Fine Tuning of Resonances and Periodic Solutions of Hamiltonian Systems Near Equilibrium

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1. Introduction

Consider in R^{2n} a Hamiltonian System which has the origin as isolated equilibrium. We use the notation $z_k \equiv (q_k, p_k)$, $k = i \cdots n$, and we assume that the Hamiltonian H is of class C^2 and that H_2, the quadratic part of H, is given by

$$H_2 = 1/2 \sum_0^n v_i z_i^2, \quad z_i^2 = q_i^2 + p_i^2, \tag{1.1}$$

with $v_i \neq 0$, $i = 1 \cdots n$. If $v_i \cdot v_j^{-1}$ is not an integer for any pair i,j $(i \neq j)$ a theorem by Lyapunov [1] states that one can find a neighborhood N of the origin such that in N there are n families of elliptic periodic solutions of (1.1); the kth family has minimal periods approximately equal to $2\pi v_k^{-1}$ and lies approximately on the hyperplane $\{z | z_h = 0 \text{ if } h \neq k\}$. One can use as parameter the distance from the origin. These results follow from an application of the Inverse Function Theorem to the periodicity condition.

General lower bounds on the number of periodic solutions on every energy surface sufficiently close to the origin have been obtained within a variational approach. These results hold independently of whether there is a resonance among the frequencies, but are often restricted to the case when H_2 is of definite sign [2] (see however [6]); also, the localization and stability of the periodic solutions are in general not known.

On the other hand, if $v_i = kv_j$ for some $i \neq j$, $k \in \mathbb{Z}$ (a resonant case) the number of families can be different from the one indicated in Lyapunov's Theorem, and some of the periodic solutions are hyperbolic. Systems with resonance have been studied extensively in phase space [3]. [4] leading to various estimates on the number of families of periodic solutions and their stability. In particular it was shown (see also [5]) that, if $n = 2$, $v_2 = 2v_1$, one has in general three families; the minimal period is approximately $2\pi v_1^{-1}$ for two of them, while it is approximately

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equal to $2\pi v_2^{-1}$ for the third. If one considers instead the case $v_2 = -2v_1$, one has in general only one family, with minimal period approximately $2\pi v_2^{-1}$.

In view of these results, it is natural to ask whether there is a discontinuity in the behaviour of the system when the set of frequencies in H_2 approaches a resonance. To answer this question, we shall study the system in a full neighborhood of a resonance, i.e. in a domain in $R^{2n} \times R^n$ of the form

$$N = \{(z, v) \mid z \in \mathbf{B}_\varepsilon^{2n}, \ v - v_0 \in (\mathbf{B}_{\alpha(\varepsilon)}^n)\},$$

where v_0 is a set of resonant frequencies, $\alpha(\varepsilon) = \varepsilon^\alpha$ for some $\alpha > 0$ to be chosen later, and $\mathbf{B}_\varepsilon \equiv \{x \in R^{2n}, |x| \leqq \varepsilon\}$.

Several aspects of this "detuning" (or bifurcation) have been studied previously by J. Duistermaat [5] using the variational approach. Many of our results overlap with those of [5], and we refer to that paper for a list of previous results in various cases of resonance.

Our aim is to give a systematic treatment within phase space; this approach, as compared to the variational one, has the advantage that one can exploit symplectic transformations, through the use of normal forms. At the same time one can apply directly in our context the estimates which have been derived within the variational approach through the use of a cohomological index [6].

We shall in fact prove, in the phase space approach, that the number of families of periodic solutions near the origin is linked to the number of critical points of a suitable function F defined on a manifold in phase space.

The function F and the manifold are closely related to the ones introduced in [6] through a reduction to a finite dimensional subspace of functions. On the other hand, the results obtained in the phase-space approach hold only "generically" (in a sense to be made precise later) since to exploit the Inverse Function Theorem one needs transversality conditions which are stronger than those required in the variational approach.

A basic role in the analysis described here is held by a lemma on commuting vector fields (Lemma 1) which may have an independent interest and is therefore formulated and proved in greater generality than needed in the rest of the paper.

The paper is organized as follows: In Sect. 2 we prove the lemma and some related results. In Sect. 3 we specialize to the case of Hamiltonian vector fields, using the theory of normal forms. Here we connect the families of periodic solutions to the critical points of a suitable function in phase space. We also discuss the stability of these families of solutions. In Sect. 4 we discuss the detuning, and, as an example, we give a detailed analysis of the cases $n = 2$, $v_2 = 2v_1$, $v_2 = -2v_1$, $2v_2 = \pm 3v_1$.

2. A Lemma on Commuting Vector Fields

We prove first a simple lemma which plays a central role in what follows. Consider in R^{2n} the vector field

$$Y_\varepsilon(z) = Az + \varepsilon G(z, \varepsilon), \quad [G(z, \varepsilon), Az]_{\mathrm{Lie}} = 0, \tag{2.1}$$

where G is Lipshitz-continuous in z in $B_a = \{z \mid z \leqq a\}$, uniformly for $0 \leqq \varepsilon \leqq \varepsilon_1$. The

matrix A is antisymmetric with eigenvalues $\pm iv_1 \cdots \pm iv_n$, which satisfy

$$v_k = s_k \underline{v}, \quad s_k \in \mathbf{Z}/\{0\}, \quad k = 1 \cdots n, \quad \underline{v} \neq 0. \tag{2.2}$$

We shall use the notation

$$\tau_k \equiv 2\pi |v_k|^{-1}, \quad \underline{\tau} \equiv 2\pi |\underline{v}|^{-1},$$

and denote by τ the largest positive number for which

$$\tau_k = m_k \tau, \quad m_k \in \mathbf{Z}. \tag{2.3}_1$$

One has then

$$\underline{\tau} = N\tau, \quad N = m_k |s_k| \quad \forall k. \tag{2.3}_2$$

For $\varepsilon \geq 0$, we denote by $\varphi_\varepsilon(t, z^*)$ the solution of

$$dz/dt = Y_\varepsilon(z), \quad \varphi_\varepsilon(0, z^*) = z^*. \tag{2.4}$$

Notice the following: if for given ε, z^* there exists $\lambda \in R$ such that

$$G(z^*, \varepsilon) = \lambda A z^*, \tag{2.5}$$

then this relation holds also for $\varepsilon, \varphi_0(t, z^*)$ $\forall t$, since the two vector fields Az and $G(z, \varepsilon)$ commute. Therefore

$$\varphi_\varepsilon(t. z^*) = \varphi_0((1 + \lambda)^{-1} t, z^*), \tag{2.6}$$

so that in particular $\varphi_\varepsilon(t, z^*)$ is periodic with minimal period

$$T_\varepsilon(z^*) = (1 + \lambda) T_0(z^*).$$

We now prove (Lemma 2.1) that for every $T > 0$, if ε is small enough, all periodic solutions of (2.4) arise in this way, i.e. have the form (2.6).

