Commun. Math. Phys. 122, 281–292 (1989)

Communications in Mathematical Physics

Compactifications of $d = 11$ Supergravity on Kähler Manifolds*

C. N. Pope[1] and P. van Nieuwenhuizen[2]

[1] Center for Theoretical Physics, Department of Physics, Texas A&M University, College Station, TX 77843-4242, USA

[2] Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, L.I., NY 11794, USA

Abstract. We consider compactifications of eleven-dimensional supergravity to five and three spacetime dimensions, on internal spaces K_6 and $K_2 \times K_6$, where K_n denotes an n-dimensional Kähler manifold. The compactifications to five dimensions yield no surviving spacetime supersymmetries. However, we find compactifications to three dimensions on $S^2 \times K_6$ and $T^2 \times K_6$ where K_6 is Ricci-flat and Kähler (a Calabi-Yau space) with $N = 4$ supersymmetry. We also discuss the massless spectrum.

The eleven-dimensional supergravity theory attracted much attention a few years ago as a possible candidate for a unified theory of all the fundamental interactions, including gravity. These original hopes seemed not to be realized in practice, and with the growing surge of interest in superstring theory, work on eleven-dimensional supergravity was largely abandoned. Recently, however, eleven-dimensional supergravity has been making a modest comeback, as a result of developments in supersymmetric theories of higher-dimensional extended objects (p-branes) [1]. In particular, the eleven-dimensional supermembrane seems to have the best chance of being consistent at the quantum level, although at this stage the evidence is only circumstantial. It has, however, survived consistency checks which appear to fail for all the other super p-brane theories. As one might expect, the eleven-dimensional supermembrane is intimately related to eleven-dimensional supergravity, in much the same way as the ten-dimensional superstring is related to ten-dimensional supergravity. In particular, it can be embedded in an eleven-dimensional supergravity background, and the three-dimensional supermembrane action exhibits a local fermionic "Siegel"-symmetry if the background satisfies the classical equations of motion of eleven-dimensional supergravity.

In the light of the renewed interest in $d = 11$ supergravity as a possible low-energy limit of the supermembrane theory, it seems appropriate to re-examine the theory. In particular, just as superstrings emphasize the importance of two-dimensional supersymmetric theories, so supermembranes emphasize the

* Research supported in part by the U.S. Department of Energy under grant No. DE-FG03-84ER40168

importance of three-dimensional supersymmetric theories. The bulk of this paper will be concerned with some new compactifications of the eleven-dimensional theory that give rise to supergravities in three dimensions. Interestingly, they involve a combination of coset spaces and Calabi-Yau spaces for the internal manifold, with the complex structure of the internal manifold playing a crucial role. We shall also consider a class of compactifications to five dimensions which were first described in [2]. These, however, appear to be of limited interest since, as we shall show, the resulting theories are not supersymmetric. Both classes of compactification illustrate a point that was not appreciated in the earlier work on $d=11$ supergravity, namely that spontaneous compactification can occur to many more spacetime dimensions than just four or seven. Our solutions make considerable use of the properties of Kähler manifolds, and some relevant definitions and results are contained in an appendix.

Our discussion of the new solutions begins by considering the equations of motion of $d=11$ supergravity for a background in which the fermions are set to zero. The bosonic equations then become

$$R_{MN}=\tfrac{1}{3}(F_{MPQR}F_N{}^{PQR}-\tfrac{1}{12}g_{MN}F^2), \tag{1}$$

$$\nabla_M F^{MPQR}=-\tfrac{1}{576}\varepsilon^{M_1\cdots M_8PQR}F_{M_1\cdots M_4}F_{M5\cdots M8}, \tag{2}$$

where $F_{MNPQ}=4\partial_{[M}A_{NPQ]}$ is the field strength of the three-index photon A_{MNP}. There are various well-known solutions to these equations, describing ground-state configurations of the form $(\mathrm{AdS})_4\times K_7$ or $(\mathrm{AdS})_7\times K_4$, where $(\mathrm{AdS})_n$ denotes n-dimensional anti-de Sitter spacetime and K_p denotes a p-dimensional compact internal space. Discussion of the known solutions may be found in [3, 4, 5].

The first class of solutions that we shall consider corresponds to ground states of the form $(\mathrm{AdS})_5\times K_6$, i.e. the product of five-dimensional anti-de Sitter spacetime and a compact internal 6-manifold. We take K_6 to be Kähler, so it admits a covariantly-constant complex structure tensor $J_m{}^n$, satisfying $J_m{}^nJ_n{}^p=-\delta_m{}^p$ and $\nabla_m J_n{}^p=0$. (A $2n$-dimensional Kähler space is a space whose holonomy group is $U(n)$ or a subgroup thereof.) Lowering the upper index of $J_m{}^n$ yields an antisymmetric tensor, the Kähler form J_{mn}, which is of course also covariantly constant. We now make the ansatz that in the ground state the metric g_{MN} takes the direct product form, with $g_{\mu\nu}$ being the metric on $(\mathrm{AdS})_5$ and g_{mn} the metric on K_6. For the antisymmetric tensor F_{MNPQ} we take all its components to be zero except when all four indices lie in the internal space, in which case

