_{h_1} = {} & \tfrac{1}{2}p_0 \otimes p_0 + \tfrac{1}{2}p_1 \otimes p_1 - \tfrac{1}{2}p_2 \otimes p_2 - \tfrac{1}{2}p_3 \otimes p_3 + \tfrac{1}{2}\textstyle\sum \tilde{x}^i_+ \otimes \tilde{x}^i_+ \\ & - \tfrac{1}{2}q_0 \otimes q_0 - \tfrac{1}{2}q_1 \otimes q_1 + \tfrac{1}{2}q_2 \otimes q_2 + \tfrac{1}{2}q_3 \otimes q_3 - \tfrac{1}{2}\textstyle\sum \tilde{x}^i_- \otimes \tilde{x}^i_-, \end{aligned} \tag{2.25a}$$

$$\begin{aligned} T_{h_2} = {} & \tfrac{1}{2}p_0 \otimes p_0 - \tfrac{1}{2}p_1 \otimes p_1 + \tfrac{1}{2}p_2 \otimes p_2 - \tfrac{1}{2}p_3 \otimes p_3 + \tfrac{1}{2}\textstyle\sum \tilde{y}^i_+ \otimes \tilde{y}^i_+ \\ & - \tfrac{1}{2}q_0 \otimes q_0 + \tfrac{1}{2}q_1 \otimes q_1 - \tfrac{1}{2}q_2 \otimes q_2 + \tfrac{1}{2}q_3 \otimes q_3 - \tfrac{1}{2}\textstyle\sum \tilde{y}^i_- \otimes \tilde{y}^i_-, \end{aligned} \tag{2.25b}$$

$$\begin{aligned} T_{h_3} = {} & \tfrac{1}{2}p_0 \otimes p_0 - \tfrac{1}{2}p_1 \otimes p_1 - \tfrac{1}{2}p_2 \otimes p_2 + \tfrac{1}{2}p_3 \otimes p_3 + \tfrac{1}{2}\textstyle\sum \tilde{z}^i_+ \otimes \tilde{z}^i_+ \\ & - \tfrac{1}{2}q_0 \otimes q_0 + \tfrac{1}{2}q_1 \otimes q_1 + \tfrac{1}{2}q_2 \otimes q_2 - \tfrac{1}{2}q_3 \otimes q_3 - \tfrac{1}{2}\textstyle\sum \tilde{z}^i_- \otimes \tilde{z}^i_-, \end{aligned} \tag{2.25c}$$

where p_0, q_0, p_α and q_α are the orthonormal elements of U'

$$p_0 = \tfrac{1}{2}(\tilde{h}_0 + \tilde{h}_1 + \tilde{h}_2 + \tilde{h}_3), \tag{2.26a}$$

$$q_0 = \tfrac{1}{2}(\tilde{g}_0 - \tilde{g}_1 - \tilde{g}_2 - \tilde{g}_3), \tag{2.26b}$$

$$p_\alpha = \tfrac{1}{2}(-\tilde{h}_0 - \tilde{h}_\alpha + \tilde{h}_\beta + \tilde{h}_\gamma), \tag{2.26c}$$

$$q_\alpha = \tfrac{1}{2}(-\tilde{g}_0 + \tilde{g}_\alpha - \tilde{g}_\beta - \tilde{g}_\gamma) \quad \alpha, \beta, \gamma \text{ permutations of } 1,2,3. \tag{2.26d}$$

The three g generators are given by

$$T_{g_1} = p_1 \otimes q_0 + p_0 \otimes q_1 + q_3 \otimes p_2 + q_2 \otimes p_3 + \sum \tilde{x}^i_+ \otimes \tilde{x}^i_-, \tag{2.27a}$$

$$T_{g_2} = p_2 \otimes q_0 + p_0 \otimes q_2 + q_3 \otimes p_1 + q_1 \otimes p_3 + \sum \tilde{y}^i_+ \otimes \tilde{y}^i_-, \tag{2.27b}$$

$$T_{g_3} = p_3 \otimes q_0 + p_0 \otimes q_3 + q_1 \otimes p_2 + q_2 \otimes p_1 + \sum \tilde{z}^i_+ \otimes \tilde{z}^i_-. \tag{2.27c}$$

In order to describe the remaining generators belonging to the vector spaces X, Y and Z we introduce, following Alekseevskii, a new product operation mapping two of the spaces X, Y, Z into the third

$$(u, v) \to u * v \equiv 2L_u v. \tag{2.28a}$$

Using Eqs. (2.13, 22) and the fact that ψ is *special* one proves the two identities

$$\langle X_- * Z_-, X_- * Z_- \rangle = \tfrac{1}{2}\langle X_-, X_- \rangle \langle Z_-, Z_- \rangle, \tag{2.28b}$$

$$\langle X_- * Y_-, X_- * Y_- \rangle = \tfrac{1}{2}\langle X_-, X_- \rangle \langle Y_-, Y_- \rangle. \tag{2.28c}$$

Then the remaining generators read (always in an orthonormal basis for U)

$$\begin{aligned} T_{x_+} = {} & p_0 \otimes \tilde{x}_+ - \tilde{x}_+ \otimes p_1 + \widetilde{J_1 x_+} \otimes q_0 - q_1 \otimes \widetilde{J_1 x_+} \\ & + \sum \widetilde{x_+ * y^i_-} \otimes \tilde{y}^i_- + \sum \widetilde{x_+ * z^i_-} \otimes \tilde{z}^i_-, \end{aligned} \tag{2.29a}$$

$$\begin{aligned} T_{y_+} = {} & p_0 \otimes \tilde{y}_+ - \tilde{y}_+ \otimes p_2 + \widetilde{J_1 y_+} \otimes q_0 - q_2 \otimes \widetilde{J_1 y_+} \\ & + \sum \widetilde{y_+ * x^i_-} \otimes \tilde{x}^i_- + \sum \widetilde{y_+ * z^i_-} \otimes \tilde{z}^i_-, \end{aligned} \tag{2.29b}$$

$$\begin{aligned} T_{z_+} = {} & p_0 \otimes \tilde{z}_+ - \tilde{z}_+ \otimes p_3 + \widetilde{J_1 z_+} \otimes q_0 - q_3 \otimes \widetilde{J_1 z_+} \\ & + \sum \widetilde{z_+ * x^i_-} \otimes \tilde{x}^i_- + \sum \widetilde{z_+ * y^i_-} \otimes \tilde{y}^i_-, \end{aligned} \tag{2.29c}$$

$$\begin{aligned} T_{x_-} = {} & - p_2 \otimes \tilde{x}_- - \tilde{x}_- \otimes p_3 - \widetilde{J_1 x_-} \otimes q_2 - q_3 \otimes \widetilde{J_1 x_-} \\ & - \sum \widetilde{x_- * y^i_-} \otimes \tilde{y}^i_- + \sum \widetilde{x_- * z^i_+} \otimes \tilde{z}^i_+, \end{aligned} \tag{2.29d}$$

$$\begin{aligned} T_{y_-} = {} & - p_1 \otimes \tilde{y}_- - \tilde{y}_- \otimes p_3 - \widetilde{J_1 y_-} \otimes q_1 - q_3 \otimes \widetilde{J_1 y_-} \\ & - \sum \widetilde{y_- * x^i_-} \otimes \tilde{x}^i_- + \sum \widetilde{y_- * z^i_+} \otimes \tilde{z}^i_+, \end{aligned} \tag{2.29e}$$

$$\begin{aligned} T_{z_-} = {} & - p_1 \otimes \tilde{z}_- - \tilde{z}_- \otimes p_2 - \widetilde{J_1 z_-} \otimes q_1 - q_2 \otimes \widetilde{J_1 z_-} \\ & - \sum \widetilde{z_- * x^i_-} \otimes \tilde{x}^i_- + \sum \widetilde{z_- * y^i_+} \otimes \tilde{y}^i_+. \end{aligned} \tag{2.29f}$$

This completes the algebraic structure of the normal quaternionic algebras, alias Q-representations of normal Kähler algebras.

The homogeneous spaces with $X = 0$ will be denoted by $K(p, q)$ where the two integers p and q are equal to

$$p = \dim Y_-, \quad q = \dim Z_-. \tag{2.30}$$

Since $K(p, q) = K(q, p)$ we can assume $p \leqq q$. These spaces have complex dimension equal to $3 + p + q$. The special cases $K(0, q)$ are symmetric manifolds,

$$K(0, q) = \{SU(1,1)/U(1)\} \otimes \{SO(q+2,2)/[SO(q+2) \otimes SO(2)]\}. \tag{2.31}$$

The associated homogeneous quaternionic manifolds are the Alekseevskii spaces $W(p, q)$ of dimension $4(4 + p + q)$. Again $p = 0$ gives symmetric spaces, namely, $SO(q+4,4)/[SO(q+4) \otimes SO(4)]$.

