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# Morse Theory Interpretation of Topological Quantum Field Theories

J. M. F. Labastida
Theory Division, CERN, CH-1211 Geneva 23, Switzerland

**Abstract.** Topological quantum field theories are interpreted as a generalized form of Morse theory. This interpretation is applied to formulate the simplest topological quantum field theory: topological quantum mechanics. The only non-trivial topological invariant corresponding to this theory is computed and identified with the Euler characteristic. Using field theoretical methods this topological invariant is calculated in different ways and in the process a proof of the Gauss–Bonnet–Chern–Avez formula as well as some results of degenerate Morse theory are obtained.

## 1. Introduction

Topological quantum field theories [1, 2, 3] may provide a very useful tool in analyzing some mathematical problems and in describing a possible unbroken phase of string theory. The essential feature of topological quantum field theories is that they possess a symmetry which leads to the formulation of possibly non-trivial topological invariants. Due to the presence of this symmetry, which will be called $Q$-symmetry, there are no physical excitations. Commuting bosonic degrees of freedom cancel against anticommuting bosonic degrees of freedom, leaving only zero modes which give rise to topological invariants. This $Q$-symmetry, as well as the content of the theory resembles the BRST quantization of some theory. Typically, topological quantum field theories consist of commuting bosonic fields and anticommuting bosonic ones making an appearance very similar to the one in the BRST quantization of a classical bosonic theory. Furthermore, the operator corresponding to the $Q$-symmetry is anticommuting and squares to zero at least in some subspace of field configurations. Ever since topological quantum field theories were created it was believed that these theories might correspond to a BRST quantization of an underline gauge invariant theory.

Canonical BRST-quantization methods have been used [4, 5] to make contact with topological Yang-Mills theory [1]. In these cases, starting with a trivial action (either a gaussian, zero, a topological invariant, or 137), and BRST-gauge-fixing deformations of the gauge potential modulo gauge transformations, a formulation very close to the one in ref. [1] was obtained. However, although close, the resulting theory and topological Yang-Mills are not equivalent. There are two reasons for this. First, the chosen "gauge fixing" leads to several Gribov copies (arguments leading to this conclusion can be found in [1, 6]) and so it is not really a "gauge fixing." Actually, it is because there are these Gribov copies that topological field theories are interesting and, in fact, that is the basis of the interpretation of these theories in terms of generalized Morse theory presented in this paper. Second, the observables of topological Yang-Mills, once the Yang-Mills gauge symmetry is fixed [7] (if one ignores for the moment possible Gribov ambiguities [8]) are not trivial and, in fact, are in correspondence with the Donaldson invariants [9] as shown in [1]. However, the observables of the theories presented in [4, 5] are trivial. For a recent analysis of this question see [10]. The $Q$-symmetry in this theory does not correspond to a BRST symmetry and it has a nature similar to supersymmetry in a supersymmetric Yang-Mills theory (as one could have concluded from the fact that topological Yang-Mills can be constructed from a truncated $N=2$ supersymmetric Yang-Mills theory [1]). In this sense, the quantization of topological Yang-Mills proposed in [7] seems to be the right one.

Similar arguments lead to the same conclusion in the case of topological sigma models [3]. The resemblance of topological quantum field theory to trivial theories in a BRST-quantized form may, however, be used to formulate new topological quantum field theories. This has been treated without much success for the moment for the case of topological gravity in four [11] and in two [12, 13] dimensions.

In this paper it is conjectured that topological quantum field theories can be interpreted as a generalized form of Morse theory. This correspondence is shown in full detail for the case of topological quantum mechanics which seems to be the simplest of such theories. A similar construction can be carried out to formulate topological Yang-Mills and topological sigma models. The connection with generalized Morse theory allows us to find out the general pattern of a topological quantum field theory and permits, in principle, its construction.

The only non-trivial topological invariant in topological quantum mechanics corresponds to the Euler characteristic. Using methods of quantum field theory one can show the invariance of the observables of the theory under continuous deformations of the potential. This allows us to obtain a proof of the Gauss–Bonnet–Chern–Avez formula. In addition, making a deformation of the potential to other adequate choices, we obtain results corresponding to ordinary degenerate Morse theory. This shows how powerful this type of theory can be from a mathematical point of view.

The paper is organized as follows. In Sect. 2 the generalized form of Morse theory is conjectured. In Sect. 3, it is applied to the case of topological quantum mechanics and its topological invariants are computed. Finally, in Sect. 4, we state our conclusions analyzing the general characteristics of topological quantum field theories and listing some of the possible theories.

## 2. Morse Theory

In this section we will briefly review the elements of Morse theory [14] which will be utilized in the paper and we will conjecture their generalization. In the next section we will discuss the validity of this generalization using methods of topological quantum field theories.

Let us consider a compact $C^\infty$ manifold $V$ and a smooth function $f$ on it. A point $p \in V$ is a critical point of $f$ if $df$ vanishes at such a point. The Hessian at $p$, $H_p f$, is a quadratic form on $T_p V$, the tangent space to $V$ at $p$. In local coordinates $\{x^i\}$ centered at $p$, the matrix of $(H_p f)_{ij}$ relative to the base $\partial/\partial x^i$ at $p$ is

$$(H_p f)_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}. \tag{2.1}$$

The point $p$ is called a non-degenerate critical point of $f$ if $\det H_p f \neq 0$. The index of $p$ is the number of negative eigenvalues of $H_p f$ and it will be denoted by $\lambda_p(f)$. One of the results of Morse theory is that if $f$ is a function with a finite number of non-degenerate critical points the Euler characteristic of the manifold, $\chi(V)$, can be written in terms of the indices associated to the critical points of $f$, namely,

$$\chi(V) = \sum_p (-1)^{\lambda_p(f)}, \tag{2.2}$$

where the sum extends over all the critical points. This equation shows that such a sum is a topological invariant. Morse theory proves stronger results than this, however, it is (2.2) that is of interest to us. Our main goal in this section is to conjecture an equation like (2.2) for a more general case. In more concrete terms we will conjecture that an expression similar in spirit to the right-hand side of (2.2) is topological invariant. In the subsequent section we will argue that the conjecture is true using methods of topological quantum field theories.

Let us introduce more notation to be able to deal with the case of having degenerate critical points [15] which in fact are the central part of our discussion. Let $\mathscr{M}_\alpha \subset V$ be a connected submanifold of $V$. We will call $\mathscr{M}_\alpha$ a non-degenerate critical submanifold for $f$ if $df = 0$ along $\mathscr{M}_\alpha$ and $H_{\mathscr{M}_\alpha} f$ is non-degenerate on the normal bundle $\nu(\mathscr{M}_\alpha)$ of $\mathscr{M}_\alpha$. If we denote by $\det' H_{\mathscr{M}_\alpha} f$ the determinant of $H_{\mathscr{M}_\alpha} f$ restricted to the normal bundle, from our definition, $\det' H_{\mathscr{M}_\alpha} f \neq 0$.