It follows from (2.2) that for every z^* in R^{2n}, $\varphi_0(t, z^*)$ is periodic. We denote by $T_0(z^*)$ its minimal period and by $K(z^*)$ the positive integer for which

$$T_0(z^*) = K(z^*)\tau. \tag{2.7}$$

From the definition of φ_0 one easily verifies that given $c_2 > 0$, one can find $\varepsilon_0 > 0$, and $c_1 > 0$, depending on a and on $v_1 \cdots v_n$, such that for each $z^* \in \mathbf{B}_a$, $0 < \varepsilon < \varepsilon_0$, $t \in R$ there exists an integer $m (z^*, \varepsilon, t)$ with the property that

$$|\varphi_0(t, z^*) - z^*| < c_1 \varepsilon \quad \text{implies} \quad |t - m(z^*, \varepsilon, t)\tau| < c_2 \varepsilon. \tag{2.8}$$

If $\varphi_\varepsilon(t, z^*)$ is periodic, we shall denote by $T_\varepsilon(z^*)$ its minimal period. Let T be an arbitrary but fixed positive number; we shall look for periodic solutions of (2.3) for which $T_\varepsilon(z^*) < T$. If $\varphi_\varepsilon(t, z^*)$ is periodic, and $|z^*| < a$, one can find $c_1 > 0$ such that

$$|z^* - \varphi_0(T_\varepsilon(z^*), z^*)| = |\varphi_\varepsilon(T_\varepsilon(z^*), z^*) - \varphi_0(T_\varepsilon(z^*), z^*)| < c_1 \varepsilon. \tag{2.9}$$

We conclude from (2.8), (2.9) that for all $0 < \varepsilon < \varepsilon_0$, for all $z^* \in \mathbf{B}_a$ for which $\varphi_\varepsilon(t, z^*)$ is periodic one can find an integer K such that

$$|K T_\varepsilon(z^*) - \underline{\tau}| < c_2 \varepsilon. \tag{2.10}$$

Define $b(a)$ to be the largest positive number for which

$$|z^*| < b(a) \Rightarrow |\varphi_0(t, z^*)| < a \quad \forall \varepsilon \in [0, \varepsilon_0] \forall t \in [0, T_\varepsilon(z^*)].$$

One can now prove

Lemma 2.1. *Let $Y_\varepsilon(z)$ be given as in (2.1), (2.2). Given $T > 0$, one can find $\gamma > 0$ with the property that, if $\varepsilon < \gamma T^{-1}$ and $z^* \in B_{b(a)}$, then $\varphi_\varepsilon(t, z^*)$ is periodic with minimal period $T_\varepsilon(z^*) < T$ if and only if there exists $\lambda(z^*, \varepsilon) \in R$ such that (2.5) holds.*

When (2.5) holds, $\varphi_\varepsilon(t, z^*)$ coincides with $\varphi_0(t, z^*)$ up to a time scale, $\quad\square$

Proof. For fixed z^*, ε define λ as follows:

$$1 + \varepsilon\lambda = \tau/KT_\varepsilon(z^*), \tag{2.11}_1$$

so that, from (2.4), (2.10)

$$\exp\{K(1 + \varepsilon\lambda)T_\varepsilon(z^*)A\} = I \tag{2.11}_2$$

Define $\psi_\varepsilon(t, z^*)$ by

$$\psi_\varepsilon(t, z^*) \equiv \exp[-K(1 + \varepsilon\lambda)tA]\varphi_\varepsilon(t, z^*). \tag{2.12}$$

It follows from (2.11) that $\psi_\varepsilon(t, z^*)$ is periodic with period $KT_\varepsilon(z^*)$. Differentiating (2.12) and using (2.3) one concludes that $\psi_\varepsilon(t, z^*)$ is the solution of

$$d\psi_\varepsilon/dt = \varepsilon\Theta(\psi_\varepsilon, \varepsilon) \quad \psi_\varepsilon(0, z^*) = z^*,$$

where

$$\Theta(z, \varepsilon) \equiv \exp[-K(1 + \varepsilon\lambda)tA]\{G(\exp[K(1 + \varepsilon\lambda)tA]\varphi_\varepsilon(t, z), \varepsilon) - \lambda Az\} = G(z, \varepsilon) - \lambda Az. \tag{2.13}$$

The last equality in (2.13) follows from the fact that the fields $G(z, \varepsilon)$ and Az commute.

Without loss of generality we can assume that

$$|\Theta(\psi_\varepsilon(t, z^*), \varepsilon)| \leq |\Theta(z^*, \varepsilon)| \quad \forall 0 \leq t \leq T_\varepsilon(z^*),$$

so that

$$|\psi_\varepsilon(t, z^*)| \leq \varepsilon t|\Theta(z^*, \varepsilon)| \quad \forall 0 \leq t \leq T_\varepsilon(z^*). \tag{2.14}$$

Since ψ_ε is periodic

$$0 = \int_0^{KT_\varepsilon(z^*)} \Theta(\psi_\varepsilon(t, z^*), \varepsilon)dt = KT_\varepsilon(z^*)\Theta(z^*, \varepsilon) + \int_0^{KT_\varepsilon(z^*)} [\Theta(\psi_\varepsilon(t, z^*), \varepsilon) - \Theta(z^*, \varepsilon)]dt,$$

so that, using (2.14) and the Lipshitz condition on G

$$KT_\varepsilon(z^*)|\Theta(z^*, \varepsilon)| \leq c_4 \int_0^{KT_\varepsilon(z^*)} |\psi_\varepsilon(t, z^*) - z^*|dt \leq 1/2c_4\varepsilon K^2 T_\varepsilon^2(z^*)|\Theta(z^*, \varepsilon)|,$$

i.e., if $|\Theta(z^*, \varepsilon)| \neq 0$,

$$2 \leq \varepsilon c_4 KT_\varepsilon(z^*). \tag{2.15}$$

We conclude that, if $\varepsilon \leq \min\{\varepsilon_0, 2[c_4 KT]^{-1}\}$, and if $T_\varepsilon(z^*) \leq T$, one must have

$|\Theta(z^*, \varepsilon)| = 0$, i.e. $G(z^*, \varepsilon) = \lambda A z^*$. In particular

$$T_\varepsilon(z^*) = (1 + \lambda\varepsilon)^{-1} T_0(z^*),$$

so that

$$T_0(z^*) = \underline{\tau} \quad \Rightarrow \quad K = 1. \tag{2.16}$$

Conversely, if (2.5) holds, then one has

$$G(\varphi_0(t, z^*), \varepsilon) = \lambda(z^*, \varepsilon) A(t, z^*),$$

since $G(z, \varepsilon)$ and Az commute. Therefore $Y_\varepsilon(z)$ is parallel to Az along the orbit of φ_0 starting at z^*, and $\varphi_0(t, z^*)$ solves the equation

$$dz/dt = (1 + \varepsilon\lambda)^{-1} Y_\varepsilon(z).$$

The remaining statement in Lemma 2.1 is obvious. □

Remark 2.2. Condition (2.10) can be written in the form

$$\lambda\nabla(Az, Az) = -2AG(z, \varepsilon). \tag{2.17}$$

Let

$$\Sigma_c \equiv \{z | (Az, Az) = c\}.$$

Then (2.17) is equivalent to the condition that z be a critical point for $(AG)_0(z)$, the projection of AG to $T\Sigma_c(z)$.

The flow $t \to \exp\{At\}$ provides a natural S^1 action on $(AG)_0$ and on the (compact) manifold Σ_c. A lower bound on the number of critical orbits for $(AG)_0$ (and therefore on the number of periodic solutions of (2.4) with period smaller than T on each Σ_c), can be obtained by the use of equivalent Morse Theory or of Cohomological Index methods [2, 6]. We shall not discuss this point here, and refer to [6] for an extensive analysis. □

Remark 2.3. Consider the case in which Y_ε is a Hamiltonian vector field. Denoting by J the standard symplectic map, one will have

$$Y_\varepsilon(z) = JdH \quad H(z) = H_2(z) + \varepsilon H'(z, \varepsilon), \tag{2.18}$$

where $JdH_2(z) = Az$ and $\{H', H_2\}_{\text{P.B.}} = 0$.