$$F_{mnpq}=3cJ_{[mn}J_{pq]}, \tag{3}$$

where c is a constant. Since J_{mn} is covariantly constant, the left-hand side of (2) is zero, while the fact that $m, \mathrm{n}\cdots$ range only over the six directions on K_6 implies that the right-hand side of (2) is zero too. Substituting (3) into the Einstein equation (1), we find that the Ricci tensors $R_{\mu\nu}$ and R_{mn} on $(\mathrm{AdS})_5$) and K_6 respectively must satisfy

$$R_{\mu\nu}=-2c^2g_{\mu\nu}, \tag{4}$$

$$R_{mn}=2c^2g_{mn}. \tag{5}$$

Thus we have a solution where the eleven-dimensional spacetime is the product of five-dimensional anti-de Sitter spacetime and an Einstein–Kähler 6-manifold K_6 with positive Ricci tensor. Examples of such manifolds are $CP^3, CP^2 \times S^2$, $S^2 \times S^2 \times S^2$ and $SU(3)/[U(1) \times U(1)]$. (The first three can be understood by observing that S^2 is CP^1 and that all CP^n are Kähler, while also direct products of Kähler spaces are Kähler. The last case is obtained from $CP^2 \times CP^2$ by imposing the holomorphic constraint $\sum_{i=1}^{3} z^i \omega^i = 0$, where z^i and w^i are the holomorphic coordinates of the two CP^2 spaces.) On the other hand, $G_2/SU(3) = S^6$, $S^3 \times S^3$ and $S^2 \times S^4$ are not Kähler. For the first two of these, this follows from the fact that their second Betti numbers are zero (recall that for a product manifold $M_1 \times M_2$, the Kunneth formula for the Betti numbers $b_p(M_1 \times M_2)$ of $M_1 \times M_2$ is $\sum_{r=0}^{r=p} b_r(M_1) b_{p-r}(M_2)$).

Since the fermion field Ψ_M is zero in the ground state, the criterion for unbroken supersymmetry is that it should remain zero after a supersymmetry transformation, i.e. $\delta \Psi_M = 0$, where

$$\delta \Psi_M = \bar{D}_M \varepsilon_{11} \equiv \nabla_M \varepsilon_{11} + \tfrac{1}{144}(\hat{\Gamma}_{MNPQR} F^{NPQR} - 8 \hat{\Gamma}_{NPQ} F_M{}^{NPQ}) \varepsilon_{11}, \tag{6}$$

and $\hat{\Gamma}_M$ denotes the Dirac matrices of eleven dimensions. We make a decomposition of the Γ-matrices appropriate to the $11 = 5 + 6$ split, by taking $\hat{\Gamma}_\mu = \gamma_\mu \otimes \Gamma_7$ and $\hat{\Gamma}_m = 1 \otimes \Gamma_m$, where γ_μ and Γ_m are the Dirac matrices on $(\mathrm{Ads})_5$ and K_6 respectively, and $\Gamma_7 = (i/6!) \varepsilon^{mnpqrs} \Gamma_{mnpqrs}$. Thus from the internal components of (6), the criterion of surviving supersymmetry requires that

$$\bar{D}_m \eta \equiv \nabla_m \eta + \tfrac{1}{144}(\Gamma_{mnpqr} F^{npqr} - 8 \Gamma_{npq} F_m{}^{npq}) \eta = 0, \tag{7}$$

where the eleven-dimensional supersymmetry parameter is now written as $\varepsilon \otimes \eta$, and F_{mnpq} is given by (3). We need not consider the spacetime part of (6), since we shall show that (7) admits only the trivial solution $\eta = 0$.

It is now a straightforward, although tedious, matter to calculate the integrability condition $[\bar{D}_m, \bar{D}_n]\eta = 0$ that follows from (7). We find that

$$\begin{aligned} M_{mn} &\equiv [\bar{D}_m, \bar{D}_n] \\ &= \tfrac{1}{4} R_{mnpq} \Gamma^{pq} - \frac{2c^2}{9} \Gamma_{mn} - \frac{c^2}{6} J_m{}^p J_n{}^q \Gamma_{pq} \\ &\quad - \frac{ic^2}{9}(J_n{}^p \Gamma_{mp} + J_m{}^p \Gamma_{pn}) - \frac{4ic^2}{9} J_{mn} \Gamma_7. \end{aligned} \tag{8}$$

For the case when K_6 is CP^3, with Einstein metric satisfying (5), the Riemann tensor takes the form [6,7]

$$R_{mnpq} = \frac{c^2}{4}(g_{mp} g_{nq} - g_{mq} g_{np} + J_{mp} J_{nq} - J_{mq} J_{np} + 2 J_{mn} J_{pq}). \tag{9}$$