In order to introduce our conjecture let us rewrite (2.2) as a sum of a ratio of determinants:

$$\chi(\mathscr{M}) = \sum_p \frac{\det H_p f}{|\det H_p f|}. \tag{2.3}$$

The possible generalizations of this sum for the case of having degenerate points should involve integrals over critical submanifolds. These integrals in general would not lead to topological invariants unless, possibly, if one introduces some $d_\alpha$-forms with $d_\alpha$ being the dimension of the critical submanifold $\mathscr{M}_\alpha$. For the case at hand, i.e., degenerated Morse theory with non-degenerate critical submanifolds, it is known that there exist $d_\alpha$-forms $\mathscr{A}^\alpha$ such that the quantity

$$\Xi = \sum_\alpha \int_{\mathscr{M}_\alpha} \mathscr{A}^\alpha \frac{\det' H_{\mathscr{M}_\alpha} f}{|\det' H_{\mathscr{M}_\alpha} f|} \tag{2.4}$$

is a topological invariant. In fact, if $f$ is a constant on $\mathscr{M}_\alpha$, $\mathscr{A}^\alpha$ is zero if $d_\alpha$ is odd, and it is the $d_\alpha/2$ exterior product of the curvature 2-form $\Omega_{ab}$ on $\mathscr{M}_\alpha$ if $d_\alpha$ is even.

Actually, we are interested in a further generalization of (2.2). So far, our discussion is based on the analysis of the critical submanifolds originated from the equation $df = 0$, with $f$ being a smooth function on a manifold $V$. A more general situation consists of the following. Let us consider two manifolds $\Sigma$ and $M$, with $\Sigma$ compact, and a topological class of smooth mappings $\phi:\Sigma \to M$. Let $V$ be the space of mappings contained on such a topological class. Notice that now $V$ does not need to be a manifold. Let us consider an operator $F$ acting on $V$ such that it maps $\phi \in V$ into sections in the pullback of the tangent bundle of $M$, $\phi^*(T(M))$, and such that $V$ and $\operatorname{Im} F$ have the same dimensionality. The kernel of $F$ defines critical subspaces $\mathscr{M}_\alpha \subset V$. Let $J_{\mathscr{M}_\alpha} F$ be the Jacobian of the operator $F$ at the critical subspace $\mathscr{M}_\alpha$. This Jacobian defines an homomorphism on sections of the pullback bundle $\phi^*(T(M))$. $J_{\mathscr{M}_\alpha} F$ will play the role of the Hessian in the previous discussion. Contrary to the case of the Hessian, $J_{\mathscr{M}_\alpha} F$ is not selfadjoint in general. We will consider situations where the Jacobian is degenerate in the normal subspace to $\mathscr{M}_\alpha$. It is typically the case that the kernel and the cokernel of this Jacobian restricted to the normal subspace are not trivial. Thus, we should not restrict ourselves to non-degenerate critical subspaces. We will try to make sense of (2.4) even in these situations.

We conjecture that given two manifolds $\Sigma$ and $M$ (being $\Sigma$ compact), a topological class of mappings $V = \{\phi \mid \phi:\Sigma \to M\}$, and an operator $F$ acting on $V$ which maps $\phi \in V$ into sections of $\phi^*(T(M))$ such that $V$ and $\operatorname{Im} F$ have the same dimensionality, there exist $d_\alpha$-forms $\mathscr{A}^\alpha$ living in the critical $d_\alpha$-dimensional subspace $\mathscr{M}_\alpha \subset V$ such that

$$\Xi = \sum_\alpha \int_{\mathscr{M}_\alpha} \mathscr{A}^\alpha \frac{\det' J_{\mathscr{M}_\alpha} F}{|\det' J_{\mathscr{M}_\alpha} F|} \tag{2.5}$$

is a topological invariant, i.e., $\Xi$ depends only on the topology of $\Sigma$ and $M$. If the kernel and cokernel of $F$ in the normal bundle to $\mathscr{M}_\alpha$ are non-trivial this expression contains ratios of zeros. This may seem rather undefined at this point. The formalism of topological quantum field theory will provide a way to make (2.5) well defined. Notice that in the case that $\Sigma$ is a point, $V$ is the space of mappings of that point into $M = V$, and $F = df$, (2.5) reduces to (2.4).

Using methods of topological quantum field theories one can show that $\Xi$ is topological invariant and, in the process, find the forms $\mathscr{A}^\alpha$. Of course (2.5) may be a vacuous statement unless a good choice of $F$ is made. The topological sigma models and the topological Yang-Mills theory formulated by Witten correspond to two cases where there is a good choice of $F$ and where the forms $\mathscr{A}^\alpha$ are constructed. In this paper we will present in detail a simpler case to show that (2.5) holds. The treatment of the topological sigma model is analogous to the one of this simpler model. The case of topological Yang-Mills has special features and deserves a separate publication. However, it corresponds also to a particular case of (2.5).

## 3. Topological Quantum Mechanics

In this section we will evaluate the right-hand side of (2.5) for a concrete choice of $\Sigma$, $M$ and $F$. This will lead to the formulation of topological quantum mechanics.

Applying methods of topological quantum field theory to this formulation we will show that indeed $\Xi$ is a topological invariant and we will construct the forms $\mathscr{A}^\alpha$ entering in (2.5). Topological quantum mechanics is a rather simple model and so it does not contain a rich quantity of topological invariants. However, the discussion of this model in the framework of Morse theory proposed in the previous section illustrates the general characteristics of this type of formulations very simply.

*3.1. The Model and its Field Theory Representation.* Let us consider a smooth Riemannian manifold $M$ with Euclidean signature and a smooth vector field $\xi^i$ (or more precisely, a section on the tangent bundle $T(M)$) on it. The space $V$ introduced in the previous section consists of all the mappings $\phi: S^1 \to M$ of a given topological class. Taking $u^i$ as local coordinates in $M$, the map $\phi$ can be locally described by functions $u^i(\tau)$, where $\tau$ denotes a point on the circle $S^1$. The operator $F$ is chosen to be

$$F^i(u^j) = \dot{u}^i + \xi^i(u^j), \tag{3.1}$$

where the dot denotes a derivative with respect to $\tau$. Notice that $V$ and $\operatorname{Im} F$ have the same dimensionality. The corresponding Jacobian at a critical subspace $\mathscr{M}_\alpha$ of (3.1) is $(J_{\mathscr{M}_\alpha} F)^i{}_j$ which, acting on an arbitrary section of the pullback bundle $\phi^*(T(M))$, $\lambda^i$, has the form:

$$(J_{\mathscr{M}_\alpha} F)^i{}_j \lambda^j = D\lambda^j + (D_j \xi^i)\lambda^j, \tag{3.2}$$

where $D_k$ is the Riemannian covariant derivative, $D_k W^i = \partial_k W^i + \Gamma^i_{kj} W^j$, $\Gamma^i_{kj}$ is the standard affine connection on $M$, and $D$ is the covariant derivative on sections of $\phi^*(T(M))$. If $\lambda^i$ is one of these sections, $D$ is defined to act as

$$D\lambda^i = \dot{\lambda}^i + \dot{u}^k \Gamma^i_{kj} \lambda^j. \tag{3.3}$$

Notice that (3.2) is correct at a critical subspace $\mathscr{M}_\alpha$. In general, $J_{\mathscr{M}_\alpha} F$ contains non-covariant looking terms which just vanish at the critical subspace. Let us restrict ourselves for the moment to the case of only one critical subspace and let us call the Jacobian at this subspace $\mathscr{D}^i_j$,

$$\mathscr{D}^i_j = \delta^i_j \frac{d}{d\tau} + \dot{u}^k \Gamma^i_{kj} + D_j \xi^i. \tag{3.4}$$

The conjectured topological invariant (2.5) can then be written as

$$\Xi = \int_{\mathscr{M}} \mathscr{A} \frac{\det' \mathscr{D}}{|\det' \mathscr{D}|}, \tag{3.5}$$

where $\mathscr{A}$ are certain forms to be constructed.