The other class of solutions is in one-to-one correspondence with Clifford modules. These are again characterized by two integers $p, q \geqq 0$. The corresponding spaces are denoted by $H(p, q)$, and have (complex) dimension $3 + p + 2qN(p)$, where $N(q)$ is the dimension of the irreducible Clifford module in q dimensions,

$$\begin{aligned} &N(1) = 1 \quad N(2) = 2 \quad N(3) = N(4) = 4, \\ &N(5) = N(6) = N(7) = N(8) = 8, \\ &N(8s + t) = (16)^s N(t) \quad \text{for} \quad s \geqq 1, \quad 1 < t \leqq 8. \end{aligned} \tag{2.32}$$

Again some special case leads to symmetric spaces. These are exactly the magic ones related to the Jordan algebras [12],

$$H(1,1) = Sp(6, \mathbb{R})/U(3), \tag{2.33a}$$

$$H(1,2) = U(3,3)/\{U(3) \otimes U(3)\}, \tag{2.33b}$$

$$H(1,4) = SO^*(12)/U(6), \tag{2.33c}$$

$$H(1,8) = E_{7(-26)}/\{E_6 \otimes SO(2)\}. \tag{2.33d}$$

Under the c-map the spaces $H(p, q)$ give the Alekseevskii spaces $V(p, q)$. This, in particular, gives us back the results of refs. [12, 3] for the magic cases

$$V(1,1) = F_{4(+4)}/[USp(6) \otimes SU(2)], \tag{2.34a}$$

$$V(1,2) = E_{6(+2)}/[SU(6) \otimes SU(2)], \tag{2.34b}$$

$$V(1,4) = E_{7(-5)}/[\mathrm{SO}(12) \otimes SU(2)], \tag{2.34c}$$

$$V(1,8) = E_{8(-24)}/[E_7 \otimes SU(2)]. \tag{2.34d}$$

3. Construction of the Coupling to $N = 2$ Supergravity

In this section we shall compute the functions F (or, equivalently S) associated to the previous homogeneous Kähler spaces, firstly in the canonical parametrization and then in the stringy one. Since all algebras W with rank less than three lead to well-known symmetric spaces [7], here we shall limit ourselves to the rank 3 case. The functions F are completely determined by duality invariance.

3.A. Duality Invariance. To write more compact formulas we shall adopt the following notation. The elements of the *orthonormal* basis of the three vector spaces X_-, Y_- and Z_- will be denoted by ${X^\mu}_{1-}$, ${X^m}_{2-}$ and ${X^M}_{3-}$, ($\mu = 1, \ldots, \dim X_-$, $m = 1, \ldots, \dim Y_-$ and $M = 1, \ldots, \dim Z_-$) and the same notation will be used for the isomorphic spaces X_+, Y_+ and Z_+. As a shorthand notation, we write $\underline{M}$ for $(1,\mu)$, $(2,m)$ or $(3,M)$ and $\underline{m}$ for μ, m or M. We define the coefficients $d^{\underline{M}\underline{N}\underline{P}}$ by the formula

$$X_+^{\underline{M}} * X_-^{\underline{N}} = d^{\underline{M}\underline{N}\underline{P}} X_+^{\underline{N}}. \tag{3.1}$$

Using the properties of the rank 4 quaternionic algebras is it easy to show that: i) the coefficients $d^{\underline{M}\underline{N}\underline{P}}$ are totally symmetric; ii) the only non-vanishing coefficients are

$$d^{(1,\mu)(2,m)(3,M)} \equiv d^{\mu m M}, \tag{3.2}$$

as well as those obtained from them by permutations of $\underline{M}, \underline{N}$ and $\underline{P}$. From the properties of the Nomizu operators, it is easy to show that

$$d^{\mu m M} = \frac{1}{\sqrt{2}} \langle \psi(X^\mu_{1-}, X^M_{3-}), X^m_{2-} \rangle. \tag{3.3}$$

As we have seen in Sect. 2.B, if $X_- \neq 0$, the isometric map ψ should be special, i.e. $\dim Y_- = \dim Z_-$. Then, in this case we replace the index M by $\dot{m}$, and we shall write $(\gamma^\mu)^{m\dot{n}}$ for $\sqrt{2} d^{\mu m \dot{n}}$. Then the condition that ψ is a special isometric map is

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\delta^{\mu\nu}, \tag{3.4}$$

where

$$\Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)^{m\dot{n}} \\ (\gamma^\mu)^{\dot{m}n} & 0 \end{pmatrix},$$

that is Γ^μ are Dirac matrices in $q = \dim X_-$ dimensions and Y_-, Z_- are (not necessarily irreducible) "chiral" spinor spaces. Such Clifford modules are fully specified by q and the "number of flavours" (i.e. the number of times the basic irreducible representation is repeated) p.

For an algebra W of complex dimension n, we have $2n+1$ equations for the function S, $2n$ from the duality invariance and one from the homogeneity condition. These are all first order differential equations, at most quadratic in the derivative of S (i.e. *at worst HJ* equations for harmonic oscillators). However, we have still to choose a canonical basis in phase-space, i.e. we have to decide what elements of U' we consider "coordinates" and "momenta" (what elements are associated to F and G field-strength, respectively). The only constraint for this choice is that the symplectic form J should be trivial in "configuration space." This choice corresponds to choosing an explicit parametrization of the function S in terms of the superconformal fields X^I (since two allowed parametrizations are related by an $Sp(2n+2,\mathbb{R})$ transformation, there is a one-to-one correspondence between canonical basis and field parametrizations).

The most convenient choice is the one which makes *linear* the largest number of differential equations. Choosing as "configuration-space" (F-field strengths) the space spanned by the orthonormal vectors

$$q_0, q_1, q_2, q_3, \widetilde{X^\mu_{1-}}, \widetilde{X^m_{2-}}, \widetilde{X^M_{3-}} \tag{3.5}$$

we have $n+1$ linear equations, i.e. those corresponding to the homogeneity condition and to the invariance under the n transformations $h_i, X^{\underline{M}}{}_+$. These $n+1$

linear equations have a simultaneous solution of the form $S_0 +$ const. S', where S_0 is a particular solution and S' is the solution of the corresponding homogeneous system, which is *unique* up to a constant factor. Therefore, the linear subsystem already determines S up to a constant (only the phase of this constant is physically relevant). Requiring invariance under *anyone* of the remaining n duality transformations fixes the constant. Then the remaining $n-1$ equations should hold automatically as a consequence of the first $n+2$, and in fact they do. This is a non-trivial check of the correctness of our identifications and of the whole philosophy underlying our classification of the homogeneous manifolds allowed in $N=2$ SUGRA.

From Eqs. (2.29, 2.33a, b, c) of Sect. 2.B we see that the linear $n+1$ equations are (we denote the various fields X^I with $q_0, q_1, q_2, q_3, X^\mu{}_1, X^m{}_2$ and $X^M{}_3$, for consistency with the notations of ref. [15])

$$h_1 : q_0 \frac{\partial S}{\partial q_0} + q_1 \frac{\partial S}{\partial q_1} - q_2 \frac{\partial S}{\partial q_2} - q_3 \frac{\partial S}{\partial q_3} + X_1^\mu \frac{\partial S}{\partial X_1^\mu} = 0, \tag{3.6a}$$

$$h_2 : q_0 \frac{\partial S}{\partial q_0} - q_1 \frac{\partial S}{\partial q_1} + q_2 \frac{\partial S}{\partial q_2} - q_3 \frac{\partial S}{\partial q_3} + X_2^m \frac{\partial S}{\partial X_2^m} = 0, \tag{3.6b}$$

$$h_3 : q_0 \frac{\partial S}{\partial q_0} - q_1 \frac{\partial S}{\partial q_1} - q_2 \frac{\partial S}{\partial q_2} + q_3 \frac{\partial S}{\partial q_3} + X_3^M \frac{\partial S}{\partial X_3^M} = 0, \tag{3.6c}$$

$$\text{homogeneity: } \sum_{I=0}^{3} q_I \frac{\partial S}{\partial q_I} + \sum_{A=I}^{3} X_A^{\underline{m}} \frac{\partial S}{\partial X_A^{\underline{m}}} = 2S, \tag{3.6d}$$

$$X_1^\mu{}_+ : q_0 \frac{\partial S}{\partial X_1^\mu} - X_1^\mu \frac{\partial S}{\partial q_1} = \frac{1}{\sqrt{2}} (\gamma^\mu)^{m\dot{n}} X_2^m X_3^{\dot{n}}, \tag{3.6e}$$

$$X_2^m{}_+ : q_0 \frac{\partial S}{\partial X_2^m} - X_2^m \frac{\partial S}{\partial q_2} = \frac{1}{\sqrt{2}} (\gamma^\mu)^{m\dot{n}} X_2^\mu X_3^{\dot{n}}, \tag{3.6f}$$

$$X_3^{\dot{n}}{}_+ : q_0 \frac{\partial S}{\partial X_3^{\dot{n}}} - X_3^{\dot{n}} \frac{\partial S}{\partial q_3} = \frac{1}{\sqrt{2}} (\gamma^\mu)^{m\dot{n}} X_1^\mu X_2^m. \tag{3.6g}$$