One may check the relative signs and coefficients in this expression by verifying that the cyclic identity is satisfied, and that J_{mn} commutes with R_{mnpq} as it must since $[\nabla_m, \nabla_n] J_{pq} = 0$. Substituting (9) into (8) yields a set of 15 matrices M_{mn}

labelled by m and n for which we need to find a zero eigenvector η, satisfying $M_{mn}\eta = 0$, if there is to be any residual supersymmetry in this case. At this point it is convenient to take the indices in (8) and (9) to be tangent-space indices, and work in a basis where the non-zero components of J_{mn} are given by $J_{12} = J_{34} = J_{56} = 1$. Taking $c^2 = 72$ for convenience, we note that in particular $M_{12} = 14\Gamma_{12} + 18\Gamma_{34} + 18\Gamma_{56} - 32i\Gamma_7$, and $M_{13} + M_{24} = -10(\Gamma_{13} + \Gamma_{24})$, together with similar expressions obtained by permuting the index pairs 12, 34 and 56. Imposing $M_{mn}\eta = 0$, one readily shows from this that $\Gamma_{12}\eta = \Gamma_{34}\eta = \Gamma_{56}\eta = i\Gamma_7\eta$, and hence from $M_{12}\eta = 0$ that the only solution is given by $\eta = 0$. Thus there are no surviving supersymmetries if K_6 is taken to be CP^3. As this is the most symmetrical of the 6-dimensional Einstein-Kähler spaces, it is presumably the one that would have been most likely to have admitted Killing spinors. Although we have not checked the other choices for K_6, we consider it unlikely that any of them would yield a supersymmetric ground state either.

We now turn to the case of compactifications of the form $(\mathrm{AdS})_3 \times K_8$, where $(\mathrm{AdS})_3$ is three-dimensional anti-de Sitter space and K_8 is some internal 8-manifold. Again, our ansatz will involve taking K_8 to be Kähler. It is tempting to try solutions analogous to those we have just been discussing, in which F_{MNPQ} is again taken to have the form (3), where now the indices $m, n, \ldots$ range over the 8 dimensions of K_8. Unfortunately this does not work; the "Maxwell"-type equation (2) is no longer satisfied. This is because although the left-hand side is still zero by virtue of the covariant constancy of J_{mn}, the right-hand side is now non-zero because the 8 antisymmetrized indices can now take 8 rather than 6 values. (The Kähler form is non-degenerate, since $J^2 = -1$, so the right-hand side, being proportional to $\det J$, is *always* non-zero if the free indices PQR lie in the $(\mathrm{AdS})_3$ spacetime directions.) We therefore look instead for solutions of a different kind, where K_8 is itself the product of a two-dimensional and a six-dimensional Kähler manifold, $K_8 = K_2 \times K_6$. We may now make an ansatz for F_{MNPQ} in which the only non-zero components are given by

$$F_{\alpha\beta cd} = c\varepsilon_{\alpha\beta} J_{cd}, \tag{10}$$

where $\varepsilon_{\alpha\beta}$ is the Levi-Civita tensor on K_2 (which is the same thing as the Kähler form on K_2) and J_{cd} is the Kähler form on K_6. Our notation is that the index M is split into μ on the three-dimensional spacetime $(\mathrm{AdS})_3$, α on K_2 and a on K_6.

The over-antisymmetrization of $\alpha, \beta \cdots$ indices on the right-hand side of (2) now ensures that the Maxwell equation is satisfied, so it only remains to substitute (10) into the Einstein equation (1). We find that the Ricci tensors $R_{\mu\nu}$, $R_{\alpha\beta}$ and R_{ab} on $(\mathrm{AdS})_3$, K_2 and K_6 must satisfy

$$R_{\mu\nu} = -2c^2 g_{\mu\nu}, \tag{11}$$

$$R_{\alpha\beta} = 4c^2 g_{\alpha\beta}, \tag{12}$$

$$R_{ab} = 0. \tag{13}$$

$(\mathrm{AdS})_3$ may thus be taken to be three-dimensional anti-de Sitter spacetime, while K_2 and K_6 are Einstein-Kähler spaces having respectively positive and zero Ricci tensor. All 2-manifolds are Kähler, since the most general holonomy group in 2 dimensions is $SO(2) \sim U(1)$, but only the 2-sphere admits on Einstein metric with positive Ricci curvature. This follows from the fact that the Euler number for 2-dimensional compact manifolds (Riemann surfaces) is given by $1/(2\pi)\int d^2x \sqrt{g}R = 2 - 2g$, where g is the genus, and is positive only for the sphere. Thus the internal manifold K_8 must be the product $S^2 \times Y$, where Y is a six-dimensional Ricci-flat Kähler manifold, i.e. a Calabi-Yau space.