We would like to express (3.5) as an integral over the whole space $V$ and not just over the critical subspace $\mathscr{M}$. To do this one could think of a delta function restricting $V$ to the critical subspace. This could be easily accomplished since such a delta function would be accompanied by the absolute value of the determinant of $\mathscr{D}$ restricted to the normal subspace of $\mathscr{M}$ and evaluated at $\mathscr{M}$ in the denominator. This is in respect to the integration region. Another problem in extending (3.5) to an integral over the full space $V$ is that the $d$-form ($d$ is the dimension of the critical subspace) $\mathscr{A}$ contains $d$ indices which should be properly contracted. There is a satisfactory form of handling this problem which involves the expression of $\det'\mathscr{D}$ in the numerator in a field theoretical language. One would be tempted to write such a determinant as a functional integration over anticommuting fields $\varrho_i$ and $\chi^i$ (which are sections on the pullback of the tangent bundle $\phi^*(T(M))$,

$$\det'\mathscr{D} \sim \int D\varrho_i D\chi^i \exp\left( - \int_{S^1} d\tau\, \varrho_i \mathscr{D}^i{}_j \chi^j \right). \tag{3.6}$$

However, the right-hand side of (3.6) vanishes whenever the operator $\mathscr{D}^i{}_j$ (or its adjoint) has zero modes. One way to remove these zeros is soaking them up. This procedure just requires the introduction of a product of $\chi^i$ and $\varrho_i$ in (3.6) in such a way that, properly normalized, (3.6) becomes an equality. However, we may also have zeros in the denominator of the integrand of (3.5) that we want to overcome so we should not soak up all of them. Let us analyze the situation in detail.

Given a solution to the equation

$$\dot{u}^i + \xi^i = 0, \tag{3.7}$$

we may ask when this solution can be deformed $u^i \to u^i + \delta u^i$ so that it remains still a solution. From (3.7) one finds that to lowest order in $\delta u^i$,

$$\mathscr{D}^i{}_j \delta u^j = 0, \tag{3.8}$$

where $\mathscr{D}$ is the operator defined in (3.4). The solutions of (3.8) would give us the dimension of the critical subspace $\mathscr{M}$. Actually, this is not true because there may be obstructions to make the infinitesimal deformation finite. Typically (although not in the case at hand), if the adjoint operator of $\mathscr{D}$, $\mathscr{D}^*$, has zero modes, not all the solutions to (3.8) can be made finite. The solutions of (3.8) constitute the kernel of the operator $\mathscr{D}$ while the zero modes of its adjoint constitute the cokernel of $\mathscr{D}$. The dimension of the critical subspace, $d$, is the difference between the dimension of the kernel and the cokernel, i.e., the index of the operator $\mathscr{D}$. If we denote by $p$ the dimension of $\ker\mathscr{D}$ and by $q$ the dimension of $\operatorname{coker}\mathscr{D}$, we have $d = p - q$. In the particular case of topological quantum mechanics the index of $\mathscr{D}$ is zero and so $p = q$. However, we will carry out the analysis for arbitrary $p$ and $q$ so that we can develop the general formalism.

Going back to (3.6) and using this reasoning we observe that the functional integral contains $p$ $\chi^i$-zero modes and $q$ $\varrho_i$-zero modes. This means that one must introduce a $p$-product of $\chi^i$ and a $q$-product of $\varrho_i$ to soak up these zero modes. However, this is not what we want to make contact with (3.5). That ratio of determinants contains ratios of zeros whenever the normal subspace to $\mathscr{M}$ is degenerate. Such is the case if $q \neq 0$. The quantity (3.5) contains $q$ zeros in the numerator and $q$ zeros in the denominator. This means that we do not have to soak up all the $\chi^i$ or $\varrho_i$ zero modes, but only $d = p - q$ of them to make contact with (3.5). This implies that (3.5) could be written as,

$$\Xi = \int Du^i D\varrho_i D\chi^i \delta(\dot{u}^i + \xi^i) \mathscr{A}_{i_1,\ldots,i_d}(u^i)\chi^{i_1}\cdots\chi^{i_d} \exp\left( - \int_{S^1} d\tau\, \varrho_i \mathscr{D}^i{}_j \chi^j \right), \tag{3.9}$$

where $\mathscr{A}_{i_1,\ldots,i_d}$ is a certain tensor. Notice that if a factor $\varrho_i$ or one more $\chi^i$ were introduced in front of the exponential, one would break the balance of zero modes in the normal subspace and the expression either becomes infinity or zero. To compute $\mathscr{A}$ in (3.5) from this expression one must expand $\chi^i$ and $\varrho_i$ in zero modes and non-zero modes,

$$\chi^i = \sum_{\mu=1}^{d} x^i_\mu \chi^\mu + \sum_{\mu=d+1}^{p} \tilde{x}^i_\mu \tilde{\chi}^\mu + \text{non-zero modes},$$
$$\varrho_i = \sum_{\mu=1}^{q} \tilde{r}^\mu_i \tilde{\varrho}_\mu + \text{non-zero modes}, \tag{3.10}$$

where $x^i_\mu$ are the $d$ zero modes which give rise to a finite deformation of a solution and so form a basis of the tangent space of $\mathscr{M}$. One of the contributions in (3.9) consists of the integration over the non-zero modes, giving the non-vanishing part of $\det' \mathscr{D}$, and the integration over the $\tilde{\chi}^\mu$ giving the product of zeros necessary to balance the product of zeros of the denominator. There are other contributions where, instead, some of the $\chi^i$ in front of the exponential become $\tilde{\chi}^\mu$. After adding all of them up and integrating $\tilde{\varrho}_\mu$ and $\tilde{\chi}^\mu$, one is left with the integration of $\chi^\mu$ which gives the form $\mathscr{A}$ in (3.5) from $\mathscr{A}_{i_1,\ldots,i_d}$. Notice that since the $x^i_\mu$, $\mu = 1,\ldots,d$ make a basis in the tangent space of $\mathscr{M}$ this way to proceed creates a canonical measure in $\mathscr{M}$, i.e., if we denote by $a^\mu$, $\mu = 1,\ldots,d$ the local coordinates in $\mathscr{M}$, the measure $\left(\prod_{\mu=1}^{d} da^\mu\right)\left(\prod_{\mu=1}^{d} d\chi^\mu\right)$ is invariant under reparametrizations of $\mathscr{M}$ since $a^\mu$ and $\chi^\mu$ transforms in opposite way as follows from (3.10). Expression (3.9), however, is still not well defined due to the presence of $q$ ratios of zeros. We will show below, using field theoretical techniques how this expression can be made well defined.

