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# The Two-Dimensional $O(N)$ Nonlinear $\sigma$-Model: Renormalisation and Effective Actions

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**Abstract.** We establish the existence of the Wilson Renormalised trajectory of the $O(N)$ sigma model in perturbation theory in the "effective charge." This yields a proof of perturbative renormalisability, and is also relevant in the "small-field" analysis of the rigorous renormalisation group construction of the continuum theory.

## 1. Introduction

The two-dimensional nonlinear $O(N)$ sigma model, with $N \geqq 3$, is perturbatively renormalisable and asymptotically free [10, 1, 2]. In this work we study the model from the Wilson Renormalisation Group (RG) viewpoint and show the existence of the renormalised trajectory in perturbation theory in the "effective charge." This yields in particular a proof of perturbative renormalisability and the expansion in the small-field region that would be part of a rigorous RG construction of the model. The approach is similar to that of J. Polchinski [9], where the $\lambda\phi^4$ theory in four dimensions is treated. There are, however, marked differences and surprising simplifications. We *do not* break the symmetry by applying a magnetic field, and the analysis is therefore not around the Gaussian fixed point. Only two marginal directions are involved, and these can be isolated very cleanly, yielding a surprisingly pleasant proof of renormalisability.

The model is the quantum field theory of maps $\mathbb{R}^2 \to S^{N-1}$. With a lattice cut-off $a$, the theory is defined by a $\mathbb{R}^N$-valued field $\phi$ on $a\mathbb{Z}^2 \subset \mathbb{R}^2$ with the constraint $\phi^2 = 1/Z_0(a)$, and the bare action

$$S(a) = \frac{Z_0(a)}{2g_0^2(a)} \sum_{\substack{x,y \in \mathbb{Z}^2 \\ x,y \text{ n.nbs}}} \frac{[\phi(x) - \phi(y)]^2}{2}.$$

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In applying Wilson renormalisation group (RG) techniques to the sigma model we immediately face the problem that the simplest and most appealing RG transformations destroy the delta-function constraint in the model. Here we adopt the following approach: drop the constraint $\phi^2 = 1/Z(\Lambda_0)$ and take as bare action

$$S(\Lambda_0) = \frac{1}{2g_0(\Lambda_0)^2} Z_0(\Lambda_0)(d\phi, F^{-1}(\sqrt{-\Delta}/\Lambda_0)d\phi) + \int \tilde{\lambda}(Z_0(\Lambda_0)\phi^2 - 1)^2 d^2x.$$

Here (,) denotes the scalar product of $\mathbb{R}^N$-valued functions, $\Delta$ is the laplacian and $\hat{F}$ is a momentum space cutoff function. The function $F$ is chosen such that it is smooth, positive, $F(0) = 1$, and $F \to 0$ at infinity; for example, we could take $F(x) = \exp - x^2$. Now that we have dropped the constraint on $\phi^2$ we can work with a momentum cutoff. This we do from now on. Then for large $\tilde{\lambda}$ we are approximating the sigma model. The idea is to let $\tilde{\lambda}$ go to infinity as $\Lambda_0 \to \infty$.

Of course, $\tilde{\lambda}$ has to go to infinity at some minimal rate. In [7] we implemented the above idea in conventional perturbation theory. After applying a magnetic field to break the symmetry and get perturbation theory started we showed that up to one loop we can recover the usual perturbative results, including the renormalisation constants, provided $\tilde{\lambda}$ goes to infinity at rate greater than or equal to

$$\tilde{\lambda} = \frac{\lambda \Lambda_0^2}{g_0(\Lambda_0)^2},$$

where $\lambda$ is an arbitrary positive cut-off independent constant.

Let us therefore rewrite the above action in terms of $\lambda$. The action then becomes:

$$S(\Delta_0) = \frac{1}{g_0(\Lambda_0)^2}\left( V(\phi, \Lambda_0) + \frac{Z_0(\Lambda_0)}{2}\left( d\phi, F^{-1}\left(\frac{\sqrt{-\Delta}}{\Lambda_0}\right)d\phi \right)\right), \tag{1-1}$$

where

$$V(\phi, \Lambda_0) = \int d^2x \lambda \Lambda_0^2 (Z_0(\Lambda_0)|\phi(x)|^2 - 1)^2. \tag{1-2}$$

The RG transformation acting on any $S(\Lambda_0)$ of the form (1-1), with $V(\Lambda_0)$ not necessarily of the form (1-2), produces for every $\Lambda < \Lambda_0$, an "effective action at scale $\Lambda$" by a process of integrating out degrees freedom. Let us denote this by $R(\Lambda, \Lambda_0)$. The RG transformations form a semigroup; we have, for $\Lambda_3 \leqq \Lambda_2 \leqq \Lambda_1$,

$$R(\Lambda_3, \Lambda_2)R(\Lambda_2, \Lambda_1) = R(\Lambda_3, \Lambda_1). \tag{1-3}$$

The problem of taking the continuum limit in field theory can be rephrased in terms of effective actions as follows: as $\Lambda_0 \to \infty$ arrange the dependence of the bare action $S(\Lambda_0)$ such that

$$S_{RN}(\Lambda) \equiv \lim_{\Lambda_0 \to \infty} R(\Lambda, \Lambda_0)S(\Lambda_0)$$

exists. Note that because of the semigroup property (1-3) of $R$ the renormalised effective actions $S_{RN}(\Lambda)$ obey

$$S_{RN}(\Lambda') = R(\Lambda', \Lambda)S_{RN}(\Lambda).$$

This is the *renormalised trajectory*.

In Sect. 2 we define the RG transformation in our context, and begin the study of the renormalised trajectory. In fact we study a differential form of RG, and continuously rescale back to unit cutoff. We write

$$S(t)=\frac{1}{g(t)^2}\left[V(\sqrt{Z(t)}\Phi,t)+\frac{Z(t)}{2}(d\Phi,F^{-1}(\sqrt{-\Delta})d\Phi)\right].$$

(We defer the definition of $g^2$ and $V$ in terms of $S$ till Sect. 2. $Z$ is so defined that $V$ takes its minimum on constant field configurations $\Phi$ with $|\Phi|=1$. This is implementable in perturbation theory.) Then $(V(t),g(t)^2,Z(t))$ satisfies a nonlinear evolution equation

$$\frac{d(V(t),g(t)^2,Z(t))}{dt}=\mathfrak{F}((V(t),g(t)^2,Z(t))). \tag{1-4}$$

We refer to this as the dimensionless form of the RG flow; we denote the field by $\Phi$ to signal the rescaling that has been done, and $t$ is defined by: $t=-\log\Lambda$. In terms of the flow (1-4) the scaling limit consists in doing the following: For $T>0$, $(T=\log\Lambda_0)$, solve (1-4) with initial condition

$$g(-T)=g_0(T),Z(-T)=Z_0(T),V(\Phi,-T)=\int d^2x\lambda(|\phi(x)|^2-1)^2. \tag{1-5}$$

Let $(V(t),g(t)^2,Z(t))_T$ denote the solution. Show that for suitable choices of $g_0(T)$ and $Z_0(T)$, both tending to 0 as $T\to\infty$,

$$(V(t),g(t),Z(t))_{RN}\equiv\lim_{T\to\infty}(V(t),g(t),Z(t))_T \tag{1-6}$$

exists. (The fact that $Z_0(T)\to 0$ is elementary because the spins $\phi$ of the original model are bounded. The fact that $g_0(T)\to 0$ is asymptotic freedom.)

Note that in (1-5) we do not have $\lambda$ depending on $T$. This is because we will find (in perturbation theory) that

i) The limit (1-6) exists for $\lambda$ any positive number.
ii) The limit is independent of the actual value of $\lambda$ chosen.

In other words $\lambda$ as well as the other couplings that arise in the RG flow hit fixed points. This is the explanation of the success of the perturbative computation of [7].

We now summarize the results of Sects. 2. to 6. We need some preliminary remarks. Fix $T>0$. Then (given $\lambda$) it is clear that the trajectories $(V(t),g(t),Z(t))_T$ are parametrised by $g_T(s)^2$ and $Z_T(s)$ for any fixed $s>-T$. Also

$$(V(t),g(t)^2,Z(t))_T\to(V(t),g(t)^2,\alpha Z(t))_T$$

with $\alpha>0$, is a symmetry of the evolution equation (1-4). This means that the trajectories, modulo this scaling of $Z$, are parametrised by $g_T(s)^2$. We can therefore write

$$V_T(t)\simeq\sum_{n=0}^{\infty}g_T(t)^{2n}V_T^{(n)}(t),$$

$$\frac{d\dfrac{1}{g_T(t)^2}}{dt} \simeq \sum_{n=0}^{\infty} g_T(t)^{2n} a_T^{(n)}(t),$$

$$\frac{1}{Z_T(t)}\frac{dZ_T(t)}{dt} \simeq g_T(t)^2 \sum_{n=0}^{\infty} g_T(t)^{2n} b_T^{(n)}(t).$$

In perturbation theory one can only gain access to the functional derivatives of $V$ at the minimum $\Phi = constant$, $|\Phi| = 1$. We shall use $V^{(n)}$ as a shorthand for "the derivatives of $V^{(n)}$ at the minimum configuration." Then the results of Sects. 3 to 5 can be summarised as follows:

i) As $T \to \infty$, $V_T^{(n)}$, $a_T^{(n)}$, and $b_T^{(n)}$ tend to finite limits which we denote by $V^{(n)*}$, $a^{(n)*}$ and $b^{(n)*}$ respectively.

ii) The leading coefficients:

$$a^{(0)*} = -\frac{(N-2)}{2\pi}, \quad b^{(0)*} = \frac{(N-1)}{2\pi},$$

are given by the standard asymptotic formulae, *independently of the specific form of F.*

We conclude with some comments and speculations in Sect. 6. In particular we make the connection with conventional perturbation theory and pose the question: how does a sigma model based on an arbitrary manifold fit into the current framework? We also show how a Wess–Zumino term can be incorporated.