The next step is to determine whether there are any surviving supersymmetries for this ground state. We begin by making an appropriate $11 = 3 + 2 + 6$ split of the eleven-dimensional Dirac matrices $\hat{\Gamma}_M$:

$$\begin{aligned} \hat{\Gamma}_\mu &= \gamma_\mu \otimes \tau_3 \otimes \Gamma_7, \\ \hat{\Gamma}_\alpha &= 1 \otimes \tau_\alpha \otimes 1, \\ \hat{\Gamma}_a &= 1 \otimes \tau_3 \otimes \Gamma_a, \end{aligned} \tag{14}$$

where γ_μ, τ_α and Γ_a are the Dirac matrices for $(\mathrm{AdS})_3$, S^2 and Y respectively; $\tau_3 = (-i/2)\varepsilon^{\alpha\beta}\tau_{\alpha\beta}$ and $\Gamma_7 = (i/6!)\varepsilon^{mnpqrs}\Gamma_{mnpqrs}$. As for the previous compactifications that we discussed, the criterion for unbroken supersymmetry of the ground state is that there exist solutions of $\tilde{D}_M \varepsilon_{11} = 0$, where $\tilde{D}_M$ is given by (7). We now decompose the super-symmetry parameter as a sum of terms of the form

$$\varepsilon_{11} = \varepsilon \otimes \zeta \otimes \eta, \tag{15}$$

where ε is an anticommuting supersymmetry parameter on $(\mathrm{AdS})_3$, and ζ and η are commuting spinors on S^2 and Y respectively.

At this stage, it is useful to recall some properties of six-dimensional Calabi–Yau spaces. Being Ricci-flat, and Kähler, the holonomy group is $SU(3)$. This means that there are two covariantly-constant spinors, one left-handed and the other right-handed. This may be seen by noting that left- and right-handed spinors in six dimensions transform respectively as the 4 and $\bar{4}$ of $SU(4)$ (which is isomorphic to Spin(6), the double cover of the $SO(6)$ tangent space group). There is only one embedding of $SU(3)$ in $SU(4)$, and the 4 and $\bar{4}$ decompose as $3 + 1$ and $\bar{3} + 1$ respectively. It is the two singlets in these decompositions that correspond to the two covariantly-constant spinors on Y. Denoting the left- and right-handed covariantly-constant spinors by η_- and η_+ respectively, we may choose conventions such that they satisfy

$$J_a{}^b \Gamma_b \eta_\pm = \pm i \Gamma_a \eta_\pm. \tag{16}$$

These equations follow from the fact that $J_a{}^b \Gamma_b \eta_\pm$ are covariantly-constant vector-spinors, and by looking at the decomposition of vector-spinors under $SU(4) \to SU(3)$, i.e. $6 \otimes 4 \to 8 + 6 + 3 + \bar{3} + \bar{3} + 1$ and $6 \otimes \bar{4} \to 8 + \bar{6} + \bar{3} + 3 + 3 + 1$, we see that they *must* simply be constant multiples of $\Gamma_a \eta_\pm$, since there is only one singlet in each decomposition. The constants can only be $\pm i$, because $J_a{}^b J_b{}^c = -\delta_a{}^c$, and hence with a suitable choice of conventions we obtain (16). A direct consequence of (16) is that

$$J_{ab}\Gamma^{ab}\eta_\pm = \pm 6i\eta_\pm = 6i\Gamma_7\eta_\pm. \tag{17}$$

With these preliminaries, we can now proceed to substitute (10) and (15) into

(6), in order to determine whether there are any surviving supersymmetries for these solutions. First we consider the case where the index M in (6) lies in the Calabi–Yau directions, i.e. $M = a$. From the decomposition (14) of the Dirac matrices, it follows that

$$\bar{D}_a = \nabla_a + \frac{ic}{12}(\Gamma_a \Gamma_{bc} J^{bc} - 6 J_{ab} \Gamma^b). \tag{18}$$

Thus using (16) and (17), we see that the Γ-matrix terms in (18) cancel if we act with $\bar{D}_a$ on $\eta_\pm$, and so since by definition $\eta_\pm$ are covariantly constant, they therefore satisfy $\bar{D}_a \eta_\pm = 0$.

Next we consider the case where the index M in (6) lies in the S^2 directions, i.e. $M = \alpha$. From (10) and (14) it follows that

$$\bar{D}_\alpha = \nabla_\alpha - ic\Gamma_7 \varepsilon_\alpha{}^\beta \tau_\beta, \tag{19}$$

so taking η in (15) to be $\eta_\pm$, and using (17), the requirement $\bar{D}_\alpha \varepsilon_{11} = 0$ reduces to

$$\bar{D}_\alpha \zeta = \nabla_\alpha \zeta \mp ic\varepsilon_\alpha{}^\beta \tau_\beta \zeta = 0. \tag{20}$$

The integrability condition for this equation is $[\bar{D}_\alpha, \bar{D}_\beta]\zeta = \frac{1}{4} R_{\alpha\beta\gamma\delta} \tau^{\gamma\delta} \zeta - 2ic^2 \varepsilon_{\alpha\beta} \tau_3 \zeta = 0$, which is indeed satisfied identically if the metric on the two-sphere is taken to have its standard maximally-symmetric form, for which $R_{\alpha\beta\gamma\delta} = 4c^2(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$. (The normalization here is fixed by Eq. (12).) Thus (20) does indeed have solutions, namely the usual Killing spinors on S^2.