The delta function entering (3.9) can be expressed as a functional integral just introducing a multiplier field $d_i$. As $\varrho_i$ and $\chi^i$, $d_i$ is a section in the pullback bundle $\phi^*(T(\mathscr{M}))$. We have,

$$\Xi = \int Du^i D\varrho_i D\chi^i \frac{Dd_i}{(2\pi)^n} \mathscr{A}_{i_1,\ldots,i_d}(u^i)\chi^{i_1}\cdots\chi^{i_d}$$
$$\cdot\exp\left( - \int_{S^1} d\tau (id_i(\dot{u}^i + \xi^i) + \varrho_i \mathscr{D}^i{}_j \chi^j) \right). \tag{3.11}$$

Notice that we have used the normalization of the delta function as it would correspond to constant configurations. In this form the expression for $\Xi$ resembles the vacuum expectation value of an operator in quantum field theory. The argument of the exponential should be identified as the action of such a theory. Once written in this form we can use the machinery of quantum field theories to study the properties of $\Xi$. The first question to ask is if the action

$$S = \int_{S^1} d\tau (id_i(\dot{u}^i + \xi^i) + \varrho_i \mathscr{D}^i{}_j \chi^j), \tag{3.12}$$

possesses some symmetries. We will find out that in fact this action has a BRST-like symmetry characteristic of topological quantum field theories. To look for this symmetry it is very useful to utilize the fact that typically these theories have a similar structure to the one of a trivial theory in which one starts with zero as a classical action and one gauge fixes deformation invariance in the gauge $\dot{u}^i + \xi^i = 0$.

Of course, this is only a similarity which helps in finding the symmetries involved in topological quantum field theories. Topological quantum field theories are non-trivial while the other ones are. This trick, however, does not always work as is the case of topological gravity [11].

Using the insight discussed in the paragraph above, we start with the BRST version of the algebra of deformations of $u^i$:

$$\delta u^i = i\varepsilon\chi^i, \quad \delta\chi^i = 0, \tag{3.13}$$

where $\varepsilon$ is a constant anticommuting parameter. For the "gauge fixing" of the deformations we should introduce an "antighost" field $\varrho_i$ and a "Nakanishi-Lautrup" one, $d_i$. The transformations of these fields are typically of the form,

$$\delta\varrho_i = \varepsilon d_i + \cdots, \quad \delta d_i = \cdots, \tag{3.14}$$

where the content of the dots is determined as follows. On one hand, one would like to have

$$\delta\left(\int_{S^1} d\tau\, i\varrho_i(\dot{u}^i + \xi^i)\right), \tag{3.15}$$

leading to (3.12). That requirement fixes uniquely the transformation of $\varrho_i$,

$$\delta\varrho_i = \varepsilon(d_i - i\varrho_j \Gamma^j_{ik}\chi^k). \tag{3.16}$$

On the other hand, one wants $\delta$ to be nilpotent so that (3.15) (and so (3.12)) is invariant. Using (3.16) that requirement determines the transformation of $d_i$,

$$\delta d_i = \varepsilon(i d_j \Gamma^j_{ik}\chi^k + \tfrac{1}{2}R_{lk}{}^j{}_i\varrho_j\chi^l\chi^k), \tag{3.17}$$

where $R_{ij}{}^k{}_l$ is the Riemann tensor $R_{ij}{}^k{}_l = \partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}$. By construction, (3.12) is invariant under (3.13), (3.16) and (3.17). The transformations (3.13), (3.16) and (3.17) preserve $U$-invariance if the fields $(u^i, \chi^i, \varrho_i, d^i)$ have the $U$-number ("ghost" number) assignment $(0, 1, -1, 0)$. The action (3.12) has $U$-number 0.

The fundamental property of the symmetry found for the action (3.12) is that the action itself can be written as a symmetry transformation. Defining the operator $Q$ such that

$$\{-i\varepsilon Q, X\} = \delta X, \tag{3.18}$$

where $X$ represents any field $u^i, \chi^i, \varrho_i$ or $d_i$, one finds,

$$S = \{-iQ, \mathscr{V}\}, \tag{3.19}$$

where

$$\mathscr{V} = i\int_{S^1} d\tau\, \varrho_i(\dot{u}^i + \xi^i). \tag{3.20}$$

Now that we have discovered this symmetry in the action (3.12), we are in the position to analyze the properties of $\Xi$ in (3.11) using methods of quantum field theory. Our aim has been all along to show that there exist some forms $\mathscr{A}$ such that $\Xi$ is topological invariant. Let us investigate which ones are the properties that $\mathscr{A}_{i_1,\ldots,i_d}$ have to satisfy for this to be true.

Given an operator $\mathcal{O}$, we define its vacuum expectation value, $\langle\mathcal{O}\rangle$ as,

$$\langle\mathcal{O}\rangle=\int(DX)\mathcal{O}e^{-S(X)}. \tag{3.21}$$

With this definition $\Xi$ in (3.11) is just the vacuum expectation value of an operator. To study when $\langle\mathcal{O}\rangle$ is topological invariant let us perform a variation of the metric in $M$ and of the vector field $\xi^i$, which we will denote by $\tilde{\delta}$. From (3.19) this variation takes the form,

$$\tilde{\delta}\langle\mathcal{O}\rangle=\int(DX)e^{-S(X)}(\{-iQ,\tilde{\delta}\mathscr{V}\}\mathcal{O}+\tilde{\delta}\mathcal{O}), \tag{3.22}$$

and therefore, if we assume that $\mathcal{O}$ does not depend on the metric of $\mathcal{M}$ neither on the vector field $\xi^i$, we conclude that $\langle\mathcal{O}\rangle$ is topological invariant if

$$\langle\{-iQ,\tilde{\delta}\mathscr{V}\}\mathcal{O}\rangle=0. \tag{3.23}$$

A sufficient condition for (3.23) to hold is,

$$\{Q,\mathcal{O}\}=0. \tag{3.24}$$

The reason for this is that if (3.24) holds then (3.23) can be written as

$$\langle\{-iQ,\tilde{\delta}\mathscr{V}\}\mathcal{O}\rangle=\langle\{-iQ,(\tilde{\delta}\mathscr{V})\mathcal{O}\}\rangle, \tag{3.25}$$

and, assuming that the measure in (3.21) is invariant under a $Q$-symmetry transformation, the right-hand side of (3.25) vanishes.

According to (3.24) and (3.25), if an operator $\mathcal{O}$ is such that $\{Q,\mathcal{O}\}=0$, then $\langle\mathcal{O}\rangle$ is topological invariant. Actually, one would like to know which ones among the operators leading to (3.24) do not lead to a trivial vacuum expectation value. As we have discussed in the previous paragraph, if an operator is $Q$-exact, i.e., there exist another operator $\lambda$ such that $\mathcal{O}=\{Q,\lambda\}$, its vacuum expectation value vanishes. Therefore, to find out the non-trivial operators leading to topological invariants we are led to study the cohomology of $Q$. This analysis is identical to the one done in [3]. Applied to this case one finds that the cohomology classes are built out of the de Rham cohomology of $M$. Given a de Rham cohomology class in $M$ with representative $A=A_{i_1,\ldots,i_n}du^{i_1}\cdots du^{i_n}$, one can build two operators. The first one,

$$\mathcal{O}_A^{(0)}=A_{i_1,\ldots,i_n}\chi^{i_1}\cdots\chi^{i_n}, \tag{3.26}$$

satisfies

$$\{Q,\mathcal{O}_A^{(0)}\}=0, \tag{3.27}$$

and constitutes the first set of $Q$-cohomology classes,

$$W_A^{(0)}=\mathcal{O}_A^{(0)}. \tag{3.28}$$

The second one is built out of the exterior derivative (in $S^1$) of the first one,

$$d\mathcal{O}_A^{(0)}=i\{Q,\mathcal{O}_A^{(1)}\}, \tag{3.29}$$

where,

$$\mathcal{O}_A^{(1)}=inA_{i_1,\ldots,i_n}du^{i_1}\chi^{i_2}\cdots\chi^{i_n}. \tag{3.30}$$