The first four of the Eq. (3.6) can be rewritten as

$$2q_1 \frac{\partial S}{\partial q_1} + X_1^\mu \frac{\partial S}{\partial X_1^\mu} = S, \tag{3.7a}$$

$$2q_2 \frac{\partial S}{\partial q_2} + X_2^m \frac{\partial S}{\partial X_2^m} = S, \tag{3.7b}$$

$$2q_3 \frac{\partial S}{\partial q_3} + X_3^{\dot{n}} \frac{\partial S}{\partial X_3^{\dot{n}}} = S, \tag{3.7c}$$

$$q_0 \frac{\partial S}{\partial q_0} = \sum_{A=1}^{3} q_A \frac{\partial S}{\partial q_A} - S. \tag{3.7d}$$

Solving Eqs. (3.6e, f, g) for $\Sigma_{\underline{m}} X^{\underline{m}}{}_A (\partial S / \partial X^{\underline{m}}{}_A)$ and inserting the result in Eqs. (3.7) we get the equations

$$[2q_0q_A+(X_A)^2]\frac{\partial S}{\partial q_A}=q_0S-(X_1,X_2,X_3) \quad (A=1,2,3), \tag{3.8}$$

where

$$(X_1,X_2,X_3)=\frac{1}{\sqrt{2}}(\gamma^\mu)^{m\dot{n}}X_1^\mu X_2^m X_3^{\dot{n}}. \tag{3.9}$$

The general solution to Eqs. (3.8) is

$$S=\frac{(X_1,X_2,X_3)}{q_0}+h(q_0,X_1,X_2,X_3)\prod_{A=1}^{3}[2q_0q_A+(X_A)^2]^{1/2}. \tag{3.10}$$

Using Eq. (3.7d), we get for the function h

$$h(q_0,X_1,X_2,X_3)=\frac{1}{q_0}H(X_1,X_2,X_3). \tag{3.11}$$

This solution for S, Eqs. (3.10, 11) can be substituted back in Eqs. (3.6e, f, g). All these equations are solved simultaneously if and only if $H=C=\text{const.}$,

$$S=\frac{(X_1,X_2,X_3)}{q_0}+C\frac{1}{q_0}\prod_{A=1}^{3}[2q_0q_A+(X_A)^2]^{1/2}. \tag{3.12}$$

The value of the constant C is fixed by requiring invariance with respect to the remaining n generators g_i, X^M_-, which imply the following six equations (compare with Eqs. (2.31, 2.33d, e, f) of Sect. 2.B).

$$g_1: \frac{\partial S}{\partial q_2}\frac{\partial S}{\partial q_3}=q_0q_1+\frac{1}{2}(X_1)^2, \tag{3.13a}$$

$$g_2: \frac{\partial S}{\partial q_1}\frac{\partial S}{\partial q_3}=q_0q_2+\frac{1}{2}(X_2)^2, \tag{3.13b}$$

$$g_3: \frac{\partial S}{\partial q_1}\frac{\partial S}{\partial q_2}=q_0q_3+\frac{1}{2}(X_3)^2, \tag{3.13c}$$

$$X_{1-}^\mu: \frac{\partial S}{\partial q_3}\frac{\partial S}{\partial X_1^\mu}-\frac{1}{\sqrt{2}}\frac{\partial S}{\partial X_3^{\dot{n}}}(\gamma^\mu)^{m\dot{n}}X_2^m-q_2X_1^\mu=0, \tag{3.13d}$$

$$X_{2-}^m: \frac{\partial S}{\partial q_3}\frac{\partial S}{\partial X_2^m}-\frac{1}{\sqrt{2}}\frac{\partial S}{\partial X_3^{\dot{n}}}(\gamma^\mu)^{m\dot{n}}X_1^\mu-q_1X_2^m=0, \tag{3.13e}$$

$$X_{3-}^{\dot{n}}: \frac{\partial S}{\partial q_2}\frac{\partial S}{\partial X_3^{\dot{n}}}-\frac{1}{\sqrt{2}}\frac{\partial S}{\partial X_2^m}(\gamma^\mu)^{m\dot{n}}X_1^\mu-q_1X_3^{\dot{n}}=0. \tag{3.13f}$$

Of course (since the integrability conditions of Sect. 2.A are fulfilled) we get the same value for all the equations, namely $C=\pm 1/\sqrt{2}$. The sign ambiguity is physically irrelevant, since it can be absorbed by a field redefinition.

It should be stressed that S is a solution to Eqs. (3.13) if and only if: i) $X_-=0$, or ii) the Dirac algebra, Eq. (3.4), holds. This is not surprising at all, since these are precisely the solutions to the integrability conditions of Sect. 2.A, as it follows from the discussion after Eq. (9.11) of ref. [15].

Therefore, in the canonical parametrization, the function S for the homogeneous spaces $K(p,q)$ reads $(m=1,\dots,p; M=1,\dots,q)$

$$S=\{q_1[2q_0q_2+(X_2)^2][2q_0q_3+(X_3)^2]/q_0\}^{1/2} \tag{3.14}$$

and for the spaces $H(p,q)(\mu=1,\dots,q, \dim X_{2-}=\dim X_{3-}=pN(q))$,

$$S=\frac{(X_1,X_2,X_3)}{q_0}+\frac{1}{\sqrt{2q_0}}\prod_{A=1}^{3}[2q_0q_A+(X_A)^2]^{1/2}. \tag{3.15}$$

This completes the proof that there exists a homogeneous holomorphic function $F(X^I)=-2iS(X^I)$ such that the corresponding Kähler metric, Eq. (2.2) describes the homogeneous Kähler manifolds associated to the rank 4 Alekseevskii spaces.

The full $N=2$ supergravity Lagrangian can be obtained by inserting this function $F(X^I)$ into the general $N=2$ supergravity Lagrangian of ref. [9].

From the above expressions it is very easy to see that in the four magical cases, Eqs. (2.37), we have actually an isometry algebra larger than W. In fact, in these cases all the three vector spaces X_{1-}, X_{2-} and X_{3-} are isomorphic (as vector spaces) to one of the four division algebras $\mathbb{A}$. The corresponding isometric mapping ψ is simply the product $\mathbb{A}\otimes\mathbb{A}\to\mathbb{A}$, which is an isometric map for the metric on $\mathbb{A}$ given by the norm. In this case the function S is completely symmetric in the three spaces X_{1-}, X_{2-} and X_{3-}. Therefore, the theory will be invariant not only with respect to the $2n$ symmetries above, but also with respect to those whose generators are obtained from the above ones by arbitrary permutations of the three spaces X_{1-}, X_{2-} and X_{3-}. A similar permutation argument shows the enhanced symmetry for the spaces $K(0,q)$ which are symmetric (and reducible), Eq. (2.35).

3.B. $F(X^I)$ in the Stringy ($\equiv$ Cubic) Parametrization. In ref. [3] (see also ref. [4]) it was shown that in type IIA superstrings we can always find a field parametrization such that the function $F(X^I)$ takes the form in Eq. (1.2). Couplings of this form have many interesting properties, for instance, in gauged $N=2$ supergravity they lead to identically vanishing scalar potential [17].

In the introduction we claimed that all the homogeneous Kähler manifolds allowed in $N=2$ SUGRA (other than hyperbolic spaces) have this property. Here we want to show this and also give the explicit form of the coefficients d_{ABC} for each homogeneous manifold.

From our previous discussion, we know that changing the parametrization of the function F amounts to changing the canonical basis in phase-space. The canonical transformation putting Eq. (3.15) into the cubic form is just the replacement of the "Lagrange coordinates" q_1, q_2, q_3, by the corresponding "conjugate momenta" p_1, p_2, p_3, which -on $\mathscr{F}[S]$- equal

$$p_A=\frac{\partial S}{\partial q_A} \quad (A=1,2,3). \tag{3.16}$$

The new S can be found exploiting the canonical transformation $q_i\to p_i$, (recall that S is *not* invariant under canonical transformations, but it can be determined by requiring that the new "momenta" are the gradient of the new S with respect the new coordinates). The result is

$$S = \frac{(X_1, X_2, X_3)}{q_0} - \frac{1}{2q_0}\left\{2p_1p_2p_3 - \sum_{A=1}^{3} p_A(X_A)^2\right\}, \tag{3.17}$$

where in the case of the $K(p,q)$ spaces, we understand $X_1 \equiv 0$. The function F is

$$F = \frac{i}{q_0}\left\{2p_1p_2p_3 - \sum_{A=1}^{3} p_A(X_A)^2 - 2(X_1, X_2, X_3)\right\}. \tag{3.18}$$

It is manifest that Eq. (3.18) correctly reproduces the known results for the factorizable symmetric models of Eqs. (2.35) as well as for the *magical* symmetric models, Eqs. (2.37). We close this subsection by recalling some useful formulae which hold for any model with a cubic F [17].

$$N_{AB} = \tfrac{3}{2} d_{ABC} y^C, \quad Y = \tfrac{1}{4} d_{ABC} y^A y^B y^C, \tag{3.19a}$$

$$(Nz)_A = \tfrac{3}{4} i d_{ABC} y^B y^C, \tag{3.19b}$$

$$y^A \equiv i(z^A - \bar{z}^A). \tag{3.19c}$$

These formulae will simplify the computations of the next subsection.