Finally, we consider the case where the index M in (6) lies in the anti-de Sitter directions, i.e. $M = \mu$. Now we find

$$\bar{D}_\mu = \nabla_\mu + \frac{ic}{12} \gamma_\mu \Gamma_7 J_{ab} \Gamma^{ab}, \tag{21}$$

and so again taking $\eta = \eta_\pm$ in (15), and using (17), it follows that the requirement $\bar{D}_\mu \varepsilon_{11} = 0$ reduces to

$$\bar{D}_\mu \varepsilon = \nabla_\mu \varepsilon - \frac{c}{2} \gamma_\mu \varepsilon = 0. \tag{22}$$

The integrability condition $[\bar{D}_\mu, \bar{D}_\nu]\varepsilon = 0$ for this equation is $\frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \varepsilon + (c^2/2)\gamma_{\mu\nu} \varepsilon = 0$, which is satisfied identically if we take the metric on $(\mathrm{AdS})_3$ to be the anti-de Sitter metric satisfying (11), so that $R_{\mu\nu\rho\sigma} = -c^2(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$. Thus there are solutions of (22) in which ε is a Killing spinor on the three-dimensional anti-de Sitter spacetime.

We have thus shown that there are indeed solutions of $\bar{D}_M \varepsilon_{11} = 0$ in our $(\mathrm{AdS})_3 \times S^2 \times Y$ solutions, and so these correspond to supersymmetric ground states of the theory. To count the number of supersymmetries, N, we note that there are two Killing spinors on S^2 satisfying (20) for each choice of sign, while on Y there are the two covariantly-constant spinors η_- and η_+, which are left- and right-handed respectively. Bearing in mind that Ψ_M and ε_{11} are Majorana spinors in eleven dimensions, while ζ and $\eta_\pm$ are complex, we must take appropriate real combinations in (15). Taking into account that the signs in (16) and (20) are correlated, the result is that the effective three-dimensional theory on $(\mathrm{AdS})_3$ has

$N = 4$ supersymmetry. Since the Killing spinors ζ on S^2 transform as doublets under the $SU(2)$ isometry group of S^2, the supersymmetry parameters in three dimensions transform as $2 + 2$ under $SU(2)$. The way in which the supersymmetry arises in these solutions, involving as it does the interplay between the Killing spinors on $(\mathrm{AdS})_3$ and S^2 and the covariantly-constant spinors on the Calabi–Yau space Y, is very different from that seen, for example, in Freund–Rubin solutions of the theory [4]. In particular, it is interesting to note that unlike the Freund–Rubin case, the *relative* sign between the two terms involving F_{MNPQ} in (6) is crucial for the existence of supersymmetry.

In three dimensions the anti-de Sitter group is $SO(2,2)$, which is isomorphic to $SO(2,1) \times SO(2,1)$. It therefore follows that the N supersymmetries in three-dimensional anti-de Sitter theories really take the form of (p,q) supersymmetries, $p + q = N$, where p and q label the representations under the two $SO(2,1)$ factors. The integers p and q corresponds to the numbers of positive and negative eigenvalues in the gravitino mass matrix, or equivalently, the numbers of positive and negative coefficients of the γ_μ term in the three-dimensional gravitino transformation rule $\delta\psi_\mu = \nabla_\mu \varepsilon - c/2\gamma_\mu \varepsilon$ (see (22)) [8]. Thus in our case, since all four Killing spinors satisfy the same equation (22), it follows that our three-dimensional theory has $(4,0)$ supersymmetry. (The choice $(4,0)$ rather than $(0,4)$ is purely conventional.)

To determine the full spectrum of massless and massive states for these compactifications would be quite an involved calculation, analogous to those that have been carried out for Freund–Rubin compactifications, except that here the more complicated structure of the solutions makes things more difficult. We shall not carry out such an analysis here. Even to determine the massless spectrum is non-trivial because unlike, for example, the seven-sphere Freund–Rubin compactification, the effective three-dimensional supergravity theories that we obtain here contain *reducible* massless representations of the super-de Sitter group. What is immediately clear is that included in the massless spectrum are the graviton, four gravitinos and the three gauge bosons of the $SU(2)$ Yang–Mills group corresponding to the isometry group of the 2-sphere. In addition, there will be massless states associated with zero modes on the Calabi–Yau space Y, together, possibly, with further massless states coming from spherical harmonics on S^2.