An earlier study, incorporating the local approximation to the RG flow, was presented at the Ringberg Workshop 1987 [7] and this work itself at the Cargèse School, 1987 [14]. In contrast to the present work, in [14] the fields $\Phi$ are rescaled $\Phi \to g_0^2 \Phi$. We have found the present parameterisation more elegant. It should be mentioned that the local approximation is similar to the RG iteration of a certain hierarchical sigma model whose continuum limit was constructed in [4]. Also, the asymptotic freedom of the $O(N)$ models has recently been called into question. See [8, 11, 6].

## 2. The RG Flow

In this section we summarise the derivation/definition of the RG flow. We work formally; a less formal derivation would start with a volume cut-off. We are, however, only aiming at perturbative results—we shall see below that no infrared divergences occur in the RG flow equations. See Remark 2c.

We first set up notation. Let $\phi$ be a real $N$-component field in two dimensions, $\tilde{\phi}$ its Fourier transform. The Einstein summation convention holds. We shall denote by $p$ both a 2-momentum and its norm except when confusion might arise, when we denote $|p|$ the norm. Let

$$KE(\phi, \Lambda) = \tfrac{1}{2}\int \frac{d^2p}{(2\pi)^2} \hat{\phi}(p)\hat{\phi}(-p)p^2 F^{-1}(p/\Lambda),$$

$$S(\phi, \Lambda) = KE(\phi, \Lambda) + \tilde{U}(\phi, \Lambda),$$

$$E(J)=\exp\left\{-\frac{1}{g^2(\Lambda)}S(\sqrt{Z(\Lambda)}\,\phi)+\int J_i(p)\hat{\phi}(-p)\frac{d^2p}{(2\pi)^2}\right\},$$

$$\mathscr{Z}(J)=\frac{\int \mathscr{D}\phi E(J)}{\int \mathscr{D}\phi E(0)},$$

$$\frac{\partial}{\partial\hat{\phi}_i(p)}=\int\frac{d^2p}{(2\pi)^2}e^{ipx}\frac{\partial}{\partial\phi_i(p)}.$$

Here $F:\mathbb{R}^+\to\mathbb{R}^+$ is the cut-off function. We shall require: $F(0)=1$, $F'\leqq 0$ and that $F$ and its derivatives fall off to zero at least exponentially at $\infty$. We shall also require $F(|p|)$ to be a smooth function of the two-momentum. (Thus $F(s)=\exp-s^2$ is okay, but not $F(s)=\exp-s$). The effect of $F$ is to damp out the contribution of the high frequency modes in the definition of $\mathscr{Z}(J)$. We will strengthen the assumptions on $F$, at the end of this section.

We define the RG flow by the Eq. (2.1) [and (2.4), (2.5) below].

$$\begin{aligned}\frac{\Lambda\partial\tilde{U}}{\partial\Lambda}(\phi,\lambda)=&-g\left(\Lambda\frac{d}{d\Lambda}\frac{1}{g^2}\right)S(\phi,\Lambda)-\frac{\Lambda}{Z}\frac{dZ}{d\Lambda}\\&\cdot\left[KE(\phi,\Lambda)+\tfrac{1}{2}\int\frac{d^2p}{(2\pi)^2}\hat{\phi}_i(p)\frac{\partial\tilde{U}(\phi,\Lambda)}{\partial\hat{\phi}_i(p)}\right]+\tfrac{1}{2}\int\frac{d^2p}{p^2}(2\pi)^2\Lambda\frac{\partial}{\partial\Lambda}(F(p/\Lambda))\\&\cdot\left[\frac{\partial\tilde{U}}{\partial\hat{\phi}_i(p)}(\phi,\Lambda)\frac{\partial\tilde{U}}{\partial\hat{\phi}_i(-p)}(\phi,\Lambda)-g^2\frac{\partial^2\tilde{U}(\phi,\Lambda)}{\partial\hat{\phi}_i(p)\partial\hat{\phi}_i(-p)}\right].\end{aligned}\tag{2.1}$$

For motivation see remarks at the end of the section. We now transform the RG flow into "dimensionless form."

We define

$$\frac{1}{2}\frac{\Lambda^2}{p^2}\Lambda\frac{\partial}{\partial\Lambda}(F(p/\Lambda))=-\frac{1}{2}\frac{\Lambda}{p}F'(p/\Lambda)\equiv\hat{K}(p/\Lambda).\tag{2.2}$$

Thus $K(x)$ will be the inverse Fourier transform of $\hat{K}$. We put $\Lambda\partial/\partial\Lambda=-\partial/\partial t$. Finally we switch to dimensionless fields $\hat{\Phi}_i(p)=\hat{\phi}_i(p\Lambda)\Lambda^2$ or equivalently, $\Phi_i(x)=\phi_i(x/\Lambda)$, and define $U(\Phi)=\tilde{U}(\phi)$. Then the RG flow becomes

$$\begin{aligned}\frac{\partial U}{\partial t}(\Phi,t)=&-g^2\left(\frac{d}{dt}\frac{1}{g^2}\right)S(\Phi,t)-\frac{1}{Z}\frac{dZ}{dt}\left[KE(\Phi)+\tfrac{1}{2}\int d^2p\,\hat{\Phi}_i(p)\frac{\partial U(\Phi,t)}{\partial\hat{\Phi}_i(p)}\right]\\&+\int d^2p\frac{\partial U(\Phi,t)}{\partial\hat{\Phi}_i(p)}\partial_\mu(p_\mu\hat{\Phi}_i(p))\\&-\int d^2p(2\pi)^2\hat{K}(p)\left[\frac{\partial U(\Phi,t)}{\partial\hat{\Phi}_i(p)}\frac{\partial U(\Phi,t)}{\partial\hat{\Phi}_i(-p)}-g^2\frac{\partial^2U(\Phi,t)}{\partial\hat{\Phi}_i(p)\partial\hat{\Phi}_i(-p)}\right]\end{aligned}$$

$$S(\Phi,t)=KE(\Phi)+U(\Phi,t),$$

with

$$KE(\Phi)=\tfrac{1}{2}\int\frac{d^2p}{(2\pi)^2}p^2F^{-1}(p)\hat{\Phi}_i(p)\hat{\Phi}_i(-p).\tag{2.3}$$

We have not yet completed the specification of the RG flow, since the evolution of $g^2$ and $Z$ is not yet given. We do that now in two steps

i) $Z$ is determined by the assumption

$$U \text{ achieves a minimum on constant field configurations } \Phi \equiv v \in \mathbb{R}^N, \quad |v| = 1. \tag{2.4}$$

Notation: From now on $v$ will denote a vector of unit length in $\mathbb{R}^N$.

ii) Define

$$\hat{\alpha}_{j_1 j_2}(p_1, p_2; t) \equiv \left.\frac{(2\pi)^4 \partial U(\Phi, t)}{\partial \hat{\Phi}_{j_1}(p_1) \partial \hat{\Phi}_{j_2}(p_2)}\right|_{\Phi \equiv v}.$$

By translational invariance

$$\hat{\alpha}_{j_1 j_2}(p_1, p_2; t) = (2\pi)^2 \delta(p_1 + p_2) \beta_{j_1 j_2}(p_2; t).$$

Further, by rotational invariance in $\mathbb{R}^2$ and $O(N)$ invariance

$$\beta_{j_1 j_2}(p; t) = \beta(|p|; t) v_{j_1} v_{j_2} + \beta_\delta(|p|; t) \delta_{j_1 j_2}.$$

Since $\Phi \equiv v$ is a minimum of $U, \beta_\delta(0; t) = 0$ [see below for a proof]. We define $g^2$ by the assumption

$$\beta_\delta(|p|; t) = O(|p|^4). \tag{2.5}$$

*Remarks.*

2a) Consider the assumption

$$F(s) = 1 \quad \text{for} \quad s \leqq 1. \tag{2.6}$$

The flow (2.1) is designed such that, under assumption (2.6),

$$\Lambda \frac{\partial z}{\partial \Lambda} \mathscr{Z}(J, \Lambda) = 0 \quad \text{if} \quad \operatorname{supp} J(p) \subset \{p \,|\, |p| < \Lambda\}. \tag{2.7}$$