3.C. Homogeneous Kähler–Einstein Spaces. There is another problem in the classification of the homogeneous Kähler spaces allowed in $N = 2$ supergravity that we want to discuss, namely, we ask ourselves what spaces in the above list are also Kähler–Einstein manifolds. This question may be relevant for string theory too. It can be shown [20, 21] that all the homogeneous Kähler manifolds allowed in $N = 2$ SUGRA are holomorphically equivalent to a bounded domain in $\mathbb{C}^n$ (in fact, they are equivalent to Siegel domains of type I). As is well known [23], all homogeneous bounded domains admit at least one Kähler–Einstein metric, i.e. the Bergmann metric. In all the exceptional spaces which are symmetric, our metric above is the Bergmann one (within a positive factor). So, at first sight, one could hope that the above homogeneous spaces are Kähler–Einstein. *But it is not so.*

It turns out that a normal homogeneous Kähler space allowed in $N = 2$ SUGRA is Kähler–Einstein if and only if it is *symmetric*. Therefore only the spaces which were already in the Cremmer-Van Proeyen list are both homogeneous and Kähler–Einstein.

It should be stressed that strictly speaking this conclusion is only proved for *normal* homogeneous Kähler spaces. However, as we mentioned in the introduction, there is a conjecture by Alekseevskii stating that all homogeneous quaternionic manifolds with negative definite Ricci curvature are normal. If this conjecture holds true, then the Alekseevskii spaces exhaust all such homogeneous quaternionic manifolds, and hence -by the c-map isomorphism- our list of the homogeneous Kähler spaces allowed in $N = 2$ SUGRA is also complete and we can omit the word *normal* in the above statement.

We prove this assertion just by computing the Ricci tensor. Obviously, the Kähler metric in Eq. (2.2) is Einstein if and only if

$$\mathrm{DET}\{G_{\alpha\bar{\beta}}\} = |h|^2 Y^{\lambda}, \tag{3.20}$$

where h is some holomorphic function.

The computation of the determinant of the metric can be simplified by exploiting the fact that all models (with the exception of the minimal coupling case) have a cubic function F. Let, as always, $G = -\ln Y$ be the Kähler potential. Since all the gauged supergravity models corresponding to cubic F have an identically vanishing scalar potential,

$$V \equiv e^{-G}(G^{\alpha\bar{\beta}} G_\alpha G_{\bar{\beta}} - 3) \equiv 0, \tag{3.21}$$

we should have

$$G^{\alpha\bar{\beta}} G_\alpha G_{\bar{\beta}} \equiv 3. \tag{3.22}$$

On the other hand,

$$G_{\alpha\bar{\beta}} = -Y^{-1} Y_{\alpha\bar{\beta}} + Y^{-2} Y_\alpha Y_{\bar{\beta}}. \tag{3.23}$$

Equations (3.22, 23) imply

$$\mathrm{DET}\{-Y^{-1} Y_{\alpha\bar{\beta}}\} = -2\,\mathrm{DET}\{G_{\alpha\bar{\beta}}\}. \tag{3.24}$$

Then, a space described by a cubic F is Einstein if and only if $\mathrm{Det}\{-Y_{AB}\}$ is proportional to some power of the function Y. Notice that in the cubic case both Y and $\mathrm{Det}\{-Y_{AB}\}$ are polynomials in the variables $y^A = -2\,\mathrm{Im}(z^A)$, and then the holomorphic function h should be a constant.

Consider the spaces $K(p,q)$. A simple computation (using Eq. (3.19c)) gives

$$\mathrm{DET}\{-Y_{AB}\} = -2[2\,\mathrm{Im}(p_2)]^p[2\,\mathrm{Im}(p_3)]^q Y. \tag{3.25}$$

Equation (3.25) implies that the spaces $K(p,q)$ are not Kähler–Einstein. Consider the special case $p=0$ which corresponds to the symmetric manifolds of Eq. (2.35). In this case the space is the direct product of two Kähler manifolds, and also Y is factorized $Y = \mathrm{Im}(p_3) Y'$, where $Y' = \{\mathrm{Im}(p_1)\mathrm{Im}(p_2) - (\mathrm{Im}\, X_3)^2\}$ and p_3 is the complex coordinate on the coset $SU(1,1)/U(1)$. Then $\mathrm{Det}[G] = (\mathrm{Im}\, p_3)^{-2}\{Y'\}^{-(2+q)}$, and the Kähler manifold is the product of two Einstein spaces but with different "cosmological constants" -2 and $-(2+q) = (n-1)$ {compare with ref. [7] and Appendix A of ref. [12]}. For a generic space $K(p,q)$, with $p, q \neq 0$, this factorization does not hold and the metric is not Einstein.

For the spaces $H(p,q)$ the direct computation of $\mathrm{Det}\{-Y_{AB}\}$ is more involved. However, for our purposes the full computation is not needed. Let us assume (absurd) that $\mathrm{Det}\{-Y_{AB}\}$ is indeed proportional to some power of Y. Since at $X_i = 0$ $(i = 1,2,3)$ Y reduces to $\mathrm{Im}\, p_1\ \mathrm{Im}\, p_2\ \mathrm{Im}\, p_3$, we should have that $\mathrm{Det}\{-Y_{AB}\}|_{Xi=0}$ is a symmetric function of $\mathrm{Im}\, p_i$. An explicit computation gives

$$\mathrm{DET}\{-Y_{AB}\}|_{X=0} = 2[-2\,\mathrm{Im}(p_3)]^{\dim X_1}[4\,\mathrm{Im}(p_2)\,\mathrm{Im}(p_3)]^{\dim X_2}. \tag{3.26}$$

Therefore, a $H(p,q)$ space can be Einstein only if $\dim X_1 = \dim X_2$. Looking at the dimensionality of the Clifford modules we see that there are only four solutions to this condition, i.e. both spaces X_1 and X_2 should be isomorphic to one of the four division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. The corresponding spaces are the symmetric magical spaces of Eq. (2.37). These magical manifolds are known to be Kähler–Einstein. By the above argument, no other $H(p,q)$ spaces can be Einstein.

These results will be recovered in Sect. 4 using more sophisticated techniques (T-algebras). There we shall explain the relation between the W-invariant Kähler metric of Eq. (2.2) and Bergmann's one, as well as other geometrical properties of the spaces $K(p,q)$ and $H(p,q)$.

4. Geometry of the $K(p,q)$ & $H(p,q)$ Spaces, Type I Siegel Domains and T-Algebras

In this section we study in detail the geometry of the homogeneous Kähler spaces $K(p,q)$ and $H(p,q)$. The main result of this section is that all the above manifolds allowed in $N=2$ SUGRA (except -of course- $\mathbb{C}H^n$) are (homogeneous) Siegel domains of rank 3 [20, 21]. More precisely, the spaces $K(p,q)$ and $H(p,q)$ are exactly those rank 3 Siegel domains whose associated isometric mapping is *special* (or degenerate).

This geometrical characterization is so important that we give three proofs of it. The last two are completely rigorous mathematical proofs based on the characterization of the (homogeneous) Siegel domains of the first type in terms of j-algebras and T-algebras, respectively.

The first argument is based on *physical* considerations. We present it in order to explain the physical meaning of this geometrical property.

In the physical literature one usually considers homogeneous manifolds of the form G/K, where $G=H\otimes U(1)^n$ with H semisimple. The generic homogeneous space, however, is not of this form. In particular, the homogeneous spaces we have found in Sect. 3 are *not* of this form. Indeed, it is easy to show that all the homogeneous Kähler manifolds allowed in $N=2$ SUGRA should be biholomorphically equivalent to bounded domains. A theorem by Borel [22] states that if a bounded domain has a semisimple group of automorphisms, the domain is *symmetric*. This is not the case for the spaces $K(p,q)$ and $H(p,q)$.

4.A. Stringy Considerations. To make a long story short, let us assume that the Gepner conjecture [2] holds true. Then, roughly speaking, the moduli of the $(2,2)$ $c=9$ superconformal theory should correspond to the Calabi data [28],

$$\{K, J, [\omega], [\tau]\} \tag{4.1}$$

for a Calabi–Yau [29] 3-fold. Here K is a complex 3-fold with complex structure J such that the canonical bundle is trivial, $[\omega]\in H^{1,1}(K)$ ($[\omega]$ real) is the Kähler class and $[\tau]\in H^2(K)$ is the "torsion" class.