In order to disentangle the various sources of massless states for these compactifications, we adopt the following strategy. First of all, we note that massless states associated with zero modes on Y can be divided into two categories. On the one hand there are those that will arise for *any* choice of Calabi–Yau space, while on the other hand there are those that depend upon the particular choice for Y that one happens to make. Those zero modes in the former category comprise the two covariantly-constant spinors $\eta_\pm$; the Kähler 2-form J_{ab}; the complex 3-form of type $(0,3)$ that corresponds to the totally antisymmetric holomorphic tensor ε_{ABC}, where $A, B, \ldots$ are 3-valued complex indices, and its $(3,0)$ complex conjugate $\varepsilon_{\bar{A}\bar{B}\bar{C}}$; the vector-spinors $\Gamma_a \eta_\pm$; and the constant scalar mode. In the second category are the additional zero modes on Y associated with additional harmonic forms of type $(1,1)$, $(1,2)$ and $(2,1)$ that are not covariantly constant. The numbers of such modes depend upon the specific choice of Calabi–Yau manifold. The modes in the former category are characterized by the fact that they are all *covariantly constant*, while those in the latter category are not. As discussed in [4], this means that one can always consistently truncate the full non-linear theory to just those states associated with the covariantly-constant modes. This is because the set of covariantly-constant harmonics is closed under multiplication, i.e. on one can never generate non-covariantly constant harmonics by multiplying covariantly-constant ones together. Thus even at the non-linear level, we may if we wish separate the discussion of the zero modes into the two disjoint categories.

The other device that we adopt in order to simplify the discussion is to note that whenever one has a lower dimensional supergravity theory obtained by compactification on a sphere, the number of states of each spin in the massless multiplet is always the same as one would obtain by making a humble torus compactification instead. Thus, for example, by compactifying from $d = 11$ to $d = 4$ on T^7 rather than S^7 (with the Freund–Rubin "mass" parameter set to zero), one obtains the same numbers $(1, 8, 28, 56, 70)$ of states of spins $(2, 3/2, 1, 1/2, 0)$, the difference being that the theory resulting from the torus compactification is ungauged rather than gauged. Of course the origins of the various massless states may differ between the two cases, but not the total numbers of states. Thus, for example, the 28 gauge bosons in the S^7 compactification all come from the $SO(8)$ Killing vectors on the seven-sphere, whilst in the T^7 compactification 7 come from the $[U(1)]^7$ Killing vectors on the torus and the remainder come from the 21 harmonic 2-forms on T^7.

Thus for our compactifications we may temporarily set the constant parameter c in (10) to zero, and take the internal space to be $T^2 \times Y$ rather than $S^2 \times Y$, in order to count the states in the massless sector of the three-dimensional theory. As discussed above, we shall truncate out the zero modes associated with non-covariantly constant harmonics on Y. Thus the counting of massless states coming from $T^2 \times Y$ is as follows: From the metric g_{MN} we obtain the graviton $g_{\mu\nu}$, two abelian vector fields associated with the two $U(1)$ Killing vectors on T^2, and four scalar fields coming from the three constant symmetric tensors on the flat T^2 metric and the metric of the Calabi–Yau space Y. From A_{MNP} we obtain $\tilde{b}_2$ vector fields corresponding to the case where two of the indices are internal, and $\tilde{b}_3$ scalar fields corresponding to all three indices being internal. Here $\tilde{b}_2$ and $\tilde{b}_3$ denote the "effective" second and third Betti numbers of $T^2 \times Y$, by which we mean the numbers of harmonic two and three forms associated with the covariantly-constant modes on Y which are the only ones we are retaining in our truncation. From the Kunneth formula for the Betti numbers of product manifolds, we see that $\tilde{b}_2 = 2$ and $\tilde{b}_3 = 4$. For the fermions, when M lies in spacetime Ψ_M yields the four gravitions discussed earlier, and when M lies in the internal space we have a total of 12 covariantly-constant vector-spinors on $T^2 \times Y$, arising as four complex vector-spinors on T^2 (namely the two Killing vectors times the 2 complex covariantly-constant spinors) times the two covariantly-constant spinors $\eta_\pm$ on Y plus the two covariantly-constant spinors on T^2 times the two vectors-spinors $\Gamma_a \eta_\pm$ on Y. This gives 12 spin 1/2 massless fields. These results are summarized in Table 1.

Bearing in mind that we are in three dimensions, so that a graviton has no

Table 1

	Numbers of massless modes	
Spin	$T^2 \times K_6$	$S^2 \times K_6$
2	1	1
3/2	4	4
1	4	9
1/2	12	12
0	8	3

degrees of freedom and a massless vector has only one degree of freedom, we see from Table 1 that the number of on-shell bosonic and fermionic degrees of freedom do indeed match, i.e. $12 + 12$. In fact a massless vector in three dimensions is really equivalent to a scalar field, since its field strength can be written as the dual of the gradient of a scalar. One might expect that in the $S^2 \times Y$ compactifications that we are really interested in, the numbers of vectors and scalars differ from Table 1, although the total number of vectors plus scalars will be the same. The 12 scalars are just what one needs for a scalar coset manifold $SO(N, n)/[SO(N) \times SO(n)]$ with $N = 4$ and $n = 3$[9]. Upon gauging one expects therefore $6 + 3 = 9$ vectors and 3 scalars. As a check on our previous results for $T^2 \times K_6$, we consider the massless spectrum of bosons for the $S^2 \times K_6$ compactification. The metric g_{MN} gives one graviton, three vectors from the Killing vectors on S^2 and one scalar from the dilation mode of K_6. From A_{MNP} we get six vectors, namely three from $J_{ab}\Phi$ and three from $\varepsilon_{\alpha\beta}\Phi$, where Φ denotes the three conformal scalars on S^2. In addition, there are two scalars from the ε_{ABC} and $\varepsilon_{\bar{A}\bar{B}\bar{C}}$ on K_6. This massless spectrum is also given in Table 1, and agrees with that on $T^2 \times K_6$, bearing in mind the equivalence of vectors and scalars discussed above. The full massless spectrum will thus comprise the states listed in the table, together with additional matter multiplets associated with the non-covariantly constant modes on the Calabi–Yau space.