We leave it to the reader to check this. Note that (2.7) implies that if $\langle \hat{\phi}_{j_1}(p_1) \cdots \hat{\phi}_{j_r}(p) \rangle_{S(\Lambda)}$ denote the correlating function evaluated with respect to $S(\Lambda)$ then

$$\frac{\Lambda \partial}{\partial \Lambda} \langle \hat{\phi}_{j_1}(p_1) \cdots \hat{\phi}_{j_r}(p_r) \rangle_{S(\Lambda)} = 0 \quad \text{if} \quad |p_i| < \Lambda \quad \text{for} \quad i = 1, \ldots, r.$$

2b) If we know the evolution of $(U(t), g^2(t), Z(t))$ with $U$ rotationally symmetric with respect to $O(N)$, under (2.3), it is easy to add a magnetic field and trace through the change. Let $U_M(t) = U + h(t) \int \Phi_1(x) d^2x = U + h(t) \hat{\Phi}_1(0)$. The reader can check that $(U_M(t), g^2(t), Z(t))$ is a solution of (2.3) provided

$$\frac{\partial h(t)}{\partial t} = -g^2 \left( \frac{d}{dt} \frac{1}{g^2} \right) h - \left( \frac{1}{Z} \frac{dZ}{dt} \right) h + 2h,$$

and assumption (2.6) holds—in fact it suffices that $\hat{K}(0) = 0$, for then the potential "cross-term" vanishes. Note that this is solved by

$$h(t) = h_r \frac{g^2(t)}{Z(t)} e^{2t},$$

where $h_r$ is the dimensionless renormalised magnetic field. In other words, the magnetic field is not independently renormalised.

2c) If condition (2-6) is not satisfied, we can still evaluate $(\Lambda\partial/\partial\Lambda)\mathscr{Z}(J)$ and show that the result vanishes as $\Lambda \to \infty$, so that (2-3) is still asymptotically valid. In the subsequent sections, *we shall in fact assume* $\hat{K}(0) \neq 0$. Also, we shall also assume $F$ is $C^\infty$ and *has compact support*. See Remarks 4b) and 4e) for an explanation. Note that $K(x \dot{-} y)$ has fast decay and thus no infrared divergences occur in the RG equation (2.3).

## 3. Statement of the Main Result, Preparations for Proof

Note that as claimed in the introduction, the RG flow has the symmetry

$$(U(t), g^2(t), Z(t)) \to (U(t), g^2(t), \alpha Z(t))$$

for any constant $\alpha > 0$. This is clear from Eq. (2.3) where $Z$ occurs only in the combination $(1/Z)(dZ/dt)$. This implies that trajectories, modulo this rescaling, are parametrized by $g^2(t)$ for any fixed $t$, so that we can write

$$U(\phi, t) \simeq \sum_{n=0}^{\infty} g(t)^{2n} U^{(n)}(\Phi, t),$$

$$\frac{d}{dt}\frac{1}{g^2(t)} \simeq \sum_{n=0}^{\infty} g(t)^{2n} a^{(n)}(t),$$

$$\frac{1}{Z(t)}\frac{dZ(t)}{dt} \simeq g^2(t) \sum_{n=0}^{\infty} g(t)^{2n} b^{(n)}(t). \tag{3.1}$$

(We shall prove below that the $(1/Z)(dZ/dt)$ is of order $g^2(t)$.) This is, of course, just a concise way of organising and working with "derivatives with respect to the trajectory," the trajectory being parametrised by $g^2(t)$ and $Z(t)$. We have suppressed reference to the initial time $t = -T$. This we shall continue to do except when necessary.

We introduce the notation

$\mathscr{D}_n(t) = \{\text{set of derivatives of } U^{(1)}, U^{(2)}, \ldots, U^{(n)} \text{ on } \Phi \equiv v\}$
$\mathscr{A}_n(t) = \{a^{(0)}(t), \ldots, a^{(n)}(t)\}$
$\mathscr{B}_n(t) = \{b^{(0)}(t), \ldots, b^{(n)}(t)\}$.

We set $\mathscr{A}_n = \mathscr{B}_n = \text{empty set for } n < 0$.
We have

**Proposition.** *The following inductive scheme holds: for* $n \geqq 0$,

i) $\{\mathscr{A}_{n-1}, \mathscr{B}_{n-i}, \mathscr{D}_n\}(t)$ *determines* $\{\mathscr{A}_n, \mathscr{B}_n, \mathscr{D}_n\}(t)$,
ii) $\{\mathscr{A}_n, \mathscr{B}_n, \mathscr{D}_n\}(t)$ *determines the evolution of* $\{\mathscr{A}_n, \mathscr{B}_n, \mathscr{D}_{n+1}\}(t)$.

It is clear from the above proposition that $\{\mathscr{A}_n, \mathscr{B}_n, \mathscr{D}_n\}(t)$ are determined by induction via the evolution equation and the initial condition

$$U = U^{(0)} = \lambda \int (|\Phi|^2 - 1)^2 d^2x \tag{3.2}$$

at $t = -T$. Note that the initial values of $g^2$ and $Z$ are *not* involved. We can now state our main result:

**Theorem.**

i) *For fixed* $T$, $\{\mathscr{A}_{n,T}(t), \mathscr{B}_{n,T}(t), \mathscr{D}_{n,T}(t)\}$ *exist for all* $t \geqq -T$.
ii) $\lim_{T\to\infty} \{\mathscr{A}_{n,T}(t), \mathscr{B}_{n,T}(t), \mathscr{D}_{n,T}(t)\} \equiv \{\mathscr{A}_n^*, \mathscr{B}_n^*, \mathscr{D}_n^*\}$ *exists and is independent of* $t$.
iii) *The limit is independent of* $\lambda$ *in (3.2)*
iv)

$$a^{*,(0)} = -\frac{(N-2)}{2\pi},$$

$$b^{*,(0)} = \frac{(N-1)}{2\pi} \tag{3.3}$$

*in agreement with conventional perturbation theory, independent of the particular choice of* $F$.

*Remarks.*

3a) The theory is renormalised by letting the dimensional coupling constant $\tilde{\lambda}$ go to infinity, while the dimensionless $\lambda$ hits a fixed point under RG. Note the similarity with the Wilson–Fisher fixed point. In our case however no mass-tuning is needed.
b) Note that we have not referred to the bare couplings $g_0$ and $Z_0$ in the statement of the theorem. Or course since this is an asymptotically free theory it is enough to choose $g_0$ and $Z_0$ as given by the two-loop $\beta$ function to stabilize the flow.
c) The theorem, as it stands, suffices to prove ultra-violet finiteness of Green's functions. We give the brief argument, assuming infra-red finiteness. Consider an invariant Green's function

$$\langle \varphi_{j_1}(x_1)\varphi_{j_1}(y_1)\cdots\varphi_{j_r}(x_r)\varphi_{j_r}(y_r)\rangle_{S^*_{(\Lambda)}} \equiv G(x_1, y_1, \ldots, x_r, y_r; \Lambda).$$

We assume this has an infrared finite (asymptotic) expansion

$$\Sigma g^*(\Lambda)^{2m} \hat{G}_m(x_1, y_1, \ldots, x_r, y_r; \Lambda).$$

If $\Lambda_f$ is any fixed momentum scale, we have

$$g^*(\Lambda)^2 = \sum_m A_m(\Lambda, \Lambda_f) g^*(\Lambda_f)^{2m},$$

so that we can write

$$G(x_1, y_1, \ldots, x_r, y_r; \Lambda) = \Sigma g^*(\Lambda_f)^{2m} G_m(x_1, y_1, \ldots, x_r, y_r; \Lambda, \Lambda_f).$$

By Remarks 2a), 2c)

$$\frac{\Lambda \partial G}{\partial \Lambda} \to 0 \quad \text{as} \quad \Lambda \to \infty,$$

and this shows that as $\Lambda \to \infty$ the $G_m$ have finite limits.

The rest of this section is devoted to setting up notation and proving the proposition.