Obviously, for a generic class, $[\omega]\in H^{1,1}(K)$, there is no representative which is the Kähler form of a positive-definite Kähler metric. Let $V\in H^{1,1}(K)$ be the subspace of the classes $[\omega]$ which are the Kähler class of some regular Kähler-metric on K (with respect the specified complex structure J). We call V the *Kähler cone.*
V has a number of properties:

o) V is open in $\mathbb{R}^n$ ($n=h_{1,1}$; hence forth $H^{1,1}(K)\cap H^2(K,\mathbb{R})$ is identified with $\mathbb{R}^n$).
i) V is a cone: if $y\in V$ then $\lambda y\in V$ for all $\lambda>0$.
ii) V is convex: if $y_1, y_2\in V$ then $y_1+y_2\in V$ (in fact y_1+y_2 is the Kähler class of the sum of the two metrics).
iii) If $y\in V$ then $\int y^3>0$ (positivity of the volume).
iv) V does not contain any complete line (not necessarily passing through the

origin). This is essentially the "archimedean" property of Sect. 18 of ref. [30]). Indeed, assume (absurd) that $y_t \equiv (ty_1 + y_2) \in V$ for all $t \in \mathbb{R}$. Letting $t \to +\infty \Rightarrow y_1 \in V$. Then $\int y_1{}^3 > 0$. Therefore $A(t) \equiv \int y_t{}^3$ is a cubic polynomial in t with a non-vanishing coefficient of t^3. But then $A(t)$ should be negative for some real values of t. Therefore using iii) we get the absurd.

v) (Riemann-Hodge quadratic relations). For a given $y \in H^{1,1}$, define the effective (or primitive) cohomology classes $w_a \in H^{1,1}$ by the property

$$\int_K y \wedge y \wedge w_A = 0.$$

Then, if $y \in V$ the quadratic form

$$Q_{AB} = \int_K y \wedge w_A \wedge w_B \tag{4.2}$$

is a non-degenerate negative definite symmetric matrix [31].

Consider the complex $(1, 1)$ class $z = x + iy$, where y is identified with the Kähler class and x is the type $(1, 1)$ part of the torsion class $[\tau]$. By the above considerations, in order for the corresponding Calabi data to correspond to some regular Kähler–Einstein metric, we should have

$$z \in \mathbb{R}^n + iV. \tag{4.3}$$

In particular, V in Eq. (4.3) has the properties o), ..., v).

The (in general, non-homogeneous) Siegel domains of the first type are defined to be domains in $\mathbb{C}^n$ of the form (4.3) where $V \in \mathbb{R}^n$ is a cone such that the properties o), i), ii) and iv) above are fulfilled. A Siegel domain of type I is homogeneous if and only if the cone V is homogeneous (with respect a group of affine automorphisms).

Let us expand the (complex) class z in a (real) basis ω_A of $H^{1,1}(K) \cap H^2(K, \mathbb{R})$,

$$z = z^A \omega_A.$$

If we compactify the heterotic string on the Calabi–Yau space specified by the above Calabi data, the z^A are promoted to chiral multiplets of the effective low-energy $N = 1$ $4D$ supergravity. These chiral multiplets should parametrize some Kähler space (in fact, a Hodge manifold [32]). The complex manifold underlying this Kählerian σ-model should be the space of the allowed values for z^A, that is $\mathbb{R}^n + iV$.

Thus the Kähler (geodesic) submanifold of the heterotic low-energy effective theory, which is parametrized by the moduli corresponding to the deformations of the Calabi data $[\omega]$ and $[\tau]$, *is a Siegel domain of the first kind.*

The same is true for a type IIA superstring, since the moduli space (and metric) is independent of the particular string one compactifies.

Therefore, a *necessary* condition for a $N = 2$ supergravity model to be a candidate for being the low-energy limit of a type IIA string is that the corresponding Kähler space *is a Siegel domain of the first kind.*

Then, saying that the models we constructed in Sect. 3 are (homogeneous) Siegel domains (of the first type) is just the same as saying that they may be relevant to string theory.

4.B. Complex Homogeneous Bounded Domains and j-Algebras. In this subsection we consider the geometry of the spaces $K(p,q)$ and $H(p,q)$ from the point of view of the classification theory for the homogeneous bounded domains developed by Gindikin, Pjateckii-Sapiro and Vinberg [20, 21]. Again, the aim is to prove that these spaces are biholomorphically equivalent to Siegel domains of type I (alias radiated tube domains). The merit of this approach is that it makes manifest the connection between duality invariance and the geometry of bounded domains.

First of all, it is quite obvious that the spaces $K(p,q)$ and $H(p,q)$ are biholomorphically equivalent to some bounded domain of $\mathbb{C}^n$. To check this, it is sufficient to use the criterion of boundeness given in ref. [21]. Four our spaces, the j-algebra is just the (normal) Kähler algebra W. This algebra, being totally solvable, does not contain any compact semisimple j-subalgebra. Therefore the spaces $K(p,q)$ and $H(p,q)$ are bounded domains.

(However, it is convenient to use non-bounded models for these bounded domains. These non-bounded models are just the Siegel spaces in their usual form $\mathbb{R}^n + iV$).

The general theorem of refs. [20, 21] states that the Kähler manifold generated by exponentiating a j-algebra is either a Siegel domain of type 2 or of type 1 (which is, in fact, a special instance of the type 2 case).

More precisely, their result can be stated as follows. $\mathscr{W} = \exp W$ is biholomorphic to a Siegel domain of the second kind if and only if in W there exists an element H such that

i) Ad_H is semisimple;
ii) The spectrum of Ad_H consists of $\lambda = 1, 0, 1/2$.
iii) $[H,x] = \lambda x \Rightarrow [H,Jx] = (1-\lambda)x$ for all $x \in W$.

Moreover, $\mathscr{W}$ is a Siegel domain of the first kind if and only if the value $1/2$ is not present in the spectrum of Ad_H.

The element $H \in W$ can be constructed from the j-algebra W by the following procedure. Consider the skew-symmetric form on $W \langle J\ldots,\ldots\rangle$. It is just the Kähler form on $T\mathscr{W}$. Then it should be cohomologous to zero on W. Therefore, there is a one-form on W, $\omega(\cdots)$, such that

$$\langle Jx, y\rangle = d\omega(x,y) = -\omega([x,y]) \quad \forall x, y \in W. \tag{4.4}$$

So, the *non-degenerate* inner product $\langle \ldots,\ldots\rangle$ has the form

$$\langle x, y\rangle = \omega([Jx,y]) \quad \forall x, y \in W. \tag{4.5}$$

Any one-form on W can be expressed as $\langle u,\ldots\rangle$ for a unique $u \in W$. Then there exists a *unique* $u \in W$ such that

$$\omega(x) = \langle u, x\rangle \quad \forall x \in W. \tag{4.6}$$

It can be shown [19, 20, 21] that the element u so defined has the properties i), ii) and iii) above and so it can be identified with H. (By the way, this shows that all homogeneous bounded domains are Siegel domains.)

In the case of the normal Kähler algebras we discussed in Sect. 2 (for rank 3 and type 1), we have

$$H = h_1 + h_2 + h_3. \tag{4.7}$$

Using Eq. (2.26) and the relations defining an elementary Kähler algebra it is easy to see that

$$[H, h_\alpha] = 0, \tag{4.8a}$$

$$[H, g_\alpha] = g_\alpha \quad \alpha = 1, 2, 3, \tag{4.8b}$$

$$[H, x^m_{A+}] = x^m_{A+}, \tag{4.8c}$$

$$[H, x^m_{A-}] = 0 \quad A = 1, 2, 3. \tag{4.8d}$$

Given that $Jh_i = g_i$, $Jx^m{}_{A+} = x^m{}_{A-}$, we see that H satisfies the conditions i), ii) and iii). Moreover, from Eq. (4.8) it is manifest that 1/2 is not present in the spectrum of Ad_H.

This completes the proof that the spaces $K(p, q)$ and $H(p, q)$ are homogeneous Siegel domains of type 1.

The above identification for H, Eq. (4.7) can also be obtained from the form ω. In each algebra W of Sect. 2, there is a *canonical linear form* ω with the property (see Corollary 6.2 of ref. [15])

$$\omega(g_\alpha) = -1, \tag{4.9a}$$

$$\omega(u) = 0 \quad \text{if} \quad u \neq g_\alpha \quad \alpha = 1, 2, 3, \tag{4.9b}$$

which, together with Eq. (2.22) implies Eq. (4.7).

Let us take a closer look at the structure of the Kähler algebra W. In agreement with the general theorems of refs. [20, 21], it should be of the form

$$W = JR + R \tag{4.10}$$

with R a *commutative ideal* of W. In fact from Eqs. (2.26) we see that R is generated by g_i and $X^m{}_{A+}$.

The automorphisms of the Siegel space $\mathbb{R}^n + iV$ are of the following two forms:

(A) $x + iy \to (x + a) + iy \quad (a \in \mathbb{R}^n)$,

(B) $x + iy \to Ax + iAy$,

where A is an (affine) automorphism of V. The automorphisms of the type (A) are exactly the "Peccei-Quinn" symmetries of ref. [3]. *From the algebra W we see that these Peccei-Quinn symmetries are just generated by the commutative ideal R.* (Notice that the existence of this ideal R—the PQ symmetry—is enough to deduce that the spaces *if homogeneous* should be Siegel type I. This is the argument we used in ref. [3]).

The automorphisms of the convex cone V can be described as follows in terms of the j-algebra W. Let y be a point in R. We define an infinitesimal affine transformation of R by

$$C_h y = [h, y] \quad h \in JR. \tag{4.11}$$

Using the axioms of a (normal) j-algebra it is easy to check that the transformations C_h, $h \in JR$, form an affine Lie algebra L. Consider the orbit of the point y under the corresponding Lie group. It is a convex cone (not containing lines) which is canonically identified with V [20, 21]. L is obviously the Lie algebra of the automorphisms of V.