Condition (2.5) becomes now

$$\lambda dH_2(z) = dH'(z, \varepsilon), \tag{2.19}$$

i.e. that z be a critical point of H' on a surface $\Gamma_c \equiv \{z | H_2(z) = c\}$, or equivalently that z be a critical point of H_2 on a surface $\Gamma'_c \equiv \{z | H'(z) = c\}$. Notice that $t \to \exp\{At\}$ defines a natural S^1 action on JdH' and on Γ_c, and also on JdH_0 and on Γ'_c. Therefore Equivalent Morse theory or a Cohomological index can be used to give a lower bound on the number of periodic solutions of (2.1), with minimal period smaller than T, on each surface Γ_c of Γ'_c. In particular, at least n such periodic solutions can be found if Γ_c or Γ'_c are convex. It should be noted that in the former case H_0 is of definite sign [2], but that the condition that Γ'_c be convex

poses no restriction of convexity on the surfaces

$$\Sigma'_c \equiv \{z \,|\, H_0(z) + \varepsilon H'(z) = c\}$$

for ε small.

More generally, it is easy to see that, if one can find a real valued function $F(x, y)$ and a real constant c such that

$$\Phi_c \equiv \{z \,|\, F(H_0(z), H'(z)) = c\} \quad \text{is convex} \tag{2.20}_1$$

$$\Phi_c \in \mathbf{B}_{a(b)} \tag{2.20}_2$$

and if F does not vanish on Φ_c, then the system (2.4) has on Φ_c at least n periodic solutions with minimal period less than T [7]. $\quad\square$

We determine now the stability of the periodic solutions described in Lemma 2.1. One has

Lemma 2.4. *The Floquet multipliers* $\{\rho_i, i = 1 \cdots n\}$ *for the periodic solutions described in Lemma 2.1 satisfy*

$$(\rho_i)^K = \exp\left[K \mu_i T_\varepsilon(z^*)\right], \tag{2.21}$$

where μ_i *are the eigenvalues of the matrix*

$$DG(z^*, \varepsilon) - \lambda(z^*, \varepsilon) A, \tag{2.22}$$

and K is defined in (2.8). In particular (see (2.16)), if $T_0(z^*) = \underline{\tau}$, *then (2.21) holds with* $K = 1$. $\quad\square$

Proof. Let $p(t)$ be a periodic solution of (2.4) of the type described in Lemma 2.1; one has

$$p(t) = \exp\{(1 + \varepsilon\lambda)At\}p(0).$$

The Floquet multipliers are the eigenvalues of the map

$$R^{2n} \ni \zeta \to \sigma(T_\varepsilon(p(0)), \zeta), \tag{2.23}$$

where $\sigma(t, \zeta)$ is the solution of

$$d\sigma/dt = (DY_\varepsilon)(p(t))\sigma, \quad \sigma(0, \zeta) = 0. \tag{2.24}$$

Define $\chi(t, \zeta)$ by

$$\chi(t, \zeta) \equiv \exp\{-(1 + \varepsilon\lambda)At\}\sigma(t, \zeta). \tag{2.25}$$

By construction

$$\exp\{K(1 + \varepsilon\lambda)AT_\varepsilon(p(0))\} = I,$$

and therefore

$$\chi(KT_\varepsilon(p(0)), \zeta) = \sigma(KT_\varepsilon(p(0)), \zeta). \tag{2.26}$$

Moreover

$$d\chi/dt = \varepsilon[DG(p(0), \varepsilon) - \lambda A]\chi, \tag{2.27}$$

since G commutes with Az, and therefore

$$\exp\{-(1+\varepsilon\lambda)At\}DG(p(t),\varepsilon)=DG(p(0),\varepsilon)\exp\{-(1+\varepsilon\lambda)At\}.$$

The conclusion of Lemma 2.4 follows immediately from (2.26), (2.27). $\square$

A special important case is the one in which the vector field $Y_\varepsilon(z)$ is Hamiltonian, so that it can be written in the form (2.18). One has then

$$d\chi/dt=DJ[dH'-\lambda dH_0]\chi \tag{2.28}$$

The vectors ∇H_0 and JdH_0 are eigenvectors of the matrix $DJ[dH'-\lambda dH_0]$ with eigenvalue zero. Define D_0JdH' as follows:

Definition 2.5. D_0JdH' is the restriction of $DJdH'$ to the orthogonal complement in R^{2n} of the subspace generated by ∇H_0 and JdH_0. $\square$
One has then

Corollary 2.6. *If the vector field has the form (2.18), and $p(t)$ is a periodic solution of (2.4) for ε sufficiently small (as described in Lemma 2.1), then $p(t)$ is elliptic precisely if all the eigenvalues of D_0JdH' are purely imaginary, while $p(t)$ is hyperbolic if all the eigenvalues of D_0JdH' have non-zero real part.* $\square$
Consider now a vector field of the form

$$Y_\varepsilon(z)=Az+\varepsilon G(z,\varepsilon)+\varepsilon^{p+\delta}R(z,\varepsilon), \tag{2.29}$$

where A and G are as in Lemma 2.1, $p\geq1$, $\delta>0$ and R is uniformly bounded for $z\in\mathbf{B}_a$, $0<\varepsilon<\varepsilon_0$ and Lipshitz continuous in z, uniformly for $z\in\mathbf{B}_a$ and $0<\varepsilon<\varepsilon_0$.
One can follow the same steps as in Lemma 2.1 and prove the following: Given $T>0$, one can find $\varepsilon_1>$ such that, if $\varepsilon<\varepsilon_1$ and $\phi(t,z_\varepsilon^*)$ is a solution of

$$d\phi/dt=A\phi+\varepsilon G(\phi,\varepsilon)+\varepsilon^{p+\delta}R(\phi,\varepsilon),\quad\phi(0,z_\varepsilon^*)=z_\varepsilon^*,$$

then one can find $c\in R$ and $\lambda\in R$ such that

$$|\lambda Az^*-G(z_\varepsilon^*,\varepsilon)|<c\varepsilon^{\delta+p-1}. \tag{2.30}$$

Under suitable assumptions on the Jacobian of the map

$$z^*\to Az^*+G(z^*,\varepsilon)$$

(essentially that its inverse be bounded by $c_5\varepsilon^{-p+1}$ uniformly for $\varepsilon<\varepsilon_1$ and z in a neighborhood of z_ε^*), condition (2.30) is necessary and sufficient in order that there exist $c_6\in R$ and, for each $\varepsilon<\varepsilon_1$, a point $y_\varepsilon^*\in R^{2n}$ such that

$$|y_\varepsilon^*-z_\varepsilon^*|<c_6\varepsilon^\delta$$

and $\phi_\varepsilon(t,y^*)$ is a periodic solution of (2.4) with minimal period approximately equal to $T_\varepsilon(z^*)$.
Also, results similar to those of Lemma 2.4 can be obtained on the linear stability of the periodic solutions $\phi_\varepsilon(t,y_\varepsilon^*)$. We shall study this problem in the next section, restricting ourselves to the case in which Y_ε is a Hamiltonian vector field.