The integral over $S^1$ of this 1-form leads to the second set of $Q$-cohomology classes,

$$W_A^{(1)}=\int_{S^1}\mathcal{O}_A^{(1)}. \tag{3.31}$$

In general, the quantities which may lead to non-trivial topological invariants from a quantum field theory point of view are vacuum expectation values of the form,

$$\left\langle \prod_i W_{A_i}^{(k_i)} \right\rangle, \tag{3.32}$$

where $k_i = 0, 1$ depending on which set of operators $W_{A_i}^{(k_i)}$ belongs.

We are now in the position to make contact with (3.11). So far we have been very vague about the operator $\mathscr{A}_{i_1,\ldots,i_d}\chi^{i_1}\cdots\chi^{i_d}$ in (3.11). We have not specified if it depends on points on $S^1$ or if it involves integrations over some regions of $S^1$. However, this vagueness is in fact natural since our aim was to construct them. Comparing (3.11) and (3.32) we observe that this construction is about to finish. Let us define the $\chi$-number of an operator as the number of times that $\chi^i$ appears in such an operator. Notice that since $\chi^i$ is anticommuting this is well defined. For example, $W_A^{(0)}$ and $W_A^{(1)}$ in (3.31) have $\chi$-numbers $m$ and $m-1$ respectively. If we denote by $n_\chi^i$ the $\chi$-number of $W_{A_i}^{(k_i)}$, among all the vacuum expectation values of products (3.32), only the ones which satisfy

$$d = \sum_i n_\chi^i \tag{3.33}$$

will lead to expressions of the form (3.11) and will be topological invariants. Notice that if (3.33) does not hold (3.32) would vanish due to the presence of zero modes. To be more precise, the operator in (3.11) which makes $\Xi$ a non-trivial topological invariant has the general form:

$$\mathscr{A}_{i_1,\ldots,i_d}\chi^{i_1}\cdots\chi^{i_d} = \prod_i W_{A_i}^{(k_i)}, \tag{3.34}$$

where $W_{A_i}^{(k_i)}$ are such that (3.33) holds.

So far we have analyzed the behavior of (3.11) and we have constructed the operators which could lead to topological invariants. However, when $q \neq 0$ the expression (3.11) is not well defined due to the fact that it contains a product of $q$ ratios of zeros. We will try to make it well defined by introducing a regulator and requiring that the answer be independent of such a regulator. Perhaps there are several ways to do this. Here we will present what we think corresponds to the simplest choice. Let us add to the action (3.12) a term of the form $(1/\beta)d_i d_j g^{ij}$ with $\beta > 0$. In the limit $\beta \to \infty$ this regulated formulation defines the previous one. Actually we would like to do better than that. We would like all our arguments leading to (3.34) to be valid in this regulated form of the functional integral (3.11). In other words, we would like to have a regulator which is compatible with the $Q$-symmetry. This certainly can be done and that will be the way we will proceed.

To have a term of the form $(1/\beta)d_i d_j g^{ij}$ in the regulated form of the action and to keep $Q$-invariance we may exploit the fact that $Q^2 = 0$ and add the following term to the action $S$ in (3.12):

$$S_\beta = S + \Delta S, \tag{3.35}$$

where,

$$\Delta S = \left\{ -iQ, \int_{S^1} d\tau \frac{1}{\beta} d_i \varrho_j g^{ij} \right\}. \tag{3.36}$$

Using (3.13), (3.16) and (3.17) one finds,

$$\Delta S = \frac{1}{\beta} \int_{S^1} d\tau (d_i d_j g^{ij} + \tfrac{1}{2} R_{kl}{}^{ij} \chi^k \chi^l \varrho_i \varrho_j), \tag{3.37}$$

and performing the change of variables

$$d_i = \tilde{d}_i - \frac{i}{2} \beta (\dot{u}^j + \xi^j) g_{ij}, \tag{3.38}$$

the new action becomes,

$$\tilde{S}_\beta = \int_{S^1} d\tau \left( \frac{1}{\beta} \tilde{d}_i \tilde{d}_j g^{ij} + \frac{\beta}{4} g_{ij} (\dot{u}^i + \xi^i)(\dot{u}^j + \xi^j) + \varrho_i \mathscr{D}^i{}_j \chi^j + \frac{1}{2\beta} R_{kl}{}^{ij} \chi^k \chi^l \varrho_i \varrho_j \right). \tag{3.39}$$

Notice that due to the presence of the new terms the product of ratios of zeros in (3.11) is not present as long as $\beta > 0$. If one rescales the fields $\chi^i$, $\varrho_i$ and $\tilde{d}_i$, $\chi^i \to \sqrt{\beta}\chi^i$, $\varrho_i \to \sqrt{\beta}\varrho_i$ and $\tilde{d}_i \to \beta \tilde{d}_i$, one finds that $\beta$ factors out of the action $\tilde{S}_\beta$,

$$\begin{aligned} \tilde{S}_\beta \to \tilde{S}_\beta = \beta \int_{S^1} d\tau (\tilde{d}_i \tilde{d}_j g^{ij} + \tfrac{1}{4} g_{ij} (\dot{u}^i + \xi^i)(\dot{u}^j + \xi^j) \\ + \varrho_i \mathscr{D}^i{}_j \chi^j + \tfrac{1}{2} R_{kl}{}^{ij} \chi^k \chi^l \varrho_i \varrho_j), \end{aligned} \tag{3.40}$$

while the measure in the functional integral times $\mathscr{A}_{i_1,\ldots,i_d} \chi^{i_1} \cdots \chi^{i_d}$ as in (3.11) remains invariant. Using (3.13), (3.16), (3.17) and (3.38), the $Q$-transformations of the fields after rescaling $\varepsilon \to \varepsilon/\sqrt{\beta}$ become,

$$\delta u^i = i\varepsilon\chi^i, \quad \delta\chi^i = 0, \quad \delta\varrho_i = \varepsilon \left( \tilde{d}_i - \frac{i}{2} (\dot{u}^j + \xi^j) g_{ij} - i\varrho_j \Gamma^j_{ik} \chi^k \right), \tag{3.41}$$

$$\begin{aligned} \delta\tilde{d}_i = \varepsilon \Big[ & i\tilde{d}_j \Gamma^j_{ik} \chi^k + \tfrac{1}{2} R_{lk}{}^j{}_i \varrho_j \chi^l \chi^k \\ & + \frac{i}{2} g_{ij} ((D\chi^j + (D_k \xi^j)\chi^k) + \tfrac{1}{2} (\dot{u}^m + \xi^m) \Gamma^j_{mk} \chi^k) \Big]. \end{aligned}$$

Since the action (3.40) is $Q$-exact, the vacuum expectation value of any operator invariant under a $Q$-transformation is independent of $\beta$. To prove this one just follows the same reasoning that led to (3.25). Notice that this procedure could be done starting with any other expression with $U$-number -1 and introducing its $Q$-transformation. The one chosen here is rather simple because it allows us to integrate out one of the fields ($d^i$).