This concludes our discussion of compactifications of $d = 11$ supergravity on Kähler manifolds. One could, of course, look directly at supergravity theories in three dimensions, and work out further details of the massless sectors for the theories that we have obtained. We shall not pursue this further here. For now, we consider that one of the most interesting aspects of this paper is the say in which various mathematical tools, including especially the use of Killing spinors, play a role in the physics of Kaluza–Klein theories.

Appendix

In this appendix, we collect together some basic facts about almost complex, complex and Kähler manifolds. We include a simple discussion, based on the use of Killing spinors, of the non-integrable almost complex structure on S^6. There are several equivalent ways in which one can describe a Kähler manifold, and depending upon the situation of interest one may find that one description is more convenient than another. Since a Kähler manifold is in particular a complex manifold, one has the option of using n complex coordinates z^α rather than $2n$ real coordinates x^m to parametrize points in the manifold. If we divide the real coordinates into two sets of n, $x^m = (x_1{}^\alpha, x_2{}^\alpha)$, we may relate the real and complex coordinate bases by choosing relations of the form $z^\alpha = x_1{}^\alpha + ix_2\alpha$. In terms of the complex basis, one may write the metric in the form $ds^2 = 2g_{\alpha\bar\beta} dz^\alpha s\bar z^{\bar\beta}$, where

$$g_{\alpha\bar\beta} = \partial_\alpha \bar\partial_{\bar\beta} K(z,\bar z), \tag{A1}$$

and $K(z,\bar z)$ is the so-called Kähler potential. The Kähler form J may then be written as $J = 2ig_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^{\bar\beta}$.

In practice, it is more convenient for our purposes to describe the Kähler manifold purely in terms of the real coordinate basis. Thus we need the following definitions. To begin, we define an *almost manifold* as a manifold of real dimension $2n$ that admits an *almost complex structure* tensor $J_m{}^n$ that satisfies

$$J_m{}^n J_n{}^p = -\delta_m{}^p. \tag{A2}$$

(Note that (A2) necessarily requires that the real dimension be even. This follows by taking the determinant of (A2); $(\det J_m{}^n)^2 = (-1)^d$. Thus since the left-hand side is positive, the dimension d must be even.) Next, we define an *almost Hermitian manifold* as an almost complex manifold which admits a metric g_{mn} such that

$$J_m{}^p J_n{}^q g_{pq} = g_{mn}. \tag{A3}$$

Combined with (A2), this amounts to the statement that the tensor J_{mn}, obtained by lowering the second index on $J_m{}^p$ with g_{pn}, is antisymmetric,

$$J_{mn} = -J_{nm}. \tag{A4}$$

This antisymmetric tensor may be viewed as a 2-form J,

$$J = \tfrac{1}{2} J_{mn} dx^m \wedge dx^n. \tag{A5}$$

We define a *complex manifold* as being an almost complex manifold for which the almost complex structure tensor is integrable, i.e. the Nijenhuis tensor, defined by

$$N_{mn}{}^p = \partial_m J_n{}^p - \partial_n J_m{}^p + J_m{}^q J_r{}^p \partial_q J_n{}^r - J_n{}^q J_r{}^p \partial_q J_m{}^r, \tag{A6}$$

vanishes. (It is not a *priori* obvious that (A6) is a tensor, but one can, for example, check that the partial derivatives may be replaced by covariant derivatives, by virtue of the connection terms cancelling.) Similarly, we define an *Hermitan manifold* as a complex manifold which admits a metric satisfying (A3).

A Kähler manifold may now be defined as an Hermitian manifold that admits a metric such that the Kähler form (A5) is closed,

$$dJ = 0. \tag{A7}$$

One can now show that the above definitions lead to the stronger result that in fact J is covariantly constant. To see this, we first observe that (A7) implies that

$$\nabla_m J_{np} - \nabla_n J_{mp} + \nabla_p J_{mn} = 0. \tag{A8}$$

Using $N_{mn}{}^{p} = 0$, we re-write the first two terms as

$$-J_m{}^q J_{rp} \nabla_q J_n{}^r - (m \leftrightarrow n) + \nabla_p J_{mn} = 0. \tag{A9}$$

Using (A2), we may write this as

$$(J_m{}^q J_n{}^r - J_n{}^q J_m{}^r)\nabla_q J_{rp} + \nabla_p J_{mn} = J_m{}^q J_n{}^r(\nabla_q J_{rp} - \nabla_r J_{qp}) + \nabla_p J_{mn} = 0. \tag{A10}$$