3. Applications to Hamiltonian Systems

We use the results of Sect. 2 to study periodic solutions near equilibrium for Hamiltonian systems when the frequencies are at or near resonance. We consider a system for which the Hamiltonian H has the form

$$H(z) = 1/2 \sum_0^n v_i z_i^2 + H^*(z), \quad z = \{z_1 \cdots z_n\}, \quad z_i = \{q_i, p_i\}, \tag{3.1}$$

where H^* is of class C^p in a neighborhood of the origin, for some $p \geq 3$, and is infinitesimal at the origin of order $q \geq 3$. We also assume that $v_k \neq 0$ for all k.

Since we are interested in the behaviour near the origin, we introduce the relevant scale through the canonical transformation

$$z_k \rightarrow \varepsilon z_k, \quad H(z) \rightarrow \varepsilon^{-2} H(\varepsilon z) \equiv H_\varepsilon(z). \tag{3.2}$$

In the new variables, Hamilton's equations read:

$$dz/dt = JdH_\varepsilon(z) = JdH_2(z) + \varepsilon^{q-2} JdQ(z, \varepsilon), \tag{3.3}$$

where

$$H_2(z) = 1/2 \sum_1^n v_k z_k^2, \quad Q(z, \varepsilon) = \varepsilon^{-q} H^*(\varepsilon z). \tag{3.4}$$

By construction, $Q(z, \varepsilon)$ is of class C^p in z in a ball $\mathbf{B}_a$, uniformly in ε near 0. We are interested in the case when the frequencies are at or near resonance. We set therefore

$$v_k = v_k^0 + \varepsilon^\alpha \mu_k, \quad k = 1 \cdots n, \quad \mu_k \in \mathbf{R}, \tag{3.5}$$

where $\{v_k^0\}$ is a set of resonating frequencies and the constant α depends on H^* and will be chosen later.

Remark 3.0. Using the Inverse Function Theorem it is not difficult to verify [1] that for any given $T > 0$ one can find positive constants c_1 and c_2 and ε_1 such that, if $|z| = a$ and $\varepsilon < \varepsilon_1$, and $\varphi_\varepsilon(t, z)$ is periodic solution of (3.3) with minimal period $T_\varepsilon < T$, then one must have

$$|T_\varepsilon - N_k \tau| < c_2 \quad \text{for some} \quad k \in \{1, , n\} \quad \text{and} \quad N_k \in Z, \tag{3.6}$$

where τ is defined in $(2.3)_1$. Moreover in the study of periodic solutions with minimal period approximately equal to $M\tau$, $M \in \mathbf{Z}$, one can without loss of generality set $z_h = 0$ if $(2\pi)^{-1} M v_h \tau \notin \mathbf{Z}$. We can therefore restrict attention to the case of full resonance, i.e.

$$v_k^0 = n_k \underline{v}, \quad n_k \in \mathbf{Z}/\{0\} \quad \underline{v} \neq 0. \tag{3.7}$$

We shall assume (3.7) from now on. Using (3.7) we write (3.3) as

$$dz/dt = JdH_0 + 1/2\varepsilon^\alpha \sum_1^n \mu_k Jdz_k^2 + \varepsilon^{q-2} JdQ(z, \varepsilon), \tag{3.8}_1$$

$$H_0 = 1/2 \sum_1^n v_k^0 z_k^2. \tag{3.8}_2$$

For the best choice of α we need a lemma on normal forms.

Lemma 3.1 [8]. *Let $H_\varepsilon(z)$ have the form*

$$H_\varepsilon(z) = H_0(z) + \varepsilon^r H(\varepsilon z), \quad z \in R^{2n}, \tag{3.9}$$

where H_0 is given in $(3.8)_2$ and $H'(y)$ is of class C^s, $s \geq 3$, uniformly in a neighborhood of the origin, and is infinitesimal at the origin of order d, with $r + 1 \leq d$ and $d \leq s$. If (3.7) holds, given $a > 0$, one can find $\varepsilon_1 > 0$ such that, for each $0 \leq \varepsilon < \varepsilon_1$ and each $m \leq s - 2$ there exists a symplectic transformation $\Phi_{m,\varepsilon}$, asymptotic to the identity when $\varepsilon \to 0$, and such that in $\mathbf{B}_a^{2n}$,

$$(H_\varepsilon \Phi_{m,\varepsilon})(z) = H_0(z) + \varepsilon^{r(m)} N_{m,\varepsilon}(z) + \varepsilon^{d-r+m} R_{m,\varepsilon}(Z), \tag{3.10}$$

where $d - r + m \geq r(m) \geq r(m - 1)$, $r(0) = d - r$, and $R_{m,\varepsilon}$ is infinitesimal in ε uniformly for $z \in B_a^{2n}$. The functions $N_{m,\varepsilon}$ and $R_{m,\varepsilon}$ are of class C^{s-m} and one has

$$\{N_{m,\varepsilon}, H_0\}_{\text{P.B.}} = 0. \tag{3.11}$$

Moreover if $N_{m,\varepsilon} \neq 0$, then $r(m') = r(m)$ for all $m' > m$. The function $N_{m,\varepsilon}$ will be called "normal form of H to order m" (relative to H_0). □

Outline of proof. The symplectic transformation $\Phi_{m,\varepsilon}$ is constructed in m steps. At step i the transformation is the time-ε map $\Phi^{(i)}$ for a suitable Hamiltonian K_i. To find K_i one has to solve

$$\{K_i, H_0\}_{\text{P.B.}} = B_{i-1}, \tag{3.12}$$

where B_{i-1} is known from step $i - 1$ and has no component in the kernel of the map $F \to \{F, H_0\}_{\text{P.B.}}$. No small denominator arises, since by (3.7), if $\sum_1^n m_k v_k \neq 0$, $m_k \in Z$, then $\left| \sum_1^n m_k v_k \right| > \underline{v}$. On the other hand, at each step one loses in general one order of differentiability, since the map $\Phi^{(i)}$ and the functions B_i are as regular as JdK_i, while the solution K_i of (3.12) has one order of differentiability less than B_{i-1}. □

We now apply Lemma 3.1 to the study of periodic solutions of (3.8). Notice that the term

$$1/2 \sum_1^n \mu_k z_k^2 \tag{3.13}$$

is in normal form with respect to H_0, and recall that

$$\{F, G\}_{\text{P.B.}} = 0 \quad \Rightarrow \quad [JdF, JdG]_{\text{Lie}} = 0.$$

Consider the Hamiltonian

$$H_\varepsilon'(z) \equiv H_0(z) + \varepsilon^{q-2} Q(z, \varepsilon).$$

By construction $Q(z, \varepsilon)$ satisfies the assumptions of Lemma 3.1 with $r = 2$, $d = q$, $s = p$. For every $m \leq p$ one can therefore write H' in new symplectic coordinates (still denoted by z) in the form

$$H_\varepsilon'(z) = H_0(z) + \varepsilon^{r(m)} N_{m,\varepsilon}'(z) + \varepsilon^{d(m)} R_{m,\varepsilon}'(z),$$

where $r(m) \geqq q - 2$, $d(m) \geqq q - 2 + m$, $N'_{m,\varepsilon}$ satisfies (3.11) and $R'_{m,\varepsilon}$ is infinitesimal in ε uniformly for $z \in \mathbf{B}_a$. We consider only the case in which one can find $\underline{m} \leqq p$ such that $N'_{\underline{m}} \neq 0$ (while $N'_m = 0$ if $m < \underline{m}$), and choose α in (3.5) as

$$\alpha = r(\underline{m}). \tag{3.14}$$

One has then, using Lemma 3.1, for all $\underline{m} \leqq m \leqq p$,

$$H_\varepsilon(z) = H_0(z) + \varepsilon^\alpha \left[1/2 \sum_1^n \mu_k z_k^2 + N_{m,\varepsilon,\mu}(z) \right] + \varepsilon^{d(m)} R_{m,\varepsilon,\mu}(z), \tag{3.15}$$

where the $N_{m,\varepsilon,\mu}$ satisfy (3.11), and $d(m) \geqq q - 2 + m$, $d(m) \geqq \alpha$ and $d(m) > \alpha$ if $m > \underline{m}$. The functions $R_{m,\varepsilon,\mu}$ are infinitesimal in ε uniformly for $z \in \mathbf{B}_a$. The best choice of $m, \underline{m} \leqq m \leqq p$, depends on the resonant set of frequencies. This will be seen explicitly in the example we shall discuss in the next section.