*Notation.* We first set up notation for functional derivatives of $U$. Given $(j_1, \dots, j_r)$, each $1 \leqq j_i \leqq N$, and points $(x_1, \dots, x_r)$, each $x_i \in \mathbb{R}^2$, we let

$$\alpha_{j_1 \cdots j_r}(x_1, \dots, x_r; t) = \left. \frac{\partial^r U(\Phi, t)}{\partial \Phi_{j_1}(x_1) \cdots \partial \Phi_{j_r}(x_r)} \right|_{\Phi \equiv v}.$$

We shall let $\hat{\alpha}$ denote the Fourier transform of $\alpha$. We have

$$\hat{\alpha}_{j_1 \cdots j_r}(p_1, \dots, p_r; t) = (2\pi)^{2r} \left. \frac{\partial^r U(\Phi, t)}{\partial \hat{\Phi}_{j_1}(-p_1) \cdots \partial \hat{\Phi}_{j_r}(-p_r)} \right|_{\Phi \equiv v}.$$

We can write, by translational invariance

$$\hat{\alpha}_{j_1 \cdots j_r}(p_1, \dots, p_r; t) = (2\pi)^2 \delta\left( \sum_{i=1}^{r} p_i \right) \beta_{j_1 \cdots j_r}(p_2, \dots, p_r; t). \tag{3.4}$$

We shall let $R$ denote the sequence $(1, 2, \dots, r)$, $R'$ the sequence $(2, \dots, r)$, and $S_1, S_2$ etc. denote subsequences of $R$, their cardinality being denoted by $|S_1|, |S_2|$ etc. Thus $r = |R|$. We will also use the abbreviations $j_R = (j_1, \dots, j_r)$, etc. Thus, for example, we can rewrite (3.4) as

$$\hat{\alpha}_{j_R}(p_R; t) = (2\pi)^2 \delta\left( \sum_i p_i \right) \beta_{j_R}(p_{R'}; t)$$

By the $O(N)$ invariance, the tensors $\beta_{j_R}$ are in the tensor algebra generated by the vectors $v$ and the identity operator—a typical such tensor being

$$t_{j_1 j_2 j_3 j_4 j_5 j_6} = v_{j_1} \delta_{j_2 j_3} \delta_{j_4 j_5} v_{j_6}.$$

We write $\beta_{j_R} = \sum_{n=0}^{|R|} \beta_{j_R}^{[a]}$, where $\beta_{j_R}^{[a]}$ is the component with "$a$" factors of $v$ in them. This is well-defined since tensors with different numbers of "$v$-factors" are linearly independent. Thus for $t$ above, $t = t^{[2]}$.

By differentiating (3.5) $r$ times, and evaluating at $\Phi \equiv v$ we get:

$$\begin{aligned}
\frac{\partial}{\partial t} \beta_{j_R}(p_{R'}; t) = & -g^2 \left( \frac{d}{dt} \frac{1}{g^2} \right) [(KE)_{j_R}(p_{R'}; t) + \beta_{j_R}(p_{R'}; t)] \\
& - \frac{1}{Z} \frac{dZ}{dt} \left[ (KE)_{j_R}(p_{R'}; t) + \tfrac{1}{2} v_i \beta_{i j_R}(p_R; t) + \frac{r}{2} \beta_{j_R}(p_{R'}; t) \right] \\
& + 2\beta_{j_R}(p_{R'}; t) - \sum_{\substack{l=2 \\ \mu=1,2}}^{r} p_{l\mu} \frac{\partial}{\partial p_{l\mu}} \beta_{j_R}(p_{R'}; t) \\
& - \sum_{S_1 \cup S_2 = R} \hat{K}\left( \left| \sum_{m \in S_1} p_m \right| \right) \beta_{i j_{S_1}}(p_{S_1}; t) \beta_{i j_{S_2}}(p_{S_2}; t) \\
& + \frac{g^2}{(2\pi)^2} \int d^2 p \hat{K}(p) \beta_{i i j_R}(p, p_R; t),
\end{aligned} \tag{3.5}$$

where

$$(KE)_{j_R}(p_{R'};t)=\begin{cases}0 & \text{if } |R|\neq 2\\ p_2^2F^{-1}(p_2)\delta_{j_1j_2} & \text{if } |R|=2\end{cases}.$$

This is the basic equation we shall deal with.

Corresponding to the expansion (3.1) we have

$$\beta_{j_R}(p_{R'};t)\simeq\sum_{n=0}^{\infty}\beta_{j_R}^{(n)}(p_{R'};t)g^{2n}(t).$$

The initial condition (3.2) translates into

$$\beta_{j_R}(p_{R'};-T)=\beta_{j_R}^{(0)}(p_{R'};-T)=\left.\frac{\partial^r u(|w|)}{\partial w_{j_1}\cdots\partial w_{j_r}}\right|_{w=v}, \tag{3.6}$$

independent of $p'_{R'}$ for

$$u(|w|)=\lambda(|w|^2-1)^2.$$

We can now give the

a) *Proof that* $(1/Z)(dZ/dt)$ *is of order* $g^2(t)$: Reading (3.5) to order 0 for $r=1$ we get $(\partial/\partial t)\beta_{j_1}^{(0)}(t)=-((1/Z)(dZ/dt))^{(0)}((2\pi)^2/2)v_i\beta_{ij_1}^{(0)})(0;t)+$ terms linear in $\beta_{j_1}^{(0)}(t)$. Requiring $\beta_{j_1}^{(0)}(t)=0$ yields $((1/Z)(dZ/dt))^{(0)}=0$ as long as $v_i\beta_{ij_1}^{(0)}(0;t)\neq 0$ which we shall show later to be the case. ■

b) *Proof of Proposition*: Part ii) is trivial. Note however that an induction on $|R|$ is required and is possible because the $|R+2|^{\text{th}}$ derivative appears in a lower order in $g^2(t)$. This is in contrast to [9] and a major simplification.

We now turn to part i) of the Proposition. Consider (3.5) for $r=1$, put $\beta_{j_1}^{(n')}=0$ for $n'\leqq n$, $j_1=1,2,\dots,N$, then:

$$\frac{\partial}{\partial t}\beta_{j_1}^{(n+1)}(t)=-\left(\frac{1}{Z}\frac{dZ}{dt}\right)^{(n+1)}\frac{(2\pi)^2}{2}v_i\beta_{ij_1}^{(0)}(0;t)+\text{terms showing } b^{(0)},\dots,b^{(n-1)}$$
$$+\frac{1}{(2\pi)^2}\int d^2p\hat{K}(p)\beta_{iij_1}^{(n)}(p,0;t)+\text{terms linear in } \beta_{j_1}^{(n+1)}(t).$$

Requiring $\beta_{j_1}^{(n+1)}(t)=0$ yields $b^{(n)}(t)$.

A similar proof now works for $a^{(n)}(t)$. ■

## 4. Convergence of $U^{(0)}$

We adopt the notation $\sigma_{j_R}(p_{R'};t)=\beta_{j_R}^{(0)}(p_{R'};t)$. We have

$$\begin{aligned}\frac{\partial\sigma_{j_R}}{\partial t}(p_{R'};t)=&-\sum_{l,\mu}p_{l\mu}\frac{\partial}{\partial p_{l\mu}}\sigma_{j_R}(p_{R'};t)+2\sigma_{j_R}(p_{R'};t)\\&-2\sum_{l\in R}\hat{K}(p_l)\sigma_{ij_l}(p_l;t)\sigma_{j_{R(l,i)}}(p_{R'};t)\\&-\sum_{\substack{S_1\cup S_2=R\\|S_1|,|S_2|\geqq 2}}\hat{K}\left(\left|\sum_{m\in S_1}p_m\right|\right)\sigma_{ij_{S_1}}(p_{S_1};t)\sigma_{ij_{S_2}}(p_{S_1};t),\end{aligned}\tag{4.1}$$

where we have used the fact that for $S_1 = R\backslash\{l\}\, \beta_{ij_{S_1}}(p_{S_1};t) = \beta_{j_{R(l,i)}}(p_{R'};t)$ with $j_{R(l,i)} = (j_1, j_2, \ldots, i \cdots j_r)$ with $i$ in the $l^{\text{th}}$ place. This follows from

$$\hat{\alpha}_{ij_1\cdots j_l \cdots j_r}(p_l, p_1, \ldots, \hat{p}_l \cdots p_{R'};t) = \hat{\alpha}_{j_1\cdots i \cdots j_r}(p_1, \ldots, p_l, \ldots, p_r;t),$$

where the ^ over $p_l$ or $j_l$ means that it is omitted.

Before we state the main results of this section we set up a little notation: the letter $\mathscr{C}$ will denote a "varying constant." independent of $t$, but in general depending on the order of perturbation in $g^2$ and number of derivatives. We will fix the constant $b$ (to appear soon) in a while; it could, for example, equal 1.

We shall also make a translation in time so that $-T \to 0$. Thus convergence will be achieved if $t \to \infty$ rather than $T \to \infty$ for fixed $t$.