The j-algebra of a Siegel domain of the first type has a natural normal decomposition (the semidirect sum decomposition of Sect. 2.B). We discuss it here since we shall need it in Sect. 4.C to construct the corresponding T-algebras.

The general theorem [21] states that there exist elements $r_i \in R$ $(i = 1, \ldots, m)$ such that the ideal R can be decomposed into a direct sum of spaces

$$R = \sum_{i \leqq j} R_{ij} \tag{4.12}$$

such that the following formulae hold [21]:

$$R_{ii} = (r_i), \tag{4.13a}$$

$$x \in R_{jk} \Rightarrow [Jr_i, x] = \tfrac{1}{2}(\delta_{ij} + \delta_{ik})x, \tag{4.13b}$$

$$x \in R_{jk} \Rightarrow [Jr_i, Jx] = \tfrac{1}{2}(\delta_{ij} - \delta_{ik})Jx, \tag{4.13c}$$

$$H = J \sum_i r_i. \tag{4.13d}$$

Comparing these equations with Eq. (2.22, 4.8) we get in the case of our algebras W, that $r_i = -g_i$ $(i = 1, 2, 3)$, and

$$R_{12} = X_{3+}, \tag{4.11a}$$

$$R_{13} = X_{2+}, \tag{4.14b}$$

$$R_{23} = X_{1+}. \tag{4.14c}$$

4.C. T-Algebras and the Geometry of the Spaces $K(p,q)$ & $H(p,q)$. In the previous subsection we proved that the homogeneous spaces we coupled to $N = 2$ SUGRA in Sect. 3 are (homogeneous) Siegel domains of the first kind. However, not all such Siegel domains can be coupled to $N = 2$ supergravity. Then, the natural problem is to give a geometrical characterization of the Siegel domains that *can* be coupled to $N = 2$ sugra.

In order to solve this problem we have to introduce the formalism of the T-algebras [14], which is also very convenient for the practical computations and will give us some extra bonus. In a certain sense, the T-algebras are a generalization of the Jordan algebras. The homogeneous models of Sect. 3 are related to the T-algebras in exactly the same way as the symmetric models are related to the Jordan algebras [12]. In this sense, this section is close in spirit to the work of Gunaydin, Sierra and Townsend [12] on the applications of the Jordan algebras to $N = 2$ SUGRA. Our construction below reduces to theirs in the particular case in which the domain is symmetric.

The T-algebras give an efficient way to describe the Bergmann metric on a homogeneous bounded domain. By "distorting" the T-algebra we can get the other homogeneous Kähler metric on the Siegel domain by the same token we used to construct the Bergmann one. In particular we get a simple and elegant interpretation of the Kähler metric relevant for $N = 2$ supergravity, Eq. (2.2).

4.C.1. T-Algebras and N-Algebras. From the considerations of Sect. 4.A.2 it is easy to compute the cones V for the *symmetric* case (see also ref. [12]). For the magical models, V is the cone of *positive-definite* Hermitian 3×3 matrices whose entries belong to one of the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. The "factorizable" models, Eq. (2.31), instead correspond to the spherical cones[4]

$$x^0 > \sqrt{(x^1)^2 + \cdots + (x^n)^2}. \tag{4.15}$$

Here we want to show that this interpretation of the cone V can be extended also to the homogeneous case. The (convex) cone V, defined by the homogeneous spaces we have found above, is the space of the, in some sense positive, 3×3 Hermitian matrices whose entries belong (quite roughly) to the Clifford modules of Sect. 2.A. Such matrix algebras are examples of a more general concept introduced by E. B. Vinberg [14]: that of T-algebras.

We begin by recalling some definitions. (The convention on the sum over repeated indices is NOT used in this and the following section). A *matrix algebra* of rank m is an algebra $\mathfrak{A}$, bigraded by subspaces $\mathfrak{A}_{ij}$ $(i, j = 1, \ldots, m)$ such that

$$\mathfrak{A}_{ij}\mathfrak{A}_{jk} \subset \mathfrak{A}_{ik}, \tag{4.16a}$$

$$\mathfrak{A}_{ij}\mathfrak{A}_{lk} = 0 \quad \text{for} \quad j \neq l. \tag{4.16b}$$

An element $a \in \mathfrak{A}$ can be represented as a matrix (a_{ij}), where a_{ij} is the projection of a into $\mathfrak{A}_{ij}$. Then the algebra product is written as the usual product for matrices. An involution * of $\mathfrak{A}$ is a linear mapping $\mathfrak{A} \to \mathfrak{A}$ such that

$$a^{**} = a \quad (ab)^* = b^* a^*, \tag{4.17a}$$

$$\mathfrak{A}^*_{ij} \subset \mathfrak{A}_{ji}. \tag{4.17b}$$

In the matrix notation this is the "Hermitian conjugate." We denote by $\mathfrak{C}$ the subalgebra of $\mathfrak{A}$ consisting of upper triangular matrices, and by $\mathfrak{X}$ the subspace of "Hermitian" matrices

$$\mathfrak{X} = \{x \in \mathfrak{A} \,|\, x^* = x\}. \tag{4.18}$$

A matrix algebra $\mathfrak{A}$ with an involution * is called a T-algebra if

1. all the subalgebras $\mathfrak{A}_{ii}$ are isomorphic to $\mathbb{R}$ (the corresponding unit element is denoted as e_i);
2. for any $a_{ij} \in \mathfrak{A}_{ij}$: $e_i a_{ij} = a_{ij} e_j = a_{ij}$;
3. There is an operation (trace) $\mathrm{Tr}: \mathfrak{A} \to \mathbb{R}$ such that
3.a $\mathrm{Tr}([a, b]) = 0$;
3.b $\mathrm{Tr}([a, b, c]) = 0$;
3.c $\mathrm{Tr}(aa^*) > 0$ if $a \neq 0$;
4. For any $t, u, w \in \mathfrak{C}$, $[t, u, w] = 0$;
5. For any $t, u \in \mathfrak{C}$, $[t, u, u^*] = 0$.

Here $[a, b, c]$ is the associator, $[a, b, c] = a(bc) - (ab)c$. In the matrix notation, the

[4] Notice that the symmetric Kähler spaces allowed in $N = 2$ SUGRA just correspond to the *classical* cones, (i.e. the homogeneous convex cones known *before* the 1960's), see ref. [14]

trace is defined as

$$\operatorname{Tr} a = \sum_{i=1}^{m} n_i a_{ii}, \tag{4.19}$$

where the n_i are positive numbers. If

$$n_i = 1 + \frac{1}{2} \sum_{s \neq i} n_{is}, \tag{4.20a}$$

$$n_{ij} = \dim \mathfrak{A}_{ij} \tag{4.20b}$$

we have

$$\operatorname{Tr} 1 = \dim \mathfrak{A}, \tag{4.21}$$

and we call $\mathfrak{A}$ a *natural* T-algebra. This trace will be called the graded trace and denoted by Tr_G. If Eq. (4.20a) does not hold, we shall speak of a *distorted* T-algebra. This process of distorting the T-algebra is analogous to defining a different metric on a coset by rescaling the generators.

Let $a = (a_{ij}) \in \mathfrak{A}$. Then we define

$$\hat{a} = \frac{1}{2} \sum a_{ii} + \sum_{i<j} a_{ij}, \tag{4.22a}$$

$$\underset{\smile}{a} = \frac{1}{2} \sum a_{ii} + \sum_{i>j} a_{ij}. \tag{4.22b}$$

$\hat{a}$ is an upper triangular matrix, $\underset{\smile}{a}$ is lower triangular, and $a = \hat{a} + \underset{\smile}{a}$. In $\mathfrak{A}$ one defines a product $\blacklozenge$

$$a \blacklozenge b = \hat{a} b + b \underset{\smile}{a}. \tag{4.23}$$

Under the $\blacklozenge$-product the space $\mathfrak{X}$ is closed.

Consider now the space $\mathscr{T}(\mathfrak{A})$

$$\mathscr{T}(\mathfrak{A}) = \{ t \in \mathfrak{C} \mid t_{ii} > 0 \, (i = 1, \ldots, m) \}. \tag{4.24}$$

$\mathscr{T}(\mathfrak{A})$ is obviously a connected Lie group. Its Lie algebra $T(\mathfrak{A})$ is just the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$. Let us consider the map $\mathscr{T}(\mathfrak{A}) \to \mathfrak{X}$ given by

$$t \to t t^* \in \mathfrak{X}. \tag{4.25}$$

The Hermitian matrices of the form tt^* are, in some sense, positive definite. Let

$$V(\mathfrak{A}) = \{ t t^* \mid t \in \mathscr{T}(\mathfrak{A}) \} \tag{4.26}$$

be the cone in $\mathfrak{X}$ of such "positive definite" Hermitian matrices. Each element of $V(\mathfrak{A})$ can be written in a unique way in the form tt^* with $t \in \mathscr{T}(\mathfrak{A})$. Therefore, $V(\mathfrak{A})$ is a one-to-one image of the Lie group $\mathscr{T}(\mathfrak{A})$. This group acts on the cone $V(\mathfrak{A})$ by

$$\pi(\omega) \colon u u^* \to (\omega u)(u^* \omega^*). \tag{4.27}$$

In fact, $\pi(\omega)$ is just a left translation on the group $\mathscr{T}(\mathfrak{A})$. Then $\mathscr{T}(\mathfrak{A})$ acts transitively on the cone $V(\mathfrak{A})$, which can be shown to have all the properties of the cone V for a Siegel domain of the first kind. Equation (4.27) implies the following action of $T(\mathfrak{A})$

on $\mathfrak{X}$

$$L_x: y \to x \blacklozenge y, \tag{4.28}$$

which is easily identified with the action of the affine algebra C_h of Eq. (4.11).