Define the Hamiltonians $\underline{H}_\varepsilon$ by

$$\underline{H}_\varepsilon(z) = H_0(z) + \varepsilon^\alpha \left[1/2 \sum_1^n \mu_k z_k^2 + N_{m,\varepsilon,\mu}(z) \right]. \tag{3.16}$$

The term in square brackets in (3.16) is by construction in normal form with respect to H_0. We are therefore in condition to apply to the Hamiltonian flow of $\underline{H}_\varepsilon$ Lemmas 2.1 and 2.4, Remark 2.3 and Corollary 2.5.

Remark 3.2. We notice in particular that $N_{m,\varepsilon,\mu}$ has by construction the form

$$\varepsilon^{r(m)} N_{m,\varepsilon,\mu}(z) = \sum_{s=3}^m \varepsilon^s P_{s,\mu}(z), \tag{3.17}$$

where $P_{s,\mu}(z)$ are homogeneous polynomials of order s, normal with respect to H_0. Therefore the solutions $z_\varepsilon, \lambda_\varepsilon, \mu_\varepsilon$ of

$$X_{m,\varepsilon,\mu}(z, T) \equiv dN_{m,\varepsilon,\mu}(z) - \lambda dH_0(z) = 0 \tag{3.18}$$

form continuous families, with parameter ε. $\square$

We have proved in Sect. 2 that for any given $T > 0$, for sufficiently small ε the φ_0-orbits of solutions of (3.18) are in one-to one correspondence with the periodic solutions of

$$dz/dt = Jd\underline{H}_\varepsilon(z) \tag{3.19}$$

with minimal period $\leqq T$.

We shall now study, still for ε sufficiently small, the periodic solutions of (3.3) with minimal period $T_\varepsilon \leqq T$. When $\mu_k = 0 \ \forall k$, this is equivalent to the study of periodic solutions of

$$dz/dt = JdH(z) \tag{3.20}$$

(with minimal period $T_\varepsilon(p(0)) \leqq T$) in a small neighborhood of the origin. When some of the μ_k are $\neq 0$, the problem we study can be regarded [5] as a bifurcation analysis for periodic solutions of (3.20).

Since we are interested in small values of ε, we shall use the Inverse Function Theorem. We make therefore an assumption on the Jacobian of the map

$z \to X_{m,\varepsilon,\mu}(z, T_\varepsilon)$, with $X_{m,\varepsilon,\mu}(z, T)$ defined in (3.18) (the relation between T and λ is given in (2.11)). This assumption holds only if p is large enough (depending on the choice of $v_1^0 \cdots v_n^0$), and, for given p, only if $N_{m,\varepsilon,\mu}$ does not belong to a subspace of codimension greater or equal to one in the vector space of polynomials of order $\leq m$ normal with respect to H_0. In this sense, condition (3.22) in the next proposition is satisfied generically.

Proposition 3.3. *Let the Hamiltonians H_ε have the form (3.15), with $t(m) > \alpha$ and $R_{m,\varepsilon,\mu}$ uniformly bounded for $z \in B_a$. Define $X_{m,\varepsilon,\mu}$ by (3.18) and denote by $\mathrm{Hess}_0 B$ the restriction of the Hessian matrix of B to the orthogonal complement in R^{2n} to the subspace spanned by ∇H_0 and JdH_0. Then, for every $\delta < d(m)$, $c_k > 0$, $k = 1, 2, 3$, $\mu \in R^n$, one can find $\varepsilon_1 > 0$ such that the following statements hold true:*

I: *if $\varepsilon < \varepsilon_1$ and $z_\varepsilon, T_\varepsilon^0$ is a solution of*

$$X_{m,\varepsilon,\mu}(z_\varepsilon \cdot T_\varepsilon^0) = 0, \tag{3.21}$$

which satisfies

$$\varepsilon^{-m} \det [\mathrm{Hess}_0 N_{m,\varepsilon,\mu}(z)]|_{z = z_r} > c_1, \tag{3.22}$$

then one can find $y_\varepsilon \in R^{2n}$ such that

  a) *$|y_\varepsilon - z_\varepsilon| < c_2 \varepsilon^{\tau(m) - \delta}$.*
  b) *$\varphi_\varepsilon(t, y_\varepsilon)$ is a periodic solution of (3.3) with period $T_\varepsilon(y_\varepsilon)$ approximately equal to the minimal period $T_{0,\varepsilon}$ of $\varphi_0(t, z_\varepsilon)$.*
  c) *$\Delta(\varphi_\varepsilon, \varphi_0) < c_3 \varepsilon^{d(m) - \delta}$,*
    where $\Delta(\varphi_\varepsilon, \varphi_0)$ is the distance of the orbits of $\varphi_\varepsilon(t, y_\varepsilon)$ and of $\varphi_0(t, z_\varepsilon)$.
The point y_ε is not unique; however the periodic solution $\varphi_\varepsilon(t, y_\varepsilon)$ is uniquely determined by the condition
  d) *$H_\varepsilon(y) = \underline{H}_\varepsilon(z)$*
    (H_ε and $\underline{H}$ are defined in (3.15), (3.16)).

II: *Conversely, if $\varepsilon < \varepsilon_1$ and $y_\varepsilon \in R^{2n}$ is such that y_ε satisfies (3.22) and $\varphi_\varepsilon(t, y_\varepsilon)$ is periodic solution of (3.3) with minimal period $T_\varepsilon(y_\varepsilon) < T$, there exists $z_\varepsilon \in R^{2n}$, $\lambda_\varepsilon \in R$ such that*

  a) *$|z_\varepsilon - y_\varepsilon| < c_2 \varepsilon^{d(m) - \delta}$.*
  b) *$X_{m,\varepsilon,\mu}(z_\varepsilon) = 0$ holds, so that $\varphi_0(t, z_\varepsilon)$ is a periodic solution of (3.19) with period T'_0 approximately equal to $T_\varepsilon(y_\varepsilon)$.*
  c) *$\Delta(\varphi_0(t, z_\varepsilon), \varphi_\varepsilon(t, y_\varepsilon)) < \varepsilon^{d(m) - \delta}$.*
    The point z_ε is not uniquely determined; however the periodic solution $\varphi_0(t, z_\varepsilon)$ is uniquely determined by the condition $\underline{H}_\varepsilon(z_\varepsilon) = H_\varepsilon(y_\varepsilon)$. $\square$