**Lemma 4.1.** *There exist $\sigma^*_{j_R}$ such that $|\sigma_{j_R}(0;t) - \sigma^*_{jR}| < \mathscr{C}e^{-bt}$.*

*Proof.* Let $\sigma_{j_R}(t) \equiv \sigma_{j_R}(p_{R'} = 0;t)$. Then

$$\frac{\partial \sigma_{j_R}}{\partial t}(t) = 2\sigma_{j_R} - \sum_{S_1 \cup S_2 = R} \hat{K}(0)\sigma_{ij_{S_1}}(t)\sigma_{ij_{S_2}}(t).$$

It is easily verified that this is solved by

$$\sigma_{j_R(t)} = \left.\frac{\partial^r u(|w|,t)}{\partial w_{j_1} \cdots \partial w_{j_r}}\right|_{w=v}$$

with

$$\frac{\partial u}{\partial t}(|w|,t) = 2u - \hat{K}(0)\left(\frac{\partial u}{\partial |w|}\right)^2, \quad u(|w|,0) = \lambda(|w|^2 - 1)^2.$$

One can show that $u \to ((|w| - 1)^2/2\hat{K}(0))$, but we shall only show convergence of derivatives at $|w| = 1$ since this is easier, and illustrative of what follows later. Let

$$d_n(t) = \left.\frac{\partial^n u}{\partial |w|^n}\right|_{|w|=1}.$$

We have then

$$\frac{dd_1(t)}{dt} = 2d_1(t) - 2d_1(t)d_2(t)\hat{K}(0),$$

and since $d_1(0) = 0$, we have $d_1(t) = 0$ for all $t \geqq 0$. Again

$$\frac{dd_2(t)}{dt} = 2d_2(t)(1 - d_2(t)\hat{K}(0)),$$

which is solved by

$$\hat{K}(0)d_2(t) = \frac{Ce^{2t}}{1 + Ce^{2t}}, \quad \text{with} \quad \frac{C}{1+C} = 8\lambda\hat{K}(0).$$

Note that $d_2 \to d_2^* \equiv (1/\hat{K}(0))$, and

$$|d_2(t) - d_2^*| \leqq \mathscr{C}e^{-2t}.$$

We shall choose $b$ to be a constant such that $1 \leqq b < 2$.

We show by induction that the higher derivatives converge: $d_n(t) \to d_n^*$, $|d_n(t) - d_n^*| \leqq \mathscr{C}e^{-bt}$. In fact $d_n^* = 0$ for $n \geqq 3$, but we will not use this. The evolution equation for $d_n(t)$, $n \geqq 3$, is

$$\frac{dd_2(t)}{dt} = 2d_n(t) - 2n\hat{K}(0)d_2(t)d_n(t) + G_n(t),$$

$$G_n(t) = -\sum_{l=2}^{n-2} \hat{K}(0)\binom{n}{l} d_{l+1}(t)d_{n-l+1}(t). \tag{4.2}$$

Assume $|d_{n'}(t) - d_{n'}^*| \leqq \mathscr{C}e^{-bt}$ for $n' < n$. Then $G_n(t) \to G_n^*$, $|G_n^{(t)} - G_n^*| < \mathscr{C}e^{-bt}$. We can write the solution to (4.2) in terms of

$$\theta_n(t) = \exp 2n \int_0^t \hat{K}(0)d_2(s)ds = \left(\frac{1 + Ce^{2t}}{1+C}\right)^n. \tag{4.3}$$

Then

$$d_n(t) = e^{2t}\theta_n^{-1}(t)\left\{\int_0^t e^{-2s}\theta_n(s)G_n(s)ds + d_n(0)\right\}$$

$$= \int_0^t e^{-2(s-t)}\left(\frac{1 + Ce^{2s}}{1 + Ce^{2t}}\right)^n F_n(s)ds + d_n(0)e^{2t}\left(\frac{1 + Ce^{2t}}{1+C}\right)^{-n}.$$

From this it trivially follows that if $d_n^* = G_n^*/2n - 2$, then

$$|d_n(t) - d_n^*| < \mathscr{C}e^{-bt}. \quad \blacksquare$$

*Remark 4a*). The above convergence was first observed as the local approximation, cf. [7].

We now prove convergence of $\sigma_{j_R}$, $|R| = 2$, at nonzero momenta.

**Lemma 4.2.** *Let us write* $\sigma_{j_1j_2}(p;t) = \sigma(p;t)v_{j_1}v_{j_2} + \Delta(p;t)\delta_{j_1j_2}$. *Then* $\Delta(p;t) = 0$ *and* $\sigma(p;t) \to \sigma^*(p)$ *uniformly on compact sets.*

*Proof.* The evolution equation for $\sigma_{j_1j_2}$ is

$$\frac{\partial}{\partial t}\sigma_{j_1j_2}(p;t) = -p\frac{\partial}{\partial p}\sigma_{j_1j_2}(p;t) + 2\sigma_{j_1j_2}(p;t) - 2\Sigma\hat{K}(p)\sigma_{j_1i}(p;t)\sigma_{ij_2}(p;t), \tag{4.4}$$

and the initial condition

$$\sigma_{j_1j_2}(p;0) = 8\lambda v_{j_1}v_{j_2}. \tag{4.5}$$

Equation (4.4) and the initial condition (4.5) imply $\Delta(p;t) = 0$. We can now write

$$\frac{\partial}{\partial t}\sigma(p;t) = -p\frac{\partial}{\partial p}\sigma(p;t) + 2\sigma(p;t) - 2\hat{K}(p)\sigma^2(p;t), \tag{4.6}$$

which gives

$$\frac{\partial}{\partial s}\frac{1}{\sigma(pe^s;t+s)} = -\frac{2}{\sigma(pe^s;t+s)} + 2\hat{K}(pe^s).$$

This can be solved to give

$$\frac{1}{\sigma(p;t)}=\frac{1}{\sigma(pe^{-t};0)}e^{-2t}+2\int_{e^{-t}}^{1}\hat{K}(qp)qdq=\frac{1}{8\lambda}e^{-2t}+\frac{1}{p^2}\{F(pe^{-t})-F(p)\}, \quad (4.7)$$

where the last equation holds for $p \neq 0$. (cf. definition of $\hat{K}$, Eq. (2.2)). We shall now prove convergence, with

$$\sigma^*(p)^{-1}=2\int_0^1 \hat{K}(qp)qdq=\frac{2}{p^2}\int_0^p \hat{K}(q)qdq.$$

We have, for $p$ ranging over a compact set, trivial inequalities $|\sigma(p,t)| \leqq \mathscr{C}e^{+2t}$, and $|(1/\sigma)-(1/\sigma^*)| \leqq \mathscr{C}e^{-2t}$ which give

$$|\sigma-\sigma^*|=|\sigma\sigma^*|\left|\frac{1}{\sigma}-\frac{1}{\sigma^*}\right| \leqq \mathscr{C}\sigma^*, \quad (4.8)$$

which in turn yields

$$|\sigma-\sigma^*| \leqq |\sigma^*|(|\sigma^*|+|\sigma-\sigma^*|)\left|\frac{1}{\sigma}-\frac{1}{\sigma^*}\right| \leqq \mathscr{C}|\sigma^*|^2 e^{-2t}. \quad \blacksquare$$

*Remarks.*

4b) The expression for $\sigma^*$ shows the necessity for the assumption $\hat{K}(0) \neq 0$. But for this the limit $\sigma^*$ would be $+\infty$ at some points. This can probably be interpreted as a $\delta$-function constraint on the low-frequency components of the field, and in fact the convergence on higher orders will hold unchanged since in their evolution equation $\sigma(p;t)$ only occurs in the combination $\hat{K}(p)\sigma(p;t)$ which has a finite limit.
4c) $\sigma(0;t)=d_2(t)$ in the notation of the proof of Lemma 4.1.
4d) Note that $\sigma$ satisfies

$$|\sigma| \leqq \mathscr{C}(1+p^2). \quad (4.9)$$

4e) In deriving (4.9) uniformly in $p$ we have used the assumptions that $F$ is of compact support. This inequality will enable us to apply the dominated convergence theorem.

**Lemma 4.2b.** *Let $D\sigma_{j_1j_2}$ denote any momentum derivative of $\sigma_{j_1,j_2}$:*

$$D\sigma_{j_1,j_2}(p;t)=\frac{\partial^{c_1+c_2}}{\partial p_1{}^{c_1}\partial p_2{}^{c_2}}\sigma_{j_1j_2}(p;t). \quad (4.10)$$

*Then $D\sigma_{j_1j_2}(p;t) \xrightarrow[t\to\infty]{} D\sigma^*_{j_1,j_2}(p)$ uniformly on compact sets. Also*

$$|D\sigma_{j_1j_2}| \leqq \mathscr{C}(1+p^2). \quad (4.11)$$

*Proof.* It is enough to show pointwise convergence and the bound (4.11). Also it is enough to show convergence on $[0,\infty]$ of $\sigma(|p|,t)$ and its derivatives and bound them because for finite time these are smooth functions of the two-momentum $p$.

By differentiating (4.7) with respect to $p$, and noting that $\sigma(p;0)$ is a constant,

$$\sigma'(p;t)=-2\sigma^2\int_{e^{-t}}^{1}\hat{K}'(qp)q^2dq,$$

which shows pointwise convergence as well as

$$|\sigma'(p,t)| \leqq \frac{\mathscr{C}\sigma^1}{1+p^3} \leqq \mathscr{C}(1+p^2).$$

Differentiating further and by induction we are done. In fact, the bound (4.11) can be much improved. ∎

We next prove convergence of the second $p$-derivative of $\sigma_{j_R}(p_{R'};t)$ at $p_{R'}=0$. For any function $f$ of $p_{R'}$ let

$$\tilde{f}(p_{R'}) = \frac{1}{2}\sum_{l,m=2}^{r} \left.\frac{\partial^2 f}{\partial p_{l\mu}\partial p_{m\mu}}\right|_{(p_{R'}=0)} p_{l\mu}p_{m\mu}.$$

Also recall the notation from Sect. 3: $\sigma_{j_R}^{[a]}$ denotes the component of $\sigma_{j_R}$ what '$a$' factors of $v$. From (4.1) we get

$$\frac{\partial\tilde{\sigma}_{j_R}^{[a]}(p_{R'};t)}{\partial t} = 2a\hat{K}(0)\sigma(t)\tilde{\sigma}_{j_R}^{[a]}(p_{R'};t) + \tilde{R}_{j_R}^{[a]}(p_{R'};t), \tag{4.12}$$

where $\tilde{R}_{j_R}^{[a]}(t)$ is a quadratic function of $\sigma_{j_S}(0;t)$, $|S|<|R|$ and $\tilde{\sigma}_{j_S}^{[b]}$ $b<a$, $|s|\leqq R$.