So far we have been discussing the case of having only one critical subspace. It is straightforward to generalize the formalism to the case of having several critical subspaces. Clearly, the field theoretical formulation is the same for such a case. In fact, the field theoretical formulation forces us to add the contributions of different critical subspaces. Otherwise topological invariance, as derived in the field theory language, is not guaranteed.

In this section we have treated topological quantum mechanics in full generality in order to develop the general formalism. However, since the index of $\mathscr{D}$ for this case is zero, the only non-trivial topological invariant corresponds to the operator 1. In the next section we will carry out the computation of its vacuum expectation value.

*3.2 Explicit Calculation of Topological Invariants.* In this subsection we are going to evaluate the vacuum expectation values (3.32) leading to topological invariants for specific choices of the vector field $\xi^i$. Using the fact that the result must be independent of these choices we will be able to prove the Gauss–Bonnet–Chern–Avez formula, and some results of degenerate Morse theory.

First, let us start with a potential $W$, i.e., a smooth scalar function on $M$, $W: M \to R$, and let us assume that $\xi^i = \partial^i W$. The topological classes of mappings from $S^1$ into $M$ correspond to the homotopy group of $M$, $\pi_1(M)$. Let us consider the class of trivial mappings, i.e., mappings which can be continuously shrinked to a point. In evaluating (3.32) we need to analyze first the equation

$$\dot{u}^i + \partial^i W = 0. \tag{3.42}$$

In particular, we have to find out the critical subspaces corresponding to this equation, and in those subspaces we have to study the eigenvectors of the Jacobian of (3.42). This, in general, may be a rather complicated problem. However, from the analysis of the previous subsection we know that the result of the evaluation of (3.32) is invariant under continuous deformations of the potential. Thus, given a potential $W$ requiring a complicated analysis one can always deform it to a more convenient one. Using this reasoning we can assume that the potential $W$ has a finite number of non-degenerate critical points. The reason for this is that any smooth scalar function on $M$ can be uniformly approximated by another one which has a finite number of non-degenerate critical points.

Let us study the solutions to (3.42). If the critical points of $W$ are labeled by $u^i_\alpha$, a set of obvious solutions to (3.42) are the constant mappings $u^i(\tau) = u^i_\alpha$. Notice that these mappings are contained in the topological class of mappings under consideration. Let us prove that in fact those are the only solutions. Let us assume that there exists a non-constant solution and let us denote it by $v^i(\tau)$. This solution generates a loop in $M$ with total length

$$\oint d\tau \dot{v}^i \dot{v}^j g_{ij} \geqq 0, \tag{3.43}$$

since the manifold $M$ has Euclidean signature. However, since $v^i$ is a solution of (3.42),

$$\oint d\tau \dot{v}^i \dot{v}^j g_{ij} = -\oint dv^i \partial_i W = 0, \tag{3.44}$$

after using Stokes theorem. Therefore, $v^i$ can be only a constant.

Let us evaluate (3.32) for the case of the operator 1, which is the only one leading to non-trivial topological invariants. Looking at the form of the action in (3.12), after integrating $d_i$ one obtains a delta function that selects the solutions to (3.42). At such solutions the Berezin integral of $\varrho_i$ and $\chi^j$ is straightforward and one obtains,

$$\Xi = \sum_\alpha \frac{\det \partial_i \partial_j W}{|\det \partial_i \partial_j W|}\bigg|_\alpha = \chi(\mathscr{M}), \tag{3.45}$$

where in the last step we have used the standard result of Morse theory discussed in (2.2) and (2.3).

We may also compute $\Xi$ using the regulated action (3.40). In fact, as we show now, in doing this computation we obtain a proof of the Gauss–Bonnet–Chern–Avez formula. The computation of the partition function corresponding to the action (3.40) is complicated for an arbitrary $W$ even in the limit $\beta \to \infty$. However, as discussed in the previous subsection, the result is invariant under continuous deformations of the potential $W$. In particular, one can make a continuous deformation of $W$ such that it vanishes everywhere in $M$. In this case the computation of $\Xi$ can be easily done in the $\beta \to \infty$ limit. Taking (3.40), the quantity to be evaluated is

$$\Xi = \int Du^i D\varrho_i D\chi^i \frac{D\tilde{d}_i}{(2\pi)^n} e^{-\tilde{S}_\beta}, \tag{3.46}$$

where

$$\tilde{S}_\beta = \beta \int_{S^1} d\tau (\tilde{d}_i \tilde{d}_j g^{ij} + \tfrac{1}{4} g_{ij} \dot{u}^i \dot{u}^j + \varrho_i D\chi^i + \tfrac{1}{2} R_{kl}{}^{ij} \chi^k \chi^l \varrho_i \varrho_j). \tag{3.47}$$

The $\tilde{d}_i$-integration is gaussian and can be easily performed,

$$\int \frac{D\tilde{d}_i}{(2\pi)^n} \exp(-\beta \tilde{d}_i \tilde{d}_j g^{ij}) = \frac{\sqrt{g}}{(4\pi\beta)^{n/2}}. \tag{3.48}$$

To carry out the rest of the functional integrations just notice that in the $\beta \to \infty$ limit there is a contribution from the constant configurations and another from the ratio of determinants for non-constant modes which is known to be 1 due to the $Q$-symmetry. Therefore, using the result (3.48), the functional integral (3.46) reduces to the computation of

$$\Xi = \frac{1}{(4\pi\beta)^{n/2}} \int_M \sqrt{g} d^n u^i \int \left( \prod_{p=1}^n d\varrho_p \right) \left( \prod_{q=1}^n d\chi^q \right) \exp(-\beta \tfrac{1}{2} R_{kl}{}^{ij} \chi^k \chi^l \varrho_i \varrho_j), \tag{3.49}$$

where the only integration left is the one corresponding to constant Grassmann variables. Using the standard Berezin integration one gets that only the term with $n$ times the variable $\varrho_i$, and $n$ times the variable $\chi^i$ contributes. This implies that for compact manifolds of odd dimension $\Xi = 0$, in agreement with the standard result for the Euler characteristic. If $n$ is even, $n = 2m$, one obtains,

$$\Xi = \frac{(-1)^m}{(4\pi)^m m! 2^m} \int_M \sqrt{g} d^n u^i \varepsilon^{k_1 l_1 \ldots k_m l_m} \varepsilon^{i_1 j_1 \ldots i_m j_m} R_{k_1 l_1 i_1 j_1} \ldots R_{k_m l_m i_m j_m}, \tag{3.50}$$

which together with the result obtained in (3.42) constitutes a proof of the Gauss–Bonnet–Chern–Avez formula [16]. $\varepsilon^{i_1 \ldots i_n}$ is normalized such that $\varepsilon^{1 \ldots n} = 1$. Notice the cancellation of the $\beta$-dependence in the calculation of (3.50), a crucial test for the correctness of our arguments. The last step of this calculation resembles the computation of the index of a classical elliptic complex using supersymmetric quantum mechanics [17].