Remark 3.4. A more careful analysis shows that if z_ε is a solution of (3.21) and if condition (3.22) is satisfied, then $\varphi_0(t, z_\varepsilon)$ has minimal period $\underline{\tau}$, i.e. the greatest among the minimal periods of the solutions of $dz/dt = JdH_0$. As a consequence, $K = 1$ in Lemma 2.1. Similarly, under condition (3.22), $\varphi_\varepsilon(t, y_\varepsilon)$ in Proposition 3.2, II, has minimal period T_ε which differs from $\underline{\tau}$ by terms of order $\varepsilon^{d(m) - \delta}$. $\square$

Proof of Proposition 3.3. We prove first part I. The condition that $\varphi_\varepsilon(t, y_\varepsilon)$ be

periodic with period T_ε is equivalent to

$$\Phi_\varepsilon(T_\varepsilon, y_\varepsilon) = \int_0^{T\varepsilon} JdH_\varepsilon(\varphi_\varepsilon(t, y))dt = 0. \qquad (3.24)$$

Since H_ε is constant along the flow of JdH_ε, it is sufficient to consider the restriction of (3.24) to

$$(y, T) \in \Sigma_\varepsilon \times R,$$

where $\Sigma_\varepsilon \equiv \{z \mid H_\varepsilon(z) = H_\varepsilon(y_\varepsilon)\}$. Moreover, since H_ε does not depend on time, it is sufficient to solve (3.24) on a Poincarè section at y_ε, in particular on the hyperplane perpendicular to JdH_ε. Proceeding as in the proof of Lemma 2.1 and using the fact that

$$\varphi_\varepsilon(s, y) - \underline{\varphi}_\varepsilon(s, y) = 0(\varepsilon^{d(m)}),$$

the restricted condition can be written as

$$\Theta'(y, T) \equiv X'_{m,\varepsilon,\mu}(y, T) + \varepsilon^{d(m)}R'_{m,\varepsilon,\mu}(y, T) = 0, \qquad (3.25)$$

where R' is uniformly bounded in the domain considered, and A' is the restriction of A to the orthogonal complement to the subspace Ξ_ε spanned by JdH_ε and ∇H_ε.

By assumption (3.22), the Jacobian of the map $y \to \Theta(y, T)$ is non-singular for $\varepsilon < \varepsilon_1$. Moreover $d\Phi(y, T)/dT = JdH_\varepsilon(y) \neq 0$, so that also the Jacobian of the map $y, T \to \Theta'(y, T)$, T is not singular. By assumption, $z_\varepsilon, \underline{T}_\varepsilon$ solve the equation

$$X''_{m,\varepsilon,\nu}(z, T) = 0, \qquad (3.26)$$

where X'' is the restriction of X to the orthogonal complement to the subspace $\underline{\Xi}_\varepsilon$ generated by $Jd\underline{H}_\varepsilon$ and $\nabla \underline{H}_\varepsilon$. Since $H_\varepsilon - \underline{H}_\varepsilon = 0(\varepsilon^{d(m)})$, there exists an orthogonal transformation Ψ_ε which is of the form $I + 0(\varepsilon^{d(m)})$ and maps Ξ_ε onto $\underline{\Xi}_\varepsilon$. Part I, a), b), c), d) are then a consequence of the Implicit Function Theorem.

To prove Part II, one follows the same steps. Given a solution $\{y, T\}$ of (3.25), one can write (3.26) in the equivalent form

$$0 = X''(z, T) \equiv \Theta'(z, T) + \varepsilon^{d(m)}R''(z, T), \qquad (3.27)$$

where R'' is defined by (3.27). Since $\{y, T\}$ solves $\Theta'(z, T) = 0$, Part II, a), b), c), follows from the Inverse Function Theorem in view of assumption (3.22). $\square$

The stability properties of the solutions of (3.3) described in Proposition 3.3 follow immediately from Corollary 2.6. One has in particular

Corollary 3.5. *The solution $\varphi_\varepsilon(t, y_\varepsilon)$ of (3.3) described in Proposition 3.3 is hyperbolic if all the eigenvalues of $D_0 JdH_\varepsilon(z_\varepsilon)$ have non-zero real part. It is elliptic if all eigenvalues of $D_0 Jd\underline{H}_\varepsilon(z_\varepsilon)$ are purely imaginary and have multiplicity one.* $\square$

4. Applications: The Cases $n = 2$, $\phi_2^0 = \pm 2\phi_1^0$, $\phi_2^0 = \pm 3/2 \; \phi_1^0$

In this section we apply the formalism of Sect. 3 to three simple examples. A partial analysis of the first and the second have been given previously [3, 4] (for an analysis of the detuning within a different approach, see e.g. [5]). We present them here

to illustrate the general formalism. The third example is discussed because it illustrates generic properties of systems for which $n = 2$, $v_1^0 = m_1 \underline{v}$, $v_2^0 = m_2 \underline{v}$, $|m_1| + |m_2| \geq 5$. The common feature of these systems is that generically $H - H_2$ has a non-trivial normal form N_4 of order four. However N_4 is invariant under a one-parameter group of transformations different from the one generated by JdH_0, so that $D_0 JdN_4$ is singular (it is in Jordan form and has zero as eigenvalue with algebraic multiplicity 2).

In order to apply the results of Sect. 3, one must assume more regularity and construct the normal form of H to the order $|m_1| + |m_2|$ (which is the first order in which this continuous symmetry is broken). We refer to [9] for other examples, in particular the cases $n = 2$, $|v_2| = kv_1$, $k \geq 4$ and $n = 3$, $|v_3| = 2|v_2| = 4v_1$; $n = 3|v_2| = 2|v_1|$ (see also [4]).

We begin now the analysis of the examples mentioned above.

A) $v_2^0 = 2v_1^0$.

We shall only discuss periodic solutions which have minimal period approximately equal to $\tau_1^0 \equiv \tau$. The periodic solutions which have minimal period approximately equal to $\tau \equiv \tau_2^0$ form a one-parameter family which depends smoothly on μ_1 and μ_2. This follows from Remark 3.0.

We assume that H is at the origin of class $C^{3+\sigma}$, $\sigma > 0$; by a change in time scale we can set $v_1^0 = 1$, $\mu_1 = 0$. We write μ for μ_2. After the symplectic transformation described in Sect. 3, and the scaling $z \to \varepsilon z$, the Hamiltonians H_ε have the form

$$H_\varepsilon = 1/2(p_1^2 + q_1^2) + (p_2^2 + q_2^2) + \varepsilon N(z_1, z_2) + \varepsilon^{1+\delta} R(z_1, z_2, \varepsilon) \qquad (4.1)$$

where $z_i = \{p_i, q_i\}$, $\delta > 0$ and

$$N(z_1, z_2) = a[(p_1^2 - q_1^2)p_2 + 2p_1 q_1 q_2] + \mu(p_2^2 + q_2^2). \qquad (4.2)$$

Remark. In the notation $\zeta_k \equiv p_k + iq_k$, the flow of JdH_0 is given by

$$\zeta_1(t) = \zeta_1(0) \exp\{iv_1 t\}, \quad \zeta_2(t) = \zeta_2(0) \exp\{iv_2 t\},$$

so that $\zeta_1^{*2}\zeta_2$ and $\zeta_1^2 \zeta_2^*$ are the only monomials of order 3 which are invariant under the flow of JdH_0. Changing the phase of ζ_1 and ζ_2 is a symplectic transformation; one can therefore choose symplectic coordinates such that $N(z_1, z_2)$ has the form given above. $\square$

We assume that $a \neq 0$ in (4.2); without loss of generality one can then take $a = 1$. (If $a = 0$, one is in the framework of example 3 below.) Since normal forms to order 3 coincide with time-averages along the flow of JdH_0, this assumption holds for all the Hamiltonians H in C^3 for which the time average of $H - H_2$ is not infinitesimal at the origin of order greater than three. In this sense our results are generic.