This shows that the cone $V(\mathfrak{A})$ is equal to the cone V of a homogeneous Siegel domain. Conversely any homogeneous Siegel cone is of the form $V(\mathfrak{A})$ for a unique (natural) T-algebra $\mathfrak{A}$ [14].

We want to define the "determinant" of an element of $\mathfrak{X}$. In order to do this, we associate to each Hermitian matrix $X \in \mathfrak{A}$ (rank m) a sequence of matrices $X^{(k)}$ of rank $k = 1, \ldots, m$

$$X^{(m)} = X, \tag{4.29a}$$

$$X^{(k-1)} = \sum_{i,j=1}^{k-1} (X^{(k)}_{kk} X^{(k)}_{ij} - X^{(k)}_{ik} X^{(k)}_{kj}), \tag{4.29b}$$

and put $\rho_k(X) = X^{(k)}{}_{kk}$, $k = 1, \ldots, m$.

The matrix $X \in \mathfrak{X}$ can be written in the form tt^* (i.e. it is "positive-definite") if and only if [14]

$$\rho_k(X) > 0 \quad \text{for} \quad k = 1, \ldots, m. \tag{4.30}$$

If Eq. (4.30) holds, the unique $t \in \mathscr{T}(\mathfrak{A})$ such that $X = tt^*$ is given by

$$t_{ik} = \frac{X^{(k)}_{ik}}{\sqrt{\prod_{s \geqq k} \rho_s(X)}}. \tag{4.31}$$

The *determinant* of a matrix $X \in \mathfrak{X}$ is then defined by the formula

$$\mathrm{DET}(X) \equiv \prod_{i=1}^{m} [\rho_i(X)]^{n_i - \sum_{s=1}^{i-1} n_s}, \tag{4.32}$$

where n_i are the same numbers as in the definition of the trace, Eq. (4.21). (In the case $x = tt^*$ we have $\mathrm{DET}(x) = \{\mathrm{DET}(t)\}^2$. $\mathrm{DET}(t) \equiv \exp \mathrm{Tr} \ln(t)$ is elementary since t is triangular and the diagonal elements are just real numbers).

Let $\mathfrak{A}$ be a T-algebra. Consider the corresponding (homogeneous) Siegel domain of the first type

$$\mathscr{D}(\mathfrak{A}) = \mathbb{R}^n + iV(\mathfrak{A}). \tag{4.33}$$

It is convenient to write a point in $\mathscr{D}(\mathfrak{A})$ as a matrix $\mathfrak{z}$ in $\mathfrak{A} \otimes \mathbb{C}$ such that

$$\mathfrak{z} \in \mathfrak{X} + iV(\mathfrak{A}). \tag{4.34}$$

The natural Kähler metric on such a domain is of course the Bergmann one. Exploiting the homogeneity of $V(\mathfrak{A})$ and the explicit form of its automorphism algebra, Eqs. (4.30), it is easy to see that the Kähler potential for the Bergmann metric is (within a positive factor)

$$B = -\ln \{\mathrm{DET}_G [2 \,\mathrm{Im}\, \mathfrak{z}]\}, \tag{4.35}$$

where DET_G is the "graded" determinant, i.e. in Eq. (4.32) the n_i are given by Eq. (4.20).

Our next problem is to classify what T-algebras $\mathfrak{A}$ correspond to homogeneous Kähler spaces which can be coupled to $N=2$ SUSY. It turns out that the most efficient way to construct a T-algebra is by starting from its nilpotent part, which is an N-algebra.

An associative algebra $\mathfrak{N}$, graded by subspaces $\mathfrak{N}_{ij}$ ($i<j, i,j=1,\ldots,m$), and equipped with a Euclidean metric $(\ldots,\ldots)$, is called an N-algebra (of rank m) if:

I) $\mathfrak{N}_{ij}\mathfrak{N}_{jk} \subset \mathfrak{N}_{ik}$,
II) $\mathfrak{N}_{ij}\mathfrak{N}_{lk} = 0$ for $j \neq l$,
III) $(\mathfrak{N}_{ij}, \mathfrak{N}_{kl}) = 0$ if $i \neq k$ or $j \neq l$,
IV) For any $a_{ij} \in \mathfrak{N}_{ij}$, $b_{jk} \in \mathfrak{N}_{jk}$:

$$(a_{ij}b_{jk}, a_{ij}b_{jk}) = \frac{1}{n_j}(a_{ij}, a_{ij})(b_{jk}, b_{jk}),$$

V) If $a_{ik} \in \mathfrak{N}_{ik}, b_{jk} \in \mathfrak{N}_{jk}$ $(i<j)$ and $(a_{ik}, \mathfrak{N} b_{jk}) = 0$

$\Rightarrow (\mathfrak{N} a_{ik}, \mathfrak{N} b_{jk}) = 0$.

To construct a T-algebra out of a given N-algebra, we write

$$\mathfrak{A} = \mathfrak{N}^* + \mathfrak{H} + \mathfrak{N}, \tag{4.36}$$

where $\mathfrak{N}^*$ is a vector space isomorphic to $\mathfrak{N}$, the involution * being the isomorphism. $\mathfrak{h}$ is the space of the diagonal matrices, which is just a direct sum of copies of $\mathbb{R}$. We have to extend the product from the space of strictly upper triangular matrices $\mathfrak{N}$ to the full T-algebra $\mathfrak{A}$. The product in $\mathfrak{N}^*$ is given by the rule $a^*b^* = (ba)^*$ $(a, b \in \mathfrak{N})$. The scalar product is extended in $\mathfrak{A}$ in such a way that Eq. (4.36) is an orthogonal decomposition. In $\mathfrak{h}$ we put $(e_i, e_i) = n_i$ in agreement with Eq. (4.19). The scalar product is extended to $\mathfrak{N}^*$ by $(a^*, b^*) = (a, b)$. The product of an element of $\mathfrak{h}$ with any element of $\mathfrak{A}$ is given by point 2 in the definition of a T-algebra. The product of an element of $\mathfrak{N}^*$ for an element of $\mathfrak{N}$ (and vice versa) can be defined by

$$ab^* = \sum_{i,j} \frac{(a_{ij}, b_{ij})}{n_i} e_i + \sum_\alpha (c^\alpha a, b)c^{\alpha *} + \sum_\alpha (c^\alpha b, a)c^\alpha, \tag{4.37a}$$

$$a^*b = \sum_{i,j} \frac{(a_{ij}, b_{ij})}{n_j} e_i + \sum_\alpha (bc^\alpha, a)c^{\alpha *} + \sum_\alpha (ac^\alpha, b)c^\alpha, \tag{4.37b}$$

where c^α ($\alpha = 1, \ldots, \dim \mathfrak{N}$) is an orthonormal basis of $\mathfrak{N}$. One shows that $\mathfrak{A}$ so constructed satisfies all the axioms of a T-algebra.

4.C.2. The T-Algebras of the Kähler Spaces $K(p,q)$ & $H(p,q)$. Having set the stage in the previous sections, the construction of the T-algebras associated to our supergravity models, $K(p,q)$ & $H(p,q)$ is quite simple.

Consider the commutative ideal RW (Sect. 4.B). We saw that the cone V is the orbit of the affine group $\exp C_h$ $(h \in JR)$. In this sense we can use the elements JR as affine coordinates for V.

Thus, the coordinates z^A are identified with the generators h_α and X_{A-}, which is just what we did in Sect. 3.B to write the function F in the cubic form. (*Recall that the cubic form is—by definition—the canonical affine parametrization of the Siegel domains of the first kind.* See Sect. 4.A.).

A natural grading for JR is induced by the grading of R given by the *normal decomposition*, Eq. (4.12) (see also Sect. 2.B).

As in Sect. 4.C.1, we construct the T-algebra out of its N-algebra.