**Lemma 4.3.**

i) *Let* $|R|>2$, $a\geqq 1$. *Then* $\tilde{\sigma}_{j_R}^{[a]}(p_{R'};t)\to \tilde{R}_{j_R}^{[a]*}/2a$. *In fact*

$$\left|\tilde{\sigma}_{j_R}^{[a]}(p_{R'};t) - \frac{\tilde{R}_{j_R}^{[a]*}(p_{R'})}{2a}\right| < \mathscr{C}|p_{R'}|^2 e^{-bt}, \tag{4.13}$$

*where* $|p'_R|^2 = \sum_{l\in R'} p_l^2$.

ii) *There exist* $\tilde{\sigma}_{j_R}^{[a]*}$ *such that* $|\tilde{\sigma}_{j_R}^{[0]}(p_{R'};t) - \tilde{\sigma}_{j_R}^{[0]*}(p_{R'})| \leqq \mathscr{C}|p_{R'}|^2 e^{-bt}$.

*Proof.* We work by induction on $|R|$ and $a$. Assume that

$$|\tilde{\sigma}_{j_S} - \tilde{\sigma}_{j_S}^*| \leqq \mathscr{C}|p_{S'}|^2 e^{-bt} \quad \text{for} \quad |S|<R$$

and

$$|\tilde{\sigma}_{j_R}^{[a']} - \tilde{\sigma}_{j_R}^{[a']*}| \leqq \mathscr{C}|p_{R'}|^2 e^{-bt} \quad \text{for} \quad a'<a.$$

Assume now $a>0$. For $a=0$ we require a different argument which we postpone. From (4.12) we have

$$\tilde{\sigma}_{j_R}^{[a]*}(p_{R'};t) = \theta_a^{-1}(t)\int_0^t ds\,\theta_a(s)\tilde{R}_{j_R}^{[a]}(s)ds,$$

where

$$\theta_a(s) = \exp 2a\int_0^t \hat{K}(0)\sigma(s)dl = \left(\frac{1+Ce^{2t}}{1+C}\right)^a.$$

We now imitate the proof of Lemma (4.1).

We now turn to case $a=0$.

For $R$ odd the result is vacuously true. Consider the case $R$ even, $\geqq 4$. From the $O(N)$ invariance of $U(\Phi,t)$ it follows that for any antisymmetric $\varepsilon_{ij}$ we have

$$\varepsilon_{ij}v_i\hat{\alpha}_{jj_2\cdots j_r}(0,p_2\cdots p_r;t)=\varepsilon_{ij_2}\hat{\alpha}_{ij_3\cdots j_r}(p_2,\dots,p_r;t)+\varepsilon_{ij_3}\hat{\alpha}_{j_2ij_3\cdots j_r}(p_2,\dots,p_r;t)$$
$$+\cdots+\varepsilon_{ij_r}\hat{\alpha}_{j_2j^3\dots i}(p_2,\dots,p_r;t). \tag{4.14}$$

By induction on $r$ the left-hand side converges. Letting $\varepsilon_{ij}=\delta_{il}\delta_{jm}-\sigma_{im}\delta_{il}$ for fixed $l,m$, we obtain one convergence of $v_m\beta_{lj_2\cdots j_r}(p_{R'};t)-v_l\beta_{mj_2\cdots j_r}(p_{R'};t)$ when $\sum\limits_{l\in R'}p_l=0$. Since these are linearly independent tensors we get convergence of $v_m\beta_{lj_2\cdots j_r}$, and hence of $\beta^{[0]}_{j_R}$ when $\sum\limits_{l\in R'}p_l=0$. This implies that $\beta^{[0]}_{j_R}$ converges when at least one of the $p_l, l\in R'$ is zero. Since $|R|\geqq 4$ this suffices to prove convergence of $\tilde{\beta}_{j_R}(p_{R'};t)$ for arbitrary $p_{R'}$. ■

*Remark 4.* b) The proof of part ii) of the lemma is written, and works, to all orders in $g^2$. We shall refer to this as the "invariance argument."

c) $\tilde{\sigma}_{jR}$ is quadratic on momenta; so a bound similar to (4.11) is trivial.

We can now complete the proof of convergence of $\sigma_{j_R}(p_{R'};t)$. Let $D=\partial^{|D|}/\partial p^{d_1}_{1\mu_1}\cdots\partial p^{d_r}_{r\mu_r}$, $\sum\limits_i d_i\equiv|D|$. Then

**Lemma 4.4.** *$D\sigma_{j_R}(p_{R'};t)\to D\sigma^*_{j_R}(p_{R'})$ uniformly on compact sets. We have the bound*

$$|D\sigma_{j_R}(p_{R'};t)|\leqq\mathscr{C}(1+|p_{R'}|^2). \tag{4.15}$$

*Proof.* Let $\rho_{j_R}(p_{R'};t)=\sigma_{j_R}(p_{R'};t)-\sigma_{j_R}(0;t)-\tilde{\sigma}_{j_R}(p_{R'};t)$.

In view of our earlier results we can assume $|R|>2$ and it is enough to prove the corresponding results of the Lemma for $\rho_{j_R}$. Also it is enough to prove pointwise convergence and the bound (4.15).

We assume the results for $|S|<|R|$, and work by induction. First, suppose $|D|=0$. We have (defining $R_{j_R}$ in the obvious way)

$$\frac{\partial\rho_{j_R}(p_{R'};t)}{\partial t}=-\sum_{l,\mu}p_{l\mu}\frac{\partial}{\partial p_{l\mu}}\rho_{j_R}(p_{R'};t)+2\rho_{j_R}(p_{R'};t)$$
$$-\sum\hat{K}(p_l)\sigma_{ij_l}(p_l;t)\rho_{j_{R(l,i)}}(p_{R'};t)+R_{j_R}(p_{R'};t),$$

which we rewrite

$$\frac{\partial\rho_R}{\partial s}(p_{R'}e^s;t+s)=-(A\rho_R)(p_{R'}e^s;t+s)+R_R(p_{R'}e^s;t+s), \tag{4.16}$$

where we mean by $\rho_R$ the vector $\{\rho_{j_R}\}_{\text{sequences}\,j_R}$ and

$$(A(p_{R'};t)\rho_R)_{j_R}=2\sum\hat{K}(p_l)\sigma_{ij_l}(p_l;t)\rho_{j_{R(l,i)}}-2\rho_{j_R}.$$

We have then

$$\rho_R(p_{R'};t)=\int_0^t\Xi_{(p_{R'};t)}(s)R_R(p_{R'}e^{-s},t-s)ds, \tag{4.17}$$

where

$$\Xi^{-1}_{(p_{R'};t)}(s)\frac{\partial}{\partial s}\Xi_{(p_{R'};t)}(s)=-A(p_{R'}e^{-s},t-s),$$

$$\Xi_{(p_{R'};t)}(0)=1.$$

Note that we have $|R_{j_R}(p_{R'};t)| \leqq \varepsilon |p_{R'}|^4$ for $|p_{R'}|$ bounded, where $\varepsilon$ may depend on the bound on $|p_{R'}|$ but not on $t$. (This is because $R_{j_R}$ is of $O(p_{R'}^4)$ for fixed $t$, and derivatives converge uniformly on compact sets by the inductive assumption.) Also $R_{j_R}(p_{R'};t) \to R^*_{j_R}(p_{R'})$ pointwise by induction. By Lemma (4.2) b

$$\Xi_{(p_{R'};t)}(s) \xrightarrow[t\to\infty]{} \Xi^*_{p_{R'}}(s).$$

Since $-A = 21 -$ (positive operator) we have

$$\| \Xi_{(p_{R'};t)}(s) \| \leqq e^{2s}.$$

We can now apply the dominated convergence theorem to (4.17) to conclude that $\rho_R(p_{R'};t)$ converges pointwise.