One is now condition to use the results of Sect. 3. One has $q = 3$, $m_0 = 1$, $r(m_0) = 1$ and in (4.2) we have already taken $\alpha = 1$. One can choose as Poincaré section the hyperplane $q_1 = 0$. The condition

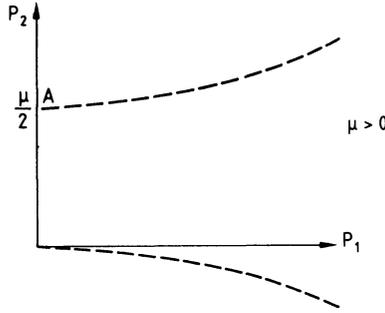$$dN = \lambda dH_0 \qquad (4.3)$$

Fig. 1

becomes then

$$2p_1p_2 = \lambda p_1, \quad p_1^2 + 2\mu p_2 = 2\lambda p_2, \quad q_2 = 0. \tag{4.4}$$

We consider only solutions for which $p_1 \neq 0$; the other solution corresponds to the family of periodic solutions with period approximately equal to τ_2^0, which is obtained in Lyapunov's analysis. Also, only the case $p_1 > 0$ will be considered, since $\{-z_1^0, z_2^0\}$ belongs to the orbit of the periodic solution of $dz/dt = JdH_0$ with initial data $\{z_1^0, z_2^0\}$. Since $p_1 \neq 0$, one has $K = 1$ in Lemma 3.4. One has then

$$\lambda^{\pm} = 2p_2 \quad p_2^{\pm} = 1/4[\mu \pm (\mu^2 + 4p_1^2)^{1/2}]. \tag{4.5}$$

The graph of $p_2^{\pm}(p_1)$ is given in Fig. 1 for fixed $\mu > 0$. It is also straightforward to compute the spectrum of $DJ(dN - \lambda dH_0)$.

For both alternatives $\pm$ in (4.5), one finds that, apart from the eigenvalue zero which has multiplicity two, there are the two eigenvalues

$$\sigma_{\pm}(p_1) = i|p_1||[p_1^2 + 8p_2^2]^{1/2}/|p_2|. \tag{4.6}$$

In particular $|(D_0JdN)^{-1}| = 1/|\sigma(p_1)|$, so that D_0JdN is not singular at the solution of (4.3), with the exception of the point A in Fig. 1.

We conclude from Proposition 3.3 that, given $a > 0$, for every sufficiently small ε there are two periodic solutions of (3.3) with period approximately equal to τ_1^0 and orbit approximately equal to the orbit of the flow of JdH_0 through the point

$$p_1 = a, \quad q_1 = 0, \quad q_2 = 0 \quad \text{and} \quad p_2^{\pm} \quad \text{as in (4.5).} \tag{4.7}$$

From (4.6) and Corollary 3.4 it follows that these periodic solutions are elliptic. Notice that the point A in Fig. 1 is a bifurcation point with period doubling for the periodic solutions of $dz/dt = JdH_{\varepsilon}$, but the same conclusion cannot be drawn in general for the periodic solutions of (3.3), since condition (3.22) fails at the point A.

Remark. For $|\mu|$ large, one of the two solutions in (4.5) (which one depends on the sign of μ) is such that $|p_2|$ becomes very small (see Fig. 1). This corresponds to the periodic solution which is found in Lyapunov analysis; in fact, large μ corresponds to a detuning which is large compared to the size of the neighborhood of the origin in R^{2n} in which one looks for periodic solutions, and corresponds therefore to a non-resonant set of frequencies. This can be explicitly seen if one

makes a different rescaling of the detuning term, i.e. writes $v_k = v_k^0 + \varepsilon^\alpha \mu_k$, with $\alpha < 1$. On this scale, the norm of the other solution in (4.5) becomes large when ε decreases, and therefore when ε is very small the Implicit Function Theorem cannot be used to deduce the existence of a corresponding periodic solution of (3.3). As a consequence, in this scaling of the detuning term, in a sufficiently small neighborhood of the origin there is in general only one family of periodic solutions with minimal period approximately equal to τ_1^0, in accordance with Lyapunov' analysis. $\square$

B) $v_2^0 = -2v_1^0$.

Also in this case we shall discuss only the periodic solutions which have minimal period approximately equal to τ_1^0. The Hamiltonians H_ε take now the form (4.1), where

$$N(z_1, z) = (p_1^2 - q_1^2)p_2 - 2p_1 q_1 q_2 + (p_2^2 + q_2^2). \tag{4.8}$$

Choosing the same Poincaré section as in the previous example, condition (4.2) becomes now

$$2p_1 p_2 = p_1, \quad p_1^2 + 2p_2 = -2p_2, \quad q_2 = 0, \quad q_1 = 0. \tag{4.9}$$

Again we consider only the solutions for which $p_1 \neq 0$. One has then

$$\lambda = 2p_2, \quad p_2^\pm = 1/4[-\mu \pm (\mu^2 - 4p_1^2)^{1/2}]. \tag{4.10}$$

From (4.10) one can see that solutions exist only if $2p_1 \leq |\mu|$; in particular there is no solution if $\mu = 0$, i.e. in the resonant case. It is also seen from (4.10) that for each $\mu > 0$ the solutions lie on a curve in the $p_1 p_2$ plane, parametrized by p_2 (see Fig. 2); as in the previous example, only the half-plane $p_1 > 0$ must be considered. The energy cannot be used as parameter (see Fig. 3).

   To study the stability of those periodic solutions of (3.3) which correspond to the solutions (4.10) of (4.9), we make use of Corollary 3.4 and compute the spectrum of $DJ(dN - dH_0)$. Apart from the eigenvalue zero, one finds the eigenvalues

$$\sigma^\pm = \pm |p_1 p_2^{-1}|(8p_2^2 - p_1^2)^{1/2}.$$

Condition (3.22) is satisfied along the curve in Fig. 1, with the exception of the point A and B. Therefore those solutions for which $8p_2^2 < p_1^2$, i.e. the part of the
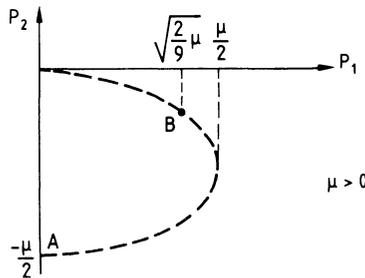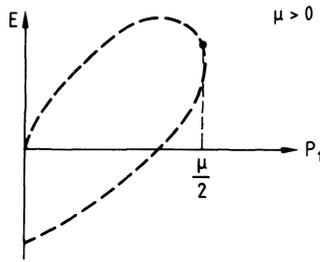

Fig. 2

Fig. 3

upper branch of the curve in Fig. 2 which lies to the left of the point B, correspond to elliptic periodic solutions, while the part of the curve in Fig. 2 which starts at the right of the point B corresponds to periodic solutions which are hyperbolic. When $8p_2^2 = p_1^2$, the matrix $D_0 J d N$ is singular, and Proposition 3.3 cannot be used. To study this case, one should make the further assumption that H is of class C^4 at the origin, and construct the normal form of H to order four.