We have been using either the regulated or the non-regulated formulations to carry out the computations, according to convenience. We may ask what happens, for example, if one tries to compute $\Xi$ for the case $\xi^i = 0$ using (3.11). That functional integral reduces in that case to a product of $n$ (notice that $q = n$) ratios of zeros which seems rather undefined. The regulated theory seems to give a definition to such a product. Furthermore, such a definition is in agreement with expectations.

On the other hand, using (3.40) one may also compute $\Xi$ very simply for the case in which $W$ has only a finite number of non-degenerate critical points $u^i_\alpha$. Let us describe this computation. The functional integration of $\tilde{d}^i$ is carried out using (3.48) as before. To perform the other integrations notice that in the limit $\beta \to \infty$ the main contribution comes from bosonic configurations which are constant $u^i = u^i_\alpha$. Let us expand $u^i$ in normal coordinates around one of these constant configurations [18]. If we denote by $\omega^i$ the tangent vector at $u^i_\alpha$ which defines the geodesic from $u^i_\alpha$ to $u^i$, we find,

$$\Xi = \sum_\alpha \int \frac{D\omega^i D\chi^i D\varrho_i}{(4\pi\beta)^{n/2}} \sqrt{g}\, e^{-S_{(\alpha)}(\omega^i,\chi^i,\varrho_i)}, \tag{3.51}$$

where,

$$S_{(\alpha)} = \beta \int_{S^1} d\tau \left(\tfrac{1}{4} g_{ij}(u^i_\alpha)(\mathscr{D}^i{}_k \omega^k)(\mathscr{D}^j{}_l \omega^l) + \varrho_i \mathscr{D}^i{}_j \chi^j + \tfrac{1}{2} R_{kl}{}^{ij} \chi^k \chi^l \varrho_i \varrho_j + \cdots\right), \tag{3.52}$$

and the dots denote terms containing at least one $\omega^i$ and two any other fields. Notice that since the critical points are not degenerate there are no $\chi^i$ or $\varrho_i$ zero modes. To proceed, let us rescale the fields $\omega^i$, $\chi^i$ and $\varrho_i$ with the vielbein and some appropriate powers of $\beta$ and other factors,

$$\omega^a = \frac{\sqrt{\beta}}{2} e^a{}_i \omega^i, \quad \chi^a = \sqrt{\beta}\, e^a{}_i \chi^i, \quad \varrho^a = \sqrt{\beta}\, e^{ai} \varrho_i. \tag{3.53}$$

We are using a notation in which tangent space indices are denoted by latin letters at the beginning of the alphabet. The functional integral (3.51) becomes in the new variables,

$$\Xi = \sum_\alpha \int \frac{D\omega^a D\chi^a D\varrho^a}{\pi^{n/2}} e^{-\tilde{S}_{(\alpha)}(\omega^a,\chi^a,\varrho^a)}, \tag{3.54}$$

where,

$$\tilde{S}_{(\alpha)} = \int_{S^1} d\tau \left( (\mathscr{D}^a_b \omega^b)(\mathscr{D}_{ac} \omega^c) + \varrho^a \mathscr{D}_{ab} \chi^b + \frac{1}{2\beta} R_{abcd} \chi^a \chi^b \varrho^c \varrho^d + \cdots \right). \tag{3.55}$$

In the large $\beta$ limit the leading contribution comes from the ratio of determinants left after the integration of $\omega^a$, $\chi^a$ and $\varrho^a$. All other terms contain powers of $\beta$ in the denominator and so they vanish in the limit $\beta \to \infty$. Taking into account (3.4) and the fact that at $u^i = u^i_\alpha$ the first derivative of the potential vanishes, from (3.54) we obtain,

$$\Xi = \sum_\alpha \left. \frac{\det \partial_i \partial_j W}{|\det \partial_i \partial_j W|} \right|_\alpha = \chi(M), \tag{3.56}$$

in agreement with (3.46).

Now that we have developed some machinery we are in the position to derive some results of degenerate Morse theory. Let us deform the potential $W$ to a new one which has critical submanifolds $\mathcal{M}_\alpha$ such that $H_{\mathcal{M}_\alpha}W$ is non-degenerate on the normal bundle $\nu(\mathcal{M}_\alpha)$. Using (3.40) we are going to derive the expression for the Euler characteristic in this case. This corresponds to a mixed situation between the two already discussed. The reason for this is that on $\mathcal{M}_\alpha$ the corresponding vector field $\partial^i W$ vanishes. Let us choose coordinates such that the metric of $M$ is box-diagonal i.e., if there are $C$ critical submanifolds each of dimension $d_\alpha$, the metric does not mix the coordinates of a critical submanifold with the coordinates of the others. The determinant of the metric then becomes a $C+1$ product: $g = g_\perp\left(\prod_{\alpha=1}^{C} g_\alpha\right)$, where $g_\alpha$ is the determinant of the metric restricted to the critical submanifold $\mathcal{M}_\alpha$ and $g_\perp$ the determinant of the metric restricted to the directions which are not tangent to any of the critical submanifolds. Let us pick one of these critical submanifolds $\mathcal{M}_\alpha$ and let us suppose that its local coordinates are $u^i_\alpha$, $i = r_\alpha$, $r_\alpha + 1, \ldots, r_\alpha + d_\alpha$. Now let us expand $u^i$ around one of these $u^i_\alpha$ in normal coordinates $\omega^i$: $u^i = u^i_\alpha + \omega^i - \frac{1}{2}\Gamma^i_{jk}(u^i)\omega^j\omega^k + \cdots$. Using this expansion the expression for $\Xi$ becomes,

$$\Xi = \sum_\alpha \int_{\mathcal{M}_\alpha} du^i_\alpha \int \frac{D\omega^i D\chi^i D\varrho_i}{(4\pi\beta)^{n/2}} \sqrt{g}\, e^{-S_{(\alpha)}(\omega^i,\chi^i,\varrho_i)}, \tag{3.57}$$

where $S_{(\alpha)}$ is given in (3.52). Now, depending on whether $\omega^i$ lies in the direction tangent to the critical submanifold $\mathcal{M}_\alpha$ or not we are in one or the other situation above. If $\omega^i$ is such that $i = r_\alpha, \ldots, r_\alpha + d_\alpha$ the contribution to $\Xi$ is dominated by the constant configurations of $\omega^i$, $\chi^i$ and $\varrho_i$. If such is the case only the term involving $R_{kl}{}^{ij}$ in (3.52) survives in the leading contribution. The integration of the non-zero modes over these $d_\alpha$ directions gives at leading order 1 because of the $Q$-symmetry. Once this is done one has still to integrate $\omega^i$ over the remaining directions. In this case one must proceed as in (3.53), i.e., one must rescale fields etc., to pick the leading contribution in $\beta$. One finds,

$$\Xi = \sum_\alpha \int_{\mathcal{M}_\alpha} du^i_\alpha \sqrt{g_\alpha} \frac{E(m_\alpha)}{(4\pi)^{m_\alpha} m_\alpha! 2^{m_\alpha}} \left.\frac{\det{}' \partial_i\partial_j W}{|\det{}' \partial_i\partial_j W|}\right|_\alpha \cdot \varepsilon^{k_1 l_1 \cdots k_{m_\alpha} l_{m_\alpha}} \varepsilon^{i_1 j_1 \cdots i_{m_\alpha} j_{m_\alpha}} R_{k_1 l_1 i_1 j_1} \cdots R_{k_{m_\alpha} l_{m_\alpha} i_{m_\alpha} j_{m_\alpha}}, \tag{3.58}$$