We will now prove the bound

$$|\rho_{j_R}(p_{R'};t)| \leqq \mathscr{C}(1 + |p_{R'}|^2).$$

We write the solution of (4.16) in the form

$$\rho_R(p_{R'}e^t, t) = e^{2t}\Xi^{-1}(p_{R'}e^t, t)\int_0^t e^{-2s}\Xi(p_{R'}e^s, s)R_R(p_{R'}e^s, s)ds, \tag{4.18}$$

where

$$\left(\Xi^{-1}\frac{\partial}{\partial s}\Xi\rho_R\right)_{j_R}(p_{R'}e^s, s) \equiv (B\rho_{R'})_{j_R} \equiv 2\sum_{l\in R}\hat{K}(p_le^s)\sigma_{ij_l}(p_le^s, s)\rho_{j_{R'(li)'}}.$$

Since $B$ is a positive operator, $\|\Xi(p_{R'}e^s, s)\|$ increases with $s$. From (4.18) one can estimate

$$\|\rho_R(p_{R'}e^t, (t)\| \leqq \mathscr{C}\left(\int_0^t e^{2(t-s)}e^{-2s}\right)|p_{R'}|^2 \leqq \mathscr{C}|p_{R'}|^2 e^{+2t}, \quad t > 0,$$

where we have used the inductive assumption to get the bound $\|R_R(p_{R'} - t)\| \leqq \mathscr{C}|p_{R'}|^2$ for large $|p_{R'}|^2$. The last inequality can be rewritten

$$\|\rho_R(p_{R'};t)\| \leqq \mathscr{C}|p_{R'}|^2 \quad \text{large} \quad |p_{R'}|^2,$$

which is what we want.

The proof for derivatives proceeds similarly. One applies $D$ to (4.1) to get, for suitable $B$ and $R_R^D$

$$\frac{\partial}{\partial s}(D\rho_R)(p_{R'}e^s, s) = -(D-2)D\rho_R((p_{R'}e^s, s) - BD\rho_R + R_R^D,$$

which can then be treated exactly as above. ∎

## 5. Proof of All Orders

For the most part the proof follows the lines of the previous section, so we give details only where necessary.

Recall the evolution equation of $\beta_{j_R}$, (3.7)

$$\frac{\partial \beta_{jR}}{\partial t}(p_{R'};t) = -g^2\left(\frac{d}{dt}\frac{1}{g^2}\right)[(KE)_{jR}(p_{R'};t) + \beta_{jR}(p_{R'};t)]$$
$$-\frac{1}{Z}\frac{dZ}{dt}\left[(KE)_{jR}(p_{R'};t) + \frac{v_i}{2}\beta_{ijR}(p_{R'};t) + \frac{r}{2}\beta_{jR}(p_{R'};t)\right]$$
$$+2\beta_{jR}(p_{R'};t) - \sum_{\substack{l=2\\ \mu=1,2}}^{r} p_{l\mu}\frac{\partial}{\partial p_{l\mu}}\beta_{jR}(p_{R'};t)$$
$$-\sum_{S_1\cup S_2=R}\hat{K}\left(\left|\sum_{m\in S_1} p_m\right|\right)\beta_{ijS_1}(p_{S_1};t)\beta_{ijS_2}(p_{S_2};t)$$
$$+\frac{g^2}{(2\pi)^2}\int d^2p\hat{K}(p)\beta_{iijR}(p,p_R;t).$$

We have, by the definition of $(1/Z)(dZ/dt)$, $\beta_j(t)=0$. Also, writing as before,

$$\beta_{j_1j_2}(p;t) = \beta(p;t)v_{j_1}v_{j_2} + \beta_\delta(p;t)\delta_{j_1j_2}.$$

In variance arguments show that $\beta_\delta(0;t)=0$, and the definition of $(d/dt)(1/g^2)$ gives $\tilde{\beta}_\delta(0;t)=0$.

Writing $\beta_{jR} = \sum \beta_{jR}^{(n)}g^{2n}$ we show convergence by induction on $n$ and $|R|$. Note that we already have the desired convergence for $\beta_{jR}^{(0)}$. Suppose now $n \geqq 1$, and we have shown convergence of $\beta_{jS}^{(n')}$ for $|S|<|R|$ and all $n$, and $S=R, n'<n$. We write (for suitably defined $R_{jR}^{(n)}$)

$$\frac{\partial}{\partial t}\beta_{jR}^{(n)}(p_{R'};t) = -\sum p_{l\mu}\frac{\partial}{\partial p_{l\mu}}\beta_{jR}^{(n)}(p_{R'};t) + 2\beta_{jR}^{(n)}(p_{R'};t)$$
$$-2\sum_{l\in R'}\hat{K}(p_l)\sigma_{ij_l}(p_l;t)\beta_{jR(l,i)}^{(n)}(p_{R'};t) + R_{jR}^{(n)}(p_{R'};t). \tag{5.1}$$

The proof of convergence differs only in the following points:

i) *Convergence at* 0-*momentum.* Here we adopt a strategy different from the proof of Lemma 4.1 because we can no longer simulate the problem by the local approximation. We follow instead the route of Lemma 4.3. We write

$$\beta_{jR}^{(n)}(0;t) \equiv \beta_{jR}^{(n)}(t) = \sum_{a=0}^{R}\beta_{jR}(n)[a]_{(t)}$$

and do an induction on $a$. The cases $|R|$ even, $a=0$ and $|R|$ odd, $a=1$ are handled by invariance arguments. We give the argument for $|R|$ odd, $a=1$. From (4.14), with $p_i=0$, we get by induction on $r$, the convergence of $\varepsilon_{ij}v_j\beta_{jj_2\cdots j_r}^{(n)[1]}(t) - v_l\beta_{m_{j_2\cdots j_r}}^{(n)[1]}(t)$. From this we see that $\beta^{(n)[1]}$ converges provided $r \geqq 3$. For $r=1$ the required convergence follows from the definition of $Z$. For $a>1$ we have

$$\frac{\partial\beta^{(n)[a]}}{\partial t}(t) = 2\beta^{(n)[a]}(t) - 2a\hat{K}(0)\sigma(t)\beta^{(n)[a]}(t) + R_{jR}^{(n),[a]}(t),$$

and since $\hat{K}(0)\sigma(t)\to 1$, arguments as before apply.

ii) In the inductive proofs we have as part of the remainder term $R^{(n)}_{jR}$ the integral

$$I^{(n)}(p_{R'};t)=\int d^2p\hat{K}(p)\beta^{(n-1)}_{iijR}(p,p_{R'};t).$$

We can use the fact that $\hat{K}$ has compact support to see that $I^{(n)}$ converges pointwise and that the bound

$$|\beta^{(n)}_{jR}(p_{R'};t)|\leqq\mathscr{C}(1+|p_{R'}|^2)$$

iterates. The same holds for derivatives of $\beta_{jR}$.

iii) The proof of convergence of $\beta^{(n)}_{j_1j_2}, n\geqq 1$ is no different from that of $\beta^{(n)}_{jR}$, $|R|>2$. Note that the only place where this would *not* work is at the $O(p^2)$ level where the invariance argument would not work for $\hat{\beta}^{(n)}_{\delta}$. But the definition of $g^2$ takes care of this.

iv) Finally, the definition of $Z$ and $g^2$ involves differentiating integrals of the type $I^{(n)}(p_{R'};t)$ above under the integral sign. The compact support of $\hat{K}$ and the bounds on derivatives make this possible.

The theorem is proved. ■

## Conclusion

i) We need to ask why the theory we have renormalised is the $O(N)$ $\sigma$-model. The reason is this: for arbitrary $\lambda>0$ in the bare trajectory, we obtain the same renormalised trajectory. But for $\lambda$ large this approximates the conventional bare trajectory since the fluctuation of the field is concentrated around the sphere, and also because leading behavior of the bare charge and wave-function renormalisation (Theorem (iv)) is the conventional one.

ii) Our approach to the $\sigma$-model is global (in contrast to usual treatments) but not intrinsic since we have chosen a particular embedding of $S^{N-1}\subset\mathbb{R}^N$. But this corresponds to choice of fundamental fields which is in any case inevitable.

iii) Note that the renormalisations are not canonically separated into a metric ("charge") and wave function renormalisation. In fact for a general submanifold $M\subset\mathbb{R}^W$, if we repeat the above procedure with the replacement

$$\frac{\lambda\Lambda_0^2}{g^2}\int(\varphi^2-1)^2d^2x\to\frac{\lambda\Lambda_0^2}{g^2}\int F(\varphi)d^2x$$

with $dF=0$ defining $M\subset\mathbb{R}^N$, we shall find that the RG flow *deforms* the minimum manifold $M$ in nontrivial ways.

iv) The specification of a continuum field theory as an "inverse limit" of cut-off theories given by the renormalised trajectory is ideally suited to the study of topological effects since the cut-off fields have good differentiability properties which make the topological terms well defined. In particular, one might investigate sigma models with Wess–Zumino terms.