Remark. For μ very large, p_2 becomes very small compared to p_1 on the elliptic branch of solutions, at least for values for p_1 small compared to μ. This branch of solutions goes over, when μ is large, to the family of solutions found in Lyapunov's analysis for the strongly detuned case, i.e. the case in which the two frequencies v_1 and v_2 are not in resonance. This is seen explicitly if one does a different rescaling of the detuning term in (4.2).

C) $v_2^0 = \pm 3/2\, v_1^0$.

It follows from Lyapunov's analysis that, for $i = 1, 2$, there exists a one-parameter family of periodic solutions, with period approximately equal to τ_i^0 and lying approximately on the hyperplane $z_j = 0, j \neq i$. We are interested in periodic solutions with minimal period longer than the one of Lyapunov's solutions. By a change in time scale, we can take $v_1^0 = 2$, $v_2^0 = \pm 3$, which corresponds to the choice $\underline{v} = 1$ in (3.7). As in the previous examples, we can also take $\mu_1 = 0$ and write μ for μ_2. In the present case, in order to satisfy (3.22)—at least generally—one must assume that H is of class $C^{4+\delta}$ at the origin, $\delta > 0$, and take $m = 2$, so that $d(m) = 2 + \delta$. The Hamiltonians H_ε can then be written in the form (4.1) where, for $v_2^0 = 3$ one has

$$N(z_1, z_2, \varepsilon) = c_1(p_1^2 + q_1^2)^2 + c_2(p_2^2 + q_2^2)^2 + \mu(p_2^2 + q_2^2)$$
$$+ \varepsilon c_3[p_1(p_1^2 - 3q_1^2)(p_2^2 - q_2^2) - 2q_1 q_2 p_2(q_1^2 - 3p_1^2)], \qquad (4.11)$$

while for $v_2^0 = -3$ the second term in the square bracket has opposite sign. *We shall assume that the constants c_1, c_2, c_3 are all different from zero.* (This condition holds generically for Hamiltonians of class C^4.) Without loss of generality one can then take $c_3 = 1$.

Making use of the same Poincaré section as in the previous two examples, condition (4.2) becomes

$$q_1 = q_2 = 0, \quad 4c_1 p_1^3 + 3p_1^3 p_2^2 = 2\lambda p_1,$$
$$4c_2 p_2^3 + 2\mu p_2 + 2\varepsilon p_1^3 p_2 = \pm 3\lambda p_2. \qquad (4.12)$$

We consider only solutions of (4.12) for which $p_1 \neq 0$ *and* $p_2 \neq 0$. Solutions which do not satisfy these two conditions correspond to the periodic solutions which are obtained through Lyapunov's analysis.

Since $\{z_1^0, -z_2^0\}$ and $\{z_1^0, z_2^0\}$ belong to the same orbit of $dz/dt = JdH_0$, we must consider only those solutions of (4.12) for which $p_2 > 0$. From (4.12) one derives

$$-p_2^2(-8c_2 \pm 9\varepsilon p_1) = \pm p_1^2(12c_1 - 4\mu p_1) - 2\mu, \quad p_2 > 0. \tag{4.13}$$

Solutions of (4.13) exist for various ranges of the parameter μ, depending on the relative sign of the coefficients c_1 and c_2, and of whether one chooses the $+$ sign or the $-$ sign in (4.13) (i.e. whether one considers the case $v_2^0 = +3/2v_2^0$ or the case $v_2^0 = -3/2v_1^0$). Since the two equations in (4.13) are interchanged if one changes sign to c_1 and μ, we shall consider only the upper alternative in (4.13), i.e. the case $v_2^0 = 3/2v_1^0$. When $\mu = 0$, $C_1/c_2 < 0$. there are no solution. When $\mu = 0$, $c_1/c_2 > 0$, the equation has two solutions for each value of p_2.

In Figs. 4 and 5 we illustrate the cases $c_1, c_2 > 0$ and $c_1 > 0$, $c_2 < 0$ respectively, for a fixed value $\mu > 0$. The case $\mu < 0$ leads to similar figures and results. At the solutions of (4.13) the matrix $DJ(dN - dH_0)$ has, in addition to the eigenvalue zero, two eigenvalues

$$\sigma^\pm = \pm |p_1 p_2|[(8c_2 + 18c_1)\varepsilon p_1 + 0(\varepsilon^2)]^{1/2}. \tag{4.14}$$

Therefore, on every point of the curve in Fig. 4, with the exception of the points A and B, condition (3.22) is satisfied if ε is sufficiently small. The corresponding periodic solutions of (3.3) are elliptic on the branch for which $(8c_2 + 18c_1)p_1 < 0$, and hyperbolic on the other branch. If $8c_2 + 18c_1 \neq 0$, the same conclusion holds for the points on the curve in Fig. 5. If the parameters in N satisfy the identity $8c_2 = 18c_1$, one has to analyze in (4.14) the terms or order two in ε. One finds that for this value of the parameters the non-zero eigenvalues zre $\sigma^\pm = \pm i3\sqrt{2\varepsilon}|p_1^2 p_2|$. Therefore also for this choice of parameters condition (3.22) is satisfied and, if ε is sufficiently small, for every point on the curve in Fig. 5, with the exception of the points A and B, one has periodic solutions of (3.3) which are elliptic.

Remark. One sees that in both cases, for every fixed $a > 0$, solutions satisfying $|z_1^2| + z_2^2| \leq a^2$ exist only if the parameter μ is in a range $\mu_1(a) \leq \mu \leq \mu_2(a)$. For every $\varepsilon > 0$, if μ is sufficiently large, one is beyond the range of applicability of the
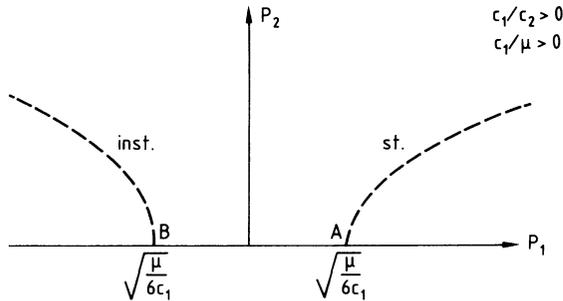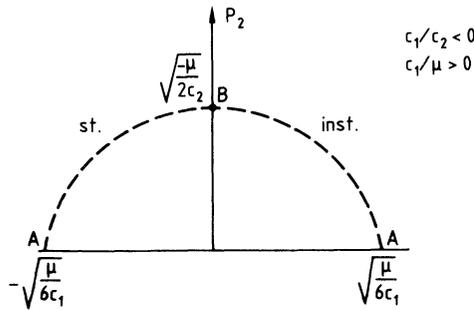

Fig. 4

P_2

$c_1/c_2 < 0$
$c_1/\mu > 0$

$\sqrt{\dfrac{-\mu}{2c_2}}$ B

st.   inst.

A   A

$-\sqrt{\dfrac{\mu}{6c_1}}$ $\sqrt{\dfrac{\mu}{6c_1}}$

Fig. 5

Inverse Function Theorem. Therefore, in the limit of "large detuning" (i.e. when the frequencies are not in resonance) one cannot conclude in general that in a neighborhood of the origin there are periodic solutions of (3.3) with minimal period not approximately equal to either τ_1^0 or τ_2^0, in accordance with Lyapunov's analysis.

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