The T-algebras of the models $K(p,q)$ and $H(p,q)$ are cooked using the following recipe:

a) The subspaces $\mathfrak{N}_{ij}$ are identified with the subspaces JR_{ij} of the normal decomposition, Eqs. (4.12, 4.14). Therefore, all the relevant T-algebras have rank 3,

$$\mathfrak{N} = J\sum_{i<j} R_{ij}. \tag{4.38}$$

b) We define the inner product $(\ldots,\ldots)$ in $\mathfrak{N}$ by the formula

$$(x_{ij}, y_{ij}) = \frac{n_i}{2}\langle x_{ij}, y_{ij}\rangle \quad x_{ij}, y_{ij} \in JR_{ij}. \tag{4.39}$$

c) The product $\mathfrak{N}_{ij} \otimes \mathfrak{N}_{jk} \to \mathfrak{N}_{ik}$ is defined as

$$x_{ij} y_{ik} = \begin{cases} x_{ij} * y_{jk} & \text{for} \quad i<j<k \\ 0 & \text{otherwise} \end{cases}, \tag{4.40}$$

where $*$ is the product whose multiplication table is given by the coefficients $d^{\mu m M}$.

We have still to check that these rules define an algebra $\mathfrak{N}$ satisfying the axiom I),…, V) of Sect. 4.C.1. The only one which deserves consideration is IV). Equation (4.16) gives the following explicit identifications for the spaces $\mathfrak{N}_{ij}$:

$$\mathfrak{N}_{12} = X_{3-}, \tag{4.41a}$$

$$\mathfrak{N}_{13} = X_{2-}, \tag{4.41b}$$

$$\mathfrak{N}_{23} = X_{1-}. \tag{4.41c}$$

Making the same identifications for the elements of each vector space, we have

$$\begin{aligned}(x_{12}y_{23}, x_{12}y_{23}) &\to \frac{n_1}{2}\langle x_{3-} * x_{1-}, x_{3-} * x_{1-}\rangle \\ &= \frac{n_1}{4}\langle x_{1-}, x_{1-}\rangle\langle x_{3-}, x_{3-}\rangle \\ &= (x_{12}, x_{12})\frac{1}{n_2}(y_{23}, y_{23}).\end{aligned} \tag{4.42}$$

Then, since ψ (see Eq. (2.25)) is an isometric map, axiom IV is fulfilled.

This proves that our spaces $K(p,q)$ and $H(p,q)$ are associated to T-algebras of rank 3 (of course, this was already obvious from the results of Sect. 4.B).

By analogy with the notations of Sect. 3, we shall denote an element X of the T-algebra as

$$X = \begin{pmatrix} p_1 & -x_3 & x_2 \\ -x_3^* & p_2 & x_1 \\ x_2^* & x_1^* & p_3 \end{pmatrix}. \tag{4.43}$$

(Each variable in this matrix should be identified with the imaginary part of the complex field denoted in Sect. 3 by the same symbol.)

From the definition of $\rho_k(X)$, Eq. (4.29), we get for rank 3 T-algebras

$$\rho_3(X) = p_3, \tag{4.44}$$

$$\rho_2(X) = p_3 p_2 - \tfrac{1}{2}(x_1)^2, \tag{4.45}$$

$$\begin{aligned}\rho_1(X) = \frac{1}{2} p_3 \left\{ 2p_1p_2p_3 - \sum_{A=1}^{3} p_A(x_A)^2 - 2(x_1, x_2, x_3) \right\} \\ + \tfrac{1}{2}\{\tfrac{1}{2}(x_1)^2(x_2)^2 - \langle x_1 * x_2, x_1 * x_2 \rangle\}.\end{aligned} \tag{4.46}$$

Notice that the functions $\rho_k(X)$ do not depend on the numbers n_i. Then, the cone of the "positive Hermitian matrices," $V(\mathfrak{A}) = \{X \in \mathfrak{X} : \rho_k(X) > 0\}$, does not depend on the numbers $n_i > 0$. Therefore, changing these numbers will modify only the metric, *not* the bounded domain itself. This elementary observation is crucial for our arguments below.

The Bergmann Kähler potential B is given by

$$e^{-B} = \mathrm{DET}_G\{i(\bar{\mathfrak{z}} - \mathfrak{z})\}, \tag{4.47}$$

where DET_G denotes the graded determinant, defined by Eq. (4.32) with n_i as in Eq. (4.20).

Here we want to show that a totally analogous formula is valid for the Kähler potential G of $N = 2$ SUGRA.

Theorem

i) *The Kähler potential G, Eq. (2.1) is given by Eq. (4.47), where the graded determinant is replaced by the naive determinant*

$$e^{-G} \equiv 2Y[i(\bar{\mathfrak{z}} - \mathfrak{z})] = \mathrm{DET}_N\{i(\bar{\mathfrak{z}} - \mathfrak{z})\}, \tag{4.48}$$

where the "naive" determinant DET_N *is defined in terms of the "naive" trace*

$$\mathrm{Tr}_N\, a = \sum_i a_{ii}, \tag{4.49}$$

i.e. *by Eq. (4.32) with* $n_i = 1$.

ii) *A Siegel domain of the first kind can be coupled to* $N = 2$ SUGRA *if and only if the "naive" determinant of the corresponding rank 3 T-matrix X is a polynomial in its components.*

iii) *The "domain of positivity" is* $\{\mathrm{Im}\, z \in V(\mathfrak{A})\}$.

Indeed, from Eqs. (4.32, 46, 49) we get

$$\mathrm{DET}_N\{i(\bar{\mathfrak{z}} - \mathfrak{z})\} = \frac{\rho_1[i(\bar{\mathfrak{z}} - \mathfrak{z})]}{\rho_3[i(\bar{\mathfrak{z}} - \mathfrak{z})]} = \frac{1}{2}\left\{ 2p_1p_2p_3 - \sum_{A=1}^{3} p_A(x_A)^2 - 2(x_1, x_2, x_3) \right\}$$

$$+\frac{1}{2}\frac{1}{p_3}\{\tfrac{1}{2}(x_1)^2(x_2)^2-\langle x_1*x_2, x_1*x_2\rangle\}$$

$$=\frac{1}{2}Y[i(\bar{\mathfrak{z}}-\mathfrak{z})]+\frac{1}{2}\frac{1}{p_3}\{\tfrac{1}{2}(x_1)^2(x_2)^2-\langle x_1*x_2, x_1*x_2\rangle\}, \quad (4.50)$$

where $Y[2\,\mathrm{Im}\,\mathfrak{z}]$ is the function Y computed from the function $F(X^0, X^A)$ in Eq. (3.18). The spurious term in Eq. (4.51)—which is not a polynomial—vanishes if and only if

$$\langle x_1*x_2, x_1*x_2\rangle=\tfrac{1}{2}(x_1)^2(x_2)^2. \quad (4.51)$$

But this is exactly the integrability condition for the HJ equations of Sect. 3 (or the closure condition for the Q-representations of Sect. 2.B). We know its solutions

a) $X_1\equiv 0 \Rightarrow$ the $K(p,q)$ spaces;
b) $\dim X_2=\dim X_3$ and ψ a *special isometric map.*
$\Rightarrow$ the $H(p,q)$ spaces.

So ii) is proved. Then i) follows from Eq. (2.1) and Eq. (4.50). iii) follows from the characterization of the "domain of positivity" in Sect. 4.A.2 and the definition of $V(\mathfrak{A})$ as the cone of the *positive* elements of $\mathfrak{X}$.

Notice that in the four magical cases all the n_i are equal to dim $\mathbb{A}$. Then, in these four cases we get

$$\mathrm{Tr}_G(\ldots)=(1+\dim\mathbb{A})\,\mathrm{Tr}_N(\ldots), \quad (4.52)$$

and therefore the SUGRA metric is proportional to the Bergmann one. In all the other cases the SUGRA metric is not proportional to the Bergmann one, and therefore cannot be Einstein (since it is homogeneous). See also Sect. 3.C.

Just as a further check, let us show that Eq. (4.48) implies that the Kähler metric $G_{\alpha\beta}$ is invariant under the full automorphism group of the cone. More generally, we show that the "Kähler metric" $K_{\alpha\beta}$ defined by the "Kähler potential" K

$$e^{-K(X)}=\mathrm{DET}_{\{n_1,n_2,n_3\}}[X] \quad X\in\mathfrak{X} \quad (4.53)$$

(where n_i are arbitrary numbers) is homogeneous. By Eq. (4.32)

$$e^{-K(X)}=[\rho_3(X)]^{n_3-(n_2+n_1)}[\rho_2(X)]^{n_2-n_1}[\rho_1(X)]^{n_1}. \quad (4.54)$$

The automorphisms of the cone $V(\mathfrak{A})$, $\omega\in\mathscr{T}(\mathfrak{A})$ act on $x\in\mathfrak{X}$ as follows:

$$\pi(\omega)\colon X\to\omega X\omega^*. \quad (4.55)$$

Or, writting $x=tt^*$, by $t\to\omega t$, where ω, t are upper triangular. Then it is easy to prove the identity

$$\mathrm{DET}_{\{n_i\}}[X]\to\mathrm{DET}_{\{n_i\}}[X]\times\{\mathrm{DET}_{\{n_i\}}[\omega]\}^2,$$

$$\mathrm{DET}_{\{n_i\}}[\omega]=\prod_{k=1}^{m}(\omega_{kk})^{n_k}. \quad (4.56)$$

Therefore,

$$K(X)\to K(X)-2\ln\mathrm{DET}_{\{n_i\}}[\omega]=K(X)+\mathrm{const.} \quad (4.57)$$

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