We briefly indicate how this may be done for the $O(4)$ sigma model, which is also the $SU(2)$-valued model. The field $\Phi$ is $\mathbb{R}^4$-valued; we add to the action a term $2\pi i n WZ(\Phi)$ defined as follows. Given a map $\mathbb{R}^2\underset{\Phi}{\to}\mathbb{R}^4$, constant outside a compact set, nowhere zero, $WZ(\Phi)$ is defined by extending this to a map $\tilde{\Phi}$ of

$\mathbb{R}^3_+ = \{x, y, t | t \geqq 0\}$ which is constant outside a compact set, and setting

$$WZ(\Phi) = \int_{\mathbb{R}^3_+} \tilde{\Phi}^* \tau, \tag{6.1}$$

where $\tau$ is the 3-form on $\mathbb{R}^4 \backslash \{0\}$ obtained by pulling back the volume-form on $S^3$ (normalised to give total volume $= 1$) by the obvious retraction $\mathbb{R}^4 \backslash \{0\} \to S^3$. This additional term is only defined mod $\mathbb{Z}$, and is not defined for all maps $\mathbb{R}^2 \to \mathbb{R}^4$. But with suitable cut-offs $\exp in2\pi n WZ(\Phi)$ will be defined a.e. in probability. We will need only

$$\delta WZ(\Phi) = \alpha \varepsilon_{ijkl} \frac{\Phi_j}{\Phi^4} \partial_x \Phi_k \partial_y \Phi_l \delta \Phi_i, \tag{6.2}$$

where $\alpha$ is a constant determined by the normalisation. We can now write the RG flow for the potential $U_w + 2\pi in WZ(\Phi)$:

$$\begin{aligned} \frac{\partial U_w(\Phi, t)}{\partial t} &= \text{(as before)} \\ &\quad - 2(2\pi in) g^2 \int d^2x d^2y K(x-y) \frac{\partial U}{\partial \Phi_i(x)} \frac{\varepsilon_{ijkl} \Phi_j(y) \partial_x \Phi_k(y) \partial_y \Phi_l(y)}{\Phi^4(y)} \\ &\quad + (2\pi in)^2 g^4 \int d^2x d^2y K(x-y) \frac{\varepsilon_{ijkl} \Phi_j(x) \partial_x \Phi_k(x) \partial_y \Phi_l(x)}{\Phi^4(x)} \\ &\quad \cdot \frac{\varepsilon_{ij'k'l'} \Phi_{j'}(y) \partial_x \Phi_{k'}(y) \partial_{l'}(y)}{\Phi^4(y)}. \end{aligned} \tag{6.3}$$

One can now proceed as before and prove the perturbative existence of the renormalised trajectory. This theory is not the Wess–Zumino–Witten model, but is asymptotically free in the ultraviolet and conformally invariant in the infra-red. It would be interesting to see if a nonperturbative study of the above flow, incorporating anomalous dimensions could yield the WZW fixed point [15].

We remark that the last term on the right-hand side of (6.3) has the effect of concentrating the field on one-dimensional submanifolds of $S^3$. This, we believe is an intriguing signal of the fact that WZW models with values in a group are the "same" as sigma-models with values in the maximal torus.

v) Finally, let us point out that the perturbative expansion implemented in this paper is just what one needs in the so-called "small-field region" of rigorous RG theory (see e.g., [5]). For a nonperturbative control of the $\Lambda_0 \to \infty$ limit one needs suitable stability estimates in the "large field region." The large field problem in this model is now under investigation.

## Appendix A. The Leading Coefficients of the Beta Functions

We will first evaluate certain expressions involving $\sigma^*_{jR} \equiv \beta^{(0)*}_{jR}$, $|R| = 3, 4$, and use these to determine $a^{(0)}$ and $b^{(0)}$. Recall $\beta^{(0)*}_{j_1 j_2}(p) \equiv \sigma^*(p)$, and we use the notation $\sigma(p) = \sigma^*(p)$ in this Appendix.

i) Write

$$\sigma_{j_1j_2j_3}(p_2,p_3)=B_1(p_2,p_3)v_{j_1}\delta_{j_2j_3}+B_1(p_3,p_1)v_{j_2}\delta_{j_3j_1}+B_1(p_1,p_2)v_{j_3}\delta_{j_1j_2}$$
$$+B_2(p_2,p_3)v'_{j_1}v'_{j_2}v'_{j_3}.$$

We have, by looking at the $v_{j_1}\delta_{j_2j_3}$ component of (4.1) for $|R|=3$, $t=\infty$,

$$p_{l\mu}\frac{\partial}{\partial p_{l\mu}}B_1(p_2,p_3)-2B_1(p_2,p_3)+2B_1(p_2,p_3)\hat{K}(p_2+p_3)\sigma(p_2+p_3)=0.$$

This is solved by

$$B_1(p,q)=\sigma(p+q). \tag{A-1}$$

The overall constant is fixed by invariance, i.e., by studying (4.17) for $r=3$, $p_i=0$.

We also have, using A-1.

$$p_{l\mu}\frac{\partial}{\partial p_{l\mu}}B_2(p_2,p_3)-2B_2(p_2,p_3)+2B_2(p_2,p_3)\left[\sum_i \hat{K}(p_i)\sigma(p_i)\right]$$
$$+2\hat{K}(p_1)\sigma(p_1)[\sigma(p_1+p_3)+\sigma(p_1+p_3)]$$
$$+2\hat{K}(p_2)\sigma(p_2)[\sigma(p_2+p_1)+\sigma(p_2+p_3)]$$
$$+2\hat{K}(p_3)\sigma(p_3)[\sigma(p_3+p_1)+\sigma(p_3+p_3)]=0.$$

We will need $B_2(p,-p)$. If we specialise the above equation to the case $p_1=0$, a little guesswork shows

$$B_2(p,-p)=-(\sigma(0)+2\sigma(p)). \tag{A-2}$$

ii) We will need a certain expression (see below) involving the fourth derivative. We write.

$$\sigma_{j_1j_2j_3j_3}(p_2,p_3,p_4)=D_1(p_2,p_3,p_4)\delta_{j_1j_2}\delta_{j_3j_4}+(\ )+(\ )$$
$$+D_2(p_2,p_3,p_4)v_{j_1}v_{j_2}\delta_{j_3j_4}+(\ )+(\ )+(\ )+(\ )+(\ )$$
$$+D_3(p_2,p_3,p_4)v_{j_1}v_{j_2}v_{j_3}v_{j_4}.$$

We will need $F^{\Delta}(p)$, where

$$F(p,q)=ND_1(-p,q,-q)+D_1(p,-p,-q)+D_1(-q,-p,q)+D_2(-p,q,-q)$$
$$\equiv ND_{1a}+D_{1b}+D_{1c}+D_{2d}$$

and

$$F^{\Delta}(p)=\Delta_q F(p,q)|_{q=0}.$$

We have

$$p\frac{\partial}{\partial p}D^{\Delta}_{1a}=0,\quad p\frac{\partial}{\partial p}D^{\Delta}_{1b}+2\Delta_p(\sigma^2\hat{K})=0.$$

By appealing to initial conditions and by guesswork again (the prime "′" denoting differentiation with respect to $p$),

$$D^{\Delta}_{1a}=0,\quad D^{\Delta}_{1b}=\sigma''+\frac{\sigma'}{p}\equiv\Delta_p\sigma,\quad D^{\Delta}_{2b}=\sigma''+\frac{\sigma'}{p}. \tag{A-3}$$

Again , by looking at the $v_{j_1}v_{j_3}\sigma_{j_3j_4}$ component we get

$$p\frac{\partial D_{2d}^{\Delta}}{\partial p} + 4D_{2d}^{\Delta}\hat{K}(p)\sigma(p) + 4(\Delta_p\hat{K})\sigma^2(p) = 0.$$

We give the solution:

$$D_{2d}^{\Delta} = \frac{2\hat{K}'\sigma'}{\hat{K}} - \frac{4(\sigma'\hat{K}' + \sigma\hat{K}')}{\hat{K}p}. \tag{A-4}$$

We finally have

$$F^{\Delta}(p) = 2\left(\sigma'' + \frac{\sigma'}{p}\right) + \frac{2\hat{K}'\sigma'}{\hat{K}} - \frac{4(\sigma\hat{K})'}{\hat{K}p}.$$

iii) We are ready to compute the wave-function renormalisation to leading order. This is given by

$$\frac{d}{dt}\beta_j^{(1)} = 0:$$

From (3.1) we get

$$b^{(0)}\frac{v_i}{2}\sigma_{ij}(0) = \frac{1}{(2\pi)^2}\int d^2p\hat{K}(p)\sigma_{iij}(p,0).$$

The results of Ai) show (recall $\sigma_{iij}(p,0) = \sigma_{jii}(p,-p)$)

$$\sigma_{iij}(p,0) = \{N\sigma(0) + \sigma(p) + \sigma(p) - (\sigma(0) + 2\sigma(p)\}v_j.$$

Also $v_i\sigma_{ij}(0) = \sigma(0)v_j$, so that finally

$$b^{(0)} = 2\frac{(N-1)}{(2\pi)^2}\int d^2p\hat{K}(p) = \frac{(N-1)}{(2\pi)},$$

using the definition of $\hat{K}$.

iv) We turn now to the computation of $a^{(0)}$. We use the definition of the $g^2$ flow, Eq. (3.1), and the results of Ai) to conclude

$$a^{(0)} = -\frac{(N-1)}{2\pi} + \frac{1}{(2\pi)^2}\int d^2p\hat{K}(p)\frac{F^{\Delta}(p)}{4}.$$

We can now use the computation in A ii) to conclude that

$$a^{(0)} = \frac{-(N-2)}{2\pi}.$$

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