From (A7), we get

$$J_m{}^q J_n{}^r(-\nabla_p J_{qr}) + \nabla_p J_{mn} = 0, \tag{A11}$$

and hence

$$J_m{}^q(\nabla_p J_n{}^r)J_{qr} + \nabla_p J_{mn} = 0. \tag{A12}$$

Finally, from (A2), we obtain

$$\nabla_p J_{mn} = 0. \tag{A13}$$

A nice example of an almost complex manifold where the almost complex structure is *not* integrable is provided by the six-sphere. With its standard $SO(7)$-invariant metric, the Riemann tensor takes the form

$$R_{mnpq} = a^2(g_{mp}g_{nq} - g_{mq}g_{np}), \tag{A14}$$

where a^{-1} is the radius of the sphere. One can show that the Killing spinor equation

$$\left(\nabla_m \pm \frac{ia}{2}\Gamma_m\right)\eta^{\pm} = 0 \tag{A15}$$

has 8 solutions η^+ and 8 solutions η^-. The proof consists of taking the commutator of Killing derivatives (A15), and substituting (A14) into the result, thereby showing that the integrability condition is identically satisfied. The Dirac matrices Γ_m of six dimensions may be chosen to be imaginary and antisymmetric, satisfying $\{\Gamma_m, \Gamma_n\} = 2g_{mn}$. The chirality operator $\Gamma_7 = i/6!\varepsilon^{m_1\cdots m_6}\Gamma_{m_1\cdots m_6}$ is then also imaginary and antisymmetric, and $\Gamma_7{}^2 = 1$. We may take the Killing spinors $\eta^{\pm}$ to be (real) Majorana spinors. In this basis for the Dirac matrices, the Majorana conjugate of a spinor η is simply η^T. Without loss of generality, we may assume that each Killing spinor is normalized to unit length, $\eta^T\eta = 1$.

In terms of one of these Killing spinors η, say one of the η^+, we construct the following real antisymmetric tensor

$$J_{mn} = i\eta^T\Gamma_7\Gamma_{mn}\eta = i\eta^T\Gamma_7\Gamma_m\Gamma_n\eta. \tag{A16}$$

One can show that any of the matrices 1, $\Gamma_7\Gamma_{mn}$ and Γ_{mnp} gives a non-zero expression when sandwiched between η^T and η, while Γ_7, Γ_m, $\Gamma_7\Gamma_m$ and Γ_{mn} give expressions that vanish. Using $\Gamma_m\Gamma_{npq}\Gamma^m = 0$, a simple Fierz rearrangement shows that (A16) implies

$$J_{mp}J^p_n = \tfrac{3}{4}g_{mn} + \tfrac{1}{8}(4g_{mr}J_{sn} + g_{mn}J_{rs})J^{rs}, \tag{A17}$$

which, after tracing to obtain $J_{mn}J^{mn} = 6$, implies that J satisfies (A2). Thus J is an almost complex structure on S^6.

From (A15), the covariant derivative of J_{np} is given by

$$\nabla_m J_{np} = a U_{mnp}; \quad U_{mnp} = \eta^T \Gamma_7 \Gamma_{mnp} \eta. \tag{A18}$$

Thus substituting into (A6), with the partial derivatives replaced by covariant derivatives, we see that the Nijenhuis tensor is given by

$$N_{ijk} = -4a T_{ijk}, \tag{A19}$$

where $T_{ijk} = U^l_{ij} J_{kl}$. Since U_{ijk} is non-vanishing, it follows that the Nijenhuis tensor is non-zero, and so the almost complex structure tensor (A16) is non-integrable. Of course this does not yet constitute a proof that S^6 is not a complex manifold, since it does not rule out the possibility that there might be another almost complex structure that is integrable; however in fact it has been proved that this is not the case.

Acknowledgements. We are very grateful to Larry Romans for discussions and for providing some calculations appearing in the Appendix, and also to Paul Townsend for discussions.

References

1. Bergshoeff, E., Sezgin, E., Townsend, P. K.: Phys. Lett. **189B**, 75 (1987)
2. Dolan, B.: Phys. Lett. **140B**, 304 (1984)
3. Castellani, L., Romans, L. J., Warner, N. P.: Ann. Phys. (NY) **157**, 394 (1984)
4. Duff, M. J., Nilsson, B. E. W., Pope, C. N.: Phys. Reps. **130**, 1 (1986); de Wit, B., Nicolai, H., Warner, N. P.: Nucl. Phys. **B255**, 29 (1985)
5. Pilch, K., Townsend, P. K., van Nieuwenhuizen, P.: Nucl. Phys. **B242**, 377 (1984)
6. Kobayashi S., Nomizu, K.: Foundations of Differential Geometry, vol 2 New York: Wiley 1969
7. Nilsson, B. E. W., Pope, C. N.: Class. Quantum Grav. **1**, 499 (1984)
8. Günaydin, M., Sierra, G., Townsend, P. K.: Class Quantum Grav. **3**, 763 (1986)
9. Marcus, N., Schwarz, J. H.: Nucl. Phys. **B228**, 145 (1983)

Communicated by A. Jaffe

Received October 17, 1988