where $m_\alpha = d_\alpha/2$ and $E(m_\alpha)$ is such that it is 1 if $m_\alpha$ is an integer and zero otherwise, and the prime denotes that the determinant is computed in the normal bundle $\nu(\mathcal{M}_\alpha)$. The ratio $\det{}' \partial_i\partial_j W / |\det{}' \partial_i\partial_j W|$ has the same value all along a critical submanifold $\mathcal{M}_\alpha$. In fact, it is related to the index of a non-degenerate critical submanifold. Given a non-degenerate critical submanifold $\mathcal{M}_\alpha$, its index, $\lambda_\alpha$, is defined as the dimension of the subspace of the normal bundle $\nu(\mathcal{M}_\alpha)$ in which $H_{\mathcal{M}_\alpha}W$ has negative eigenvalue. In terms of this index,

$$\left.\frac{\det{}' \partial_i\partial_j W}{|\det{}' \partial_i\partial_j W|}\right|_\alpha = (-1)^{\lambda_\alpha}. \tag{3.59}$$

Using this expression together with (3.46) and (3.50) we finally obtain,

$$\chi(M) = \sum_\alpha (-1)^{\lambda_\alpha} \chi(\mathcal{M}_\alpha), \tag{3.60}$$

which is one of the standard results of degenerate Morse theory [15]. Notice that if we were in the situation in which $W$ has non-degenerate critical submanifolds as well as non-degenerate critical points, (3.60) would hold defining $\chi(\mathscr{M}_\alpha)$ as 1 when $\mathscr{M}_\alpha$ is a point.

In this paper we have concentrated on obtaining results of Morse theory using topological quantum mechanics. Similarly, one could study the case of vector fields instead of potentials. In this case one may also obtain standard results. For example, if one takes a vector field which contains isolated zeros, repeating the steps that led to (3.46) one obtains a proof of the Poincaré-Hopf theorem [19].

## 4. Conclusions

In this paper we have given an interpretation of topological quantum field theories in terms of a generalized form of Morse theory. This connection may be very helpful in deriving results using either techniques of Morse theory or the much less well defined techniques of quantum field theory. As a first example of this interplay we have considered the case of topological quantum mechanics. Although simple, this model has been shown to reveal some of the virtues of topological quantum field theories. We have been able to obtain a new proof of the Gauss–Bonnet–Chern–Avez formula and some results of standard degenerate Morse theory. There are two lessons that we learn from this type of analysis. First, topological quantum field theories formulate invariants in a much larger range of situations than other theories. Notice for example that Morse theory is able to prove the result (2.2) for a special case of functions $W$. However, from the point of view of topological field theories that result is valid for a wider range of potentials. A problem of a different kind is to determine which $W$ is more suitable to carry out the functional integral. As we have seen, and this leads us to the second lesson, even in the case when the potential $W$ is deformed to zero we still get the right result. But more important, in computing the topological invariant in two different ways we are able in this case to prove a classical result: the Gauss–Bonnet–Chern–Avez formula. There may be other situations in which this kind of analysis may lead to new theorems relating topological invariants. The case of topological quantum mechanics considered in this paper is rather simple and one does not expect big surprises. In more complicated situations topological quantum field theories may lead to either new invariants or to a better defined formulation of known ones, along with new theorems relating them.

The main ingredients of a topological field theory once the manifolds $\Sigma$ and $M$ have been chosen, are the topological class of mappings $V$ and the operator $F$. This operator is such that $V$ and $\mathrm{Im}\, V$ have the same dimensionality. For the case of topological sigma models [3], $\Sigma$ is a Riemann surface and $M$ is an almost Hermitian manifold. The operator $F$ transforms a mapping $\phi \in V$, defined by local coordinates $u^i$ into $\partial_\alpha u^i + \varepsilon_\alpha{}^\beta J^i{}_j \partial_\beta u^j$, where $J^i{}_j$ is the almost complex structure of $M$ and $\varepsilon_\alpha{}^\beta$ is the complex structure of $\Sigma$. This mapping is $n$ to $n$ if $n$ is the dimension of $M$ because the operator $F$ has the properties of a projector. This case is much richer in topological invariants than topological quantum mechanics since the index of the operator obtained from $F(\phi)$ after a variation of $\phi$ is in general different from zero.

Topological Yang-Mills with gauge group $G$ correspond to a situation in which $\Sigma$ is a compact 4-dimensional Riemannian manifold and $M$ is a vector bundle over the four-dimensional manifold with structure group $G$. The space of mappings $V$ is made out of gauge connections modulo gauge transformations on that bundle. The operator $F$ is such that to each gauge connection it associates the self-dual part of its field strength. The operator $F$ is $3 \times N$ to $3 \times N$, where $N$ is the dimension of the Lie group $G$. On one hand, the dimensionality of the space of gauge connections modulo gauge transformations is $(4-1) \times N$, on the other hand the independent components of the self-dual part of the field strength are $\frac{1}{2}(\frac{1}{2}4 \times 3) \times N$.

A similar analysis shows that topological gravity [11] might exist in four dimensions. In this case $\Sigma$ is again a four-dimensional compact Riemannian manifold and $V$ is made out of metrics modulo reparametrizations and Weyl transformations. The operator $F$ is such that it transforms a metric into the self-dual part of the corresponding Weyl tensor. The dimensionality of $V$ is $10\text{-}4\text{-}1 = 5$, while the number of independent components of the self-dual part of the Weyl tensor is $\frac{1}{2}10 = 5$. However, for the moment, no successful topological quantum field theory for topological gravity has been obtained [11]. It seems that it is not possible to obtain an action with a $Q$-symmetry (where $Q$ squares to zero modulo reparametrizations and Weyl transformations) which is also reparametrization and Weyl invariant.

In two dimensions there are situations where the counting of dimensionalities indicates that there may be interesting topological quantum field theories. For the case of gravity this analysis indicates why the attempts in [12, 13] have been unsuccessful. In two dimensions, the space of metrics modulo reparametrizations and Weyl transformations has zero dimensionality, $3\text{-}2\text{-}1 = 0$. However, Yang-Mills in two dimensions seems to be very promising. In this case the space $V$ is made out of gauge connections of a given vector bundle with structure group $G$ modulo gauge transformations, which has dimensionality $2\text{-}1 = 1$. The operator $F$ is the one that transforms a gauge connection into its field strength, which has only one independent component. The construction of the topological quantum field theory for this case is entirely analogous to topological Yang-Mills in four dimensions. One does not expect to obtain new topological invariants but it would be a very interesting exercise to work out this simple case in detail and to identify the resulting topological invariants. It may provide some insight about how to perform explicit calculations in topological quantum field theory. Furthermore, the extension of this case to consider non-compact gauge groups, in a similar spirit to the one in the formulation of $2+1$ dimensional gravity in [20], may lead one to obtain topological invariants related to the Mumford classes (which was one of the motivations in [12]) within the framework of topological quantum field theories.

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